SUPERSYMMETRY, WITTEN COMPLEX AND ASYMPTOTICS FOR DIRECTIONAL LYAPUNOV EXPONENTS IN $\mathbf{Z}^d$

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ABSTRACT. By using a supersymmetric Gaussian representation, we transform the averaged Green's function for random walks in random potentials into a 2-point correlation function of a correponding lattice field theory. We study the resulting lattice field theory using the Witten Laplacian formulation. We obtain the asymptotics for the directional Lyapunov exponents.

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I. Introduction

Beginning in this paper, we start our quantitative study of ranom walks in random potentials in dimensions $d \geq 1$. In analytical language, which is the language we adopt, this corresponds to the study of

$$H = \Delta + \gamma V, \text{ on } \ell^2(\mathbf{Z}^d),$$

where

$$\begin{aligned} \Delta_{ij} &= 1 \quad |i-j|_{\ell_1} = 1 \\ &= 0 \quad \text{otherwise;} \end{aligned} \tag{1.1}$$

γ is a positive parameter, V the potential function is a diagonal matrix: $V = \mathrm{diag}(v_j)$, $j \in \mathbf{Z}^d$, where $\{v_j\}$ is a family of independently identically distributed (iid) real random variables with distribution g. From now on, we write $|\,|$ for $|\,|_{\ell_1}$. ℓ_2 norms will be denoted by $\|\,\|$. We take our probability measure to be $P = \prod_{j \in \mathbf{Z}^d} g(v_j)$. We use the notation $\langle\,\rangle_g$ to denote the expectation value with respect to P.

As is well known, the spectrum $\sigma(\Delta) = [-2d, 2d]$. (Note that the spectrum of the Laplacian defined here defers from the one usually used by the probabilists by the constant $2d$.) Let supp g be the support of g, then we have the well established fact (See e.g., [CFKS,PF].) that

$$\sigma(H) = [-2d, 2d] + \text{supp } g$$

almost surely. Assume $E \notin \sigma(H)$, then we define $G(E) = (E - H)^{-1}$ and

$$G(E, i, j) = (\delta_i, (E - H)^{-1}\delta_j)$$

to be the Green's function or the correlation function of H at energy E.

In this paper, we take $E \in \mathbf{R} \notin \sigma(H)$ and study the asymptotics of $\langle G(E, \mu, \nu)\rangle_g$, particularly as $|\mu - \nu| \to \infty$. $\langle G(E, \mu, \nu)\rangle_g$ is the expected (with respect to the random potential V) probability of finding the random walk at site ν conditioned that it starts at site μ. (Recall that the Green's function is the integral over all time t of the heat kernel.) For more details, see e.g., [L,MS]. We study the case of small variations: $\gamma << 1$. The only condition that is needed on g is to ensure that the resolvent set: $\mathbf{R} \backslash \sigma(H) \neq \emptyset$. This is satisfied if supp g is bounded from either below or above. For simplicity, we assume g is such that g has compact support. Without loss of generality, we may then assume:

(H1) g is centered, i.e., $\langle v \rangle_g = 0$,

(H2) supp $g \subseteq [-1, 1]$, where -1 or 1 is reached.

(H1) can be compensated by a shift in "E". Due to the parameter γ in front of V, once we assume g has compact support, then we can assume (H2) without loss of generality. As we will see, the construction presented here carry over to the case where supp g is only bounded on one side and that g has finite n^{th} moments for some $n \geq 2$. The Bernoulli distribution:

$$g(v) = \frac{\delta(v+1) + \delta(v-1)}{2} \tag{1.2}$$

is among one of the most distinguished examples in this class of distributions.

In computing $\langle G(E, \mu, \nu) \rangle$ for small variations: $\gamma << 1$, the interesting case is when E is not "too far" from the spectrum of H. (The case $|E| >> 1$ can also be computed using the method below. In this case, the condition $\gamma << 1$ is no longer required.) So we fix δ_0, δ, such that $0 < \delta_0 < \delta < 1$ and define

$$I = (-2d - \delta, -2d - \delta_0) \cup (2d + \delta, 2d + \delta_0). \tag{1.3}$$

Assuming (H1,2), our main result is

Theorem 7.2. *There exists $\gamma_0 > 0$, such that for all $0 < \gamma < \gamma_0$, all $E \in I$, all μ, $\nu \in \mathbf{Z}^d$,*

$$\begin{aligned} \log\langle G(E, \mu, \nu) \rangle =& \log \left((E - \Delta - \gamma^2 \langle v^2 \rangle_g [(E-\Delta)^{-1}(0,0)] - \gamma^3 \langle v^3 \rangle_g [(E-\Delta)^{-1}(0,0)]^2 \right)^{-1} (\mu - \nu, 0) \\ &+ \mathbf{O}(\gamma^4)(|\mu - \nu| + 1) \\ =& \log(\widetilde{E} - \Delta)^{-1}(\mu - \nu, 0) + \mathbf{O}(\gamma^4)(|\mu - \nu| + 1), \end{aligned}$$

where

$$\widetilde{E} \stackrel{\text{def}}{=} E - \gamma^2 \langle v^2 \rangle_g [(E-\Delta)^{-1}(0,0)] - \gamma^3 \langle v^3 \rangle_g [(E-\Delta)^{-1}(0,0)]^2 \notin [-2d, 2d]. \tag{1.4}$$

Note the interesting $\mathbf{O}(\gamma^2)$, $\mathbf{O}(\gamma^3)$ terms, which we will give a simple explanation later in the introduction. Note also that the rate of decay is better than the naive one; recall that the almost sure spectrum is contained in $[-2d - \gamma, 2d + \gamma]$. Hence in some sense, the "effective" spectrum is further away than the almost sure one. This is related to the so called Lifshitz tails for the density of states in random Schrödinger operators, see e.g., [PF].

Define

$$\widetilde{\beta}_E(j) = \lim_{|j| \to \infty} - \frac{\log \langle G(E, j, 0) \rangle_g}{|j|}.$$

Then

$$\beta_E(j) = \widetilde{\beta}_E(j) j \in \mathbf{Z}^d$$

is just the directional Lyapunov exponent as defined in [Sz,Z]. Define

$$D_{|\widetilde{E}|} = \{ y \in R^{d^*} | \sum_{i=1}^d \cosh y_i < |\widetilde{E}| \}.$$

$D_{|\widetilde{E}|}$ is a convex set. We then obtain as a direct consequence of Theorem 7.2

Corollary.

$$\widetilde{\beta}_E(j) = \sup_{y \in \bar{D}_{|\widetilde{E}|}} y \cdot \frac{j}{|j|} + \mathbf{O}(\gamma^4) \tag{1.5}$$

and is analytic in E for $E \in I$.

From the Corollary, to $\mathbf{O}(\gamma^4)$ the Lyapunov exponent is none other than the support function of a convex set corresponding to the effective convolution matrix: $\widetilde{E} - \Delta$. Hence the main work is in proving Theorem 7.2 , the Corollary follows by using convex analysis, see e.g., [H,Sj].

Remark. Hopefully it will become clear later that to all orders in γ, the Lyapunov exponent is the support function of a convex set corresponding to an effective convolution matrix by iterating the procedure used here.

We emphasize that we do not require any regularity conditions on g for Theorem 7.2 to hold, in marked contrast to the random Schrödinger case , see e.g., [SW1,2]. So in this sense, our theorem is very general. The main reason for this contrast as we will see in (1.7,1.8) is that when $E \in \mathbf{R}\backslash\sigma(H)$, it is only the Laplace transform of g that is relevant; while when $E \in \mathbf{C}\backslash\sigma(H)$, $\Re E \in \sigma(H)$, it is the Fourier transform of g.

It is generally known in the probabilistic literature [Sz,Z] that β_E exists and defines a norm. For Brownian motions in Poissonian potentials, there are also upper and lower bounds on β_E [Sz]. But it is our understanding from [Sz] that beyond $d = 1$, very little is known about β_E quantitatively. So it is our hope that Theorem 7.2 and the Corollary contribute toward our understanding in that direction.

We now describe briefly the construction of the proof of Theorem 7.2. As usual we use finite dimensional approximations and take Λ to be a finite set in $\mathbf{Z}^d$ and define H_Λ to be the restricted operator with appropriate boundary conditions. For concreteness, assume $E > 2d + \gamma$. Our starting point as in [SW1]is the following Gaussian integral representation of $G_\Lambda(E, \mu, \nu)$:

$$G_\Lambda(E, \mu, \nu) = \int x_\mu \cdot x_\nu [\det(E - H_\Lambda)]^{-1} e^{-\sum_{i,j}(E-H_\Lambda)x_i \cdot x_j} \prod_{i \in \Lambda} \frac{d^2 x_i}{\pi}$$

where $x_i \in \mathbf{R}^2$. Using Grassmann variables (see sect. II), the determinant can be further expressed as a Gaussian integral (of Grassmann variables). This is the so called supersymmetry. Using this representation, we can explicitly take the expectation value with respect to g. Let

$$\hat{g}(t) = \int e^{-tv} g(v) dv$$

be the Laplace transform of g. Since $\hat{g}(t) > 0$, we define $k(t) = \log \hat{g}(t)$. $k(t)$ is real analytic in t as g has compact support. We obtain (see (2.2,2.3))

$$(1.7) \qquad \langle G_\Lambda(E,\mu,\nu)\rangle_g = \int x_\mu \cdot x_\nu e^{-2\phi(x)} \prod_{i\in\Lambda} \frac{d^2 x_i}{\pi},$$

where
(1.8)
$$\phi(x) = \frac{1}{2}[\sum_{i,j}(E-\Delta_\Lambda)x_i\cdot x_j - \sum_{j\in\Lambda} k(\gamma x_j\cdot x_j) - \log\det(E-\Delta_\Lambda - \gamma \mathrm{diag}\, k'(\gamma x_j\cdot x_j))].$$

We see that the above supersymmetric-Gaussian transform has mapped the averaged correlation function of a random walk in random potentials to a correlation function of statistical mechanics. The measure $\prod_{i\in\Lambda} g(v_i)$ on $\mathbf{R}^\Lambda$ has been transformed to $e^{-2\phi}$ on $\mathbf{R}^{2\Lambda}$. From now on we denote by $\langle\,\rangle$ without subscript, the expectation with respect to the measure $e^{-2\phi}$. By a now well known integration by parts initiated in [Sj], which for completeness we rederive in sect. II, we have further (formally)

$$(1.9) \qquad \langle G_\Lambda(E,\mu,\nu)\rangle_g = \sum_{i=1}^{2} \int (e^{-\phi} dx_\mu^{(i)}, (\Delta_\phi^{(1)})^{-1}(e^{-\phi} dx_\nu^{(i)})),$$

where
(1.10)
$$\Delta_\phi^{(1)} = \Delta_\phi^{(0)} \otimes I + 2\phi'' \qquad \text{on } \mathbf{C}_0^\infty(\mathbf{R}^{2\Lambda}; \wedge \mathbf{R}^{2\Lambda})$$
$$\Delta_\phi^{(0)} = \sum z_j^* z_j, \text{where } z_j = \frac{\partial}{\partial x_j} + \frac{\partial \phi}{\partial x_j}, \quad z_j^* = -\frac{\partial}{\partial x_j} + \frac{\partial\phi}{\partial x_j} \qquad \text{on } \mathbf{C}_0^\infty(\mathbf{R}^{2\Lambda}),$$

z_j^* is the formal adjoint of z_j, we have identified 1-forms with functions with values in $\mathbf{R}^{2\Lambda}$ and $(\ ,\ )$ denotes the inner product on $L^2(\mathbf{R}^{2\Lambda}; \wedge\mathbf{R}^{2\Lambda})$. (We will address the questions of conditions on ϕ, domaines of $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ etc. in sect. III. At this stage it suffices to mention that under conditions (H1,2) on g, $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ have self-adjoint extensions. $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ are respectively the Witten Laplacians on 0, 1-forms. Note that when ϕ is quadratic, $\Delta_\phi^{(0)}$ is just the usual harmonic oscillator on $L^2(\mathbf{R}^{2\Lambda})$. More generally

$$\Delta_\phi = d_\phi^* d_\phi + d_\phi d_\phi^*,$$
$$d_\phi = \sum_j z_j dx_j^\wedge, \quad d_\phi^* = \sum_j z_j^* dx_j^\llcorner,$$

d_ϕ^* is the formal adjoint of d_ϕ. The spectrum of $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ play a crucial role in our construction. For now it suffices to mention that $\Delta_\phi^{(0)} \geq 0$ and that $\Delta_\phi^{(0)}$ has 0 as an eigenvalue with $e^{-\phi}$ the unique eigenfunction. If ϕ is strictly convex, i.e., $\phi'' > c > 0$ as an operator, we then obtain that $\Delta_\phi^{(1)} \geq c > 0$. This is the so called spectral gap, which is responsable for the exponential decay in Theorem 7.2

. Hence to obtain asymptotics for $\langle G_\Lambda\rangle$, we need to compute this spectral gap as an asymptotic series in γ. This requires precise control over the spectrum *beyond* the gap. The main difficulty here is that we need estimates which are uniform in Λ, so that we can pass to the limit $\Lambda \nearrow \mathbf{Z}^d$. This is achieved in sects. IV, V by using appropriate weignted spaces. We also note that for $E \notin \sigma(H)$,

$$\lim_{\Lambda \nearrow \mathbf{Z}^d} \langle G_\Lambda(E,\mu,\nu)\rangle_g = \langle G(E,\mu,\nu)\rangle_g$$

exists apriori by spectral theory, so we only need to ensure that we have uniform estimates in this paper.

We use the Grushin problem to reduce the study of $\Delta_\phi^{(1)}$ to an effective operator valid for the lower part of the spectrum of $\Delta_\phi^{(1)}$. From the representation in (1.6), we are only interested in $(\Delta_\phi^{(1)} - z)^{-1}$ for $z = 0$. We show that to order $\mathbf{O}(\gamma^4)$ in appropriate weighted spaces, this effective operator is just $\langle 2\phi''\rangle = \int e^{-2\phi}2\phi''$. It is instructive to compare this with the expression for $\Delta_\phi^{(1)}$ in (1.10). It is crucial that the estimates are in appropriate weighted spaces as we are interested in the asymptotics as $|\mu-\nu| \to \infty$. Using standard results on the inverse of a convolution matrix, we then obtain Theorem 7.2.

In the Grushin reduction, we used in a crucial way that we study the spectrum of a complex, i.e., if u is an eigenfuction of $\Delta_\phi^{(0)}$ with eigenvalue $\lambda \neq 0$, then $d_\phi u$ is an eigenform of $\Delta_\phi^{(1)}$ with eigenvalue λ. This manifests itself under various guises as we shall see. Note also that for γ small, ϕ is close to a quadratic form, $e^{-2\phi}$ is close to a Gaussian. So what we did can be seen as a perturbation theory around an "infinite" dimensional Gaussian or harmonic oscillator. In other words, what we are using is a Witten complex formulation of Euclidean lattice field theory, see e.g. [Sj].

After we finished the paper, we realized that the connections between random walks and field theory are in fact well known, see e.g., [FFS], in particular [BF1,2], where they used the term oscillation modes, which roughly corresponds to the eigenfunctions of $\Delta_\phi^{(1)}$ in our vocabulary. The novelty here is that we have an *explicit* transformation of random walks in random potentials to a specific operator: $\Delta_\phi^{(1)}$, where ϕ is determined by the random walk and the random potential as in (1.8). $\Delta_\phi^{(1)}$ can in turn be interpreted as giving rise to a field theory. It is due to this equality in (1.9) that we were able to *compute* various quantities.

We end the introduction by pointing out that Theorem 7.2 extends the perturbation series obtained in field theory, which is valid only for *fixed* $|\mu-\nu|$. By the resolvent series,

(1.11)

$$\begin{aligned} G(E,\mu,\nu) =& G_0(\mu,\nu) + \gamma(G_0VG_0)(\mu,\nu) + \gamma^2(G_0VG_0VG_0)(\mu,\nu) \\ &+ \gamma^3(G_0VG_0VG_0VG_0)(\mu,\nu) + \gamma^4(G_0VG_0VG_0VG_0VG)(\mu,\nu), \end{aligned}$$

where $G_0 = (E-\Delta)^{-1}$. Note that the last factor in order $\mathbf{O}(\gamma^4)$ is G itself. So

$$\langle G(E,\mu,\nu)\rangle_g = G_0(\mu,\nu)+\gamma^2\langle v^2\rangle_g G_0(0,0)(G_0G_0)(\mu,\nu)+\gamma^3\langle v^3\rangle_g (G_0(0,0))^2(G_0G_0)(\mu,\nu)+\mathbf{O}(\gamma^4),$$

where we used $\langle v\rangle_g = 0$. We hence obtain that for all μ, ν with $|\mu-\nu|$ fixed

$$\langle G(E,\mu,\nu)\rangle_g = \big(E-\Delta-\gamma^2\langle v^2\rangle_g(E-\Delta)^{-1}(0,0)-\gamma^3\langle v^3\rangle_g((E-\Delta)^{-1}(0,0))^2\big)^{-1}(\mu,\nu)+\mathbf{O}(\gamma^4).$$

The trouble comes when we let $|\mu-\nu|\to\infty$, as the estimate for the remainder $\mathbf{O}(\gamma^4)$ is not in the appropriate weighted space due to the appearance of G there. Therefore in order to obtain the correct behaviour for $\langle G(E,\mu,\nu)\rangle$, one needs to expand the infinite series and then resum it. The advantage of our method is that one knows by the Grushin reduction the existence of an effective convolution matrix. It is only when one wants to derive an expression for this convolution matrix that one expands. Therefore to obtain an accuracy of the given order, one only needs to expand a finite number of terms. It is *not* an infinite series.

Finally, we like to add that we also hope to address the almost sure (quenched) behaviour of G by similar constructions in the future. One can analoguesly define the quenched Lyapunov exponent α_E as in [Sz,Z]. It is conjectured that $\alpha_E=\beta_E$ a.s., for $d>2$. If it is true then the Lyapunov exponents obtained here are also the Lyapunov exponent in the quenched case for $d>2$.

II. The Averaged Green's Function

We start by recapturing the problem that we study:

$$H=\Delta+\gamma V, \text{ on } \ell^2(\mathbf{Z}^d),$$

where

$$\begin{aligned}\Delta_{ij} &= 1 \quad |i-j|_{\ell_1}=1\\ &= 0 \quad \text{otherwise;}\end{aligned}$$

γ is a positive parameter, $(Vu)_j = v_ju_j$, v_j is a family of iid random variables with common distribution g satisfying (H1,2) of sect. I. The spectrum of H: $\sigma(H)=[-2d,2d]+\gamma\,\mathrm{supp}\,g$. Let $E\in\mathbf{R}\backslash\sigma(H)$. For definitiveness assume $E>0$. ($E<0$ works the same way.) We are interested in computing the asymptotics of $\langle G(E,\mu,\nu)\rangle_g = \langle (E-H)^{-1}_{\mu\nu}\rangle_g$ as $|\mu-\nu|\to\infty$ for γ sufficiently small.

We use finite dimensional approximations. We take Λ to be a finite set in $\mathbf{Z}^d$ or a large torus of the form $(\mathbf{Z}/N\mathbf{Z})^d$. We denote by $|\Lambda|$ the number of lattice points of $\mathbf{Z}^d$ in Λ. We identify Λ with the set $\{1,2\cdots|\Lambda|\}$. The corresponding discrete Laplacian Δ_Λ is then defined as in ( ) for i, $j\in\Lambda$. Similarly we define V_Λ. Define

$$H_\Lambda=\Delta_\Lambda+V_\Lambda$$

on $\ell^2(\Lambda)$. Clearly for all Λ, $\sigma(H_\Lambda) \subseteq \sigma(H)$. So $G_\Lambda = (E - H_\Lambda)^{-1}$ is well defined. As mentioned earlier in sect. I, by using spectral theory $(E-H)^{-1}$ exists apriori. So to infer the behavior of $\langle (E-H)^{-1}_{\mu\nu} \rangle$, it is sufficient to obtain estimates on $\langle (E - H_\Lambda)^{-1}{}_{\mu\nu}) \rangle$ which are uniform in Λ. For the rest of the paper, we will only work with H_Λ etc., so for simplicity of notations, we will drop the subscript Λ. We reserve Δ (or Δ_Λ) to denote the discrete Laplacians only; the Witten Laplacians will always be denoted as Δ_ϕ.

The main goal of this section is to obtain a convenient representation of $\langle (E - H)^{-1}_{\mu\nu} \rangle_g$. This is achieved in Proposition 2.1. Our principal tools are the Gaussian integrals. For any symmetric $n \times n$ matrix A with $\Re A > 0$, we recall the following well-known formula

$$A^{-1}_{\mu\nu} = \int x_\mu \cdot x_\nu \det A e^{-\sum_{i,j=1}^n A_{ij} x_i \cdot x_j} \prod_{i=1}^n \frac{d^2 x_i}{\pi},$$

where $x_i \in \mathbf{R}^2$ and $x_i \cdot x_j$ is the usual scalar product in $\mathbf{R}^2$. Substituting $E - H$ for A in the above formula, we then have

$$G(E, \mu, \nu) = \int x_\mu \cdot x_\nu \det(E-H) e^{-\sum_{i,j\in\Lambda} (E-H)_{ij} x_i \cdot x_j} \prod_{i\in\Lambda} \frac{d^2 x_i}{\pi}. \tag{2.1}$$

Let $\hat{g}$ be the Laplace transform of g:

$$\hat{g}(t) = \int e^{-tv} dg(v).$$

For example, for the Bernoulli measure:

$$g(v) = \frac{\delta(v+1) + \delta(v-1)}{2}$$

introduced earlier, $\hat{g}(t) = \cosh t$. In general, since g has compact support, $\hat{g}$ is a real analytic function. Moreover as g is a probability measure, $g \geq 0$. Hence $\hat{g}(t) > 0$ for $t \in \mathbf{R}$. We write

$$\hat{g}(t) = e^{k(t)}$$

with k a real analytic function. Note also that $|k(t)| = \mathbf{O}(1)|t|$ and $|k^{(n)}(t)| = \mathbf{O}(1)$, $(n \geq 1)$. For the Bernoulli measure, $k(t) = \log \cosh t$.

