Stabilization of The Elastic Wave Equation

by

Mourad Bellassoued
Université de Paris Sud,
Mathématiques, Bât. 425, 91405 Orsay Cedex, France.
mourad.bellassoued@math.u-psud.fr

Abstract

In this paper, we study the stabilization problem for the elastic Wave equation in a bounded domain in two different situations. The boundary stabilization is considered with dissipative boundary condition are introduced and the internal stabilization problem. Providing regular initial data we prove, without any assumption on the dynamics, that the energy decay at last the logarithm. In order to prove that result we bound from below the spectrum of the infinitesimal generator of associated semi-group.
Key words. energy decay, Elastic wave equation.
AMS Subject Classification. 35 S 15, 35 P 15, 73 C 02.

1 Introduction and Main Results

We are mainly interested in the decaying mode of the energy (stabilization) of the solution of the initial boundary value problem in a connected and compact manifold with compact boundary for the elastic wave equation as time tends to infinity. The goal of this paper is to study the stabilization of the elastic wave equations. We will consider both the boundary stabilization problems with a boundary damping term supported in the boundary or the internal stabilization with a damping term supported in the interior of the domain. It is well known (see Lagnese [L]) that the energy decreases to zero when the dissipation is on a non empty set. Since the works of Bardos-Lebeau-Rauch [BLR], we have a necessary and sufficient condition

which guarantee an exponentially decay rate of the energy for any initial data for general second order scalar boundary value problems. Since singularities propagate along generalized bicharacteristic rays, this can be done whenever a bicharacteristic hits the part which can be controlled. Taylor [T1] gives a rigorous treatment of the singularity for the elastic wave equation with Neumann boundary condition, he prove there are three types of rays that carry singularities. The first types are classical rays reflecting at the boundary according to the lows of geometrical optics. The third type of rays lie on the boundary and singularities propagate along them with a slower propagation speed $C_R > 0$ (the Rayleigh speed). In the present paper we show that even if the geometrical control condition is not fulfilled (i.e. without any assumption on the dynamics) then the energy decays with respect to time at least as fast as the inverse of the logarithm, providing the initial data belong to the domain of A^k (A stays for the infinitesimal generator of the evolution equation). In order to prove that result we bound from below the spectrum of A. This bound is obtained by using a Carleman type estimate for the resolvent of A.
Let (M, g) be a Riemannian, compact manifold with smooth and compact boundary ∂M. We set

$$L(x, D) = \mu\Delta + (\mu + \lambda)\nabla(\operatorname{div} .) = \Delta_e \tag{1.1}$$

where μ and λ a real constant satisfied $2\mu + 3\lambda > 0$, and Δ, ∇ and div are the Laplacian, gradian and divergence operator associated to metric g. We consider two classical examples of the elastic wave equation. The first damped by a boundary scalar velocity feedback $a_0 \in C^\infty(\partial M)$ and $a_0 \geq 0$

$$\begin{cases} \partial_t^2 u - L(x, D)u = 0 & \textit{in } M \times \mathbb{R} \\ u(x, 0) = u_0, \quad \partial_t u(x, 0) = u_1 & \textit{in } M \\ Nu(x, t) + a_0(x)\partial_t u = 0 & \textit{on } \partial M \times \mathbb{R} \end{cases} \tag{1.2}$$

The second damped system by a internal matrix feedback $a \in C^\infty(M)$ satisfies for any $z \in \mathbb{C}$, $a(x)z\bar{z} \geq 0$ and the set $\left\{x \in M;\ a(x)z\bar{z} \geq \delta|z|^2\right\}$ is non empty

$$\begin{cases} \partial_t^2 u - L(x, D)u + a(x)\partial_t u = 0 & \textit{in } M \times \mathbb{R} \\ u(x, 0) = u_0, \quad \partial_t u(x, 0) = u_1 & \textit{in } M \\ Bu(x, t) = 0 & \textit{on } \partial M \times \mathbb{R} \end{cases} \tag{1.3}$$

Here the boundary operator is of the form

$$B(x, D)u = \begin{cases} u & (\textit{Dirichlet Condition}) \\ Nu = \sigma(u).n & (\textit{Neumann Condition}) \end{cases} \tag{1.4}$$

Where $n(x)$ is the unit outer normal to M at $x \in \partial M$. In the above

$$\sigma(u) = \lambda(\operatorname{div} u)Id + 2\mu\varepsilon(u) \tag{1.5}$$

is the stress tensor where $\varepsilon(u) = (\nabla u + {}^t\nabla u)$ the strain tensor.
For $(u_0, u_1) \in H = H^1 \oplus L^2$ we define the energy for the solution of (1.2) or (1.3) as

$$E(u,t) = \frac{1}{2}\int_M |\partial_t u|^2 + \lambda|\operatorname{div} u|^2 + \mu\sum|\varepsilon_{ij}(u)|^2 dx \tag{1.6}$$

and let

$$A_0 = \begin{pmatrix} 0 & -iId \\ -i\Delta_e & 0 \end{pmatrix}. \tag{1.7}$$

Then the system (1.3) is transformed into

$$\frac{dv(t)}{dt} = iA_0 v(t), \quad v(0) = v_0, \quad v = {}^t(u(t,x), \partial_t u(t,x)) \tag{1.8}$$

The domain of A_0 is defined by

$$D(A_0) = \Big\{u = (u_0, u_1) \in H / A_0 u \in H, \;\; N(u_0) + a_0 u_1 = 0 \; in \; \partial M\Big\}. \tag{1.9}$$

Then, A_0 is a closed operator in H; furthermore, A_0 is skew self-adjoint in H and the immersion $D(A_0) \hookrightarrow H$ is compact. Furthermore $Imz < 0$ is in the set resolvent of A_0 and the spectrum $Sp(A_0)$ of A_0 is discrete.
We can see easily by integration by parts

$$E(u,t) - E(u,0) = -\int_0^t \int_{\partial M} a_0(x)|\partial_s u(x,s)|^2 d\sigma ds \tag{1.10}$$

Now we can able to state our first results

Theorem 1.1 *There exist $C > 0$ such that if $z \in Sp(A_0)\backslash\{0\}$ then we have $Im(z) > e^{-C|Rez|}$ and for $Imz \leq e^{-C|Rez|}$, $|z| > 1$ we have*

$$||(A_0 - z)^{-1}||_{\mathcal{L}(H,H)} \leq Ce^{C|Rez|}. \tag{1.11}$$

Theorem 1.2 *For any $k > 0$ There exist $C > 0$ such that for any initial data $(u_0, u_1) \in D(A_0^k)$ the solution $u(t,x)$, of (1.2) satisfy*

$$E(u,t) \leq \frac{C}{Ln(2+t)^{2k}}\Big|\Big|(u_0, u_1)\Big|\Big|^2_{D(A_0^k)}. \tag{1.12}$$

Let us introduce the operator A defined as

$$A = \begin{pmatrix} 0 & -iId \\ -i\Delta_e & ia \end{pmatrix} \tag{1.13}$$

and let $H_B^1 = \Big\{(u_0, u_1) \in H^1; \; Bu_0 = 0 \; on \; \partial\Omega\Big\}$. The domain of A is defined by $D(A) = (H_B^1 \cap H^2) \oplus L^2$. Then we have $(A - z)$ is bijective from $D(A)$ onto H

for z in $\left\{z \in \mathbb{C}\, ; Imz \notin [0, ||a||_{L^\infty}]\right\}$. The immersion $D(A) \hookrightarrow H$ is compact, the spectrum of A is constituted a sequence z_j such that $Im(z_j) \in [0, ||a||_{L^\infty}]$.
By integration by parts we have

$$E(u,t) - E(u,0) = -\int_0^t \int_M a(x)\partial_s u(x,s)\overline{\partial_s u(x,s)}dxds \tag{1.14}$$

similarly we have the following Theorems

Theorem 1.3 *There exists $C > 0$ such that if $z \in Sp(A)\backslash\{0\}$ then we have $Im(z) > e^{-C|Rez|}$ and for $Imz \leq e^{-C|Rez|}$, $|z| > 1$ we have*

$$||(A-z)^{-1}||_{\mathcal{L}(H,H)} \leq Ce^{C|Rez|}. \tag{1.15}$$

Theorem 1.4 *For any $k > 0$ There exists $C > 0$ such that for any initial data $(u_0, u_1) \in D(A^k)$ the solution $u(t,x)$, of* (1.3) *satisfy*

$$E(u,t) \leq \frac{C}{Ln(2+t)^{2k}}\left|\left|(u_0,u_1)\right|\right|^2_{D(A^k)}. \tag{1.16}$$