Proposition 2.1.

$$\langle G(E, \mu, \nu) \rangle_g = \int x_\mu \cdot x_\nu e^{-2\phi(x)} \prod_{i\in\Lambda} \frac{d^2 x_i}{\pi}, \tag{2.2}$$

where

(2.3)

$$\phi(x) = \frac{1}{2}[\sum_{i,j} (E-\Delta) x_i \cdot x_j - \sum_{j\in\Lambda} k(\gamma x_j \cdot x_j) - \log\det(E - \Delta - \gamma\, diag\, k'(\gamma x_j \cdot x_j))],$$

$x_i \in \mathbf{R}^2$.

Starting from sect. III, we will be using the representation in (2.2) to study $\langle (E-H)^{-1}_{\mu\nu})\rangle_g$. The method that we will use (aside from some estimates) is essentially independent of the derivation of (2.2). So those readers who are not particularly interested in the derivation could skip the following proof and proceed directly to sect. III. The proof that we give below is almost identical to that given in [SW1]. For completeness, we resummarize it here:

Proof of Proposition 2.1. There are presumably a few ways to derive (2.2). We continue to use the method in [SW1], which uses Grassmann algebra. For a proper formulation, see e.g., [Be,V]. In the following we will only write the few lines that are absolutely necessary for the proof of (2.2).

Let $|\Lambda|$ be the number of points in Λ. We use the Grassmann algebra of $2|\Lambda|$ generators to express $\det[E-H]$. This algebra is generated by $2|\Lambda|$ anticommuting variables ξ_i, η_i, $i \in \Lambda$ satisfying the relations:

$$\begin{aligned} [\xi_i, \eta_j] &= \xi_i\eta_j + \eta_j\xi_i = 0, \\ [\xi_i, \xi_j] &= \xi_i\xi_j + \xi_j\xi_i = 0, \\ [\eta_i, \eta_j] &= \eta_i\eta_j + \eta_j\eta_i = 0, \end{aligned} \tag{2.4}$$

where we write $[a, b] = ab+ba$ for the anti-commutator. It is denoted by $\mathcal{A}[\xi_1, \eta_1, .., \xi_{|\Lambda|}, \eta_{|\Lambda|}]$ (if we identify Λ with $\{1, .., |\Lambda|\}$). From (2.4), we see in particular that $\xi_i^2 = \eta_i^2 = 0$. We write these anticommutative variables collectively as ξ, η.

To make connections with better known objects, ξ_j, η_j can be thought of as the two 1-forms on the j^{th} copy of $\mathbf{R}^2$. "C^∞ functions" $F(\xi_i, \eta_j)$ of these anticommuting variables are defined by Taylor's formula at $(0, 0)$ which contains a finite number of terms because of nilpotency. In this way $F(\xi, \eta)$ becomes an element of the Grassmann algebra. For example if

$$F(\xi, \eta) = e^{A_{ij}\xi_i\eta_j}, \tag{2.5}$$

then

$$F(\xi, \eta) = 1 + A_{ij}\xi_i\eta_j. \tag{2.6}$$

This is the function that we need in writing the determinant. We also need to define the notions of differentiation and integration. Define:

$$\begin{aligned} \frac{\partial}{\partial \xi_i}(\xi_i) &= 1, \\ \frac{\partial}{\partial \eta_i}(\eta_i) &= 1. \end{aligned}$$

We require further that these differentiations be linear operators and that Leibnitz' rule hold. We can then define integrals (with respect to ∂) as follows:

$$\int 1 d\xi_i = 0, \ \int \xi_i d\xi_i = 1, \ \int 1 d\eta_i = 0, \ \int \eta_i d\eta_i = 1. \tag{2.7}$$

A multiple integral is defined to be a repeated integral. For example,

$$\int \xi_i \eta_j d\xi_i d\eta_j = -\int \eta_j \xi_i d\xi_i d\eta_j = -\int \eta_j d\eta_j = -1. \tag{2.8}$$

Using (2.5)–(2.8), we obtain

$$\det(E-H) = \int e^{-\sum_{j,k\in\Lambda}(E-H)_{jk}\eta_j\xi_k} \prod_{j\in\Lambda}(d\eta_j d\xi_j), \tag{2.9}$$

$$(E-H)^{-1}{}_{\mu\nu} \det(E-H) = \int \xi_\mu \eta_{nu} e^{-\sum_{j,k\in\Lambda}(E-H)_{jk}\eta_j\xi_k} \prod_{j\in\Lambda}(d\eta_j d\xi_j), \tag{2.9}$$

We illustrate (2.9) in the case where $(E-H) = M$ is a 2×2 matrix. The integrand in the RHS of (2.9) is

$$e^{-\sum_{j,k=1,2} M_{j,k}\eta_j\xi_k} = \prod_{j,k=1,2} (1 - M_{jk}\eta_j\xi_k), \tag{2.10}$$

where we used the commutativity of $e^{-M_{j,k}\eta_j\xi_k}$ and (2.5,2.6). Doing the integral in the RHS of (2.10) upon using (2.7) and (2.8), we obtain

$$\text{RHS} = M_{11}M_{22} - M_{12}M_{21} = \det M,$$

as expected. Similarly, we obtain for the RHS of ( ):

$$RHS = (-1)^{|\mu|+|\nu|}\det(M^{\mu\nu}) = M^{-1}{}_{\mu\nu}\det M, \qquad \mu,\, \nu = 1,\, 2,$$

where $M^{\mu\nu}$ is the comatrix with the μth row and νth column removed. (In this case it is just the one matrix element that is left.)

Combining (2.1) with (2.9), we obtain the following expression:

$$G(E,\mu,\nu) = \int x_\mu \cdot x_\nu e^{-\sum_{j,k\in\Lambda}(E-H)_{j,k}X_j\cdot X_k} \prod_{j\in\Lambda} d^2X_j,$$

where

$$\begin{aligned} x_j &\in \mathbf{R}^2, \\ X_j &= (x_j, \xi_j, \eta_j), \\ X_j \cdot X_k &= x_j \cdot x_k + \frac{1}{2}(\eta_j\xi_k + \eta_k\xi_j), \\ d^2X_j &= \frac{d^2x_j}{\pi} d\eta_j d\xi_j. \end{aligned}$$

Hence,
(2.12)

$$\langle G(E,\mu,\nu)\rangle_g = \int x_\mu \cdot x_\nu e^{-(\sum_{j\in\Lambda} EX_j\cdot X_j - \sum_{j,k\in\Lambda,|j-k|_1=1} X_j\cdot X_k - \gamma\sum_{j\in\Lambda} v_j X_j\cdot X_j)} \prod_{j\in\Lambda} dg(v_j) \prod_{j\in\Lambda} d^2X_j$$
$$= \int x_\mu \cdot x_\nu e^{-(\sum_{j\in\Lambda} EX_j\cdot X_j - \sum_{j,k\in\Lambda,|j-k|_1=1} X_j\cdot X_k - \sum_{j\in\Lambda} k(\gamma X_j\cdot X_j))} \prod_{j\in\Lambda} d^2X_j$$
$$= \int x_\mu \cdot x_\nu e^{-2\Phi} \prod_{j\in\Lambda} d^2X_j,$$

where

$$k(\gamma X_j\cdot X_j) = k(\gamma x_j\cdot x_j + \gamma\eta_j\xi_j)$$
$$= k(\gamma x_j \cdot x_j) + \gamma k'(x_j\cdot x_j)\eta_j\xi_j,$$

by Taylor series and anticommutativity, (Recall (2.6).) and

$$2\Phi = \sum_{j\in\Lambda} EX_j\cdot X_j - \sum_{j,k\in\Lambda,|j-k|_1=1} X_j\cdot X_k - \sum_{j\in\Lambda} k(\gamma X_j\cdot X_j).$$

Using (2.12) and integrating out the anticommutative variables ξ, η, we then obtain the proposition. □

We notice from the proof of Proposition 2.1 that in (2.11), apart from the factor $x_\mu\cdot x_\nu$, the integrand is only a "function" of the $X_j\cdot X_k$. Such "functions" are called supersymmetric functions. They have the important property that: If F is supersymmetric, then

(2.13)
$$\int F(X) \prod_{j\in\Lambda} d^2X_j = F(0).$$

(For more details, see e.g., [K,SW].) This is related to the so called "localization formula" in geometry, see e.g., [BGV]. Let $F(X) = X_\mu \cdot X_\nu e^{-2\Phi}$, we then obtain from (2.12, 2.13) and the fact that $F(0)=0$ that

$$\langle G(E,\mu,\nu)\rangle_g = \int \xi_\mu\eta_\nu e^{-2\Phi}\prod_{j\in\Lambda} d^2X_j$$
$$= \int (E-\Delta-\gamma k')^{-1}{}_{\mu\nu} e^{-2\phi} \prod_{j\in\Lambda} \frac{d^2x_j}{\pi},$$

where we used ( ) to reach the last equality.

( ), ( ) and ( ) will be our main tool in estimating various integrals with respect to the measure $e^{-2\phi}$. It helps us to obtain the correct order of magnitude in γ. It also allows us to skirt the use of maximum principles as in [Sj] (or Witten Laplacians as in [BJS].) In spite of all of its virtues, supersymmetry is not central for the main constructions in this paper. So if the readers wish, they could skip the details of such estimates of the integrals and only use the end results. Finally we remark that the measure $e^{-2\phi}$ defined in (2.3) is normalized for all Λ as $\prod_{j\in\Lambda} g(v_j)$ is normalized, which can also be verified by using (2.12).

III. The Witten Laplacian Representation of the Averaged Green's Function

Recall that in the Gaussian integral representation (2.1), we doubled the dimension, we therfore introduce the set $\bar{\Lambda} = \Lambda \times \{1,2\}$, which we identify with $\{1,2\cdots|\Lambda\|\} \times \{1,2\}$. Our main aim is to write the correlation

$$\langle G(E,\mu,\nu)\rangle_g = \int x_\mu \cdot x_\nu e^{-2\phi(x)} \prod_{i\in\Lambda} d^2x_i$$

in (2.2) using Witten Laplacians as initiated in [Sj]. From (H2) we know that $k(t) = \mathbf{O}(|t|)$ for $t >> 1$ and that it is analytic in a cone around the real axis. Hence

$$|k^{(n)}(t)| = \frac{\mathbf{O}_n(1)}{(1+|t|)^{n-1}}, \quad n \geq 1. \tag{3.1}$$

It follows then

$$\phi(x) = \mathbf{O}(1+x^2),\ \phi'(x) = \mathbf{O}(1+|x|),\ \phi^{(n)}(x) = \mathbf{O}_n(1)\,\text{for}\,n \geq 2. \tag{3.2}$$

The results to be stated below (except the last subsection when ϕ is an exact quadratic form) are general and in particular do not require that $\phi'' > c > 0$.

For notational simplicity, we make a slight change of point of view. We make the identification:

$$(\mathbf{R}^2)^\Lambda \sim (\mathbf{R})^{2\Lambda} \sim (\mathbf{R})^{\bar{\Lambda}}.$$

(In fact, careful readers probably have already remarked that we made this identification earlier in the introduction.) Hence unless specified otherwise, the i, j, $k\cdots$ etc. will now be in $\bar{\Lambda}$ and x_i, x_j, $x_k\cdots$ etc. will now be in $\mathbf{R}$. We will generally keep this point of view unless we need to estimate integrals with respect to $e^{-2\phi}$, where it is more convenient to revert back to the point of view of $(\mathbf{R}^2)^\Lambda$ and use supersymmetry.

The Witten complex.

We introduce the Witten complex, (originally introduced in [W]), as in [Sj] (cf. also [BJS,HS]):

$$d_\phi = e^{-\phi} de^\phi = d + d\phi^\wedge = \sum_{j\in\bar{\Lambda}} z_j dx_j, \tag{3.3}$$

acting (for the moment) on $C_0^\infty(\mathbf{R}^{2\Lambda};\wedge^\ell\mathbf{R}^{2\Lambda})$ $(0\le\ell\le 2|\Lambda|)$, where $d=\sum_{j\in\bar\Lambda}\partial_{x_j}dx_j^\wedge$ is the usual exterior differentiation and

$$z_j=\frac{\partial}{\partial x_j}+\frac{\partial\phi}{\partial x_j} \tag{3.4}$$

can be viewed as an annihilation operator. Let

$$d_\phi^*=e^\phi d^* e^{-\phi}=\sum_{j\in\bar\Lambda} z_j^* dx_j^\rfloor, \tag{3.5}$$

be the formal comlex adjoint, where

$$z_j^*=-\frac{\partial}{\partial x_j}+\frac{\partial\phi}{\partial x_j} \tag{3.6}$$

can be viewed as a creation operator. We have the commutation relation:

$$[z_j,z_k^*]=2\partial_j\partial_k\phi, \tag{3.7}$$

which plays an important role in our later constructions. We check easily that indeed

$$d_\phi d_\phi={d_\phi}^*{d_\phi}^*=0. \tag{3.8}$$

Using d_ϕ, ${d_\phi}^*$, we define the Witten Laplacian, which is a deformed Hodge Laplacian:

$$\Delta_\phi={d_\phi}^*d_\phi+d_\phi{d_\phi}^*, \tag{3.9}$$

on $C_0^\infty(\mathbf{R}^{2\Lambda};\wedge^\ell\mathbf{R}^{2\Lambda})$ $(0\le\ell\le 2|\Lambda|)$. Notice that

$$d_\phi\Delta_\phi=\Delta_\phi d_\phi,\ d_\phi^*\Delta_\phi=\Delta_\phi d_\phi^* \tag{3.10}$$

by using (3.8). If we let $\Delta_\phi^{(\ell)}$ be the restriction of Δ_ϕ to forms of degree ℓ, we obtain more precisely:

$$d_\phi\Delta_\phi^{(\ell)}=\Delta_\phi^{(\ell+1)}d_\phi,\ d_\phi^*\Delta_\phi^{(\ell+1)}=\Delta_\phi^{(\ell)}d_\phi^*. \tag{3.11}$$

This is the same as in the case of Hodge Laplacians which corresponds to taking $\phi=0$. We have explicitly

$$\Delta_\phi^{(0)}={d_\phi}^*d_\phi=\sum_{j\in\bar\Lambda}z_j^*z_j=-\sum_{j\in\bar\Lambda}\frac{\partial^2}{\partial x_j^2}+\|d\phi\|^2-\operatorname{Tr}\operatorname{Hess}\phi. \tag{3.12}$$

For example, if ϕ is an exact non-degenerate quadratic form, then $\Delta_\phi^{(0)}$ is a 2Λ-dimensional harmonic oscillator. z_j and z_j^* are just the famous annihilation and creation operators for the harmonic oscillator. More generally, we have

$$
\begin{aligned}
\Delta_\phi &= \sum\sum z_j z_k^* dx_j^\wedge dx_k^\rfloor + \sum\sum z_k^* z_j dx_k^\rfloor dx_j^\wedge \\
&= \sum\sum z_k^* z_j (dx_j^\wedge dx_k^\rfloor + dx_k^\rfloor dx_j^\wedge) + [z_j, z_k^*] dx_j^\wedge dx_k^\rfloor \\
&= \sum z_j^* z_j + 2\sum\sum \partial_{x_j}\partial_{x_k}\phi dx_j^\wedge dx_k^\rfloor \\
&= \Delta_\phi^{(0)} \otimes I + 2\sum\sum \partial_{x_j}\partial_{x_k}\phi dx_j^\wedge dx_k^\rfloor,
\end{aligned}
\tag{3.13}
$$

where to obtain the third line from the second, we used (3.7). In particular, if we identify 1-forms with $\mathbf{R}^{2\Lambda}$ valued functions, we obtain

$$
\Delta_\phi^{(1)} = \Delta_\phi^{(0)} \otimes I + 2\phi''. \tag{3.14}
$$

Also if we identify 2-forms with functions valued in the space of real anti-symmetric matrices, we have

$$
\Delta_\phi^{(2)} u = \Delta_\phi^{(0)} u + 2(\phi'' \circ u + u \circ \phi''). \tag{3.15}
$$

Since ϕ is real analytic under the condition (H2) and furthermore $(\Delta_\phi^{(\ell)} u, u) \geq 0$ for all $u \in C_0^\infty(\mathbf{R}^{2\Lambda}; \wedge^\ell(\mathbf{R}^{2\Lambda})^*)$, we can define $\Delta_\phi^{(\ell)}$ as a self-adjoint operator by taking the Frederichs extension. Using (3.2), we also see that $\Delta_\phi^{(\ell)}$ has compact resolvent. Hence it has discrete spectrum contained in $[0, \infty)$. The lowest eigenvalue of $\Delta_\phi^{(0)}$ is zero and a corresponding eigenfuction is $e^{-\phi}$, since this function is annihilated by d_ϕ. This eigenvalue is simple, for if u is another eigenfunction associated to the same eigenvalue, then $0 = (\Delta_\phi^{(0)} u, u) = \|d_\phi u\|^2$ and hence $d_\phi u = 0$, which means precisely that u is a multiple of $e^{-\phi}$.

We next repeat the argument given in [Sj] to show that the lowest eigenvalue of $\Delta_\phi^{(1)}$ is > 0. Assume that $u \in \mathcal{D}(\Delta_\phi^{(1)})$ and that $\Delta_\phi^{(1)} u = 0$. Then we have $d_\phi u = 0$, $d_\phi^* u = 0$. Since we are in $\mathbf{R}^{2\Lambda}$, the first equation implies that $e^{-\phi} u$ is an exact form: $e^\phi u = d(e^\phi v)$, i.e., $u = d_\phi v$, where v is given by

$$
v(x) = \int_0^x e^{-(\phi(x)-\phi(y))} u(y),
$$

a line integral. From (3.2) it follows by using weighted Lithner-Agmon estimates as in [HS] that $u \in \mathcal{S}(\mathbf{R}^{2\Lambda})$ and then using (3.16) and (3.2) once more, that $v \in \mathcal{S}(\mathbf{R}^{2\Lambda})$. Using that $d_\phi^* u = 0$, we obtain $\Delta_\phi^{(0)} v = 0$, so $v = \text{const} e^{-\phi}$. This implies that $u = d_\phi v = 0$.

Remark. The above argument clearly does not require ϕ to be convex. It holds for all Λ which are finite subsets of $\mathbf{Z}^d$. The point is that the lower bound obtained above is *not* uniform in Λ.

From the relation (3.11), we have the well known fact that the spectra of Witten Laplacians on various forms are inter-related. In particular, if $\lambda \neq 0$ is an eigenvalue for $\Delta_\phi^{(\ell)}$ with eigenform $u \in \mathcal{S}(\mathbf{R}^{2\Lambda}; \wedge^\ell \mathbf{R}^{2\Lambda})$, then $d_\phi u \in \mathcal{S}(\mathbf{R}^{2\Lambda}; \wedge^{\ell+1}\mathbf{R}^{2\Lambda})$ and $d_\phi^* u \in \mathcal{S}(\mathbf{R}^{2\Lambda}; \wedge^{\ell-1}\mathbf{R}^{2\Lambda})$ (if they are non-zero) are eigenforms for $\Delta_\phi^{(\ell+1)}$ and $\Delta_\phi^{(\ell-1)}$ respectively. (When $\lambda = 0$, both $d_\phi u$ and $d_\phi^* u$ are zero, so there is no such straightforward relation between the zero eigenvalue. These are basic properties of a complex, see e.g., [ ]. The following observation, which will be used later in the paper, is a manifestation of the above fundamental fact. Let $\lambda > 0$ be the smallest eigenvalue of $\Delta_\phi^{(1)}$. Then the gap between the first eigenvalue of $\Delta_\phi^{(0)}$ which is zero, and the second one is $\geq \lambda$. This is because if $\mu < \lambda$ is the second eigenvalue for $\Delta_\phi^{(0)}$ with eigenfunction $u \in \mathcal{S}(\mathbf{R}^{2\Lambda}; \mathbf{R})$, then $0 \neq d_\phi u \in \mathcal{S}(\mathbf{R}^{2\Lambda}; \mathbf{R}^{2\Lambda})$ is an eigenform for $\Delta_\phi^{(1)}$ with eigenvalue $\mu < \lambda$, which is in contradiction with the assumption. The Grushin reductions in sect. IV can be seen as using the above argument for higher parts of the spectrum.

We mentioned earlier that in the case of a Gaussian (ϕ an exact quadratic form), $\Delta_\phi^{(0)}$ is just a $2|\Lambda|$-dimensional harmonic oscillator. This observation can be generalized to note that if $e^{-\phi}$ is the eigenfunction of the Schrödinger operator

$$-\sum \frac{\partial^2}{\partial x_j^2} + V$$

for the lowest eigenvalue λ, then

$$-\sum \frac{\partial^2}{\partial x_j^2} + V - \lambda = \Delta_\phi^{(0)}.$$

From this point of view, the Witten Laplacian can be seen as, in some sense, a natural generalization of harmonic oscillators.

The Witten Laplacian Representation.

We begin by rewriting the averaged Green's function in (3.1) in the new notation as

$$\begin{aligned} \langle G(E, \mu, \nu) \rangle_g &= 2 \int x_\mu x_\nu e^{-2\phi(x)} \prod_{i \in \bar{\Lambda}} \frac{dx_i}{\sqrt{\pi}} \\ &= 2(x_\mu e^{-\phi}, x_\nu e^{-\phi}), \end{aligned} \tag{3.17}$$

where $\mu, \nu \in \Lambda \subset \bar{\Lambda}$, x_μ, x_ν, $x_i \in \mathbf{R}$ and the extra factor of 2 comes from symmetry. We note that due to symmetry $(e^{-\phi}, x_\nu e^{-\phi}) = 0$ for all $\nu \in \Lambda$. We look for a 1-form ω, such that

$$x_\nu e^{-\phi} - (e^{-\phi}, x_\nu e^{-\phi}) = x_\nu e^{-\phi} = d_\phi^* \omega.$$

Further we look for a particular type of 1-form $\omega = d_\phi u$. (3.17) then leads to the following equation for u

$$x_\nu e^{-\phi} = \Delta_\phi^{(0)} u$$

where $u \in \mathcal{D}(\Delta_\phi^{(0)})$. Using (3.19) in (3.17), we obtain

$$\begin{aligned} \langle G(E,\mu,\nu)\rangle_g &= (x_\mu e^{-\phi}, \Delta_\phi^{(0)} u) \\ &= (e^{-\phi} dx_\mu, d_\phi u). \end{aligned} \tag{3.20}$$

Applying d_ϕ to (3.19), we obtain

$$e^{-\phi} dx_\nu = \Delta_\phi^{(1)}(d_\phi u).$$

It is easy to show that $d_\phi u \in \mathcal{D}(\Delta_\phi^{(1)})$ (see [Sj,BJS]). Hence

$$d_\phi u = (\Delta_\phi^{(1)})^{-1}(e^{-\phi} dx_\nu). \tag{3.21}$$

Using (3.21) in (3.20), we obtain finally

$$\langle G(E,\mu,\nu)\rangle_g = 2(e^{-\phi} dx_\mu, (\Delta_\phi^{(1)})^{-1}(e^{-\phi} dx_\nu)), \tag{3.22}$$

where $x_\mu,\ x_\nu \in \mathbf{R}$.

Remark. For all finite sets $\Lambda \subset \mathbf{Z}^d$, $\Delta_\phi^{(1)} > 0$ as discussed previously. So (3.22) is well defined. The main work starting in sect. IV is to study the lower part of the spectrum of $\Delta_\phi^{(1)}$ *uniformly* in Λ as $\Lambda \nearrow \mathbf{Z}^d$. Looking for a 1-form ω as in (3.18) is dual to looking for a vector field (for integration by parts) as in [HS,SW]. In fact $e^\phi \Delta_\phi^{(1)} e^{-\phi}$ is the vector heat equation obtained there after identifying 1-forms with vector fields.

$\Delta_\phi^{(0)}$ and $\Delta_\phi^{(1)}$ for ϕ an exact quadratic form–the Fock space representation.

When ϕ is an exact non-degenerate quadratic form, $\Delta_\phi^{(0)}$ is a 2Λ-dimensional harmonic oscillator, as mentioned earlier. Its properties are well-known. However, since most of our insights in the general case come from the quadratic case, below we mention briefly the spectral properties of $\Delta_\phi^{(0)}$ and $\Delta_\phi^{(1)}$ when $\gamma = 0$, i.e., for

$$\phi = \frac{1}{2}(E-\Delta)x\cdot x - \frac{1}{2}\log\det(E-\Delta), \tag{3.23}$$

where $E \notin \sigma(\Delta)$, and for definitiveness we have assumed $E > 0$; the second term in ϕ is just the normalization constant. We will also express $\Delta_\phi^{(0)}$, $\Delta_\phi^{(1)}$ in the basis provided by the associated Hermite functions and thus obtain the so called Fock space representations in (3.27),(3.30).

As we will realize later, most of the essential features in the general case are already present in the quadratic case. Since in the latter, ϕ decouples into the sum of two identical terms corresponding to the two copies of $\mathbf{R}$, for simplicity, in the rest of this subsection (and only there) we assume $j \in \Lambda$ instead of $\bar{\Lambda}$.

Substituting (3.23) in (3.12) and (3.14), we have

$$\begin{aligned} \Delta_\phi^{(0)} &= -\sum_{j\in\Lambda} \frac{\partial^2}{\partial x_j^2} + \sum_{j,k\in\Lambda} [(E-\Delta)^2]_{jk} x_j x_k - 2|\Lambda| E \\ \Delta_\phi^{(1)} &= \Delta_\phi^{(0)} \otimes I + 2(E-\Delta), \end{aligned} \tag{3.24}$$

where as earlier, we have identified 1-forms with $\mathbf{R}^\Lambda$ valued functions and we now further identify $\mathbf{R}^\Lambda$ valued functions with $\ell^2(\Lambda)$.

Let ω_i, $(i = 1, \cdots, |\Lambda|)$ be the eigenvalues of $E - \Delta$, in increasing order. Clearly $E - 2d \leq \omega_1 < \cdots < \omega_{|\Lambda|} \leq E + 2d$. As $\Lambda \nearrow \mathbf{Z}^d$, the equalities are reached and in the limit the spectrum $\sigma(E-\Delta) = [E-2d, E+2d]$. For any finite Λ, the spectrum of $\Delta_\phi^{(0)}$ is discrete. Its eigenvalues are:

$$\sum_i^{|\Lambda|} 2m_i\omega_i \qquad m_i = \{0, 1\cdots\}.$$

The first spectral gap $= 2\omega_1 > 0$ and approaches $E - 2d > 0$ as $|\Lambda| \to \infty$. The higher gaps generally diminish in size as $|\Lambda|$ increases and the ω_i's approach a continum.

Let v_i be the eigenvector of $(E - \Delta)$ corresponding to the eigenvalue ω_i $(i = 1, \cdots \Lambda)$. Then clearly $e^{-\phi} v_i$ is an eigenform of $\Delta_\phi^{(1)}$ with eigenvalue $2\omega_i$. Let ψ_1 denote the eigenfunction of $\Delta_\phi^{(0)}$ corresponding to the eigenvalue $2\omega_1$. Similarly, we have that $\psi_1 v_i$ (component-wise multiplication) are eigenforms for $\Delta_\phi^{(1)}$ corresponding to eigenvalues $2\omega_1 + 2\omega_i \geq 4\omega_1$ $(i = 0, 1 \cdots |\Lambda|)$. Clearly, we can keep on iterating and obtain the whole spectrum of $\Delta_\phi^{(1)}$ this way.

Let Π be the projection onto the subspace in $L^2(\mathbf{R}^\Lambda; \mathbf{R}^\Lambda)$ spanned by $e^{-\phi} v_i$. The feature to notice is that

$$\begin{aligned} \Delta_\phi^{(1)} &\geq 2\omega_1 \\ \Delta_\phi^{(1)}(1 - \Pi) &\geq 4\omega_1. \end{aligned}$$

Customarily, $2\omega_1$ is called "the gap" and $4\omega_1$ "the upper gap". Note that the upper gap is twice the gap. As we will see these two features remain in the general case and manifest themselves when we compute the correlation function. In fact computing the asymptotics of the correlation function is equivalent to have precise control over the spectrum beyond the gap. For example, in our case, to order $\mathbf{O}(\gamma^2)$, it is the upper gap; and to $\mathbf{O}(\gamma^4)$, it is the spectrum beyond the upper gap.

Coming back to the quadratic case, where ϕ is as in (3.23), we know that the Hermite functions provide a basis for $L^2(\mathbf{R}^\Lambda)$. It is instructive to express $\Delta_\phi^{(0)}$ in that basis. Let

$$\zeta_j^* = \sum_k [(2\phi'')^{-1/2}]_{jk} z_k^*.$$

The relevant orthonormal set of Hermite functions are:

$$e^{-\phi};\ \zeta_j^* e^{-\phi},\ (j \in \Lambda);\ \zeta_j^*\zeta_k^* e^{-\phi},\ (j,k \in \Lambda, j > k),\ \frac{(\zeta_j^*)^2}{\sqrt{2}} e^{-\phi},\ (j \in \Lambda), \cdots$$

We further identify $\mathbf{R}e^{-\phi}$ with $\mathbf{R}$, the span of $\{\zeta_j^* e^{-\phi}\}$ with $\mathbf{R}^\Lambda \sim \ell^2(\Lambda)$, the span of$\{\zeta_j^*\zeta_k^* e^{-\phi},\ (j > k),\ \frac{(\zeta_j^*)^2}{\sqrt{2}} e^{-\phi}\}$ with $\ell^2(\Lambda) \otimes_s \ell^2(\Lambda)$, $\cdots$ etc., where $\otimes_s$ denotes the symmetric tensor product as $[\zeta_j^*, \zeta_k^*] = 0$ for all $j,\ k \in \Lambda$.

Let $L^2(\mathbf{R}^\Lambda)_n$ be the span of $\zeta_{j_1}^* \cdots \zeta_{j_n}^* e^{-\phi}$. We have the decomposition (Hilbert direct sum):

$$\begin{aligned} L^2(\mathbf{R}^\Lambda) &= \oplus_{n=0}^\infty L^2(\mathbf{R}^\Lambda)_n \\ &= \oplus_{n=0}^\infty \ell^2(\Lambda)^{\otimes_s n} \\ &= \mathbf{R} \oplus_{n=1}^\infty \ell^2(\Lambda)^{\otimes_s n}. \end{aligned}$$

For any vector space E, we consider $E \otimes_s E$ as a subspace of $E \otimes E$ Therefore $\ell^2(\Lambda)^{\otimes_s n}$ is a subspace of $\ell^2(\Lambda)^{\otimes n}$. Since $\Delta_\phi^{(0)}$ leaves each $L^2(\mathbf{R}^\Lambda)_n$ invariant, we write $\Delta_{\phi,n}^{(0)}$ for the restriction. We identify $\Delta_{\phi,n}^{(0)}$ with an element of $\left(\ell^2(\Lambda)^{\otimes_s n}\right)^* \otimes \left(\ell^2(\Lambda)^{\otimes_s n}\right)$. Therefore $\Delta_{\phi,n}^{(0)}$ can be seen as a tensor in $\otimes^{2n}\ell^2(\Lambda)$, (seperately) symmetric in the first n indices and the last n indices.

By direct computation we have that

$$\begin{aligned} \Delta_{\phi,0}^{(0)} =& 0 \\ \Delta_{\phi,1}^{(0)} =& E - \Delta \\ \Delta_{\phi,2}^{(0)} =& I \otimes (E - \Delta) + (E - \Delta) \otimes I \\ & \text{restricted} \quad \text{to} \ell^2(\lambda) \otimes_s \ell^2(\Lambda) \\ \ddots & \end{aligned}$$

We summarize by writing $\Delta_\phi^{(0)}$ as a block diagonal matrix:

$$\Delta_\phi^{(0)} = 2 \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & (E - \Delta) & 0 & \cdots \\ 0 & 0 & (E - \Delta) \otimes_s I + I \otimes_s (E - \Delta) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.27}$$

where $(E-\Delta)$, I are once contra, once covariante tensors. In fact, we have the identification

$$(E-\Delta)^i_j = (E-\Delta)_{ij}; \quad I^i_j = I_{ij} = \delta_{ij}; \tag{3.28}$$

$\otimes_s$ (with a slight abuse of notation) denotes the restriction of the tensors to the symmetric subspace. For example

$$\begin{aligned}
&((E-\Delta)\otimes_s I + I\otimes_s(E-\Delta))^{ij}_{kl}\\
=&(E-\Delta)^i_k\delta^j_\ell + (E-\Delta)^i_\ell\delta^j_k + \delta^i_k(E-\Delta)^j_\ell\\
&+\delta^i_\ell(E-\Delta)^j_k \qquad (i\neq j, k\neq \ell)\\
=&(E-\Delta)^i_k\delta^j_\ell + \delta^i_\ell(E-\Delta)^j_k \qquad (i\neq j, k=\ell)\\
=&(E-\Delta)^i_k\delta^i_\ell + \delta^i_k(E-\Delta)^i_\ell \qquad (i=j, k\neq\ell)\\
=&(E-\Delta)^i_i\delta^i_i\\
=&(E-\Delta)^i_i \qquad (i=j=k=\ell).
\end{aligned}\tag{3.29}$$

By using the identification (3.28), we have the same equality as in (3.27) with all the indices down. (3.27) is just the usual Fock space representation of $\Delta^{(0)}_\phi$. (See e.g., [Be].)

By the same reasoning, $\Delta^{(1)}_\phi$ operates on

$$L^2(\mathbf{R}^\Lambda;\mathbf{R}^\Lambda)\sim\left(\mathbf{R}\oplus_{n=1}^\infty \ell^2(\Lambda)^{\otimes^n_s}\right)\otimes\ell^2(\Lambda),$$

where the tensor product outside of the parenthesis is unsymmetrized. It has the following Fock space representation:

(3.30)

$$\Delta^{(1)}_\phi = 2\begin{pmatrix}
(E-\Delta) & 0 & 0 & \cdots\\
0 & \begin{matrix}(E-\Delta)\otimes I\\ +I\otimes(E-\Delta)\end{matrix} & 0 & \cdots\\
0 & 0 & \begin{matrix}[(E-\Delta)\otimes_s I + I\otimes_s(E-\Delta)]\otimes I\\ +(I\otimes_s I)\otimes(E-\Delta)\end{matrix} & \cdots\\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.$$

We notice also that the Witten Laplacian representation gives in this case

$$\langle G(E,\mu,\nu)\rangle_g = 2(e^{-\phi}dx_\mu,(\Delta^{(1)}_\phi)^{-1}(e^{-\phi}dx_\nu)) = (E-\Delta)^{-1}(\mu,\nu), \tag{3.31}$$

which is undoubtedly the correct answer.

We end this section by commenting that for given μ, ν, the sign of $G(E;\mu,\nu)$ is independent of V for $E\notin[-2d,2d]+\gamma\,\mathrm{supp}\,g$. Hence $\langle G(E;\mu,\nu)\rangle_g$ has statistical significance. This can be seen as follows:

(i) $E > 0$, $E \notin [-2d, 2d] + \gamma\,\mathrm{supp}\, g$

$$G = (E - H)^{-1} = (E - \Delta - V)^{-1}$$

Since

$$\frac{\partial}{\partial E} G(E, \mu, \nu) = -\sum_k G(\mu, k) G(k, \nu),$$

we see that it is enough to prove $G(E; \mu, \nu) > 0$ for some large enough E. For such E's, we can simply use the Neumann series about $(E - V)^{-1}$ to conclude that $G(E; \mu, \nu) > 0$.

(ii) $E < 0$, $E \notin [-2d, 2d] + \gamma\,\mathrm{supp}\, g$

Let U be the unitary multiplication operator: $(Uf)(i) = (-1)^{|i|_1} f(i)$. Then

$$U^{-1} H U = -\Delta + V.$$

Hence

$$G(E; \mu, \nu) = (E - H)^{-1}(\mu, \nu) = (-1)^{|\mu|_1 - |\nu|_1} (E - V + \Delta)^{-1}(\mu, \nu).$$

Now $(E - V + \Delta)^{-1}(\mu, \nu) < 0$ by using the same argument as before. So for given μ, ν, $G(E; \mu, \nu)$ has a sign which is independent of V.

IV. Grushin Problems and the Spectrum of $\Delta_\phi^{(1)}$

We start the study of $\Delta_\phi^{(1)}$, where ϕ is as defined in (2.3). Recall that E is such that, uniformly in Λ

$$\text{(4.1)} \qquad \begin{aligned} &\phi''(x) = \mathbf{O}(1) \quad \text{in}\, \mathcal{L}(\ell^2(\Lambda); \ell^2(\Lambda)) \\ \text{and} \quad &\phi''(x) \geq r_E > 0, \end{aligned}$$

where r_E is independent of Λ. Using (2.3), we have the following expression for $2\phi''$, where for simplicity of presentation we temporarily (in (4.2)) revert back to the point of view of $(\mathbf{R}^2)^\Lambda$:

(4.2)

$$\begin{aligned} (2\phi'')_{i^{(\alpha)} j^{(\beta)}} =& 2E\delta_{ij}\delta_{\alpha\beta} - 2\Delta_{ij}\delta_{\alpha\beta} - 2\gamma k'(\gamma x_i \cdot x_i)\delta_{ij}\delta_{\alpha\beta} - 4\gamma^2 x_i^{(\alpha)} x_i^{(\beta)} k''(\gamma x_i \cdot x_i)\delta_{ij} \\ &+ 2\gamma^2 k''(\gamma x_i \cdot x_i)(E - \Delta - \gamma \mathrm{diag}\, k')^{-1}_{ii} \delta_{ij}\delta_{\alpha\beta} \\ &+ 4\gamma^3 x_i^{(\alpha)} x_i^{(\beta)} k'''(\gamma x_i \cdot x_i)(E - \Delta - \gamma \mathrm{diag}\, k')^{-1}_{ii} \delta_{ij} \\ &+ 4\gamma^4 x_i^{(\alpha)} x_j^{(\beta)} k''(\gamma x_i \cdot x_i) k''(\gamma x_j \cdot x_j)[(E - \Delta - \gamma \mathrm{diag}\, k')^{-1}_{ij}]^2, \end{aligned}$$

where $k = \log \hat{g}$ and $\hat{g}$ is the Laplace transform of g; i, $j \in \Lambda$, x_i, $x_j \in \mathbf{R}^2$, α, $\beta \in \{1, 2\}$. Unless specified otherwise, all our estimates below are uniform in Λ.

From (3.22), we see that we are mainly interested in the spectrum of $\Delta_\phi^{(1)}$ around 0. Hence we use the method of Grushin problems to obtain an effective operator (around 0). Similar to the quadratic case, the family of functions: $e^{-\phi}$, $\{z_j^* e^{-\phi}\ j \in \bar\Lambda\}$, $\{z_j^* z_k^* e^{-\phi},\ j,k \in \bar\Lambda,\ j \geq k\} \cdots$ etc., after orthonormalization will play a special role and provide a "basis" for the Grushin problems.

Our main goal is to compute the asymptotics for the averaged Green's function in (2.2) as $|\mu-\nu| \to \infty$. In order to capture the effect of randomness, we need the error term to be of at most $\mathbf{o}(\gamma^2)$. (To $\mathbf{O}(1)$, the asymptotics only reflects $E-\Delta$. There is no $\mathbf{O}(\gamma)$ term as the probability distribution g is assumed centered : $\langle v\rangle_g = 0$). In the following we study $\Delta_\phi^{(1)}$ to $\mathbf{O}(\gamma^4)$. However, it will become clear (we hope) that the procedure that we describe below can be iterated to get the asymptotics to all orders.

1. The initial Grushin problem for $\Delta_\phi^{(1)}$ and approximations to $\mathbf{O}(\gamma^2)$.

We begin our study of the spectrum $\Delta_\phi^{(1)}$ around 0. From (3.14,3.12,4.1) we have, a priori, for all u,

$$(\Delta_\phi^{(1)} u, u) \geq 2r_0 ||u||^2 \tag{4.3}$$

for some $r_0 \geq r_E > 0$. Hence the second eigenvalue (spectral gap) of $\Delta_\phi^{(0)} \geq r_0 > 0$. (Recall that we are working with a complex). Therefore if $u \perp e^{-\phi} dx_j$ for all $j \in \bar\Lambda$, then

$$(u, \Delta_\phi^{(1)} u) \geq 2(r_0 + r_E)\|u\|. \tag{4.4}$$

As we will see by the end of the section in order that the error in the decay rate in the asymptotics of $\langle G_\Lambda(E,\mu,\nu)\rangle$ as $|\mu-\nu| \to \infty$ is of $\mathbf{o}(\gamma^2)$, we need to use not only $e^{-\phi}$, but also $\{z_j^* e^{-\phi},\ j \in \bar\Lambda\}$, $\{z_j^* z_k^* e^{-\phi},\ j,k \in \bar\Lambda,\ j \geq k\}$ (properly orthonormalized) as our Grushin basis. In order to get a more optimal ("correct") lower bound than in (4.4) for u in the corresponding orthogonal space, we need to do a preliminary study of $\Delta_\phi^{(1)}$ around 0. To that end, we pose our first Grushin problem.

Let $e^{-\phi} dx_j$ (all $j \in \bar\Lambda$) be our initial Grushin basis. We introduce the projection operator $R_+ : L^2(\mathbf{R}^{2\Lambda}; \wedge \mathbf{R}^{2\Lambda}) \to \ell^2(\bar\Lambda)$ by

$$R_+ u(j) = (u, e^{-\phi} dx_j) = (u_j, e^{-\phi}), \quad j \in \bar\Lambda,$$

where $u = \sum_{j\in\bar\Lambda} u_j dx_j$. Let $R_- = R_+^t : \ell^2(\bar\Lambda) \to L^2(\mathbf{R}^{2\Lambda}; \wedge \mathbf{R}^{2\Lambda})$:

$$R_- u_- = \sum_j e^{-\phi} u_-(j) dx_j. \tag{4.6}$$

Note that $R_+R_- = I$. We pose our initial Grushin problem :

$$\begin{pmatrix} \Delta_\phi^{(1)} - z & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{4.7}$$

where $u \in \mathcal{D}(\Delta_\phi^{(1)})$, $v \in L^2$, $u_-, v_+ \in \ell^2$. $\begin{pmatrix} v \\ v_+ \end{pmatrix}$ is the given and $\begin{pmatrix} u \\ u_- \end{pmatrix}$ is the unknown.

Using (4.2), we have the following proposition which can be proved essentially as in [Sj].

Proposition 4.1. *For every $C > 0$, there exists $\gamma_0 > 0$ such that the problem (4.4) is well posed for all $z \leq 2\big(r_0 + r_E - \frac{1}{C}\big)$, $0 \leq \gamma \leq \gamma_0$. Moreover*

$$\|u\| + \|u_-\| \leq \mathbf{O}(1)(\|v\| + \|v_+\|). \tag{4.8}$$

The proof of the above proposition (and indeed the study of $\Delta_\phi^{(1)}$ to $\mathbf{O}(\gamma^2)$, see Proposition 4.3) depends on the following fundamental fact.