Remarks 1.1

i) Theorems 1.1 and 1.3 are the analogous of LEBEAU *[L] and* LEBEAU-ROBBIANO *[LR] results in the case of the scalar wave equation. Our method is different and consist to use the Carleman estimate directly for the stationary operator without using the interpolation inequality.*

ii) Theorems 1.2 and 1.4 follows from Theorem 1.1 and 1.3, this is proved in the general case by BURQ *(see Theorem 3 of [Bu])*

iii) Theorems 1.1 and 1.3 are optimal if we do not assume condition on the dynamic. We can even show that Theorem 0.1 is optimal under reasonable geometric framework, and the damped term $a(x)$ is positive on the whole domain M. Indeed, in this case the Railegh's rays on the boundary never hits the damped term (see KAWASHITA *[K])*

This paper is organized as follow, in section 2 we gives a Carleman inequality adapted to our case, in section 3 we prove our results and the prove of Carleman estimate is in the section 4.

Finally I would like to thank Professor L.Robbiano for useful discussion during the preparation of this work.

2 Carleman Estimates

To prove Theorem 1.1 and Theorem 1.3, we use Carleman estimates to obtain information about the resolvent in bounded domain. The cost is to use phases functions satisfying Hörmander's (see [H]) assumption and thus growing fast.
The point is to show estimations in a bounded domain Ω for the solutions of

$$(2.1)\qquad \left\{ \begin{array}{ll} P(x,D)u = f & in\ \Omega \\ B(x,D)u = g & on\ \partial\Omega \end{array} \right.$$

where $P(x,D)$ the differential operator with principal symbol

$$(2.2)\qquad P(x,\xi) = \mu^t\xi.\xi + (\mu+\lambda)\xi^t\xi - \tau^2 Id$$

and the boundary operator

$$(2.3)\qquad B(x,D)u = \left\{ \begin{array}{ll} u & (Dirichlet\ Condition) \\ \sigma(u).n + ia_0\tau u & (Neumann\ Condition) \end{array} \right.$$

Let $\varphi(x)$ a real function in $C^\infty(\mathbb{R}^n)$ we define the operator

$$(2.4)\qquad P(x,D,\tau) = e^{\tau\varphi}P(x,D)e^{-\tau\varphi}$$

with principal symbol given by

$$(2.5)\qquad P(x,\xi,\tau) = P(x,\xi + i\tau\varphi').$$

We introduce the scalar partial differential operator

$$(2.6)\qquad a_\gamma(x,D) = \Delta + \frac{\tau^2}{\gamma}, \qquad \gamma \in \{\mu, 2\mu+\lambda\},$$

and we define $a_\gamma(x,D,\tau)$ by $e^{\tau\varphi}a_\gamma(x,D)e^{-\tau\varphi}$ with principal symbol

$$(2.7)\qquad a_\gamma(x,\xi,\tau) = a_\gamma(x,\xi+i\tau\varphi').$$

We assume that φ satisfies Hörmander's assumption for the operator a_γ

$$(2.8)\ \exists\ C\ \forall x \in M,\ \forall\,\xi \in T^*M\backslash\{0\};\quad a_\gamma(x,\xi;\tau) = 0 \Longrightarrow \Big\{Rea_\gamma; Ima_\gamma\Big\} > C,$$

where $\{,\}$ denote the Poisson brackets, Furthermore we assume that

$$(2.9)\qquad \nabla\varphi \neq 0, \quad in\ \overline{M} \quad and \quad \partial_n\varphi \neq 0 \quad on\ \partial M.$$

Finally we set $B(x,D,\tau) = e^{\tau\varphi}B(x,D)e^{-\tau\varphi}$. Then we have the following Carleman type estimates

Proposition 2.1 *Let φ satisfy* (2.8) *and* (2.9) *there exists $C>0$, and τ_0 such that for any $u \in C^\infty(\overline{M})$ we have*

$$(2.10)\quad \int_M |P(x,D,\tau)u|^2 + \tau\int_{\partial M}(\tau^2|u|^2+|\nabla u|^2)d\sigma \geq C\tau\int_M(\tau^2|u|^2+|\nabla u|^2)$$

for large $\tau > \tau_0$.

Proposition 2.2 *Furthermore if we assume that $\partial_n\varphi < -C_0$ (where $C_0>0$ large constant fixed in section 4) in $\Sigma \subset \partial M$ and $Bu = g$ on Σ, then there exist $C>0$ and τ_0 such that for any $u \in C^\infty(\overline{M})$ we have*

$$(2.11)\quad \int_M |P(x,D,\tau)u|^2 + \tau\int_{\partial M\setminus\Sigma}\tau^2|u|^2 + \tau\int_\Sigma |g|^2 d\sigma + \tau\int_{\partial M\setminus\Sigma}|B(x,D,\tau)u|^2 d\sigma \geq C\tau\int_M(\tau^2|u|^2+|\nabla u|^2)$$

for large enough $\tau > \tau_0$.

2.1 Construction and properties of the phases functions.

The purpose of this section is to construct two phases φ_1, φ_2 which satisfy the Hörmander's assumption, excepting in a finite number of balls, such that on a ball where one of them do not satisfies these conditions the second does and is strictly greater.

Proposition 2.3 *(see* Burq *[Bu] and [Be]) Let Σ_1 and Σ_2 two non empty parts of the boundary ∂M such that $\Sigma_1 \cup \Sigma_2 = \partial M$. There exist two functions $\psi_{1,2} \in C^\infty(\bar{M})$ satisfying $\partial_n\psi_{i|\Sigma_1} < 0$ and $\partial_n\psi_{i|\Sigma_2} > 0$, having only no degenerate critical points, such that when $\nabla\psi_i = 0$ then $\nabla\psi_{i+1} \neq 0$ and $\psi_{i+1} > \psi_i$ (where $\psi_3 = \psi_1$).*

Then there exist a finite number of points $x_{ij} \in \Omega$; $i = 1,2$ $j = 1,...,N_i$ and $\varepsilon > 0$ such that $B(x_{ij},2\varepsilon) \subset \Omega$, $B(x_{1j},2\varepsilon) \cap B(x_{2j},2\varepsilon) = \emptyset$ and $\psi_{i+1} > \psi_i$ (where $\psi_3 = \psi_1$). Denote $\Omega_i = \Omega \cap (\cup_j B(x_{ij},\varepsilon))^c$, and we take $\varphi_i = e^{\beta\psi_i}$ then for large β the phases φ_i satisfy Hörmander assumption.

3 Proof of Mains Results.

3.1 Proof of Theorem 1.3

We prove here the existence and estimation of the norm of the resolvent $(A-z)^{-1}$ for $z \in \left\{z \in \mathbb{C}; 0 < Im(z) < \frac{1}{C}e^{-C|Rez|};\ |z| > 1\right\}$.