Lemma 4.2. *Let $w \in \ell^2$, R_- be as defined in (4.4). Then*

$$\|(\phi'' - \langle \phi'' \rangle)R_- w\| \leq \mathbf{O}(\gamma^2)\|w\|. \tag{4.9}$$

Proof.

$$\|(\phi'' - \langle \phi'' \rangle)R_- w\|^2 = \sum_{ijk} \Big((\phi''_{ik} - \langle \phi''_{ik} \rangle)e^{-\phi} w_k, (\phi''_{ij} - \langle \phi''_{ij} \rangle)e^{-\phi} w_j \Big).$$

We now use our fundamental philosophy, which is to write $\phi''_{ik} e^{-\phi} - \langle \phi''_{ik} \rangle e^{-\phi} = \Delta_\phi^{(0)} \tilde{u}$ for some $\tilde{u}$. Proceeding exactly as (3.17), (3.22) in sect. III, we obtain

$$\begin{aligned} RHS &= \sum_{ijk} \Big(e^{-\phi} d\phi''_{ik}, (\Delta_\phi^{(1)})^{-1} e^{-\phi} d\phi''_{ij} \Big) w_j w_k \\ &= \mathbf{O}(1) \sum_{j,k} w_j w_k \sum_i \|e^{-\phi} d\phi''_{ik}\| \;\; \|e^{-\phi} d\phi''_{ij}\|, \end{aligned} \tag{4.10}$$

where we used (4.3) to bound $\|(\Delta_\phi^{(1)})^{-1}\|$.

To estimate the RHS of (4.10), we need to estimate $\langle (\phi''')^2 \rangle$. For this purpose, it is convenient to revert back to the point of view of $(\mathbf{R}^2)^\Lambda$ and use supersymmetry. (We will have more occasions to do such computations in sect. VII, where we need more precise results.) Hence elements of $\bar{\Lambda}$ will be denoted by $i^{(\alpha)}$, $i \in \Lambda$, $\alpha \in \{1, 2\}$ and $x_i \in \mathbf{R}^2$. (Those readers who are not particularly interested in the details of this estimate could go directly to the end result ( ) and continue with the reading of the proof.)

From (4.2), there are 7 terms in $2\phi''$, which we denote by a_n, $n = 1 \cdots 7$. So

() $$\langle (\phi''')^2 \rangle = \langle (\sum_{n=1}^{7} a'_n)^2 \rangle = \sum_{n=3}^{7} \langle (a'_n)^2 \rangle + 2 \sum_{n>m>2} \langle a'_n a'_m \rangle$$

as a_1, a_2 are constants. To estimate $\langle (a'_3)^2 \rangle$, we need to estimate

$$\langle ({x_i^{(\alpha)}}^2 [k''(\gamma x_i \cdot x_i)]^2 \rangle = \frac{1}{2} \int x_i \cdot x_i [k''(\gamma x_i \cdot x_i)]^2 e^{-2\phi}.$$

Using (2.13-2.15), we have

() $$\int X_i \cdot X_i [k''(\gamma X_i \cdot X_i)]^2 e^{-2\Phi} \prod_{j \in \Lambda} d^2 X_j = 0,$$

where

() $$\begin{aligned} X_i \cdot X_i &= x_i \cdot x_i + \eta_i \xi_i \\ k''(\gamma X_i \cdot X_i) &= k''(\gamma x_i \cdot x_i) + \gamma \eta_i \xi_i k'''(\gamma x_i \cdot x_i) \end{aligned}$$

and Φ is as in ( ).

Using ( ) to write out ( ), we have

$$\int x_i \cdot x_i [k''(\gamma x_i \cdot x_i)]^2 e^{-2\Phi} \prod_{j \in \Lambda} d^2 X_j + 2\gamma \int \eta_i \xi_i x_i \cdot x_i k''(\gamma x_i \cdot x_i) k'''(\gamma x_i \cdot x_i) e^{-2\Phi} \prod_{j \in \Lambda} d^2 X_j$$
$$+ \int \eta_i \xi_i [k''(\gamma x_i \cdot x_i)]^2 e^{-2\Phi} \prod_{j \in \Lambda} d^2 X_j = 0,$$

where we used anticommutativity: $(\eta_i \xi_i)^2 = 0$. Using (2.9) to do the $d\eta$, $d\xi$ integration, we obtain

$$\begin{aligned} &\gamma \int \eta_i \xi_i x_i \cdot x_i k''(\gamma x_i \cdot x_i) k'''(\gamma x_i \cdot x_i) e^{-2\Phi} \prod_{j \in \Lambda} d^2 X_j \\ =&\gamma \int x_i^2 k''(\gamma x_i \cdot x_i) k'''(\gamma x_i \cdot x_i) e^{-2\phi_0} \int \eta_i \xi_i e^{-(\sum_{j \in \Lambda} E \eta_j \xi_j - \sum_{j,k \in \Lambda, |j-k|_1 = 1} \eta_j \xi_k - \gamma \sum_{j \in \Lambda} k'(\gamma x_j \cdot x_j) \eta_j \xi_j)} \prod d\eta \\ =&\gamma \int x_i \cdot x_i k''(\gamma x_i \cdot x_i) k'''(\gamma x_i \cdot x_i) \big(E - \Delta - \gamma k'(\gamma x_i \cdot x_i)\big)^{-1}_{ii} \prod_j \frac{d^2 x_j}{\pi} \\ &= \mathbf{O}(1), \end{aligned}$$

uniformly in Λ, where we used ( ) to expand 2Φ:

$$2\phi_0 = \sum_{j \in \Lambda} E x_j \cdot x_j - \sum_{j,k \in \Lambda, |j-k|_1 = 1} x_j \cdot x_k - \sum_{j \in \Lambda} k(\gamma x_j \cdot x_j)$$

, we used ( ) and the definition of ϕ in ( ) to reach the second equality, and finally (3.1) and the fact that $E \in I$ defined in (1.3), to reach the last estimate. Similarly we obtain

$$\int \eta_i \xi_i [k''(\gamma x_i^2)]^2 e^{-2\Phi} \prod_{j\in\Lambda} d^2 X_j = \mathbf{O}(1),$$

uniformly in Λ.

Integrating out the anti-commutative variables in the first integral in ( ) and combinning with the estimates in ( ) and ( ), we then obtain that

$$\int x_i^2 [k''(\gamma x_i^2)]^2 e^{-2\phi} \prod \frac{d^2 x_j}{\pi} = \mathbf{O}(1)$$

uniformly in Λ. Therefore $\langle (a_3')^2 \rangle = \mathbf{O}(\gamma^4)\delta_{ij}\delta_{\alpha\beta}$.

Since

$$\begin{aligned}&\frac{\partial}{\partial x_\ell^{(\alpha)}}(E-\Delta-\gamma \operatorname{diag} k')^{-1}_{ij}\\ =&2(E-\Delta-\gamma \operatorname{diag} k')^{-1}_{i\ell}\gamma^2 x_\ell^{(\alpha)} k''(\gamma x_\ell \cdot x_\ell)(E-\Delta-\gamma \operatorname{diag} k')^{-1}_{\ell j},\end{aligned}$$

we see that the other terms in ( ) can be estimated in the same way by iterating the resolvent identity:

$$(E-\Delta-\gamma \operatorname{diag} k')^{-1} = (E-\Delta)^{-1} + \gamma(E-\Delta)^{-1}(\operatorname{diag} k'(\gamma x_j \cdot x_j))(E-\Delta-\gamma \operatorname{diag} k')^{-1},$$

and supersymmetric equalities similar to ( ). (In fact for the purpose of the proof of this Lemma, we do not need to consider terms containing $(E-\Delta-\gamma \operatorname{diag} k')^{-1}$, as they are of order $\mathbf{o}(\gamma^2)$ pointwise, even before integration. However we will have the need to estimate them in Proposition 4.4 later.)

Let π: $\bar{\Lambda} = \Lambda \times \{1,2\} \to \Lambda$. Since $(E-\Delta-\gamma \operatorname{diag} k')^{-1}$ and also its various derivatives are pointwise uniformly bounded operators in appropriate exponentially weighted space for $E \in I$ defined in (1.3), combinning ( )-( ), we obtain

$$\|e^{-\phi} d\phi''_{ik}\| = \mathbf{O}(\gamma^2) e^{-\alpha|i'-k'|} \tag{4.11}$$

for some $\alpha > 0$ uniform in Λ, where

$$\begin{aligned} i' &= \pi i \in \Lambda \\ k' &= \pi k \in \Lambda. \end{aligned}$$

Note that by using supersymmetry we gained $\mathbf{O}(\gamma^{1/2})$ in ( ); the naive estimate would have been $\mathbf{O}(\gamma^{3/2}$.

Using (4.11) in (4.10), we obtain

$$\|(\phi'' - \langle\phi''\rangle)R_- w\|^2 = \mathbf{O}(\gamma^4)\sum_{j,k} w_j w_k e^{-\alpha|j-k|} = \mathbf{O}(\gamma^4)\|w\|^2.$$

Taking the square root on both sides, we obtain the lemma. □

Remark. Lemma 4.2 should be seen as a statement of the "small" variation of ϕ'', and indeed any function f whose derivative f is "small". We will see more general versions of (4.7) later in Lemma 4.8.

Proof of Proposition 4.1. Let

$$\mathcal{P}(z) \stackrel{\text{def}}{=} \begin{pmatrix} \Delta_\phi^{(1)} - z & R_- \\ R_+ & 0 \end{pmatrix}. \tag{4.12}$$

It is a self-adjoint operator with domain $\mathcal{D}(\Delta_\phi^{(1)}) \times \ell^2$. Hence it is enough to establish (4.6). We write out (4.7) separately :

$$\begin{aligned} (\Delta_\phi^{(1)} - z)u + R_- u_- &= v \\ R_+ u &= v_+. \end{aligned} \tag{4.13}$$

We first look at the special case $v_+ = 0$. (Later we reduce the general case to the special case).

Taking the inner product with u in the first equation in (4.13), we obtain

$$\big(u, (\Delta_\phi^{(1)} - z)u\big) + (u, R_- u_-) = (u, v). \tag{4.14}$$

Using the second equation in (4.13) with $v_+ = 0$ and the fact that $R_-^* = R_+$, we see that $(u, R_- u_-) = 0$ and we have

$$\big(u, (\Delta_\phi^{(1)} - z)u\big) = (u, v).$$

So $\frac{1}{\mathbf{O}(1)}\|u\|^2 \le \big(u, (\Delta_\phi^{(1)} - z)u\big) = (u, v) \le \|u\|\ \|v\|$ where we used (4.4) and the assumption on z. Hence

$$\|u\| \le \mathbf{O}(1)\|v\|. \tag{4.15}$$

We now take the inner product with $R_- u_-$ in the first equation in (4.13) :

$$\big(R_- u_-, (\Delta_\phi^{(1)} - z)u\big) + (R_- u_-, R_- u_-) = (R_- u_-, v). \tag{4.16}$$

Since $(\Delta_\phi^{(0)} \otimes I)R_- u_- = 0$, $\|(\phi'' - \langle\phi''\rangle)R_- u_-\| = \mathbf{O}(\gamma^2)\|u_-\|$ by using Lemma 4.2, we have

$$\begin{aligned} \big(R_- u_-, (\Delta_\phi^{(1)} - z)u\big) &= (\langle\phi''\rangle R_- u_-, u) + \mathbf{O}(\gamma^2)\|u_-\|\ \|u\| \\ &= \mathbf{O}(\gamma^2)\|u_-\|\ \|u\| \end{aligned} \tag{4.17}$$

where we used the fact that $R_+\langle\phi''\rangle u = \langle\phi''\rangle R_+ u = 0$. Hence (4.14) gives

$$\|u_-\|^2 - \mathbf{O}(\gamma^2)\|u_-\| \ \|u\| \leq \|u_-\| \ \|v\|$$

where we used $\|R_- u_-\|^2 = \|u_-\|^2$. So

$$\begin{aligned} \|u_-\| &\leq \|v\| + \mathcal{O}(\gamma^2)\|u\| \\ &= \mathbf{O}(1)\|v\| \end{aligned} \tag{4.18}$$

We now consider (4.13) in general. Since $R_+R_- = I$, we obtain

$$\begin{aligned} (\Delta_\phi^{(1)} - z)(u - R_-v_+) + R_-u_- &= v + (z-\phi'')R_-v_+ \\ R_+(u - R_-v_+) &= 0 \end{aligned} \tag{4.19}$$

which is of the type (4.13) with $v_+ = 0$.

Applying the estimates (4.13) and (4.16) to (4.17), we have

$$\begin{aligned} \|u - R_-v_+\| + \|u_-\| &= \mathbf{O}(1)(\|v + (Z-\phi'')R_-v_+\|) \\ &= \mathbf{O}(1)(\|v\| + \|v_+\|). \end{aligned}$$

Since

$$\begin{aligned} \|u - R_-v_+\| \geq& \|u\| - \|R_-v_+\| \\ \geq& \|u\| - \|v_+\|, \end{aligned}$$

we obtain

$$\|u\| + \|u_-\| = \mathbf{O}(1)(\|v\| + \|v_+\|). \tag{4.20}$$

□

The above proposition shows that the self-adjoint Grushin operator

$$\mathcal{P} = \begin{pmatrix} \Delta_\phi^{(1)} - z & R_- \\ R_+ & 0 \end{pmatrix}$$

introduced in (4.12) is invertible for such z, γ. Let $\mathcal{E}$ denote the inverse of $\mathcal{P}$:

$$\mathcal{E} \stackrel{\text{def}}{=} \mathcal{P}^{-1} \stackrel{\text{def}}{=} \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \tag{4.21}$$

where $E_- = E_+^*$, $\mathcal{E} \in \mathcal{L}\big(L^2(\mathbf{R}^{2\Lambda}, \wedge\mathbf{R}^{2\Lambda}); L^2(\mathbf{R}^{2\Lambda}, \wedge\mathbf{R}^{2\Lambda})\big)$.

We then have the following identity :

$$(\Delta_\phi^{(1)} - z)^{-1} = E - E_+E_{-+}^{-1}E_-, \tag{4.22}$$

where E, E_+, E_- and E_{-+} are analytic in z. Therefore (4.22) implies that

$$z \in \sigma(\Delta_\phi^{(1)}) \text{ iff } 0 \in \sigma(E_{-+}).$$

E_{-+} is our effective operator.

Using Lemma 4.2, we have moreover the following approximations for E_{-+}, E_+, E_- :

Proposition 4.3. *For every $C > 0$, there exists $\gamma_0 > 0$, such that for all $z \leq 2\big(r_0 + r_E - \frac{1}{C}\big)$, $|z| = \mathbf{O}(1)$, all $0 \leq \gamma \leq \gamma_0$,*

$$\begin{aligned} &\|E_{-+} - (z - 2\langle \phi'' \rangle)\|_{\mathcal{L}(\ell^2,\ell^2)} = \mathbf{O}(\gamma^2), \\ &\|E_{+} - R_{-}\|_{\mathcal{L}(\ell^2,L^2)} = \mathbf{O}(\gamma^2), \\ &\|E_{-} - R_{+}\|_{\mathcal{L}(L^2,\ell^2)} = \mathbf{O}(\gamma^2). \end{aligned} \tag{4.23}$$

Proof. From (4.21)

$$\begin{aligned} u &= Ev + E_+ v_+ \\ u_- &= E_- v + E_{-+} v_+. \end{aligned} \tag{4.24}$$

Take $v = 0$ and let

$$\begin{aligned} E_+^0 &= R_- \\ E_-^0 &= R_+ \\ E_{-+}^0 &= z - 2\langle \phi'' \rangle. \end{aligned} \tag{4.25}$$

Take as approximate solutions :

$$\begin{aligned} u^0 &= E_+^0 v_+ = R_- v_+ \\ u_-^0 &= E_{-+}^0 v_+ = (z - 2\langle \phi'' \rangle) v_+. \end{aligned} \tag{4.26}$$

We then have

$$\begin{aligned} (\Delta_\phi^{(1)} - z)(u^0 - u) + R_-(u_-^0 - u_-) &= (\Delta_\phi^{(1)} - z) R_- v_+ + R_-(z - 2\langle \phi'' \rangle) v_+ \\ &= 2(\phi'' - \langle \phi'' \rangle) R_- v_+, \end{aligned} \tag{4.27}$$

where we used the fact that $v = 0$, and

$$R_+(u^0 - u) = R_+(R_- v_+ - u) = 0. \tag{4.28}$$

Applying Proposition 4.1 to the system of equations comprised of (4.27) and (4.28), we have

$$\begin{aligned} \|u^0 - u\| + \|u_-^0 - u_-\| &= \mathbf{O}(1) \|(\phi'' - \langle \phi'' \rangle) R_- v_+\| \\ &= \mathbf{O}(\gamma^2) \|v_+\| \end{aligned} \tag{4.29}$$

where we used Lemma 4.2 to obtain the last inequality. Combining (4.29) with (4.24) and (4.26), we then obtain the Lemma. $\square$

2. A first Grushin problem for $\Delta_\phi^{(0)}$.

As mentioned earlier, in order to obtain an accuracy which is of $\mathbf{O}(\gamma^2)$, we need to also use (a side from $e^{-\phi}dx_j$) $z_k^* e^{-\phi}dx_j$, $z_k^* z_\ell^* e^{-\phi}dx_j$ $(k \geq \ell)$ (after orthonormalization) as our Grushin basis. We do this in two steps. We first add $z_k^* e^{-\phi}dx_j$ to our basis. In order to obtain the correct analogue of the lower bound in (4.4) for $u \perp e^{-\phi}dx_j$, $z_k^* e^{-\phi}dx_j$, for all j, $k \in \bar{\Lambda}$, we need to study more in depth the spectrum of $\Delta_\phi^{(0)}$. As before, we shall exploit in an essential way the relation between the spectra of $\Delta_\phi^{(0)}$ and $\Delta_\phi^{(1)}$.

We start by posing a variant of the Grushin problem in (4.7) :

$$() \qquad \begin{pmatrix} \Delta_\phi^{(1)} - z & \langle 2\phi''\rangle^{-1} d_\phi d_\phi^* R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}$$

where $\langle 2\phi''\rangle^{-1}$ means the inverse of the matrix $\langle 2\phi''\rangle$ (Recall also that we have identified 1-forms with $\mathbf{R}^{\bar{\Lambda}}$ valued functions.); $u \in \mathcal{D}(\Delta_\phi^{(1)})$, $v \in L^2$, $u_-, v_+ \in \ell^2$; R_+, R_- are as defined before in (4.5,4.6). The interest of the above Grushin problem is that once its well posedness is established, it implies the well posedness of a related Grushin problem for $\Delta_\phi^{(0)}$ (Proposition 4.5), which in turn gives us the desired spectral information on $\Delta_\phi^{(0)}$.

Proposition 4.4. *For every $C > 0$, there exists $\gamma_0 > 0$ such that the problem (4.30) is well posed for all $z \leq 2\big(r_0 + r_E - \frac{1}{C}\big)$, $0 \leq \gamma \leq \gamma_0$. Moreover*

$$\|u\| + \|u_-\| \leq \mathbf{O}(1)(\|v\| + \|v_+\|). \tag{4.31}$$

Proof. We prove the proposition by reference to the well-posedness of the Grushin problem in Proposition 4.1.

(4.30) can be written as

$$\begin{cases} (\Delta_\phi^{(1)} - z)u + R_- u_- = v + (1 - \langle 2\phi''\rangle^{-1} d_\phi d_\phi^*) R_- u_- \\ \qquad\qquad\qquad R_+ u = v_+. \end{cases} \tag{4.32}$$

From Proposition 4.1, we have

$$\|u\| + \|u_-\| \leq \mathbf{O}(1)\big(\|v\| + \|(1 - \langle 2\phi''\rangle^{-1} d_\phi d_\phi^*) R_- u_-\| + \|v_+\|\big). \tag{4.33}$$

Since

$$\begin{aligned} d_\phi d_\phi^* R_- u_- &= d_\phi^* d_\phi R_- u_- + [d_\phi, d_\phi^*] R_- u_- \\ &= 2\phi'' R_- u_-, \end{aligned} \tag{4.34}$$

the second term in the RHS can be written as

$$\begin{aligned}(4.35)\qquad \|(1-\langle 2\phi''\rangle^{-1}d_\phi d_\phi^*)R_-u_-\| &= \|\langle 2\phi''\rangle^{-1}(\langle 2\phi''\rangle - 2\phi'')R_-u_-\| \\ &\le \|\langle \phi''\rangle^{-1}\|\ \|(\phi''-\langle\phi''\rangle)R_-u_-\| \\ &\le \mathbf{O}(\gamma^2)\|u_-\|\end{aligned}$$

where we used Lemma 4.2 to reach the last inequality. Using (4.35) in (4.33), we have

$$\|u\| + \|u_-\| \le \mathbf{O}(1)(\|v\| + \|v_+\|) + \mathbf{O}(\gamma^2)\|u_-\|.$$

So

$$\|u\| + \|u_-\| \le \mathbf{O}(1)(\|u\| + \|v_+\|)$$

for $z \le 2\big(r_0 + r_E - \frac{1}{C}\big)$ and small enough γ. This proves uniqueness. To prove existence, we write

(4.36)

$$\begin{aligned}\begin{pmatrix} \Delta_\phi^{(1)} - z & \langle 2\phi''\rangle^{-1}d_\phi d_\phi^* R_- \\ R_+ & 0\end{pmatrix} &= \begin{pmatrix} \Delta_\phi^{(1)} - z & R_- \\ R_+ & 0\end{pmatrix} + \begin{pmatrix} 0 & \langle 2\phi''\rangle^{-1}d_\phi d_\phi^* R_- - R_- \\ 0 & 0\end{pmatrix} \\ &\stackrel{\text{def}}{=} \mathcal{P} + A.\end{aligned}$$

Since $\mathcal{P}$ is invertible with $\|\mathcal{P}^{-1}\| = \mathbf{O}(1)$ from Proposition 4.1, and $\|A\| = \mathbf{O}(\gamma^2)$, from Lemma 4.2, we obtain the existence and boundedness of

$$\begin{pmatrix} \Delta_\phi^{(1)} - z & \langle 2\phi''\rangle^{-1}d_\phi d_\phi^* R_- \\ R_+ & 0\end{pmatrix}^{-1},$$

hence the existence of solutions to (4.30). □

We now come to the main reason why we posed (4.30) in the first place, which is to use (4.30) to establish the well-posedness of the Grushin problem for $\Delta_\phi^{(0)}$ in the following Proposition 4.5.