Let B_{4r} a ball of radius $4r > 0$, in M such that $a(x) > 0$ in B_{4r} we note $\Omega_0 = M \backslash \overline{B_r}$. Let $f = (f_0, f_1) \in H$ and $u = (u_0, u_1) \in D(A)$ such that $(A - z)u = f$ then we have

$$\left\{ \begin{array}{l} -zu_0 - iu_1 = f_0 \\ -i\Delta_e u_0 + ia(x)u_1 - zu_1 = f_1 \end{array} \right. \tag{3.1}$$

then we get

$$\left\{ \begin{array}{l} (\Delta_e + z^2 - iaz)u_0 = g_0 \\ u_1 = if_0 + izu_0. \end{array} \right. \tag{3.2}$$

where $g_0 = if_1 + iaf_0 - zf_0$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ such that χ supported in B_{2r}^c equal to 1 in B_{3r}^c, denote $v_0 = \chi u_0$, then we get

$$(\Delta_e + z^2 - iaz)v_0 = \chi g_0 + [\Delta_e, \chi]u_0 := \widetilde{g}_0 \tag{3.3}$$

and $v_0 \equiv 0$ on ∂B_r. Now let $\Sigma_1 = \partial M$ and $\Sigma_2 = \partial B_r$ thus $\partial\Omega_0 = \Sigma_1 \cup \Sigma_2$. Let χ_i, $i = 1, 2$, two cutoff functions equal to 1 in $\left(\bigcup_j B(x_{ij}, 2\varepsilon)\right)^c$ and supported in $\left(\bigcup_j B(x_{ij}, \varepsilon)\right)^c$. We applique Proposition 2.2 to the function $e^{\tau\varphi_i}\chi_i u_0$ in a domain $\Omega_i = \Omega_0 \cap \left(\bigcup_j B(x_{ij}, \varepsilon)\right)^c$ we get, for $\tau^2 = Re(z^2)$, and

$$g = [\Delta_e, \chi]e^{\tau\varphi}\chi v_0 + \chi\widetilde{g_0} - iIm(z^2)v_0 \tag{3.4}$$

the following estimate

$$\int_{\Omega_i} e^{2\tau\varphi_i}|g|^2 \geq C\tau \int_{\Omega_i} (\tau^2|\chi_i v_0|^2 + |\nabla(\chi_i v_0)|^2)e^{2\tau\varphi_i}. \tag{3.5}$$

Using the properties of the phases $\varphi_i > \varphi_{i+1}$ in $\left(\bigcup_j B(x_{i,j+1}, 2\varepsilon)\right)$ then we have for large τ

$$\begin{aligned} &\int_{\Omega_0} (e^{2\tau\varphi_1} + e^{2\tau\varphi_2})(|\widetilde{g}|^2 + \\ &\qquad +2|Imz||Rez||v_0|^2) \geq C\tau \int_{\Omega_0} (\tau^2|v_0|^2 + |\nabla v_0|^2)(e^{2\tau\varphi_1} + e^{2\tau\varphi_2}), \end{aligned} \tag{3.6}$$

and for $|Imz| \leq C$ we can have

$$\begin{aligned} \int_M |u_0|^2 + |\nabla u_0|^2 &\leq Ce^{C\tau}\Big(\int_M (|g|^2 + \int_{B_{4r}} |u_0|^2 + \int_{\Omega_0} |[\Delta_e, \chi]u_0|^2\Big) \\ &\leq Ce^{C\tau}\Big(|g|^2 + \int_M a(x)u_0.\overline{u}_0 + \int_{\Omega_0} |[\Delta_e, \chi]u_0|^2\Big) \end{aligned} \tag{3.7}$$

We set $\widetilde{\chi}$ a cutoff function equal to 1 in a neighborhood of B_{3r} and supported in B_{4r} then we have

$$(-1 + \Delta_e)\chi u_0 = \widetilde{\chi}(z^2 - 1 + iazu_0) + \widetilde{\chi}g_0 + [\Delta_e, \widetilde{\chi}]u_0 \tag{3.8}$$

and hence we get

$$
\begin{aligned}
(3.9) \qquad ||u_0||^2_{H^1(B_{3r})} &\le C||(-1+\Delta_e)\widetilde{\chi}u_0||^2_{H^{-1}} \\
&\le C\Big(||g_0||^2_{L^2} + (1+|z|^2)^2||u_0||^2_{L^2(B_{4r})}\Big).
\end{aligned}
$$

On the other hand integrating by parts gives

$$
\begin{aligned}
(3.10) \qquad \int_M g_0\overline{u}_0 &= \int_M (\Delta_e + z^2 - iaz)u_0\overline{u}_0 \\
&= \int_M z^2|u_0|^2 + \int_M E(u_0,\overline{u}_0) - 2iz\int_M a(x)u_0\overline{u}_0
\end{aligned}
$$

This implies

$$
(3.11) \qquad \tau\int_M a(x)u_0\overline{u}_0 \le 2|Imz||Rez|||u_0||^2_{L^2} + ||g_0||_{L^2}||u_0||_{L^2}
$$

Combining (3.11), (3.9), and (3.7), we obtain

$$
(3.12) \qquad ||u_0||^2_{H^1_0} \le Ce^{C\tau}\Big(||g_0||^2_{L^2} + ||g_0||_{L^2}||u_0||_{L^2} + |Imz||Rez|||u_0||^2_{L^2}\Big)
$$

these imply that for $|Imz| \le \frac{1}{2C}e^{-C\tau}$

$$
(3.13) \qquad ||u_0||^2_{H^1_0} \le Ce^{C\tau}\Big(||f_0||^2_{H^1_0} + ||f_1||^2_{L^2}\Big).
$$

Using (3.2) we can get also

$$
(3.14) \qquad ||u_1||^2_{L^2} \le Ce^{C\tau}\Big(||f_0||^2_{H^1_0} + ||f_1||^2_{L^2}\Big).
$$

Which completes the proof.

3.2 Proof of Theorem 1.1

Let $f = (f_0, f_1) \in H$, $u \in D(A_0)$ then $u = (u_0, u_1)$ such that $N(u_0) + a_0u_1 = 0$ on ∂M. The equation $(A_0 - z)u = f$ is equivalent to

$$
(3.15) \qquad \begin{cases} -zu_0 - iu_1 = f_0 \\ -i\Delta_e u_0 - zu_1 = f_1 & in \;\; M \\ N(u_0) + a_0u_1 = 0 & on \;\; \partial M \end{cases}
$$

then we have

$$
(3.16) \qquad \begin{cases} (\Delta_e + z^2)u_0 = \widetilde{g}_1 & in \;\; M \\ N(u_0) + a_0u_0 = \widetilde{g}_0 & on \;\; \partial M \\ u_1 = if_0 + izu_0. \end{cases}
$$

where $\widetilde{g}_0 = -ia_0 f_0$. Let $\Sigma_2 \subset \Big\{x \in \partial M; a_0(x) > \delta\Big\}$ and $\Sigma_1 = \partial M \backslash \Sigma_2$, and χ_i, $i = 1, 2$, two cutoff functions equal to 1 in $\Big(\bigcup_j B(x_{ij}, 2\varepsilon)\Big)^c$ and supported in $\Big(\bigcup_j B(x_{ij}, \varepsilon)\Big)^c$. We applique Proposition 2.2 to the function $e^{\tau\varphi_i}\chi_i u_0$ in a domain $\Omega_i = M \cap \Big(\bigcup_j B(x_{ij}, \varepsilon)\Big)$ we get for $\tau^2 = Re(z^2)$ and

$$(3.17) \qquad \begin{cases} g_i = [\Delta_e, \chi_i]e^{\tau\varphi_i}\chi_i u_0 + \chi_i \widetilde{g}_1 - iIm(z^2)u_0 \\ \\ N(u_0) + ia_0\tau u_0 = \widetilde{g}_0 + Im(z)a_0 u_0 := h_0 \quad on \;\; \partial M \end{cases}$$

the following inequality

$$(3.18) \int_{\Omega_i} (e^{\tau\varphi_1} + e^{\tau\varphi_2})(|\widetilde{g}_1|^2 + |Im(z^2)u_0|^2) + \tau \int_{\Sigma_1} (e^{\tau\varphi_1} + e^{\tau\varphi_2})|h_0|^2 +$$
$$+ \tau \int_{\Sigma_2} \tau^2|u_0|^2(e^{\tau\varphi_1} + e^{\tau\varphi_2}) \geq C\tau \int_{\Omega_i} (\tau^2|u_0|^2 + |\nabla u_0|^2)(e^{\tau\varphi_1} + e^{\tau\varphi_2}).$$