Let

$$(4.37)\qquad \begin{aligned} e_0 &= e^{-\phi} \\ \begin{pmatrix} e_1 \\ \vdots \\ e_{2\Lambda}\end{pmatrix} &= (2\langle\phi''\rangle)^{-1/2}\begin{pmatrix} z_1^* e^{-\phi} \\ \vdots \\ z_{2\Lambda}^* e^{-\phi}\end{pmatrix}\end{aligned}$$

be the new Grushin basis. Note that $(e_j, e_k) = \delta_{jk}$, $j,k \in \{0, \bar{\Lambda}\}$. Then we have

Proposition 4.5. *For every $C > 0$, there exists $\gamma_0 > 0$ such that the following Grushin problem for $\Delta_\phi^{(0)}$:*

$$(\Delta_\phi^{(0)} - z)u + \sum_{j=0}^{2\Lambda} u_-(j)e_j = v$$

$$(u, e_j) = v_+(j), \quad j \in \{0, \bar{\Lambda}\} \tag{4.38}$$

where $u \in \mathcal{D}(\Delta_\phi^0)$, $v \in L^2(\mathbf{R}^{2\Lambda})$, u_-, $v_+ \in \big(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\big)$ *is well posed for all* $z \leq 2\big(r_0 + r_E - \frac{1}{C}\big)$, $0 \leq \gamma \leq \gamma_0$. *Moreover*

$$\|u\| + \|u_-\| \leq \mathbf{O}(1)(\|v\| + \|v_+\|). \tag{4.39}$$

To prove Proposition 4.5, we specialize to the case where v is a closed form : $d_\phi v = 0$, $v \in \mathcal{S}$ in (4.38). We first need to prove :

Lemma 4.6. *For* C, γ_0, z *such that Proposition 4.4 holds, if* v *is a closed form :* $d_\phi v = 0$, $v \in \mathcal{S}$, *then so is* $u : d_\phi u = 0$.

Proof. Applying d_ϕ to the first equation in (4.30), we obtain

$$(\Delta_\phi^{(2)} - z) d_\phi u = 0. \tag{4.40}$$

Assuming $d_\phi u \neq 0$, applying d_ϕ^* to (4.40), we have that

$$(\Delta_\phi^{(1)} - z) d_\phi^* d_\phi u = 0. \tag{4.41}$$

But $(d_\phi^* d_\phi u, e^{-\phi} dx_j) = 0$, which implies

$$d_\phi^* d_\phi u \perp e^{-\phi} dx_j, \quad \text{for all } \ j \in \overline{\Lambda}. \tag{4.42}$$

(4.41) thus contradicts the established inequality (4.4). We therefore conclude that

$$d_\phi u = 0.$$

$\square$

Proof of Proposition 4.5. Let v be a closed 1−form in (4.30). Since $d_\phi \stackrel{\text{def}}{=} e^{-\phi} d e^{\phi}$ and we work in $\mathbf{R}^{2\Lambda}$, closed forms are exact. Therefore if $d_\phi v = 0$, then $v = d_\phi \tilde{v}$, for some $\tilde{v} \in \mathcal{S}$. Lemma 4.6 then implies that $u = d_\phi \tilde{u}$ for some $\tilde{u} \in \mathcal{S}$. Note there is no uniqueness for $\tilde{u}$, if $\tilde{u}$ is a solution, then so is $\tilde{u} + C e^{-\phi}$, for all C. Therefore in order to specify $\tilde{u}$ uniquely, we need to specify C.

Let $\tilde{v}$ be a given function in $\mathcal{S}(\mathbf{R}^{2\Lambda})$, v_+ a given vector in $\mathbf{R} \oplus \ell^2(\bar{\Lambda})$. We consider the Grushin problem as in (4.30) with $v = d_\phi \tilde{v}$. Let w be a 1−form $\in \mathcal{S}$. Taking the scalar product of the first equation in (4.30) with w, we obtain

$$(w, (\Delta_\phi^{(1)} - z) d_\phi \tilde{u}) + (w, \langle 2\phi'' \rangle^{-1} d_\phi d_\phi^* R_- u_-) = (w, d_\phi \tilde{v}). \tag{4.43}$$

Note that there is an ambiguity when we write the solution u as $d_\phi \tilde{u}$, which we clarify by imposing the condition

$$R_+ \tilde{u} = (e^{-\phi}, \tilde{u}) = v_+(0). \tag{4.44}$$

For notational simplicity, we now drop the tilde. We have from (4.43)

$$(4.45)\qquad (d_\phi^* w, (\Delta_\phi^{(0)} - z)u) + (d_\phi^* w, \langle 2\phi''\rangle^{-1} d_\phi^* R_- u_-) = (d_\phi^* w, v)$$

for all 1−forms w. Since d_ϕ^*: $\mathcal{S}(\wedge \mathbf{R}^{2\Lambda}) \to \mathcal{S}(\mathbf{R}^{2\Lambda})$ and $d_\phi^* w = 0$ iff $w = 0$, and $(d_\phi^* w, f) = 0$ for all w, iff $f = Ce^{-\phi}$, we obtain from (4.45) the following equation

$$(4.46)\qquad (\Delta_\phi^{(0)} - z)u + (\langle 2\phi''\rangle^{-1} d_\phi^*) R_- u_- + R_- u_-(0) = v.$$

The second equation in (4.30) gives

$$(4.47)\qquad R_+ d_\phi u = v_+.$$

The well-posedness of the Grushin problem (4.30), then implies the well posedness of the Grushin problem composed of (4.46), (4.47) and (4.44). Let e_0 denote the function $e^{-\phi}$, then

$$(4.48)\qquad (\langle 2\phi''\rangle^{-1} d_\phi^*) R_- u_- + R_- u_-(0) = \sum_{j,k} \langle 2\phi''\rangle_{jk}^{-1} z_k^* e_0 u_-(j) + e_0 u_-(0).$$

Define $\begin{pmatrix} e_1 \\ \vdots \\ e_{2\Lambda} \end{pmatrix} = [\langle 2\phi''\rangle^{-1/2}] \begin{pmatrix} z_1^* e_0 \\ \vdots \\ z_{2\Lambda}^* e_0 \end{pmatrix}$, from (4.46), (4.47) and (4.48) we then get the well-posedness of the Grushin problem (4.38), with the $v_+(0)$ there equals to the $v_+(0)$ in (4.44) and the $v_+(j)$, $(j \neq 0)$ there equals to $\sum_k \langle 2\phi''\rangle_{jk}^{-1/2} v_+(k)$ of the $v_+(k)$ defined in (4.30). $\square$

Remark. Although we did not state it that way, the earlier remark that the second eigenvalue of $\Delta_\phi^{(0)} \geq 2r_0$ implies the well-posedness of

$$(4.49)\qquad \begin{cases} (\Delta_\phi^{(0)} - z)u + u_-(0)e_0 = v \\ \qquad\qquad (u, e_0) = v_+(0) \end{cases}$$

for $z \leq 2\big(r_0 - \frac{1}{\mathbf{O}(1)}\big)$ and sufficiently small γ. This leads to the well-posedness of the Grushin problem for $\Delta_\phi^{(1)}$ in (4.7) for z in *the larger domain* : $z \leq 2\big(r_0 + r_E - \frac{1}{\mathcal{O}(1)}\big)$ (Proposition 4.1), which in turn implies the well-posedness of the Grushin problem for $\Delta_\phi^{(0)}$ in (4.38) in the larger domain : $z \leq 2\big(r_0 + r_E - \frac{1}{\mathbf{O}(1)}\big)$. Clearly, we have started an iteration process.

3. The second Grushin problem for $\Delta_\phi^{(1)}$.

Bearing in mind the preceeding remark, the proposition below should come as no surprise.

Let $e_\alpha dx_j$, (all $\alpha \in \{0,\bar\Lambda\}$, $j \in \bar\Lambda$) be our Grushin basis. (From now on, to avoid confusion we use latin letters only to denote an element of $\bar\Lambda$. For elements of other larger sets, we generally use greek letters). Define $R_+ : L^2(\mathbf{R}^{2\Lambda}) \to \big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda)$ by

$$(R_+u)(j,\alpha) = (u, e_\alpha dx_j) = (u_j, e_\alpha),\ \ j\in\bar\Lambda,\ \ \alpha\in\{0,\bar\Lambda\}, \tag{4.50}$$

where $u = \sum_{j\in\bar\Lambda} u_j dx_j$. Let $R_- = R_+^*$ as usual. Note that $\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda)$ is of dimension $2|\Lambda|(2|\Lambda|+1)$. We have

Proposition 4.7. *For every $C>0$, there exists $\gamma_0>0$ such that the Grushin problem*

$$\begin{pmatrix} \Delta_\phi^{(1)} - z & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} \tag{4.51}$$

where $u\in\mathcal{D}(\Delta_\phi^{(1)})$, $v\in L^2(\mathbf{R}^{2\Lambda};\wedge\mathbf{R}^{2\Lambda})$, $u_-, v_+ \in \big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda)$ is well posed for all $z\le 2\big(r_0+2r_E-\frac{1}{C}\big)$, $0\le\gamma\le\gamma_0$. Moreover

$$\|u\|+\|u_-\| \le \mathbf{O}(1)(\|v\|+\|v_+\|). \tag{4.52}$$

In order to prove the above Proposition, we need the following lemma on small variations (It is instructive to compare with Lemma 4.2) :

Lemma 4.8. *Let $u_-\in\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda)$. Define*

$$\overline{\phi}'' \in \mathcal{L}\left(\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda); \big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)\otimes\ell^2(\bar\Lambda)\right)$$

to be

$$\overline{\phi}''_{j,\beta;k,\alpha} = (e_\beta, \phi''_{jk}e_\alpha),\ \ j,k\in\bar\Lambda,\ \ \alpha,\beta\in\{0,\bar\Lambda\}. \tag{4.53}$$

Define $M\in\mathcal{L}\big(\mathbf{R}\oplus\ell^2(\bar\Lambda);\mathbf{R}\oplus\ell^2(\bar\Lambda)\big)$ to be

$$M_{\alpha\beta} = (e_\alpha, \sum_{j\in\bar\Lambda} z_j^* z_j e_\beta),\ \ \alpha,\beta\in\{0,\bar\Lambda\} \tag{4.54}$$

(Note that $M_{\alpha\beta}=0$ if $\alpha=0$ or $\beta=0$ or both.)

We have

$$\|\phi'' R_- u_- - R_-\overline{\phi}'' u_-\| = \mathbf{O}(\gamma^2)\|u_-\| \tag{4.55}$$

where $R_-=R_+^*$, R_+^* is as defined in (4.50) and

$$\Big\|\Big(\sum z_j^* z_j\Big)e_\alpha - \sum_{\beta\in\{0,\bar\Lambda\}} M_{\alpha\beta}e_\beta\Big\| = \mathbf{O}(\gamma^2) \tag{4.56}$$

Proof. Define $\tilde{\phi}'' \in \mathcal{L}\left(\left(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\right) \otimes \ell^2(\bar{\Lambda}); \left(\mathbf{R} \oplus \ell^2(\bar{\Lambda})\right) \otimes \ell^2(\bar{\Lambda})\right)$ to be

$$\tilde{\phi}''_{j,\beta,k,\alpha} = \langle \phi''_{jk} \rangle \delta_{\beta\alpha}. \tag{4.57}$$

We first show that

$$\|R_-(\overline{\phi}'' - \tilde{\phi}'')u_-\| = \|(\overline{\phi}'' - \tilde{\phi}'')u_-\| = \mathbf{O}(\gamma^2)\|u_-\|. \tag{4.58}$$

Assume $\alpha \neq 0$, $\beta \neq 0$,

$$\begin{aligned} \overline{\phi}''_{j,\beta;k,\alpha} &= (e_\beta, \phi''_{jk} e_\alpha) \\ &= \sum_{\omega,\kappa} \langle 2\phi'' \rangle^{-1/2}_{\beta\omega} \langle 2\phi'' \rangle^{-1/2}_{\alpha\kappa} (z^*_\omega e_0, \phi''_{jk} z^*_\kappa e_0) \\ &= \sum_{\omega,\kappa} \langle 2\phi'' \rangle^{-1/2}_{\beta\omega} \langle 2\phi'' \rangle^{-1/2}_{\alpha\kappa} (e_0, z_\omega \phi''_{jk} z^*_\kappa e_0) \\ &= \sum_{\omega,\kappa} \langle \phi'' \rangle^{-1/2}_{\beta\omega} \langle \phi'' \rangle^{-1/2}_{\alpha\kappa} \left(\langle \phi''_{jk} \phi''_{\omega\kappa} \rangle + \frac{1}{2} \langle \phi^{(4)}_{jk\omega\kappa} \rangle \right). \end{aligned} \tag{4.59}$$

So

$$\begin{aligned} (\overline{\phi}'' - \tilde{\phi}'')_{j,\beta;k,\alpha} = &\sum_{\omega,\kappa} \langle \phi'' \rangle^{-1/2}_{\beta\omega} \langle \phi'' \rangle^{-1/2}_{\alpha\kappa} \left(\langle \phi''_{jk} \phi''_{\omega\kappa} \rangle - \langle \phi''_{jk} \rangle \langle \phi''_{\omega\kappa} \rangle \right) \\ &+ \frac{1}{2} \sum_{\omega,\kappa} \langle \phi'' \rangle^{-1/2}_{\beta\omega} \langle \phi'' \rangle^{-1/2}_{\alpha\kappa} \langle \phi^{(4)}_{jk\omega\kappa} \rangle \qquad (\alpha \neq 0,\ \beta \neq 0), \end{aligned} \tag{4.60}$$

(4.61)
$$(\overline{\phi}'' - \tilde{\phi}'')_{j,0;k,0} = 0, \ \ (\overline{\phi}'' - \tilde{\phi}'')_{j,0;k,\ell} = (\overline{\phi}'' - \tilde{\phi}'')_{j,\ell;k,0} = \sum_m \langle 2\phi'' \rangle^{-1/2}_{\ell m} \langle \phi'''_{jmk} \rangle = 0,$$

where to reach the last equality we used the fact that $\phi(x) = \phi(-x)$ and hence $\phi'''(x) = -\phi'''(-x)$. Using Lemma 4.2 to approximate the first term in (4.60) and a straight forward computation to approximate the second term in (4.60), and since $\phi^{(4)} = \mathbf{O}(\gamma^2)$ pointwise, we obtain (4.57).

Using (4.57) (4.58) in (4.55), (4.6), we see that it is sufficient to prove

$$\|(\phi'' - \langle \phi'' \rangle) R_- u_-\| = \mathbf{O}(\gamma^2)\|u_-\|. \tag{4.61a}$$

Write

$$\|(\phi'' - \langle \phi'' \rangle) R_- u_-\|^2 = \left((\phi'' - \langle \phi'' \rangle) R_- u_-, (\phi'' - \langle \phi'' \rangle) R_- u_- \right). \tag{4.62}$$

Once again, we use our time-honored philosophy to write

$$(\phi'' - \langle \phi'' \rangle) R_- u_- - (e_0, (\phi'' - \langle \phi'' \rangle) R_- u_-) e_0 = \Delta^{(0)}_\phi \tilde{u} \tag{4.63}$$

for some $\tilde{u}$. Let A denote the LHS of (4.63). Since the second term in A is $\mathbf{O}(\gamma^2)\|u_-\|$, in order to prove (4.61a), it is sufficient to prove

$$\|A\| = \mathbf{O}(\gamma^2)\|u_-\|. \tag{4.64}$$

Using our usual strategy, we write

(4.65)

$$\begin{aligned}\|A\|^2 &= \sum_{j,k,\ell}\Big(\sum_\alpha u_-(k,\alpha)d_\phi[(\phi''_{jk}-\langle\phi''_{jk}\rangle)e_\alpha],(\Delta^{(1)}_\phi)^{-1}\Big(\sum_\beta u_-(\ell,\beta)d_\phi[(\phi''_{j\ell}-\langle\phi''_{j\ell}\rangle)e_\beta]\Big)\Big)\\ &\le \mathbf{O}(1)\sum_{j,k,\ell}\Big\|\sum_\alpha u_-(k,\alpha)d_\phi[(\phi''_{jk}-\langle\phi''_{jk}\rangle)e_\alpha]\Big\|\;\Big\|\sum_\beta u_-(\ell,\beta)d_\phi[\phi''_{j\ell}-\langle\phi''_{j\ell}\rangle e_\beta]\Big\|\end{aligned}$$

where we used the fact that $\|(\Delta^{(1)}_\phi)^{-1}\| = \mathbf{O}(1)$.

Define $\omega_\alpha \stackrel{\text{def}}{=} d_\phi[(\phi''_{jk}-\langle\phi''_{jk}\rangle)e_\alpha]$ ($\alpha\in\{0,\bar\Lambda\}$). Then

$$\begin{aligned}\omega_\alpha &= \sum_i z_i[(\phi''_{jk}-\langle\phi''_{jk}\rangle)e_\alpha]dx_i\\ &= \sum_i\sum_{\alpha'}(\phi''_{jk}-\langle\phi''_{jk}\rangle)\langle\phi''\rangle^{-1}_{\alpha\alpha'}\phi''_{i\alpha'}e_0dx_i + \sum_i\phi''_{ijk}e_\alpha dx_i \ \text{ if } \alpha\ne 0\\ &= \sum_i \phi'''_{ijk}e_0dx_i \ \text{ if } \alpha = 0.\end{aligned}\tag{4.66}$$

Define $W_{\alpha\beta} = (\omega_\alpha,\omega_\beta)$, $\alpha,\beta\in\{0,\bar\Lambda\}$. Then applying our usual philosophy and proceed as in Lemma 4.2 to the presently slighty more complicated situation, we obtain

$$\|W\| = \mathbf{O}(\gamma^4)e^{-2\delta|j-k|} \ \text{ on } \ (\mathbf{R}\oplus\ell^2(\bar\Lambda)) \tag{4.67}$$

for some $\delta>0$ (uniformly in Λ). Let $\tilde{u}_-(k) = \sqrt{\sum_\alpha u_-^2(k,\alpha)}$, ($k\in\bar\Lambda$). Then substituting (4.67) and (4.66) in (4.65), we obtain

$$\begin{aligned}\|A\|^2 &\le \mathbf{O}(\gamma^4)\sum_{j,k,\ell}\tilde{u}_-(k)\tilde{u}_-(\ell)e^{-\delta|j-k|}e^{-\delta|j-\ell|}\\ &\le \mathbf{O}(\gamma^4)\|u_-\|^2.\end{aligned}\tag{4.68}$$

Taking the square root on both side, we obtain (4.64) and hence (4.55).

We now prove (4.56). The case $\alpha = 0$ is trivially true. We therefore assume $\alpha\ne 0$. We first compute $M_{\alpha\beta}$ for $\alpha\ne 0$, $\beta\ne 0$.

$$\begin{aligned}M_{\alpha\beta} &= (e_\alpha,\sum z_j^* z_j e_\beta)\\ &= \sum_{\alpha',\beta'}\langle\phi''\rangle^{-1/2}_{\alpha\alpha'}\langle\phi''\rangle^{-1/2}_{\beta\beta'}\sum_j(z^*_{\alpha'}e_0,\ z_j^*z_jz^*_{\beta'}e_0)\\ &= 2\sum_{\alpha',\beta'}\langle 2\phi''\rangle^{-1/2}_{\alpha\alpha'}\langle 2\phi''\rangle^{-1/2}_{\beta\beta'}\sum_j\langle\phi''_{j\alpha'}\phi''_{j\beta'}\rangle\\ &= 2\sum_{\alpha',\beta',j}\langle\phi''\rangle^{-1/2}_{\alpha\alpha'}\langle\phi''\rangle^{-1/2}_{\beta\beta'}\langle\phi''_{j\alpha'}\rangle\langle\phi''_{j\beta'}\rangle + \mathbf{O}(\gamma^2)e^{-\delta|\alpha-\beta|}\\ &= 2\langle\phi''\rangle_{\alpha,\beta} + \mathbf{O}(\gamma^2)e^{-\delta|\alpha-\beta|},\ \ (\alpha\ne 0,\ \ \beta\ne 0),\end{aligned}\tag{4.69}$$

where to reach the fourth line from the third, we used Lemma 4.2. Let $F = (\sum z_j^* z_j)e_\alpha - \sum M_{\alpha\beta}e_\beta$. We first note that $(F, e_0) = 0$. Therefore $F = \Delta_\phi^{(0)}\tilde{u}$ for some $\tilde{u}$. Proceeding as usual, we see that

$$\begin{aligned}(F,F) &= (\Delta_\phi^{(0)}\tilde{u}, F)\\ &(d_\phi \tilde{u}, d_\phi F)\\ &((\Delta_\phi^{(1)})^{-1} d_\phi F, d_\phi F).\end{aligned}$$

So

$$\|F\| \le \mathbf{O}(1)\|d_\phi F\|, \tag{4.71}$$

where

$$\begin{aligned} d_\phi F &= \sum z_i\Big(\Big(\sum z_j^* z_j\Big)e_\alpha - \sum M_{\alpha\beta}e_\beta\Big)dx_i \\ &\stackrel{\text{def}}{=} \sum g_i dx_i. \end{aligned} \tag{4.72}$$

Since

$$\begin{aligned} \Big(e_0, z_i\Big(\sum z_j^* z_j\Big)e_\alpha\Big) &= 2^{3/2} \sum_{\alpha'\neq 0, j} \langle\phi''\rangle_{\alpha\alpha'}^{-1/2}(e_0, \phi''_{ij}\phi''_{j\alpha'}e_0) \\ &= 2^{3/2} \sum_{j,\alpha'\neq 0} \langle\phi''\rangle_{\alpha\alpha'}^{-1/2}\langle\phi''_{ij}\rangle\langle\phi''_{j\alpha'}\rangle + \mathbf{O}(\gamma^2)e^{-\delta|i-\alpha|} \\ &= \langle 2\phi''\rangle_{i\alpha}^{3/2} + \mathbf{O}(\gamma^2)e^{-\delta|i-\alpha|} \qquad (\alpha \neq 0), \end{aligned} \tag{4.73}$$

and

$$\begin{aligned} z_i \sum_{\beta\neq 0} M_{\alpha\beta}e_\beta &= \sum_{\beta\beta'\neq 0} M_{\alpha\beta} z_i \langle 2\phi''\rangle_{\beta\beta'}^{-1/2} z_{\beta'}^* e_0 \\ &= \langle 2\phi''\rangle_{i\alpha}^{3/2} e_0 + \mathbf{O}(\gamma^2)e^{-\delta|i-\alpha|} \qquad (\alpha \neq 0), \end{aligned} \tag{4.74}$$

where we used (4.54) to reach the last equality, we obtain that

$$(g_i, e_0) = \mathbf{O}(\gamma^2)e^{-\delta|i-\alpha|}.$$

So

$$\begin{aligned} (g_i, g_i) &= ((\Delta_\phi^{(1)})^{-1} d_\phi g_i, d_\phi g_i) + \mathbf{O}(\gamma^4)e^{-2\delta|i-\alpha|} \\ &= \mathbf{O}(\gamma^4)e^{-2\delta|i-\alpha|}, \end{aligned}$$

$$\|g_i\| = \mathbf{O}(\gamma^2)e^{-\delta|i-\alpha|},$$

by proceeding in a similar way as in the proof of Lemma 4.2. Hence $\|d_\phi F\| = \mathbf{O}(\gamma^2)$. Using (4.72),(4.75) and summing over i, we obtain (4.56). □

Proof of Proposition 4.7. We proceed exactly as in the proof of Proposition 4.1, Proposition 4.5 in place of (4.4) and Lemma 4.8 in place of Lemma 4.2. □

4. The second Grushin problem for $\Delta_\phi^{(0)}$.

We proceed as in subsect. 2 by first posing a variant of the Grushin problem in (4.51). Let

$$\mathcal{M}(j,\beta,k,\alpha) \stackrel{\text{def}}{=} (e_\beta, z_j z_k^* e_\alpha),$$

$$\mathcal{M} \in \mathcal{L}\left(\left(\mathbf{R}\oplus\ell^2(\bar\Lambda)\right)\otimes\ell^2(\bar\Lambda);\left(\mathbf{R}\oplus\ell^2(\bar\Lambda)\right)\otimes\ell^2(\bar\Lambda)\right).$$

Let

$$\tilde{\mathcal{M}}(j,\beta;k,\alpha) = \begin{cases} 2\langle\phi''\rangle_{jk}, & \alpha=\beta=0 \\ 0 & \alpha=0 \text{ or } \beta=0 \text{ but not both} \\ 2(\langle\phi''\rangle_{\beta k}^{1/2}\langle\phi''\rangle_{\alpha j}^{1/2}+\delta_{\alpha\beta}\langle\phi''_{jk}\rangle), & \alpha\neq 0,\ \beta\neq 0. \end{cases} \tag{4.77}$$

Then $\tilde{\mathcal{M}} \in \mathcal{L}\left(\left(\mathbf{R}\oplus\ell^2(\bar\Lambda)\right)\otimes\ell^2(\bar\Lambda);\left(\mathbf{R}\oplus\ell^2(\bar\Lambda)\right)\otimes\ell^2(\bar\Lambda)\right)$ and $\|\mathcal{M}-\tilde{\mathcal{M}}\| = \mathbf{O}(\gamma^2)$. Clearly $\tilde{\mathcal{M}}^{-1}$ exists and $\tilde{\mathcal{M}}^{-1} \in \mathcal{L}\left(\left(\mathbf{R}\oplus\ell^2(\bar\Lambda)\right)\otimes\ell^2(\bar\Lambda);\left(\mathbf{R}\oplus\ell^2(\bar\Lambda)\right)\otimes\ell^2(\bar\Lambda)\right)$. So $\mathcal{M}^{-1}$ exists and $\mathcal{M}^{-1} \in \mathcal{L}\left(\left(\mathbf{R}\oplus\ell^2(\bar\Lambda)\right)\otimes\ell^2(\bar\Lambda);\left(\mathbf{R}\oplus\ell^2(\bar\Lambda)\right)\otimes\ell^2(\bar\Lambda)\right)$.

Proposition 4.9. *For every $C>0$, there exists $\gamma_0>0$ such that the Grushin problem*

$$\begin{pmatrix} \Delta_\phi^{(1)}-z & d_\phi d_\phi^* R_-\mathcal{M}^{-1} \\ R_+ & 0 \end{pmatrix}\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} 4.78$$

where $u\in\mathcal{D}(\Delta_\phi^{(1)})$, $v\in L^2(\mathbf{R}^{2\Lambda};\wedge\mathbf{R}^{2\Lambda})$, u_-, $v_+ \in \left(\mathbf{R}\oplus\ell^2(\bar\Lambda)\right)\otimes\ell^2(\bar\Lambda)$ is well posed for all $z\leq 2\left(r_0+2r_E-\frac{1}{C}\right)$, $0\leq\gamma\leq\gamma_0$. Moreover

$$\|u\|+\|u_-\| \leq \mathbf{O}(1)(\|v\|+\|v_+\|). \tag{4.79}$$

Proof. Similar to the proof of Proposition 4.4, it is enough to prove that for all u_-,

$$\|R_-u_- d_\phi d_\phi^* R_-\mathcal{M}^{-1}u_-\| = \mathbf{O}(\gamma^2)\|u_-\|. \tag{4.80}$$

Since both $\mathcal{M}$, $\mathcal{M}^{-1} \in \mathcal{L}\left(\left(\mathbf{R}\oplus\ell^2(\bar\Lambda)\right)\otimes\ell^2(\bar\Lambda);\left(\mathbf{R}\oplus\ell^2(\bar\Lambda)\right)\otimes\ell^2(\bar\Lambda)\right)$, this is equivalent to prove

$$\|R_-u_- d_\phi d_\phi^* R_-\mathcal{M}^{-1}u_-\| = \mathbf{O}(\gamma^2)\|u_-\|. \tag{}$$

The LHS of () without the norm sign, equals to

(4.82)
$$\sum_{j,k,\alpha,\beta}\mathcal{M}(j,\beta;k,\alpha)u_-(k,\alpha)e_\beta dx_j - \sum_{j,k,\alpha,\beta} z_j z_k^*(u_-(k,\alpha)e_\alpha)dx_j \stackrel{\text{def}}{=} \sum_{j\in\bar\Lambda}\psi_j dx_j.$$

Using our basic philosophy, we proceed essentially as in the proof of Lemma 4.8, which we summarize as follows :

$$
\begin{aligned}
\sum_{j\in\bar\Lambda}\|\psi_j\| &\le \mathbf{O}(1)\sum_{j\in\bar\Lambda}\|d\psi_j\| + \mathbf{O}(\gamma^2)\|u_-\| \\
&\stackrel{\text{def}}{=} \mathbf{O}(1)\sum_{j\in\bar\Lambda}\Big\|\sum\overline{\psi}_{j\ell}dx_\ell\Big\| + \mathbf{O}(\gamma^2)\|u_-\| \\
&= \mathbf{O}(1)\sum_{j,\ell\in\bar\Lambda}\|\overline{\psi}_{j\ell}\| + \mathbf{O}(\gamma^2)\|u_-\| \\
&\le \mathbf{O}(1)\sum_{j,\ell\in\bar\Lambda}\|d\overline{\psi}_{j\ell}\| + \mathbf{O}(\gamma^2)\|u_-\| \\
&\le \mathbf{O}(\gamma^2)\|u_-\|
\end{aligned}
\tag{4.83}
$$

the proposition then follows immediately by repeating the arguments in the proof of Proposition 4.4. □

Let $u_j = z_j^* e_0$, $j\in\bar\Lambda$, $u_0 = e_0$. Define

$$
\bar{\mathcal{M}}(k,\alpha;j,\beta) \stackrel{\text{def}}{=} \frac{1}{2}(z_k^* u_\alpha, z_j^* u_\beta), \qquad k,\, j\in\bar\Lambda, \quad \alpha,\,\beta\in\{0,\bar\Lambda\}.
$$

We consider $\bar{\mathcal{M}}$ as an element of

$$
\big(\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)]\big)^*\otimes\big(\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)]\big),
$$

which in turn can be seen as a tensor in $\big(\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes\ell^2(\bar\Lambda)]\big)^{\otimes^2}$, (separately) symmetric in the first set of indices and the last set of indices (cf. sect III).

$\bar{\mathcal{M}}$ is the restriction to the symmetric subspace: $\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)]$ of an operator, which (with a slight abuse of notation) we also denote by

$$
\bar{\mathcal{M}}\in\mathcal{L}\big(\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes\ell^2(\bar\Lambda)];\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes\ell^2(\bar\Lambda)]\big).
$$

it is easy to see by perturbation theory that $\bar{\mathcal{M}}^{-1}$ exists and

$$
\bar{\mathcal{M}}^{-1}\in\mathcal{L}\big(\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes\ell^2(\bar\Lambda)];\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes\ell^2(\bar\Lambda)]\big),
$$

satisfying

$$
\bar{\mathcal{M}}\bar{\mathcal{M}}^{-1} = \bar{\mathcal{M}}^{-1}\bar{\mathcal{M}} = I\oplus I\otimes I,
$$

where $I_k^i = \delta_{ik}$, $(I\otimes I)_{k\ell}^{ij} = \delta_{ik}\delta_{j\ell}$. When restricted to the symmetric subspace: $\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)]$, we have that (with the same abuse of notation)

$$
\bar{\mathcal{M}}\bar{\mathcal{M}}^{-1} = \bar{\mathcal{M}}^{-1}\bar{\mathcal{M}} = I\oplus I\otimes_s I,
$$

where

$$
(I\otimes_s I)_{k\ell}^{ij} = \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}.
$$

Since $\bar{\mathcal{M}}^{-1} > 0$ as an operator and moreover $\bar{\mathcal{M}}^{-1}$ is a symmetric operator, $\bar{\mathcal{M}}^{-1/2}$ is also well defined:

$$\bar{\mathcal{M}}^{-1/2}\bar{\mathcal{M}}^{-1/2} = \bar{\mathcal{M}}^{-1}.$$

Define

$$e_{k,\alpha} \stackrel{\text{def}}{=} \sum_{j,\beta} 2\mathcal{M}^{-1/2}(k,\alpha;j,\beta) z_j^* u_\beta, \quad (k,j = 1,\ldots,2|\Lambda|, \quad \alpha,\beta = 0,1,\ldots,2|\Lambda|,\ k \neq \alpha),$$

$$e_{k,k} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \sum_{j,\beta} 2\mathcal{M}^{-1/2}(k,k;j,\beta) z_j^* u_\beta.$$

$e_{k,\alpha}$ $(k \geq \alpha)$ are orthonormalized by using ( ). Let $e_{k,\alpha}$ $(k \in \bar{\Lambda}, \alpha \in \{0, \bar{\Lambda}\}, k \geq \alpha)$, e_0 be our Grushin basis. Similar to Proposition 4.5, we arrive at the main conclusion of this subsection :

Proposition 4.10. *For every $C > 0$, there exists $\gamma_0 > 0$ such that the following Grushin problem for $\Delta_\phi^{(0)}$:*

$$\begin{cases} (\Delta_\phi^{(0)} - z)u + \displaystyle\sum_{k \in \bar{\Lambda}, \alpha \in \{0, \bar{\Lambda}\ k \geq \alpha\}} u_-(k,\alpha) e_{k,\alpha} + u_-(0) e_0 = v \\ \qquad (u, e_{k,\alpha}) = v_+(k,\alpha) \\ \qquad (u, e_0) = v_+(0) \end{cases}$$

where $u \in \mathcal{D}(\Delta_\phi^{(0)})$, $v \in L^2(\mathbf{R}^{2\Lambda})$, $u_-, v_+ \in \mathbf{R} \oplus \ell^2(\bar{\Lambda}) \oplus [\ell^2(\bar{\Lambda}) \otimes_s \ell^2(\bar{\Lambda})]$,which is of dimension $1 + 2|\Lambda| + |\Lambda|(2|\Lambda| + 1)$, is well posed for all $z \leq 2\big(r_0 + 2r_E - \frac{1}{C}\big)$, $0 \leq \gamma \leq \gamma_0$. Moreover

$$\|u\| + \|u_-\| \leq \mathbf{O}(1)(\|v\| + \|v_+\|).$$

(The proof is the same as the proof for Proposition 4.5, so we omit it).

5. The third Grushin problem for $\Delta_\phi^{(1)}$ and approximations to $\mathbf{O}(\gamma^4)$.

Let $e_{k,\alpha} dx_j$, $e_0 dx_j$, $(\alpha \in \{0, \bar{\Lambda}\}, k, j \in \bar{\Lambda}, k \geq \alpha)$ defined in ( ) be our Grushin basis. Define $R_+ : L^2(\mathbf{R}^{2\Lambda}; \mathbf{R}^{2\Lambda}) \to \big(\mathbf{R} \oplus \ell^2(\bar{\Lambda}) \oplus [\ell^2(\bar{\Lambda}) \otimes_s \ell^2(\bar{\Lambda})]\big) \otimes \ell^2(\bar{\Lambda})$ by

$$(R_+ u)(j,k,\alpha) = (u, e_{k,\alpha} dx_j) = (u_j, e_{k,\alpha})$$
$$(R_+ u)(j,0) = (u, e_0 dx_j) = (u_j, e_0)$$

where $u = \sum_{j \in \bar{\Lambda}} u_j dx_j$. Let $R_- = R_+^*$ as usual. As a direct consequence of Propositon 4.10, we have

Proposition 4.11. *For every $C>0$, there exists $\gamma_0>0$ such that the Grushin problem*

$$\begin{pmatrix} \Delta_\phi^{(1)}-z & R_- \\ R_+ & 0 \end{pmatrix} \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}$$

where $u\in\mathcal{D}(\Delta_\phi^{(1)})$, $v\in L^2(\mathbf{R}^{2\Lambda};\mathbf{R}^{2\Lambda})$ $u_-,v_+\in\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)]\big)\otimes\ell^2(\bar\Lambda)$, is well posed for all $z\le 2\big(r_0+3r_E-\frac{1}{C}\big)$, $0\le\gamma\le\gamma_0$. Moreover

$$\|u\|+\|u_-\|\le\mathbf{O}(1)(\|v\|+\|v_+\|).$$

As in the proof of Proposition 4.7, we need the following

Lemma 4.12. *Let $u_-\in\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)]\big)\otimes\ell^2(\bar\Lambda)$. Define $\overline{\phi}''$:*

$$\mathcal{L}\left(\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)]\big)\otimes\ell^2(\bar\Lambda);\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)]\big)\otimes\ell^2(\bar\Lambda)\right)$$

to be

$$\begin{aligned}\overline{\phi}''_{j\ell\beta;km\alpha}&=(e_{\ell,\beta},\phi''_{jk}e_{m\alpha}),\quad (j,k,\ell,m\in\bar\Lambda,\alpha,\beta\in\{0,\bar\Lambda\})\\ \overline{\phi}''_{j0;km\alpha}&=(e_0,\phi''_{jk}e_{m\alpha}),\end{aligned}$$

$$\begin{aligned}\overline{\phi}''_{j\ell\beta,k0}&=(e_{\ell\beta},\phi''_{jk}e_0)\\ \overline{\phi}''_{j0,k,0}&=(e_0\phi''_{jk}e_0)=\langle\phi''\rangle_{jk}.\end{aligned}$$

Define

$$M_{k\alpha,j\beta}=\Big(e_{k,\alpha}\sum_{\ell\in\bar\Lambda}z_\ell^*z_\ell e_{j,\beta}\Big).$$

Note that

$$M_{k\alpha,0}=M_{0,k\alpha}=M_{0,0}=0.$$

We have

$$\|\phi''R_-u_--R_-\overline{\phi}''u_-\|=\mathbf{O}(\gamma^2)\|u_-\|$$

and

$$\Big\|\Big(\sum z_\ell^*z_\ell\Big)e_{k\alpha}-\sum_{\substack{j\in\bar\Lambda\\ \beta\in\{0,\bar\Lambda\}}}M_{k\alpha,j\beta}e_{j,\beta}\Big\|=\mathbf{O}(\gamma^2).$$

Proof. The proof proceeds exactly as in the proof of Lemma 4.8. We note that

$$\begin{aligned}\overline{\phi}''_{j\ell\beta;km\alpha}&=(e_{\ell\beta}\phi''_{jk}e_{m\alpha})\\ &=\bar{\mathcal{M}}^{-1/2}(\ell\beta,\ell'\beta')\bar{\mathcal{M}}^{-1/2}(m\alpha,m'\alpha')(z^*_{\ell'}u_{\beta'},\phi''_{jk}z^*_{m'}u_{\alpha'})\\ &=\bar{\mathcal{M}}^{-1/2}(\ell\beta,\ell'\beta')\bar{\mathcal{M}}^{-1/2}(m\alpha,m'\alpha')(u_{\beta'},z_{\ell'}\phi''_{jk}z^*_{m'}u_{\alpha'})\\ &=\langle\phi''_{jk}\rangle(\delta_{\alpha\beta}\delta_{\ell m}+\delta_{\alpha\ell}\delta_{\beta m})+\bar{\mathcal{M}}^{-1/2}(\ell\beta,\ell'\beta')\bar{\mathcal{M}}^{-1/2}(m\alpha,m'\alpha')\\ &\qquad(z^*_{\ell'}e_{\beta'},(\phi''_{jk}-\langle\phi''_{jk}\rangle)z^*_{m'}e_{\alpha'})\end{aligned}\tag{}$$

where we used ().

Using by now the standard argument (e.g., as in () to ()), it can be easily shown that the second term in the last line of () correspond to an operator which is of $\mathbf{O}(\gamma^2)$ on $\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)]\big)\otimes\ell^2(\bar\Lambda)$.

Let

$$\begin{aligned}
\tilde\phi_{j\ell\beta,km\alpha} &= \langle\phi''\rangle_{jk}(\delta_{\ell m}\delta_{\alpha\beta}+\delta_{\ell\alpha}\delta_{\beta m}) = \langle\phi''\rangle_{jk}(I\otimes_s I)_{\ell\beta m\alpha}\\
\tilde\phi_{j\ell\beta,k0} &= 0\\
\tilde\phi_{j0,km\alpha} &= 0\\
\tilde\phi_{j0,k0} &= \langle\phi''_{jk}\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
\|R_-(\overline{\phi}''-\tilde\phi'')u_-\| &= \|(\overline{\phi}''-\tilde\phi'')u_-\|\\
&= \mathbf{O}(\gamma^2)\|u_-\|,
\end{aligned}$$

where we used the fact that the operators defined in the second and third equations of () are of $\mathbf{O}(\gamma^2)$. We may now follow the proof of Lemma 4.8 to obtain (). Similarly we obtain (), by using the same type of arguments that we used to prove () of Lemma 4.8. □

The proof of Proposition 4.11 now follows immediately by applying Lemma 4.12, in the same way that the proof of Proposition 4.1 follows from Lemma 4.2.

Remark. It is worth noticing that in Lemma 4.2, 4.8 and 4.12, the estimates on the RHS's are all of $\mathbf{O}(\gamma^2)\ldots$. This is fundamentally due to the fact that $[z_j,z_k^*]=2\phi''_{jk}$ and $\sqrt{\langle(\phi''')^2\rangle}=\mathbf{O}(\gamma^2)$ and hence the variation of ϕ''_{jk} is of $\mathbf{O}(\gamma^2)$ i.e., $\sqrt{\langle(\phi''_{jk})^2\rangle-\langle\phi''_{jk}\rangle^2}=\sqrt{\langle(\phi''_{jk}-\langle\phi''_{jk}\rangle)^2\rangle}=\mathbf{O}(\gamma^2)$. (This is of course the content of Lemma 4.2). If ϕ'' were a constant : $\phi'''=0$ then all the estimates on the RHS would be replaced by 0 this corresponds to a high-dimentional harmonic oscillator. The point of this paper is to do perturbation theory around the harmonic oscillator as the dimension goes to infinity.

Using Lemma 4.12 and proceeding exactly as in the proof of Proposition 4.3, we obtain the following approximations :

Proposition 4.13. *For every $C>0$, there exists $\gamma_0>0$, such that for all $z\le 2\big(r_0+3r_E-\frac{1}{C}\big)$, $|z|=\mathbf{O}(1)$, all $0\le\gamma\le\gamma_0$,*

$$\|E_{-+}-(z-M-2\overline{\phi}'')\|_{\mathcal{L}(\ell^2,\ell^2)}=\mathbf{O}(\gamma^2)$$

$$\|E_+-R_-\|_{\mathcal{L}(\ell^2,L^2)}=\mathbf{O}(\gamma^2)$$

$$\|E_--R_+\|_{\mathcal{L}(L^2,\ell^2)}=\mathbf{O}(\gamma^2)$$

where $M, \overline{\phi}''$ are defined in (), ℓ^2 is for $\big(\mathbf{R} \oplus \ell^2(\bar\Lambda) \oplus [\ell^2(\bar\Lambda) \otimes_s \ell^2(\bar\Lambda)]\big) \otimes \ell^2(\bar\Lambda)$ and L^2 for $L^2(\mathbf{R}^{2\Lambda}; \mathbf{R}^{2\Lambda})$.