Using the properties of the phases $\varphi_i > \varphi_{i+1}$ in $\Big(\bigcup_j B(x_{i,j+1}, 2\varepsilon)\Big)$ and $Bu_0 + ia_0\tau u_0 = \widetilde{g}_0 + Im(z)a_0 u_0$ we have for large τ

$$(3.19) \qquad \begin{aligned} \int_M |u_0|^2 + |\nabla u_0|^2 &\leq Ce^{C\tau}\Big(||\widetilde{g}_1||^2 + ||f_0||^2 + \int_{\Sigma_2} |u_0|^2\Big) \\ &\leq Ce^{C\tau}\Big(||\widetilde{g}_1||^2_{L^2} + ||f_0||^2_{H^1} + \int_{\partial M} a_0(x)|u_0|^2\Big) \end{aligned}$$

On the other hand, by integration by part we get

$$(3.20) \quad \begin{aligned} \int_M \widetilde{g}_1\overline{u}_0 &= \int_M (\Delta_e + z^2)u_0\overline{u}_0 \\ &= z^2\int_M |u_0|^2 + \int_M E(u_0, \overline{u}_0) + \int_{\partial M} Nu_0\overline{u}_0 + z^2||u_0||^2_{L^2} + \\ &\quad + \int_M E(u_0, \overline{u}_0) - iz\int_{\partial M} a_0(x)|u_0|^2 + \int_{\partial M} \widetilde{g}_0.\overline{u}_0. \end{aligned}$$

Then we have

$$(3.21) \qquad \tau\int_{\partial M} a_0(x)|u_0|^2 d\sigma \leq \Big(||\widetilde{g}_1||_{L^2}||u_0||_{L^2} + 2|Imz||Rez|||u_0||^2_{L^2} +$$
$$+||g_0||_{H^1}||u_0||_{H^1}\Big).$$

Combining (3.21) with(3.19) we obtain

$$||u_0||^2_{H^1} \leq Ce^{C\tau}\Big(||\widetilde{g}_1||^2_{L^2} + ||f_0||^2_{H^1} + ||\widetilde{g}_1||_{L^2}||u_0||_{L^2} + ||g_0||_{H^1}||u_0||_{H^1} + 2|Imz||Rez|||u_0||^2_{L^2}\Big).$$
(3.22)

Now assume that $|Imz| \leq \frac{1}{C}e^{-C\tau}$ then we have by the previous estimate

$$||u_0||^2_{H^1} \leq Ce^{C\tau}\Big(||f_0||^2_{H^1} + ||f_1||^2_{L^2}\Big). \tag{3.23}$$

Using (3.16) we can get

$$\begin{aligned} ||u_1||^2_{L^2} &\leq ||f_1||^2_{L^2} + |z|^2||u_0||^2_{L^2} \\ &\leq Ce^{C\tau}||f||^2_H \end{aligned} \tag{3.24}$$

these imply that

$$||u||^2 \leq Ce^{C|Rez|}||(A_0 - z)||^2 \tag{3.25}$$

then $(A_0 - z)$ is injection then bijective in $D(A_0)$ and we get

$$||(A_0 - z)^{-1}||_{\mathcal{L}(H,H)} \leq Ce^{-C|Rez|} \tag{3.26}$$

for any $z \in \Big\{z \in \mathbb{C}, \quad |Imz| < \frac{1}{C}e^{-C|Rez|}; \quad |Rez| > 1\Big\}$. The proof is complete.

4 Proof of Carleman Estimates

This section is devoted to prove estimates of Carleman's type near the boundary for solutions to boundary value problem of the form

$$\begin{cases} P(x,D)u = f & in \;\; \Omega_0 \\ B(x,D)u = g & on \;\; \partial\Omega_0 \end{cases} \tag{4.1}$$

Where $P(x,D)$ a partial differential operator with principal symbol given by

$$P(x,\xi) = \mu|\xi|^2 + (\mu+\lambda)\xi^t\xi - \tau^2 Id \tag{4.2}$$

and $B(x,D)u = Nu + ia_0\tau u$. For Dirichlet condition or $Bu = Nu$ the proposition 2.2 is proved in [Be] (see Theorem 0.3)

4.1 Reduction of the Problems.

4.1.1 Reduction of the Laplacian.

Let Ω_0 be a bounded smooth domain of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class C^∞. In a neighborhood of a given $x_0 \in \partial\Omega_0$ we denote by $x = (x', x_n)$ the system of normal geodesic coordinates where $x' \in \partial\Omega_0$ and $x_n \in \mathbb{R}$ are characterized by

$$|x_n| = dist(x, \partial\Omega_0); \quad \Omega_0 = \{x_n > 0\}; \quad dist(x', x) = dist(x, \partial\Omega_0). \tag{4.3}$$

In this system of coordinates there exist $\ell(x,\xi)$ such that

$$\ell(x,\xi)=\ell_0(x)\xi_n+\ell_1(x,\xi') \tag{4.4}$$

and the principal symbol of Laplace operator takes the form

$$\widetilde{a}={}^t\ell(x,\xi)\ell(x,\xi)=\xi_n^2+r(x,\xi') \tag{4.5}$$

where $r(x,\xi')$ is a quadratic form, such that there exist $C>0$

$$r(x,\xi')\geq C|\xi'|^2, \qquad for\ any\ x\in K,\ \ \xi'\in T^*(\partial\Omega_0) \tag{4.6}$$

where K is a fixed compact in $\overline{\Omega}_0$ finally we have

$${}^t\ell_1\ell_1=r(x,\xi'),\ \ {}^t\ell_0\ell_1=0 \tag{4.7}$$

Let $\varphi(x)$ be a $C^\infty(\mathbb{R}^n)$ function with values in $\mathbb{R}$, defined in a neighborhood of K. We define the operator

$$a(x,D,\tau)=e^{\tau\varphi}\widetilde{a}(x,D)e^{-\tau\varphi}:=op(a) \tag{4.8}$$

Denote by

$$a(x,\xi,\tau)=a(x,\xi+i\tau\varphi'). \tag{4.9}$$

the principal symbol of the operator, and we set

$$op(\widetilde{q}_2)=\frac{1}{2}(op(a)+op(a)^*); \qquad op(\widetilde{q}_1)=\frac{1}{2i}(op(a)-op(a)^*), \tag{4.10}$$

its real and imaginary part. Then we have

$$\left\{\begin{array}{l} op(a)=op(\widetilde{q}_2)+iop(\widetilde{q}_1) \\ \widetilde{q}_2=\xi_n^2+q_2(x,\xi',\tau); \qquad \widetilde{q}_1=2i\tau\xi_n\varphi'_{x_n}+2q_1(x,\xi',\tau). \end{array}\right. \tag{4.11}$$

Where q_1 and q_2 a tangential symbols of order respectively 1 and 2 given by

$$\left\{\begin{array}{l} q_2(x,\xi,\tau)=r(x,\xi')-\tau^2{\varphi'_{x_n}}^2-\tau^2r(x,\varphi'_{x'}) \\ q_1(x,\xi,\tau)=\widetilde{r}(x,\xi',\tau\varphi'_{x'}) \end{array}\right. \tag{4.12}$$

and $\widetilde{r}(x,\xi',\eta')$ the bilinear form attached to the quadratic form $r(x,\xi')$.

4.1.2 Reduction of the elasticity system

In the system of normal geodesic coordinates the principal symbol of elasticity operator can be written as

$$L(x,\xi)=\mu^t\ell(x,\xi)\ell(x,\xi)Id+(\lambda+\mu)\ell(x,\xi)^t\ell(x,\xi) \tag{4.13}$$

where $\ell(x,\xi)$ defined by (4.5) and $\ell(x,\xi)^t\ell(x,\xi)$ the orthogonal projection onto the space spanned by $\ell(x,\xi)$. The principal symbol of $P(x,D,\tau)=e^{\tau\varphi}L(x,D)e^{-\tau\varphi}-\tau^2 Id$ is given by

$$P(x,\xi,\tau)=\sum_{j=0}^{2}P_{2-j}(x,\xi',\tau)\xi_n^j \tag{4.14}$$

where $P_j(x,\xi',\tau)$ a tangential symbols in $S^j(\mathbb{R}^n\times\mathbb{R}^{n-1})$ and defined by

$$\begin{cases} P_0=\mu Id+(\mu+\lambda)\ell_0^t\ell_0 \\ \\ P_1=2i\tau\varphi'_{x_n}P_0+(\mu+\lambda)(\ell_0^t\ell_1+\ell_1^t\ell_0) \\ \\ P_2=\mu\ell_1^t\ell_1+i\tau\varphi'_{x_n}P_1(x,\xi)-\tau^2\varphi'^2_{x_n}P_0+(\mu+\lambda)\ell_1^t\ell_1-\tau^2 Id. \end{cases} \tag{4.15}$$

For fixed $(x,\xi')\in T^*(\partial\Omega_0)$ let $\alpha(x,\xi',\tau)\in\mathbb{C}$ such that

$$a(x,\xi,\tau)=(\xi_n+i\tau\varphi'_{x_n}+i\alpha)(\xi_n+i\tau\varphi'_{x_n}-i\alpha) \tag{4.16}$$

then we have also by

$$r(x,\xi'+i\tau\varphi'_{x'})=-(i\alpha)^2={}^t\ell(x,\xi'+i\tau\varphi'_{x'})\ell(x,\xi'+i\tau\varphi'_{x'}). \tag{4.17}$$

the determinant of $P(x,\xi,\tau)$ is given by

$$det(P(x,\xi,\tau))=\mu^{n-1}(2\mu+\lambda)a_\mu(x,\xi,\tau)a_{2\mu+\lambda}(x,\xi,\tau). \tag{4.18}$$

where $a_\gamma(x,\xi,\tau)=a(x,\xi,\tau)-\frac{\tau^2}{\gamma}$.

4.2 Study of the eigenvalues

The proof of Carleman's estimate rely on a cutting argument based on the nature of the roots with respect ξ_n of $a_\gamma(x,\xi',\xi_n,\tau)$.
Let use now introduce the following micro-local regions:

$$E^+=\left\{(x,\xi',\tau)\in K\times S^{n-1};\ q_2-\frac{\tau^2}{\mu}+\frac{q_1^2}{\tau^2{\varphi'_{x_n}}^2}>0\right\}, \tag{i}$$

$$Z_\gamma=\left\{(x,\xi',\tau)\in K\times S^{n-1}; q_2-\frac{\tau^2}{\gamma}+\frac{q_1^2}{\tau^2{\varphi'_{x_n}}^2}=0\right\}, \tag{ii}$$

$$E^-=\left\{(x,\xi',\tau)\in K\times S^{n-1};\ q_2-\frac{\tau^2}{2\mu+\lambda}+\frac{q_1^2}{\tau^2{\varphi'_{x_n}}^2}<0\right\}, \tag{iii}$$

(iv) $M = \left\{(x,\xi',\tau) \in K \times S^{n-1};\ q_2 - \frac{\tau^2}{\mu} + \frac{q_1^2}{\tau^2 {\varphi'_{x_n}}^2} < 0 < q_2 - \frac{\tau^2}{2\mu+\lambda} + \frac{q_1^2}{\tau^2 {\varphi'_{x_n}}^2}\right\}.$

And for fixed (x,ξ',τ) let $\alpha_\gamma(x,\xi',\tau) \in \mathbb{C}$ such that

$$\begin{aligned} a_\gamma(x,\xi,\tau) &= a(x,\xi,\tau) - \frac{\tau^2}{\gamma} \\ &= (\xi_n + i\tau\varphi'_{x_n} + i\alpha_\gamma)(\xi_n + i\tau\varphi'_{x_n} - i\alpha_\gamma). \end{aligned} \tag{4.19}$$

Taking into account (4.16) (4.23) we have

$$(i\alpha_\gamma)^2 = (i\alpha)^2 + \frac{\tau^2}{\gamma}. \tag{4.20}$$

then we get by (4.12) and (4.17)

$$\alpha_\gamma^2 = \tau^2 {\varphi'_{x_n}}^2 + q_2 - \frac{\tau^2}{\gamma} + 2i\tau q_1 \tag{4.21}$$

For a fixed (x,ξ',τ) decompose $a_\gamma(x,\xi,\tau)$ as polynomial in ξ_n then we have the following Lemma

Lemma 4.1

1) For any $(x,\xi',\tau) \in E^+$ the roots of a_μ and $a_{2\mu+\lambda}$ denote by $z_1^\pm, z_2^\pm$ satisfying $\pm Im z_k^\pm > 0$.

2) For any $(x,\xi',\tau) \in Z_\gamma$, one of the roots of a_γ is real.

3) For any $(x,\xi',\tau) \in E^-$ the roots of a_γ for $\gamma \in \{\mu, 2\mu+\lambda\}$ has in the upper half-plane if $\varphi'_{x_n} < 0$ and in the lower half-plane if $\varphi'_{x_n} > 0$.

4) For any $(x,\xi',\tau) \in M$ the roots of $a_{2\mu+\lambda}$ satisfying $\pm Im z_2^\pm > 0$ and the roots of a_μ satisfy 3).

Proof . The imaginary parts of the roots of a_γ are

$$-\tau\varphi'_{x_n} \pm Re(\alpha_\gamma) \tag{4.22}$$

but the lines $Re(z) = \pm\tau\varphi'_{x_n}$ can be transformed by $z \to z' = z^2$ in the parabola defined by $Rez' = \tau^2 {\varphi'_{x_n}}^2 - \frac{|Imz'|^2}{4{\varphi'_{x_n}}^2}$. Then the Lemma is proved if we change z' by α_γ^2 and using (4.21) •

4.3 Proof of the Carleman estimate in the region E^+

The purpose of this section is to proof the Carleman estimate in the region E^+. First let us introduce the following norms in the Sobolev spaces $H^k(\Omega)$ and $H^k(\partial\Omega)$

$$(4.23) \qquad ||u||^2_{k,\tau} = \sum_{j=0}^{k} \tau^{2(k-j)} ||u||^2_{H^j(\Omega)} \qquad |u|^2_{k,\tau} = \sum_{j=0}^{k} \tau^{2(k-j)} |u|^2_{H^j(\partial\Omega)}$$

and we set

$$(4.24) \qquad |u|^2_{1,0,\tau} = |u|^2_{1,\tau} + \left|\frac{\partial u}{\partial n}\right|^2_{L^2}.$$

Let $\chi(x,\xi',\tau)$ a homogeneous cutoff function of degree 0 in the region E^+.

Proposition 4.1 *There exist $C_0 > 0$ and $C_1 >$ such that if $\varphi'_{x_n} > C_0$ for any large enough τ we have*

$$(4.25) \quad ||P(x,D,\tau)u||^2 + \tau|B(x,D,\tau)u|^2_{0,\tau} + \\ + ||u||^2_{1,\tau} + |u|^2_{1,\tau} \geq C_1\tau(||op(\chi)u||^2_{1,\tau} + \tau|op(\chi)u|^2_{1,\tau})$$

whenever $u \in C_0^\infty(K)$.

4.3.1 Preliminaries

We consider the boundary operator $B(x,D)u = \sigma(u).n + a_0 i\tau$ and the conjugate operator $e^{\tau\varphi}B(x,D)e^{-\tau\varphi}$ with principal symbol given by

$$(4.26) \qquad B(x,\xi,\tau) = B_0(x)\xi_n + B_1(x,\xi',\tau)$$

where B_0 and B_1 two tangential symbols define by

$$(4.27) \qquad \begin{cases} B_0 = \mu Id + (\mu+\lambda)\ell_0^t\ell_0 \\ \\ B_1 = i\tau\varphi'_{x_n}B_0 + \mu\ell_1^t\ell_0 + \lambda\ell_0^t\ell_1 - a_0\tau Id. \end{cases}$$

Let $u \in C_0^\infty(K)$ denote

$$(4.28) \qquad \widetilde{u} = op(\chi)u \qquad \widetilde{f} = op(P)\widetilde{u}$$

where $op(P)$ the differential operator with principal symbol

$$(4.29) \qquad P(x,\xi,\tau) = P_0(x)\xi_n^2 + P_1(x,\xi',\tau)\xi_n + P_2(x,\xi',\tau)$$