Let $\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \begin{pmatrix} v \\ v_+ \end{pmatrix}$ be the solution to the Grushin problem in Proposition 4.11. We specialize to the case $v = 0$ and $v_+ \in \ell^2(\bar\Lambda)$, i.e. $v_+ = \{v_+(j,0), j \in \bar\Lambda)\}$. Then

$$u = E_+ v_+$$
$$u_- = E_{-+} v_+.$$

Define Λ_+ to be the projection :

$$\text{()} \qquad \Lambda_+ : \big(\mathbf{R} \oplus \ell^2(\bar\Lambda) \oplus [\ell^2(\bar\Lambda) \otimes_s \ell^2(\bar\Lambda)]\big) \otimes \ell^2(\bar\Lambda) \longrightarrow \mathbf{R} \otimes \ell^2(\bar\Lambda) \sim \ell^2(\bar\Lambda),$$

$\forall u : (\Lambda_+ u)(j,0) = u(j,0)$, $(\Lambda_+ u)(j,k,\alpha) = 0$ for all $k \in \bar\Lambda$, $\alpha \in \{0, \bar\Lambda\}$. Assume γ, z are such that Propositions 4.11, 4.13 hold and $|z| = \mathbf{O}(1)$. We then have the following improved approximations to E_+, E_- and E_{-+} in this special case.

Proposition 4.14. *For $|z| = \mathbf{O}(1)$, $z \le 2\big(r_0 + 3r_E - \frac{1}{\mathbf{O}(1)}\big)$*

$$\|(E_{-+} - (z - 2\overline{\phi}''))\Lambda_+\|_{\mathcal{L}(\ell^2, \ell^2)} = \mathbf{O}(\gamma^4)$$

$$\|(E_+ - R_-)\Lambda_+\|_{\mathcal{L}(\ell^2, L^2)} = \mathbf{O}(\gamma^4)$$
$$\|\Lambda_+(E_- - R_+)\|_{\mathcal{L}(L^2, \ell^2)} = \mathbf{O}(\gamma^4)$$

where $\overline{\phi}''$ is as defined in () in Lemma 4.12, ℓ^2 is short for $\big(\mathbf{R} \oplus \ell^2(\bar\Lambda) \oplus [\ell^2(\bar\Lambda) \otimes_s \ell^2(\bar\Lambda)]\big) \otimes \ell^2(\bar\Lambda)$ and L^2 for $L^2(\mathbf{R}^{2\Lambda}; \mathbf{R}^{2\Lambda})$.

Proof. Let

$$u^0 = R_- v_+$$
$$u^0_- = (z - 2\overline{\phi}'')v_+$$

(where $v_+ \in \ell^2(\bar\Lambda)$) be approximations to u, u_- in (). We then obtain that

$$(\Delta^{(1)}_\phi - z)(u^0 - u) + R_-(u^0_- - u_-) = 2(\phi'' - \overline{\phi}'')R_- v_+$$
$$R_+(u^0 - u) = 0,$$

where we used the fact that $(\Delta^{(0)}_\phi \otimes I)(R_- v_+) = 0$ for $v_+ \in \ell^2(\bar\Lambda)$. Applying Proposition 4.11 to the Grushin problem in (), we obtain

$$\|u^0 - u\| + \|u^0_- - u_-\| \le 2\|(\phi'' - \overline{\phi}'')R_- v_+\|.$$

Therefore in order to prove the proposition, it is sufficient to prove that

$$\|(\phi'' - \overline{\phi}'')R_- v_+\| = \mathbf{O}(\gamma^4)\|v_+\|$$

for $v_+ \in \ell^2(\bar\Lambda)$. (Compare () with () and ())

$$\|(\phi'' - \overline{\phi}'')R_- v_+\|^2 = \sum_{j,k,p} \big[(\phi''_{jk} e_0 - \langle \phi''_{jk}\rangle e_0 - \sum_{\ell,\beta,\ell \geq \beta} (e_{\ell\beta}, \phi''_{jk} e_0) e_{\ell\beta}\big] v_+(k)$$
$$\big[(\phi''_{jp} e_0 - \langle \phi''_{jp}\rangle e_0 - \sum_{u,\omega,u \geq \omega} (e_{u,\omega}, \phi''_{jp} e_0) e_{u\omega}\big] v_+(p)$$

where $e_{\ell\beta}$ is as defined earlier in ().

Let $A_1 \stackrel{\text{def}}{=} \phi''_{jk} e_0 - \langle \phi''_{jk}\rangle e_0 - \sum_{\ell \geq \beta} (e_{\ell\beta} \phi''_{jk} e_0) e_{\ell\beta}$. Since $(A_1, e_0) = 0$, $A_1 = \Delta_\phi^{(0)} \tilde u_1$. By using our usual mechanism, we need to estimate $\|dA_1\|$. Hence we need to estimate

$$\begin{aligned}
A_2 \stackrel{\text{def}}{=} z_n A_1 &= z_n(\phi''_{jk} e_0 - \langle \phi''_{jk}\rangle e_0) - \sum_{\ell \geq \beta} (e_{\ell\beta} \phi''_{jk} e_0) e_{\ell\beta} \\
&= \phi'''_{jkn} e_0 - \sum_{\ell,0} (e_{\ell 0} \phi''_{jk} e_0) z_n e_{\ell 0} - \sum_{\ell \geq \beta > 0} (e_{\ell\beta} \phi''_{jk} e_0) z_n e_{\ell\beta} \\
&= \phi'''_{jkn} e_0 - \langle 2\phi''\rangle^{-1/2}_{\ell\ell'} (z^*_{\ell'} e_0, \phi''_{jk} e_0) z_n \langle 2\phi''\rangle^{-1/2}_{\ell\ell''} z^*_{\ell''} e_0 \\
&\quad - \sum_{\ell \geq \beta > 0} (e_{\ell\beta} \phi''_{jk} e_0) z_n e_{\ell\beta} \\
&= \phi'''_{jkn} e_0 - \langle 2\phi''\rangle^{-1}_{\ell'\ell''} \langle \phi'''_{jk\ell'}\rangle (2\phi''_{n\ell''}) e_0 \\
&\quad - \sum_{\ell \geq \beta > 0} (e_{\ell\beta} \phi''_{jk} e_0) z_n e_{\ell\beta} \\
&= \phi'''_{jkn} e_0 - \sum_{\ell \geq \beta > 0} (e_{\ell\beta} \phi''_{jk} e_0) z_n e_{\ell\beta},
\end{aligned}$$

where we used the fact that $\langle \phi'''_{jk\ell}\rangle = 0$ to reach the last equality. Since $(A_2, e_0) = 0$, $A_2 = \Delta_\phi^{(0)} \tilde u_2$. We need to estimate $\|dA_2\|$. Hence we define

$$A_3 = z_m A_2 = \phi^{(4)}_{jknm} e_0 - \sum_{\ell \geq \beta > 0} (e_{\ell\beta} \phi''_{jk} e_0) z_m z_n e_{\ell\beta}$$

$$(A_3, e_0) = \langle \phi^{(4)}_{jknm}\rangle - \sum_{\ell \geq \beta > 0} (e_{\ell\beta}, \phi''_{jk} e_0)(e_0, z_m z_n e_{\ell\beta}).$$

Since $(e_0, z^*_m z^*_n e_0) = 0$ and

$$\|\phi''_{jk} e_0 - \sum_{\ell \geq \beta} (e_{\ell\beta}, \phi''_{jk} e_0) e_{\ell\beta} - \langle \phi''_{jk}\rangle e_0\| = \mathbf{O}_{jk}(\gamma^2)$$

$$\|z^*_m z^*_n e_0 - \sum_{\ell \geq \beta} (e_{\ell\beta}, z^*_m z^*_n e_0) e_{\ell\beta}\| = \mathbf{O}_{mn}(\gamma^2),$$

we obtain

$$(A_3, e_0) = \langle \phi^{(4)}_{jknm}\rangle - (e_0, z_m z_n \phi''_{jk} e_0) + \mathbf{O}_{jknm}(\gamma^4) = \mathbf{O}_{jknm}(\gamma^4).$$

Therefore A_3 can be written as

$$A_3 \stackrel{\text{def}}{=} A_4 + \mathbf{O}_{jknm}(\gamma^4)$$

where $A_4 = \Delta_\phi^{(0)} \tilde{u}_3$. We are only left to estimate $\|dA_4\|$; to that end, we define

$$A_5 = z_s A_4 = \phi^{(5)}_{jknms} e_0 + \mathbf{O}_{jknms}(\gamma^4).$$

Since $\|\phi^{(5)}_{jknms} e_0\| = \mathbf{O}_{jknms}(\gamma^4)$ by using similar arguments as ( )-( ) in the proof of Lemma 4.2 (cf. sect VII for more of such computations.), we obtain that $\|A_5\| = \mathbf{O}_{jknms}(\gamma^4)$. Using now the fact that ϕ'' and all higher Hessians are bounded operators in appropriate spaces, we obtain () by combining the estimates on $A_1, A_2, \dots A_5$. We therefore obtain the Proposition. □

V. Grushin Problems in Weighted Spaces

Beginning in this section, we study the Grushin problem in Proposition 4.11 in appropriate weighted spaces. This is the necessary final step toward obtaining the asymptotics of $\langle G(E;\mu,\nu)\rangle_g$ as $|\mu-\nu| \to \infty$. In this section, we study the general classes of weights that we are allowed to take. In sect. VI, we further specify a class of such weights, using the convolution structure that is inherent in the problem. Propositions 4.11, 4.13 as they stand are not sufficient for obtaining the asymptotics. However, from (3.22), we see that we are only interested at $z = 0$. When $z = 0$, using Propositions 4.10, 4.11, 4.13, we deduce our desired weighted estimates. But first we need to specify a class $\mathcal{W}$ of weights.

$\mathcal{W}$ consists of weights ρ: $\bar\Lambda \to (0,\infty)$ such that if $\rho \in \mathcal{W}$, then

(W1) $\rho\phi''(x)\rho^{-1}$, $\rho^{-1}\phi''(x)\rho = \mathbf{O}(1)$ in $\mathcal{L}\big(\ell^2(\bar\Lambda);\ell^2(\bar\Lambda)\big)$ for all x, where $\mathbf{O}(1)$ is uniform in ρ, x. Further ρ is such that $\rho_{i^{(1)}} = \rho_{i^{(2)}} = \rho_i$, where $i^{(1)} \in \Lambda \times \{1,2\} = \bar\Lambda$, $i^{(2)} \in \Lambda \times \{1,2\} = \bar\Lambda$, $i \in \Lambda$.

(W2) If $\rho \in \mathcal{W}$, then $\rho^{-1} \in \mathcal{W}$, and ρ satisfies

$$(\rho\phi''(x)\rho^{-1}t, t) \ge (\frac{1}{\mathbf{O}(1)} - r_0 - 2r_E)\|t\|^2, \quad t \in \ell^2(\bar\Lambda), \tag{5.1}$$

for all x, where $\mathbf{O}(1)$ is some positive constant uniform in ρ, x; r_0 is taken to be $r_0 =$ the smallest eigenvalue of $\langle\phi''\rangle - \frac{1}{C}$ and $r_E = E - 2d - \frac{1}{C}$ with arbitrarily large $C > 0$ and small enough γ depending on C.

Remark. $\mathcal{W}$ becomes useful only when $\mathbf{O}(1)$ is such that the RHS of (5.1) is negative, i.e., there exists $\rho \in \mathcal{W}$ satisfying (W1), such that

$$(\frac{1}{\mathbf{O}(1)} - r_0 - 2r_E)\|t\|^2 = -|\frac{1}{\mathbf{O}(1)} - r_0 - 2r_E|\|t\|^2 \le (\rho\phi''(x)\rho^{-1}t,t) < -c\|t\|^2 < 0.$$

(See (5.6).) From the expression for ϕ'' in (4.2), we see that in order for ρ to verify (W1), we only need to check the last term, the $\mathbf{O}(\gamma^4)$ term. To $\mathbf{O}(\gamma)$, that term decays in $|i-j|$ at approximately *twice* the rate of $(E-\Delta)^{-1}$ and hence of $(\phi''(x))^{-1}$. So for sufficiently small γ, this does not cause any problems, and there exist $\rho \in \mathcal{W}$ such that (5.2) is satisfied. For more precise statement, see (6.17,6.18).

We now use the weights in $\mathcal{W}$ to obtain the weighted analogues of Propositions 4.11, 4.13 at $z=0$. We first establish that the Grushin problem in Proposition 4.11 at $z=0$ is well posed for $\rho \in \mathcal{W}$. We proceed as in the proof of Proposition 4.11. We first look at the special case $v_+ = 0$ and write

$$\begin{aligned} \Delta_\phi^{(1)} u + R_- u_- &= v \\ R_+ u &= 0. \end{aligned}$$

Multiplying (5.3) by ρ, we obtain

$$\begin{aligned} \rho\Delta_\phi^{(1)} u + \rho R_- u_- &= \rho v \\ \rho R_+ u &= 0, \end{aligned} \tag{5.4}$$

where ρv for v a 1-form is defined to be $\rho v = \sum \rho_j v_j dx_j$, $j \in \bar{\Lambda}$; ρR_+ is defined to be: $\rho R_+ = (R_-\rho)^t = (\rho R_-)^t$. Taking the scalar product with ρu in the first equation of (5.4), we have

$$(\rho u, \rho\Delta_\phi^{(1)} u) + (\rho u, \rho R_- u_-) = (\rho u, \rho v). \tag{5.5}$$

Using the second equation of (5.4) and the fact that ρ and R_- commute, we have

$$\begin{aligned} &(\rho u, \rho\Delta_\phi^{(1)} u) + (\rho u, \rho R_- u_-) \\ =&(\rho u, (\Delta_\phi^{(0)} \otimes I)\rho u) + 2(\rho u, \rho\phi''\rho^{-1}\rho u) \\ \geq& 2(r_0 + 2r_E)\|\rho u\|^2 + 2(\frac{1}{\mathbf{O}(1)} - r_0 - 2r_E)\|\rho u\|^2 \\ \geq& \frac{1}{\mathbf{O}(1)}\|\rho u\|^2, \end{aligned}$$

where r_0, r_E are as in (4.1,4.3) and we used Proposition 4.10. Combinning (5.6) and (5.5), we obtain

$$\|\rho u\| \leq \mathbf{O}(1)\|\rho v\|. \tag{5.7}$$

We next take the scalar product of the first equation in (5.4) with $\rho R_- u_-$:

$$(\rho R_- u_-, \rho\Delta_\phi^{(1)} u) + (\rho R_- u_-, \rho R_- u_-) = (\rho R_- u_-, \rho v). \tag{5.8}$$

So

$$(\rho(\Delta_\phi^{(0)} \otimes I)R_- u_-, \rho u) + 2(\rho\phi''\rho^{-1}\rho R_- u_-, \rho u) + \|\rho u_-\|^2 \leq \|\rho u_-\|\|\rho v\|. \tag{5.9}$$

Using the weighted analogue of Lemma 4.12 and the second equation of (5.4), we obtain

$$\mathbf{O}(\gamma^2)\|\rho u_-\|\,\|\rho u\| + \|\rho u_-\|^2 \leq \|\rho u_-\|\,\|\rho v\|.$$

Therefore

$$\|\rho u_-\| \leq \mathbf{O}(1)\|\rho v\|. \tag{5.10}$$

We now consider (5.4) in general. Since $R_+R_- = I$, we have from (5.4):

$$\begin{aligned} \rho\Delta_\phi^{(1)}(u - R_-v_+) + \rho R_-u_- &= \rho(v - \Delta_\phi^{(1)}R_-v_+) \\ \rho R_+(u - R_-v_+) &= 0. \end{aligned} \tag{5.11}$$

Applying the estimates (5.7) and (5.10) to (5.11), we obtain

$$\begin{aligned} \|\rho(u - R_-v_+)\| + \|\rho u_-\| &\leq \mathbf{O}(1)\|\rho(v - \Delta_\phi^{(1)}R_-v_+)\| \\ &\leq \mathbf{O}(1)(\|\rho v\| + \|\rho v_+\|), \end{aligned} \tag{5.12}$$

where we used the weighted analogue of Lemma 4.12. Since

$$\|\rho(u - R_-v_+)\| \geq \|\rho u\| - \|\rho v_+\|, \tag{5.13}$$

we obtain from (5.12) that

$$\|\rho u\| + \|\rho u_-\| \leq \mathbf{O}(1)(\|\rho v\| + \|\rho v_+\|), \tag{5.14}$$

which implies that

$$\|\rho E(0)\rho^{-1}\|_{\mathcal{L}(L^2,L^2)},\ \|\rho E_+(0)\rho^{-1}\|_{\mathcal{L}(\ell^2,L^2)},\ \|\rho E_-(0)\rho^{-1}\|_{\mathcal{L}(L^2,\ell^2)},\ \|\rho E_{-+}(0)\rho^{-1}\|_{\mathcal{L}(\ell^2,\ell^2)} = \mathbf{O}(1), \tag{5.15}$$

where L^2 is shorthand for $L^2(\mathbf{R}^{2\Lambda};\mathbf{R}^{2\Lambda})$ and ℓ^2 is $\big(\mathbf{R}\oplus\ell^2(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)]\big)\otimes\ell^2(\bar\Lambda)$.

It is also easy to see that all the steps in the proof of Propositions 4.13, 4.14 go through in the weighted space. We hence have the following weighted analogues:

$$\begin{aligned} &\|\rho(E_{-+}(0) + (M + 2\bar\phi''))\rho^{-1}\|_{\mathcal{L}(\ell^2,\ell^2)} = \mathbf{O}(\gamma^2) && \|\rho(E_{-+}(0) + 2\bar\phi'')\rho^{-1}\Lambda_+\|_{\mathcal{L}(\ell^2,\ell^2)} = \mathbf{O}(\gamma^4) \\ &\|\rho(E_+(0) - R_-)\rho^{-1}\|_{\mathcal{L}(\ell^2,L^2)} = \mathbf{O}(\gamma^2) && \|\rho(E_+(0) - R_-)\rho^{-1}\Lambda_+\|_{\mathcal{L}(\ell^2,L^2)} = \mathbf{O}(\gamma^4) \\ &\|\rho^{-1}(E_-(0) - R_+)\rho\|_{\mathcal{L}(L^2,\ell^2)} = \mathbf{O}(\gamma^2) && \|\Lambda_+\rho^{-1}(E_-(0) - R_+)\rho\|_{\mathcal{L}(L^2,L^2)} = \mathbf{O}(\gamma^4), \end{aligned} \tag{5.16}$$

where Λ_+ is the projection as defined in ( ); M, $\bar\phi''$ are as in Propositions 4.12, 4.13.

We end this section by summarizing the above results as

Theorem 5.1. *At $z = 0$, the Grushin problem in Proposition 4.11 is well posed for $\rho \in \mathcal{W}$, with the solution:*

$$\begin{aligned} u &= E(0)v + E_+(0)v_+ \\ u_- &= E_-(0)v + E_{-+}(0)v_+, \end{aligned}$$

where $E(0)$, $E_+(0)$, $E_-(0)$, $E_{-+}(0)$ satisfy (5.15) and (5.16), uniformly in $\rho \in \mathcal{W}$.

VI. The Inverse of E_{-+} and Convolution Equations

Starting from this section, we use the symmetry in the problem to facilitate some of the computations. Naturally similar results hold in the general case. Recall from (2.3) that $\phi(x) = \phi(-x)$. Therefore $\langle \phi^{(n)}(x) \rangle = 0$ for n odd. (This was already used earlier in ( ) for $n = 3$.) $\Delta_\phi^{(1)}$ is parity conserving. Using this reflection symmetry, the Grushin basis for $\Delta_\phi^{(0)}$ constructed earlier in ( ) and ( ) simplifies into:

$$\begin{aligned} e_0 &= e^{-\phi} \\ e_k &= \sum_j (2\langle \phi'' \rangle)^{-1/2}{}_{kj} (z_j^* e^{-\phi}) \quad (k, j \in \bar\Lambda) \\ e_{kk'} &= \sum_{jj'} (2\bar{\mathcal{M}})^{-1/2}{}_{kk'jj'} (z_j^* z_{j'}^* e^{-\phi}) \quad (k, k', j, j' \in \bar\Lambda, k \geq k') \\ e_{kk} &= \frac{1}{\sqrt{2}} \sum_{jj'} (2\bar{\mathcal{M}})^{-1/2}{}_{kkjj'} (z_j^* z_{j'}^* e^{-\phi}) \quad (k, j, j' \in \bar\Lambda) \end{aligned} \tag{6.1}$$

where

$$\bar{\mathcal{M}}_{kk'jj'} = 2\langle \phi'' \rangle_{kj} \langle \phi'' \rangle_{k'j'} + 2\langle \phi'' \rangle_{kj'} \langle \phi'' \rangle_{k'j} + \langle \phi^{(4)} \rangle_{kk'jj'}.$$

(The simplification comes from the fact that $\langle \phi'''(x) \rangle = 0$.) In Proposition 4.13, using $e_0 dx_j$, $e_k dx_j$, $e_{kk'} dx_j$ ($k, k', j \in \bar\Lambda$; $k \geq k'$) as the Grushin basis, we obtained the effective operator $E_{-+}(z)$ of $\Delta_\phi^{(1)} - z$. Recall the identifications:

$$\begin{aligned} &\sum_j e_0 dx_j \in \mathbf{R}^{2\Lambda} \sim \ell^2(\bar\Lambda) \\ &\sum_{k,j} e_k dx_j \in \mathbf{R}^{2\Lambda} \otimes \mathbf{R}^{2\Lambda} \sim \ell^2(\bar\Lambda) \otimes \ell^2(\bar\Lambda) \\ &\sum_{k \geq k', j} e_{kk'} dx_j \in \left(\mathbf{R}^{2\Lambda} \otimes_s \mathbf{R}^{2\Lambda}\right) \otimes \mathbf{R}^{2\Lambda} \sim \left(\ell^2(\bar\Lambda) \otimes_s \ell^2(\bar\Lambda)\right) \otimes \ell^2(\bar\Lambda); \end{aligned} \tag{6.3}$$

and $E_{-+}(z)$ is therefore a bounded operator in

$$\begin{aligned} &\left(\mathbf{R} \oplus \ell^2(\bar\Lambda) \oplus [\ell^2(\bar\Lambda) \otimes_s \ell^2(\bar\Lambda)]\right) \otimes \ell^2(\bar\Lambda) \\ =&\ell^2(\bar\Lambda) \oplus [\ell^2(\bar\Lambda) \otimes \ell^2(\bar\Lambda)] \oplus \left[\left(\ell^2(\bar\Lambda) \otimes_s \ell^2(\bar\Lambda)\right) \otimes \ell^2(\bar\Lambda)\right] \\ \stackrel{\text{def}}{=}&F^1 \oplus F^2 \oplus F^3 = F. \end{aligned} \tag{6.4}$$

Furthermore, Theorem 5.1 implies that at $z=0$, $E_{-+}(z=0)$ is a bounded operator in

$$\ell^2_\rho(\bar\Lambda)\oplus[\ell^2(\bar\Lambda)\otimes\ell^2_\rho(\bar\Lambda)]\oplus\left[\left(\ell^2(\bar\Lambda)\otimes_s\ell^2(\bar\Lambda)\right)\otimes\ell^2_\rho(\bar\Lambda)\right]\stackrel{\text{def}}{=}F^1_\rho\oplus F^2_\rho\oplus F^3_\rho=F_\rho,$$

where $\rho\in\mathcal{W}$ satisfies (W1,2).