It is easy to see that

$$(4.30) \qquad \begin{cases} op(P)\widetilde{u} = \widetilde{f} \quad in \quad x_n > 0 \\ \\ op(B)\widetilde{u} = \widetilde{g} \quad on \quad x_n = 0 \end{cases}$$

where $\widetilde{f} = op(\chi)f + [op(P); op(\chi)]u$ and $\widetilde{g} = op(\chi)g + [op(B); op(\chi)]u$.
Let us reduce the problem (4.30) to a first order system. Put $v = {}^t(< D', \tau > \tilde{u}, D_n\tilde{u})$ then the system (4.30) is reduced to the following first system

$$(4.31) \qquad \begin{cases} D_n v - op(\mathcal{A})v = F & in \quad x_n > 0 \\ op(\mathcal{B})v = \widetilde{g} & on \quad x_n = 0 \end{cases}$$

Where the principal symbol of $op(\mathcal{A})$ is given by

$$(4.32) \qquad \mathcal{A} = \begin{pmatrix} 0 & - < \xi', \tau > \\ - < \xi', \tau >^{-1} P_0^{-1}P_2 & -P_0^{-1}P_1 \end{pmatrix}$$

and $\mathcal{B}$ the tangential symbol

$$(4.33) \qquad \mathcal{B} = (< \xi', \tau >^{-1} B_1, B_0)$$

with B_0 and B_1 are defined by (2.3) and $F = {}^t(0, \tilde{f})$.
For (x_0, ξ_0', τ_0) fixed in $\mathrm{supp}(\chi)$ then the eigenvalues of the matrix $\mathcal{A}$ are $z_1^\pm = -i\tau\varphi'_{x_n} \pm i\alpha_\mu$ and $z_2^\pm = -i\tau\varphi'_{x_n} \pm i\alpha_{2\mu+\lambda}$ with $\pm Im(z_k^\pm) > 0$.
Denote by $S = (s_1^+, ..., s_n^+, s_1^-, ..., s_n^-)$ a basis of the generalized eigenspace of the matrix $\mathcal{A}(x_0, \xi_0', \tau_0)$ corresponding to eigenvalues with positive or non positive imaginary parts. Then we can extended S as a smooth positively homogeneous function of degree zero in a small conic neighborhood of (x_0, ξ_0', τ_0) . Let $S(x, D_{x'}, \tau)$ the pseudo-differential operator with principal symbol $S(x, \xi', \tau)$. Then by the argument in Taylor [T2] (see also Yamamoto [Y]) there exist a pseudo-differential operator $K(x, D_{x'}, \tau)$ of order -1 such that the boundary value problem (4.35) is reduced to the following

$$(4.34) \qquad \begin{cases} D_n w - op(\mathcal{H})w = \widetilde{F} & in \ x_n > 0 \\ \\ op(\widetilde{\mathcal{B}})w = \widetilde{g}_0 & on \ x_n = 0 \end{cases}$$

Where $w = (I + K)^{-1}S^{-1}v$, $\widetilde{F} = (I + K)^{-1}S^{-1}F$, $\widetilde{\mathcal{B}} = \mathcal{B}S(I + K)^{-1}$
and $op(\mathcal{H})$ a tangential operator of order 1 with principal symbol $\mathcal{H} = diag(\mathcal{H}^+, \mathcal{H}^-)$ such that

$$(4.35) \qquad \pm Im(\mathcal{H}^\pm) \geq C < \xi', \tau > .$$

We can now state the Lemma which will be the key ingredient in the proof of the Proposition 4.4.

Lemma 4.2 *There exist a hermitian matrix* $\mathcal{R}(x, \xi', \tau)$, C_1,$C_2 > 0$ *such that*

$$(4.36) \qquad Im(\mathcal{R}\mathcal{H}) = e(x, \xi', \tau)$$

and we have the following properties

i) $e(x, \xi', \tau) \geq C_1 I \qquad in \ \ \mathrm{supp}(\chi)$

$ii) -\mathcal{R}(x,\xi',\tau) + C_2\widetilde{\mathcal{B}}^*\widetilde{\mathcal{B}} \geq CI \qquad on \ \ \{x_n = 0\} \cap \mathrm{supp}(\chi)$

Proof . Let $\widetilde{\mathcal{B}}$ the principal symbol of $op(\widetilde{\mathcal{B}})$ where $\widetilde{\mathcal{B}} = (\widetilde{\mathcal{B}}^+, \widetilde{\mathcal{B}}^-)$, and $\widetilde{\mathcal{B}}^+ = \widetilde{\mathcal{B}}S^+$ the projection of $\widetilde{\mathcal{B}}$ in eigenspace generated by the vectors of S^+. First we prove $\widetilde{\mathcal{B}}^+$ is an isomorphism. The eigenvalues of $\mathcal{A}$ are $z_1^{\pm} = -i\tau\varphi'_{x_n} \pm i\alpha_\mu$ and $z_2^{\pm} = -i\tau\varphi'_{x_n} \pm i\alpha_{2\mu+\lambda}$ such that $\pm Im(z_k^{\pm}) > 0$ and with multiplicity respectively $(n-1)$ and 1.
Now let $X = (X_1, X_2) \in \mathbb{C}^n \oplus \mathbb{C}^n$ eigenvector of $\mathcal{A}$ associated to z_0 then X satisfy

$$\left\{ \begin{array}{l} <\xi',\tau> X_2 = z_0 X_1 \\ P(z_0).X_1 = 0 \end{array} \right. \tag{4.37}$$

a) Calculus of eigenvector associated to z_1^+:
Denote by $\Big\{\omega_j\Big\}_{j=1,...,n-2}$ a basis of $\Big\{\ell_0, \ell_1\Big\}^\perp$ then we have

$$P(z_1^+)\omega_j^+ = 0 \qquad for \ j \in \Big\{1, ..., n-2\Big\} \tag{4.38}$$

where

$$P(z_1^+) = (\mu+\lambda)(\ell_1 + i\alpha_\mu\ell_0)^t(\ell_1 + i\alpha_\mu\ell_0) \tag{4.39}$$

Now we set the following vector in $\mathbb{C}^n$

$$\omega_{n-1} =<\xi',\tau>^{-1}((i\alpha)^2\ell_0 + i\alpha_\mu\ell_1) \tag{4.40}$$

then we have by a simple calculation

$$P(z_1^+)\omega_{n-1}^+ = 0 \tag{4.41}$$

b) Calculus of eigenvector associated to z_2^+:
We get

$$P(z_2^+) = -\frac{\mu+\lambda}{2\mu+\lambda}Id + (\mu+\lambda)(\ell_1 + i\alpha_{2\mu+\lambda}\ell_0)^t(\ell_1 + i\alpha_{2\mu+\lambda}\ell_0). \tag{4.42}$$

Let now

$$\omega_n^+ = (i\alpha_{2\mu+\lambda})\ell_0 + \ell_1 \tag{4.43}$$

then we have $P(z_2^+)\omega_n^+ = 0$, using $(3,14)$ we denote

$$\left\{ \begin{array}{l} s_j^+ = (\omega_j^+; <\xi',\tau>^{-1} z_1^+\omega_j^+) \quad j \in \Big\{1, ..., n-1\Big\} \\ \\ s_n^+ = (\omega_n^+; <\xi',\tau>^{-1} z_2^+\omega_n^+) \end{array} \right. \tag{4.44}$$

and by $(b_1^+, ..., b_n^+, b_1^-, ..., b_n^-)$ the principal symbol of $\widetilde{\mathcal{B}}$. By elementary calculus we get $\widetilde{\mathcal{B}}^+ = \widetilde{\mathcal{B}}S^+ = (b_1^+, ..., b_n^+)$ where

$$(4.45) \qquad \begin{cases} b_j^+ = (\mu(i\alpha_\mu) - a_0\tau)\omega_j^+ \\ \\ b_{n-1}^+ = [2\mu(i\alpha_\mu) - a_0\tau)](i\alpha)^2\ell_0 + [2\mu(i\alpha)^2 - a_0\tau(i\alpha_\mu) + \tau^2]\ell_1 \\ \\ b_n^+ = [2\mu(i\alpha)^2 - a_0\tau(i\alpha_{2\mu+\lambda}) + \tau^2]\ell_0 + [2\mu(i\alpha_{2\mu+\lambda}) - a_0\tau)]\ell_1 \end{cases}$$

Then we get

$$(4.46) \qquad det(\widetilde{\mathcal{B}}^+) = (\mu(i\alpha_\mu) - a_0\tau)^{n-1} R_0\Big(\frac{\sqrt{\mu}}{\tau}\alpha(x, \xi', \tau)\Big)$$

where R_0 is a function given by

$$(4.47) \qquad \begin{aligned} R_0(s) \;=\; & (1-2s^2)^2 + (\frac{a_0^2}{\mu} - 4s^2)\Big(\sqrt{s^2-1}\sqrt{s^2 - \frac{\mu}{2\mu+\lambda}}\Big) \\ & - \frac{a_0^2}{\mu}s^2 + \frac{ia_0}{\sqrt{\mu}}(1-2s^2)\Big(\sqrt{s^2-1} + \sqrt{s^2 - \frac{\mu}{2\mu+\lambda}}\Big). \end{aligned}$$

We define the characteristic manifold by

$$(4.48) \qquad \Sigma = \Big\{(x', \xi', \tau) \in T^*\partial\Omega, \quad R_0\Big(\frac{\sqrt{\mu}}{\tau}\alpha(x, \xi', \tau)\Big) = 0\Big\}.$$

Therefore $\widetilde{\mathcal{B}}^+$ is elliptic outside Σ.
Let z_0 a root of R_0 such that $\frac{\sqrt{\mu}}{\tau}\alpha(x, \xi', \tau) = z_0$ then we have

$$(4.49) \qquad \begin{aligned} \alpha^2(x, \xi', \tau) \;&=\; \tau^2 {\varphi'_{x_n}}^2 + q_2 + 2i\tau q_1 \\ &=\; \frac{\tau^2}{\mu} z_0^2 \end{aligned}$$

then we have in the region E^+.

$$(4.50) \qquad \frac{\tau^2}{\mu} Rez_0^2 - \tau^2 {\varphi'_{x_n}}^2 - \frac{\tau^2}{\mu} + \frac{(Imz_0^2)^2}{4{\varphi'_{x_n}}^2}\tau^2 > 0.$$

this is impossible if we put $\varphi'_{x_n} > C_0$ for large $C_0 > 0$. Then $E^+ \cap \Sigma = \emptyset$, and $\mathcal{B}^+$ is an isomorphism in the elliptic region E^+.
Now let us search that the symmetrieser $\mathcal{R}$ in the form $diag(Id, \rho Id)$, then we have

$$(4.51) \qquad \begin{aligned} Im(\mathcal{R}\mathcal{H}) \;&=\; diag\Big(Im(\mathcal{H}^+), -\rho Im(\mathcal{H}^-)\Big) \\ &:=\; e(x, \xi', \tau) \end{aligned}$$

where $e(x, \xi', \tau) \geq CId_{2n}$.
Let $w = (w^+, w^-) \in \mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ then we have $\widetilde{\mathcal{B}}w = \widetilde{\mathcal{B}}^+w^+ + \widetilde{\mathcal{B}}^-w^-$ and there exist $C > 0$ such that for any $w \in \mathbb{C}^{2n}$ we get

$$(4.52) \qquad |w^+|^2 \leq C(|w^-|^2 + |\mathcal{B}w|^2)$$

we deduce

$$(4.53)\qquad \begin{aligned} -(\mathcal{R}w,w) &= -|w^+|^2+\rho|w^-|^2 \\ &\geq |w^+|^2+(\rho-2C')|w^-|^2-2C'|\mathcal{B}w|^2. \end{aligned}$$

Then we have the desirable estimation, for large ρ $\bullet$

4.3.2 Proof of Proposition4.4

Let G the function define by

$$(4.54)\qquad G(x_n)=\frac{d}{dx_n}(op(\mathcal{R})w,w)_{L^2(\mathbb{R}^{n-1})}.$$

Using $D_nw-op(\mathcal{H})w=\widetilde{F}$ then we obtain

$$(4.55)\ G(x_n)=-2Im(op(\mathcal{R})op(\mathcal{H})w,w)+2Im(op(\mathcal{R})w,\widetilde{F})+(op(\mathcal{R}'_{x_n})w,w)$$

The integration in the normal direction gives

$$(4.56)\qquad \begin{aligned} (op(\mathcal{R})w,w)_0 &= 2\int_0^\infty Im(op(\mathcal{R})op(\mathcal{H})w,w)dx_n \\ &\quad -2Im\int_0^\infty (op(\mathcal{R})w,\widetilde{F})-\int_0^\infty op(\mathcal{R}'_{x_n})w,w)dx_n \end{aligned}$$

Taking into account Lemma 4.2 i) we have

$$(4.57)\qquad Im(op(\mathcal{R})op(\mathcal{H})w,w)\geq C\tau||w||^2$$

and by (ii) we obtain

$$(4.58)\qquad (op(\mathcal{R})w,w)_0\leq C|\mathcal{B}w|^2-C'|w|^2.$$

Combining (4.62) (4.61) with (4.60) we get

$$(4.59)\qquad C\tau||w||^2+C'|w|_0^2\leq||\widetilde{F}||_0^2+\tau|\mathcal{B}w|^2.$$

this complete the proof of Proposition4.4.

Remarks 4.1 *In the elliptic region E^+, τ can be degenerate to zero. But for $\tau=0$ the problem $L(x,D)u=\mu\Delta u+(\mu+\lambda)\nabla(\operatorname{div}u)$ with Neumann boundary condition $N(u)=\sigma(u).n$ is a coercive elliptic boundary problem, and we can prove the Lemma 4.2 in this case (see Lemma 5.2 of [Be]).*

4.4 Carleman estimate in a non elliptic regions

The purpose of this section is to get the Carleman estimates in the non elliptic region. Precisely we take a cutoff function $\chi_0(x,\xi',\tau)$ homogeneous of degree zero in a neighborhood of the regions $Z_\gamma \cup M$. Our purpose here is to prove the following Lemma.

Lemma 4.3 *There exist $C>0$ such that for any large enough τ we have*

$$||P(x,D,\tau)u||^2 + \tau|u|^2_{1,0,\tau} + ||u||^2_{1,\tau} \geq C\tau||op(\chi_0)u||^2_{1,\tau} \tag{4.60}$$

whenever $u \in C_0^\infty(K)$.
Furthermore if we assume that $\varphi'_{x_n} > C_0$ on $\{x_n=0\}\cap supp\chi_0$ then we have

$$||P(x,D,\tau)u||^2 + ||u||^2_{1,\tau} + |u|^2_{1,\tau} + \tau|B(x,D,\tau)u|^2 \geq C\Big(\tau||op(\chi_0)u||^2_{1,\tau} + \tau|op(\chi_0)u|^2_{1,\tau}\Big) \tag{4.61}$$

whenever $u \in C_0^\infty(K)$.

The first part of the previous Lemma is proved in [Be] (see proposition 3.1). Then we study only the case when the cutoff function supported in a neighborhood of the region Z_γ. For the proof of the second part we need the next Lemma.

Lemma 4.4 *Assume that $\varphi'_{x_n} > C_0$ on $\{x_n=0\}\cap supp\chi_0$ then we have*

$$||P(x,D,\tau)u||^2 + \tau|B(x,D,\tau)u|^2 + ||u||^2_{1,\tau} + |u|^2_{1,\tau} \geq C\tau|op(\chi_0)u|^2_{1,0,\tau} \tag{4.62}$$

whenever $u \in C_0^\infty(K)$.

Let

$$\widetilde{u} = op(\chi_0)u, \quad \widetilde{f} = op(P)\widetilde{u} \tag{4.63}$$

by an argument similar to the one of section 4.3, we have

$$\begin{cases} D_n w - op(\mathcal{H})w = \widetilde{F} & in\ x_n > 0 \\ op(\widetilde{\mathcal{B}})w = \widetilde{g}_0 & on\ x_n = 0 \end{cases} \tag{4.64}$$

Where $w = (I+K)^{-1}S^{-1}v$, $\widetilde{F} = (I+K)^{-1}S^{-1}F$, $\widetilde{\mathcal{B}} = \mathcal{B}S(I+K)^{-1}$
and $op(\mathcal{H})$ is an $n\times n$ square matrix, whose components are tangential pseudo-differential operators of order 1 with principal symbol $\mathcal{H} = diag(\mathcal{H}^+,\mathcal{H}^-)$, such that $\mathcal{H}^-$ satisfy $-Im(\mathcal{H}^-) \geq C <\xi',\tau>$. The proof of Lemma 4.4 is based on the next Lemma

Lemma 4.5 *Let $\mathcal{R} = diag(0,-\rho Id_n)$ then there exist $C>0$ such that*

i) $Im(\mathcal{R}\mathcal{H}) = diag(0, e(x,\xi',\tau))$

ii) $e(x,\xi',\tau) \geq C <\xi',\tau>$ *in* $supp\chi_0$

iii) $-\mathcal{R} + \widetilde{\mathcal{B}}^*\widetilde{\mathcal{B}} \geq C diag(0, Id)$ *on* $\{x_n=0\}\cap supp\chi_0$

4.4.1 Proof of Lemma 4.4

Denote the function define $G(x_n) = \frac{d}{dx_n}\Big(op(\mathcal{R})w, w\Big)_{L^2(\mathbb{R}^{n-1})}$.
Taking into account $D_n w - op(\mathcal{H})w = \widetilde{F}$ then we obtain

(4.65) $G(x_n) = -2Im(op(\mathcal{R})op(\mathcal{H})w, w) + 2Im(op(\mathcal{R})w, \widetilde{F}) + (op(\mathcal{R}'_{x_n})w, w)$

The integration in the normal direction gives

$$\begin{aligned}(4.66)\quad (op(\mathcal{R})w, w)_0 &= 2\int_0^\infty Im(op(\mathcal{R})op(\mathcal{H})w, w)dx_n \\ &\quad -2Im\int_0^\infty (op(\mathcal{R})w, \widetilde{F}) - \int_0^\infty op(\mathcal{R}'_{x_n})w, w)dx_n\end{aligned}$$

Taking into account Lemma 4.5 and Gärding inequality we have for large τ

$$(4.67)\qquad Im(op(\mathcal{R})op(\mathcal{H})w, w) \geq C\tau||w^-||^2$$

and further, for any ε

$$(4.68)\qquad \int_0^\infty |(op(\mathcal{R})w, \widetilde{F})|dx_n \leq \varepsilon C\tau||w^-||^2 + \frac{C_\varepsilon}{\tau}||\widetilde{f}||^2.$$

Apply now Lemma 4.5 we obtain

$$(4.69)\qquad -(op(\mathcal{R})w, w) + C|\widetilde{\mathcal{B}}w|^2 \geq C|w|^2.$$

Combining (4.73) (4.72) with (4.71) we get

$$(4.70)\qquad C\tau||w^-||^2 + C'|w|_0^2 \leq \frac{C}{\tau}||\widetilde{f}||_0^2 + |\widetilde{\mathcal{B}}w|^2.$$

This complete the proof of Lemma 4.4.

4.5 Carleman estimate in the region E^-

The purpose of this section is to proof the Carleman estimate in the region E^-. Let the cutoff function $\chi_1(x, \xi', \tau)$ homogeneous of degree zero in the region E^-, then we have the following Lemma.

Lemma 4.6 *Assume that $\varphi'_{x_n} > 0$ on $supp\chi_1 \cap \{x_n = 0\}$, then there exist $C > 0$ and τ_0 such that for any $\tau \geq \tau_0$ we have*

$$(4.71)\quad ||P(x, D, \tau)u||^2 + ||u||_{1,\tau}^2 + |u|_{1,\tau}^2 \geq C\tau\Big(||op(\chi_1)u||_{1,\tau}^2 + |op(\chi_1)u|_{1,\tau}^2\Big),$$

whenever $u \in C_0^\infty$.

Proof . Let

$$\widetilde{u} = op(\chi_1)u, \qquad \widetilde{f} = op(P)\widetilde{u}, \tag{4.72}$$

by an argument similar to the one of section 4.3, we have

$$\begin{cases} D_n w - op(\mathcal{H}^-)w = \widetilde{F} & in\ x_n > 0 \\ \\ op(\widetilde{\mathcal{B}})w = \widetilde{g}_0 & on\ x_n = 0 \end{cases} \tag{4.73}$$

Where $w = (I+K)^{-1}S^{-1}v$, $\tilde{F} = (I+K)^{-1}S^{-1}F$, $\tilde{\mathcal{B}} = \mathcal{B}S(I+K)^{-1}$
and $op(\mathcal{H}^-)$ is an $n \times n$ square matrix, whose components are tangential pseudo-differential operators of order 1 with principal symbol $\mathcal{H}^-$, such that $Im(\mathcal{H}^-) \geq C < \xi', \tau >$, let $G(x_n) = -\frac{d}{dx_n}(w,w)$, using 4.73 then we have

$$G(x_n) = -2Im(op(\mathcal{H}^-)w,w) + 2Im(w,\widetilde{F}). \tag{4.74}$$

The integration in x_n gives

$$-|w|_0^2 = 2\int_0^\infty -Im(op(\mathcal{H}^-)w,w)dx_n - 2Im\int_0^\infty (w,\tilde{F}). \tag{4.75}$$

By Gärding inequality we have

$$-Im(op(\mathcal{H}^-)w,w) \geq C\tau||w||^2 \tag{4.76}$$

Furthermore, for any ε

$$\int_0^\infty |(w,\widetilde{F})|dx_n \leq \varepsilon C\tau||w||^2 + \frac{C_\varepsilon}{\tau}||\widetilde{f}||^2. \tag{4.77}$$

Combining (4.77) (4.76) with (4.75) we get

$$C\tau||w||^2 + C'|w|_0^2 \leq \frac{C}{\tau}||\widetilde{f}||_0^2 + |\widetilde{\mathcal{B}}w|^2. \tag{4.78}$$

This complete the proof.
Combining Lemma 4.6, Lemma 4.4 and Proposition 4.4 we get Proposition 2.2 for large τ •

References

[BLR] Bardos, G. Lebeau and J. Rauch: *Contrôle et stabilisation dans les problèmes hyperboliques.* Appendix, in Lions, Tome 1, Controlabilite exacte, stabilisation et perturbations des systemes distribués, Masson, RMA 8, 1988.

[B] M. Bellassoued: *Distribution of Resonances and Decay Rate of the Local energy for the Elastic Wave Equation.* Preprint, Universite de Paris-Sud.

[Bu] N. Burq: *Décroissance de l'energie locale de l'equation des ondes pour le problème exterieur et absance de résonances au voisinage du réel*, Acta Math. 180, 1998, 1,pp 1-29.

[H] L. Hörmander: *The analysis of linear partial differential operators*, Springer Verlag Tomes 1,2,3.

[K] M. Kawashita: *On the Local Energy Decay Property for the Elastic Wave equation with Neumann boundary condition*, Duke Math. J., Vol.67, 1992, pp 333-351.

[L] J. Lagnese: *Boundary Stabilization of Linear Elastodynamic systems.* SIAM J. Control and Optimization, Vol 21, 6, 1983, pp 968-984.

[Le] G. Lebeau: *Equation des ondes amorties.* In A. Boutet de Monvel and V. Marchenko, editors, Algebraic and Geometric Methods in Mathematical Physics. Kluwer Academic, the Netherlands, 1996, pp 73-109.

[LR] G. Lebeau and L.Robbiano: *Stabilisation de l'equation des ondes par le bord*, Duke math.J., Vol. 86, 1997, 3, pp 465-491.

[T1] M. Taylor: *Rayleigh waves in linear elasticity as a propagation of singularities phenomenon*, Proceedings of the conference on Partial Diff. Equat. Geo., Marcel Dekker, New York, 1979, pp 273-291.

[T2] M. Taylor: *Reflection of singularities of solution to systems of differential equations*, Comm. Pure App. Math., 29, 1976, pp 1-38.

[Y] K. Yamamoto: *Singularities of solutions to the boundary value problems for elastic and Maxwell's equations*, Japan. J. Math. Vol 14, 1, 1988, pp 119-163.