We now look more closely at (W1,2), taking into account the special form of ϕ''. Recall from (4.2) that

(6.6)

$$\begin{aligned}(2\phi''(x))_{i^{(\alpha)}j^{(\beta)}}=&2E\delta_{ij}\delta_{\alpha\beta}-2\Delta_{ij}\delta_{\alpha\beta}-2\gamma k'(\gamma x_i\cdot x_i)\delta_{ij}\delta_{\alpha\beta}-4\gamma^2x_i^{(\alpha)}x_i^{(\beta)}k''(\gamma x_i\cdot x_i)\delta_{ij}\\&+2\gamma^2k''(\gamma x_i\cdot x_i)(E-\Delta-\gamma\operatorname{diag}k')^{-1}_{ii}\delta_{ij}\delta_{\alpha\beta}\\&+4\gamma^3x_i^{(\alpha)}x_i^{(\beta)}k'''(\gamma x_i\cdot x_i)(E-\Delta-\gamma\operatorname{diag}k')^{-1}_{ii}\delta_{ij}\\&+4\gamma^4x_i^{(\alpha)}x_j^{(\beta)}k''(\gamma x_i\cdot x_i)k''(\gamma x_j\cdot x_j)[(E-\Delta-\gamma\operatorname{diag}k')^{-1}_{ij}]^2,\end{aligned}$$

where $k=\log\hat g$ and $\hat g$ is the Laplace transform of g; and we used the point of view of $(\mathbf{R}^2)^\Lambda$, i.e., $i,\ j\in\Lambda$, $\alpha,\ \beta\in\{1,2\}$, $x_i,\ x_j\in\mathbf{R}^2$. Δ_{ij} is as defined in (1.1), for $i,\ j\in\Lambda$ and $\Delta=\Delta_\Lambda$ is the matrix of Δ_{ij}. To avoid confusion, we now write Δ_Λ for the finite volume Laplacian and reserve Δ for the corresponding one on $\ell^2(\mathbf{Z}^d)$. The spectrum of Δ_Λ is contained in $[-2d,2d]$.

At this point it is good to recall the problem that we started with: We study

$$H=\Delta+\gamma V,\ \text{on}\ \ell^2(\mathbf{Z}^d),$$

as defined in (1.1). In particular, we are interested in the asymptotics of $\langle G(E,\mu,\nu)\rangle_g$ as $|\mu-\nu|\to\infty$, where

$$E\in I=(-2d-\delta,-2d-\delta_0)\cup(2d+\delta,2d+\delta_0)\tag{6.7}$$

as in (1.3) with $0<\delta<\delta_0<1$ fixed. We proceed by using finite volume approximations:

$$\langle G(E,\mu,\nu)\rangle_g=\lim_{\Lambda\nearrow\mathbf{Z}^d}\langle G_\Lambda(E,\mu,\nu)\rangle_g,\tag{6.8}$$

where $\Lambda\ni\mu,\nu$ is a finite subset of $\mathbf{Z}^d$ and we take the Dirichlet boundary conditions. For given $\mu,\ \nu$, we take Λ sufficiently large, so that $\mu,\ \nu$ is "well" in the interior of Λ. Assume $0<\gamma<\delta_0$, so that $E\notin\sigma(H)$. Then using spectral theory, the limit in (6.8) exists apriori. We further specify Λ to be a cube of side length $N>>1$.

We now make it more precise the class of weights $\mathcal{W}$ that we shall take, using the fact that $R-\Delta$ ($R\in\mathbf{R}$) is a convolution matrix on $\ell^2(\mathbf{Z}^d)$:

$$\mathcal{F}(R-\Delta)(\xi)=R-\sum_{i=1}^d\cos\xi_i,\quad \xi=\{\xi_i\}\in\mathbf{T}^{d^*}.$$

(We abused slightly the notation and wrote $\Delta(j-k)$ for Δ_{jk}.) Let $R > 2d$, $y = \{y_i\} \in \mathbf{R}^{d^*} \sim \mathbf{R}^d$. Define

$$D_R = \{y_i | \sum_{i=1}^{d} \cosh y_i < R\} \subset \mathbf{R}^d; \tag{6.9}$$

D_R is a strictly convex, compact domain in $\mathbf{R}^d$. Let

$$F(y) = \sum_{i=1}^{d} \cosh y_i .$$

We define the support function of D_R:

$$H_{D_R}(k) = \sup_{y \in \bar{D}_R} y \cdot k$$

for $k \in \mathbf{Z}^d/\{0\}$. $H_{D_R}(k)$ is even, convex and homogeneous of degree 1. Therefore $H_{D_R}(k)$ defines a distance on $\mathbf{Z}^d$: $d_R(k,\ell) = H_{D_R}(k-\ell)$ for k, $\ell \in \mathbf{Z}^d$.

Let ρ: $\mathbf{Z}^d \to (0,\infty)$ be a class of weights such that

$$\frac{\rho(k)}{\rho(\ell)} \le e^{d_R(k,\ell)}. \tag{6.10}$$

(This is always possible by a standard modification of the weight at infinity.)

Let $2d < r \le R$ and let ρ satisfy (W1,2) and

$$\max\left(\left|\log \frac{\rho(k)}{\rho(\ell)} - y(k)\cdot(k-\ell)\right|, \left|\log \frac{\rho(k)}{\rho(\ell)} - y(\ell)\cdot(k-\ell)\right|\right) \le \epsilon \qquad (\epsilon << 1) \tag{6.11}$$

for $|k-\ell| \le 1$, $y(k)$, $y(\ell) \in D_r$. Then we have

Proposition 6.1. *If ρ satisfies (6.10), (6.11), then*

$$(\rho(R-\Delta)t, \rho t) \ge (R - r - \tilde{\epsilon}(\epsilon))\|\rho t\|_{\ell^2}^2, \qquad t \in \ell^2(\mathbf{Z}^d; \mathbf{R}) \tag{6.12}$$

where $\tilde{\epsilon} \to 0$ as $\epsilon \to 0$.

proof. It is equivalent to prove that

$$(\rho \Delta t, \rho t) \le (r + \tilde{\epsilon}(\epsilon))\|\rho t\|_{\ell^2}^2 .$$

Therefore it suffices to prove

$$\|\frac{1}{2}(\rho\delta\rho^{-1} + \rho^{-1}\delta\rho)\|_{\ell^2} \le r + \tilde{\epsilon}(\epsilon). \tag{6.13}$$

To prove (6.13), we estimate

$$\begin{aligned}
&\sup_k \sum_\ell \frac{1}{2}\left(\frac{\rho(k)}{\rho(\ell)} + \frac{\rho(\ell)}{\rho(k)}\right)\Delta(k-\ell)\\
\le&\sup_k \sum_{|k-\ell|=1} e^\epsilon \cosh(y(k)\cdot(k-\ell))\\
=&e^\epsilon F(y(k))\\
\le&e^\epsilon r.
\end{aligned}$$

We have the same estimate for $\sup_\ell \sum_k \cdots$ and the proposition follows.

□

Remark. From now on we will have the need to distinguish operators on $\ell^2(\bar\Lambda)$ from their counterparts on $\ell^2(\mathbf{Z}^d)$. So we systematically put back the subscript Λ, when we refer to the former.

From (6.6), (6.7), $2\phi''_\Lambda$ (previously denoted $2\phi''$) is a bounded operator on $\ell^2(\bar\Lambda)$. Denote the last term in the RHS of (6.6) by S_Λ and the first six terms by T_Λ, we have

$$2\phi''_\Lambda = T_\Lambda + S_\Lambda, \tag{6.14}$$

where T_Λ can be further written as

$$T_\Lambda = (E-\Delta_\Lambda)\otimes\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \mathbf{O}(\gamma). \tag{6.15}$$

Let ρ_E: $\mathbf{Z}^d \to (0,\infty)$ be a class of weights satisfying (6.10) (6.11) with R replaced by E and r strictly less than E. Let $\rho_{\Lambda,E}$: $\bar\Lambda \to (0,\infty)$ be the restriction of ρ_E to $\bar\Lambda$ such that

$$\rho_{\Lambda,E}(i^{(\alpha)}) = \rho_{\Lambda,E}(i^{(\beta)}) = \rho_{\Lambda,E}(i) \tag{6.16}$$

for $i\in\Lambda$, α, $\beta\in\{1,2\}$ and $i^{(\alpha)}$, $i^{(\beta)}\in\bar\Lambda$. Proposition 6.1 implies that for γ sufficiently small

$$T_\Lambda^{-1}\in\mathcal{L}(\ell^2_{\rho_{\Lambda,E}}(\bar\Lambda);\ell^2_{\rho_{\Lambda,E}}(\bar\Lambda)) \tag{6.17}$$

and that

$$S_\Lambda\in\mathcal{L}(\ell^2_{(\rho_{\Lambda,E})^2}(\bar\Lambda);\ell^2_{(\rho_{\Lambda,E})^2}(\bar\Lambda)), \tag{6.18}$$

where we used the fact that $\Delta(k) \geq 0$ for all $k \in \mathbf{Z}^d$. Notice that it is the weight $(\rho_{\Lambda,E})^2$ that appears in (6.18). In view of (6.18) and the fact that $0 < \gamma < \delta < E - 2d < \delta_0$, where $\delta_0 < 1$. We may replace $(\rho_{\Lambda,E})^2$ by $\rho_{\Lambda,\bar{E}}$ where

$$\bar{E} - 2d = 4(E - 2d) - \frac{1}{\mathbf{O}(1)}$$

for γ sufficiently small. Using (6.18), (W1,2), the class of weights $\rho_{\bar{E}}$: $\mathbf{Z}^d \to (0, \infty)$ that we shall take can then be specified further as satisfying (6.10) and (6.11) with R replaced by

$$\bar{E} = E + 3(E - 2d) - \frac{1}{\mathbf{O}(1)} \quad (E > 2d). \tag{6.19}$$

Let

$$H_\Lambda = M_\Lambda + 2\overline{\phi''_\Lambda}, \tag{6.20}$$

where M_Λ, $\overline{\phi''_\Lambda}$ are as in ( ) and ( ) (without the subscript Λ). Then $H_\Lambda \in \mathcal{L}(F_{\rho_\Lambda}; F_{\rho_\Lambda})$ for $\rho \in \mathcal{W}$ satisfying (6.10) (6.11) and is an approximation to $E_{-+,\Lambda}(0)$ satisfying (5.15) and (5.16). Let Λ_+^j $(j = 1, 2, 3.)$ be the projections:

$$\begin{aligned} \Lambda_+^1 F &= F^1 = \ell^2(\bar{\Lambda}) \\ \Lambda_+^2 F &= F^2 = \ell^2(\bar{\Lambda}) \otimes \ell^2(\bar{\Lambda}) \\ \Lambda_+^3 F &= F^3 = [\ell^2(\bar{\Lambda}) \otimes_s \ell^2(\bar{\Lambda})] \otimes \ell^2(\bar{\Lambda}). \end{aligned} \tag{6.21}$$

(Λ_+^1 was denoted earlier by Λ_+.) Let

$$\begin{aligned} H_{j,\Lambda} &= \Lambda_+^j H_\Lambda \Lambda_+^j \quad (j = 1, 2, 3.), \\ H_{jk,\Lambda} &= \Lambda_+^j H_\Lambda \Lambda_+^k \quad (j \neq k,\ j, k = 1, 2, 3.); \\ Q_{j,\Lambda} &= \Lambda_+^j E_{-+,\Lambda}(0) \Lambda_+^j \quad (j = 1, 2, 3.), \\ Q_{jk,\Lambda} &= \Lambda_+^j E_{-+,\Lambda}(0) \Lambda_+^k \quad (j \neq k,\ j, k = 1, 2, 3.); \\ D_\Lambda &= \oplus_{j=1}^3 Q_{j,\Lambda} \\ R_\Lambda &= E_{-+,\Lambda}(0) - D_\Lambda. \end{aligned} \tag{6.22}$$

Since parity is conserved,

$$H_{12,\Lambda} = H_{21,\Lambda} = H_{23,\Lambda} = H_{32,\Lambda} = 0;$$

similarly,

$$Q_{12,\Lambda} = Q_{21,\Lambda} = Q_{23,\Lambda} = Q_{32,\Lambda} = 0.$$

Then by using the resolvent identity of $[E_{-+,\Lambda}(0)]^{-1}$, we have

$$
(6.23)\qquad \begin{aligned}
&\Lambda_+^1[E_{-+,\Lambda}(0)]^{-1}\Lambda_+^1\\
=&\Lambda_+^1 D_\Lambda^{-1}\Lambda_+^1 + \Lambda_+^1 D_\Lambda^{-1} R_\Lambda D_\Lambda^{-1}\Lambda_+^1 + \Lambda_+^1 D_\Lambda^{-1} R_\Lambda [E_{-+,\Lambda}(0)]^{-1}\Lambda_+^1\\
=&(Q_{1,\Lambda} - Q_{13,\Lambda}Q_{3,\Lambda}^{-1}Q_{31,\Lambda})^{-1}\\
\overset{\text{def}}{=}&\tilde{Q}_\Lambda^{-1}.
\end{aligned}
$$

Furthermore, from Theorem 5.1, we have

$$
(6.24)\qquad \begin{aligned}
\|Q_{1,\Lambda} - H_{1,\Lambda}\|_{\mathcal{L}(F^1_{\rho_\Lambda},F^1_{\rho_\Lambda})} &= \mathbf{O}(\gamma^4)\\
\|Q_{13,\Lambda} - H_{13,\Lambda}\|_{\mathcal{L}(F^3_{\rho_\Lambda},F^1_{\rho_\Lambda})} &= \mathbf{O}(\gamma^4)\\
\|Q_{31,\Lambda} - H_{31,\Lambda}\|_{\mathcal{L}(F^1_{\rho_\Lambda},F^3_{\rho_\Lambda})} &= \mathbf{O}(\gamma^4)\\
\|Q_{3,\Lambda} - H_{3,\Lambda}\|_{\mathcal{L}(F^3_{\rho_\Lambda},F^3_{\rho_\Lambda})} &= \mathbf{O}(\gamma^2).
\end{aligned}
$$

From the definition of $\tilde{Q}_\Lambda$ in the RHS of (6.23), the only operator that we need to look into is $Q_{3,\Lambda}^{-1}$. Using (6.24), we see that to order $\mathbf{O}(\gamma^4)$, it is sufficient to study $H_{13,\Lambda}H_{3,\Lambda}^{-1}H_{31,\Lambda}$. To that end, we compute $H_{3,\Lambda}$ as defined by (6.20), (6.21). To gain experience and also to see the structure, we start by computing $H_{1,\Lambda}$, $H_{2,\Lambda}$.

By definition:

$$
(6.25)\qquad \begin{aligned}
(H_{1,\Lambda})_{ij} &= \langle e^{-\phi}, (\Delta_\phi^{(0)}\delta_{ij} + 2\phi''_{\Lambda,ij})e^{-\phi}\rangle\\
&= \langle 2\phi''_\Lambda\rangle_{ij},\\
\text{so}\quad H_{1,\Lambda} &= \langle 2\phi''_\Lambda\rangle.
\end{aligned}
$$

(The ϕ's that appear below should all be understood as ϕ_Λ's. For notational simplicity, we usually only put the subscript back on the end results.)

(6.26)

$$
\begin{aligned}
(H_{2,\Lambda})_{ii'jj'} &= (e_i, (\Delta_\phi^{(0)}\delta_{i'j'} + 2\phi''_{i'j'})e_j)\\
&= (2\langle\phi''\rangle)^{-1/2}_{ik}(2\langle\phi''\rangle)^{-1/2}_{j\ell}\Big((e_0, z_k(\sum z_n^* z_n)z_\ell^* e_0)\delta_{i'j'} + 2(e_0, z_k\phi''_{i'j'}z_\ell^* e_0)\Big)\\
&= (2\langle\phi''\rangle)^{-1/2}_{ik}(2\langle\phi''\rangle)^{-1/2}_{j\ell}\\
&\big(4(e_0, (\sum_n \phi''_{kn}\phi''_{n\ell})e_0)\delta_{i'j'} + 4(e_0, \phi''_{i'j'}\phi''_{k\ell}e_0) + 2(e_0, \phi^{(4)}_{i'j'k\ell}e_0)\big)\\
&= (2\langle\phi''\rangle)^{-1/2}_{ik}(2\langle\phi''\rangle)^{-1/2}_{j\ell}[4(\sum_n(\langle\phi''\rangle_{kn}\langle\phi''\rangle_{n\ell})\delta_{i'j'} + 4\langle\phi''\rangle_{i'j'}\langle\phi''\rangle_{k\ell}\\
&\quad + 2\langle\phi^{(4)}_{i'j'k\ell}\rangle + 4\sum_n(e^{-\phi}d\phi''_{kn}, (\Delta_\phi^{(1)})^{-1}(e^{-\phi}d\phi''_{n\ell}))\delta_{i'j'}\\
&\quad + 4(e^{-\phi}d\phi''_{i'j'}, (\Delta_\phi^{(1)})^{-1}(e^{-\phi}d\phi''_{k\ell}))].
\end{aligned}
$$

So

$$
(6.27)\qquad H_{2,\Lambda} = \langle 2\phi''_\Lambda\rangle \otimes I + I \otimes \langle 2\phi''_\Lambda\rangle + \mathbf{O}(\gamma^2),
$$

where $\mathbf{O}(\gamma^2)$ is in the sense of bounded operators in $F^2_{\rho_\Lambda}$. Similarly

$$H_{3,\Lambda} = [\langle 2\phi''_\Lambda\rangle \otimes_s I] \otimes I + [I \otimes_s \langle 2\phi''_\Lambda\rangle] \otimes I + [I \otimes_s I] \otimes \langle 2\phi''_\Lambda\rangle + \mathbf{O}(\gamma^2), \tag{6.28}$$

where $\mathbf{O}(\gamma^2)$ is in the sense of bounded operators in $F^3_{\rho_\Lambda}$.

Let P denote the operator in the parenthesis in the third equation of (6.30). Then $P^{-1} \in \mathcal{L}(F^3_{\rho_\Lambda}; F^3_{\rho_\Lambda})$, where $\rho \in \mathcal{W}$ satisfies (6.10) (6.11) with $\bar{E}$ in (6.19) further lowered to $E+2(E-2d)-1/\mathbf{O}(1)$. Using (6.28), we have that $H^{-1}_{3,\Lambda} \in \mathcal{L}(F^3_{\rho_\Lambda}; F^3_{\rho_\Lambda})$. Using (6.24), we therefore have that

$$Q^{-1}_{3,\Lambda} \in \mathcal{L}(F^3_{\rho_\Lambda}; F^3_{\rho_\Lambda}). \tag{6.31}$$

We now compute H_{13}:

$$\begin{aligned}(H_{13})_{jkmn} &= 2(e_0, \phi''_{jk} e_{mn})\\ &= \sum_{m',n'} (2\mathcal{M})^{-1/2}_{mnm'n'}(e_0, \phi''_{jk} z^*_{m'} z^*_{n'} e_0)\\ &= \sum_{m',n'} (2\mathcal{M})^{-1/2}_{mnm'n'}\langle \phi^{(4)}_{jkm'n'}\rangle\\ &= \mathbf{O}(\gamma^2),\end{aligned} \tag{6.32}$$

where we used an argument similar to ( )-() in the proof of Lemma 4.2 to reach the last estimate. (cf. sect. VII for more of such computations.) Therefore by the usual argument, we obtain that $H_{13} = \mathbf{O}(\gamma^2) \in \mathcal{L}(F^3_{\rho_\Lambda}; F^3_{\rho_\Lambda})$. Using (6.24) (6.31), we obtain that

$$\|Q_{13} Q_3^{-1} Q_{31}\|_{\mathcal{L}(\ell^2_\rho(\bar{\Lambda}), \ell^2_\rho(\bar{\Lambda}))} = \mathbf{O}(\gamma^4). \tag{6.33}$$

Combinning (6.33) with (6.24), we then obtain that

$$\|\tilde{Q}_\Lambda - H_{1,\Lambda}\|_{\mathcal{L}(\ell^2_\rho(\bar{\Lambda}); \ell^2_\rho(\bar{\Lambda}))} = \mathbf{O}(\gamma^4), \tag{6.34}$$

where $\tilde{Q}_\Lambda$ is as defined in (6.23). Therefore to $\mathbf{O}(\gamma^4)$, in order to study $\Lambda^1_+ E^{-1}_{-+}(0)\Lambda^1_+$, we only need to study $H_{1,\Lambda} = \langle 2\phi''_\Lambda\rangle$.

VII. Asymptotics for the Averaged Green's Function

Recall from sect. I that $\sigma(H) \subset [-2d, 2d] + \gamma \operatorname{supp} g = I$ and that $E \notin I$. Therefore

$$(E-H)^{-1} = \lim_{\Lambda \nearrow \mathbf{Z}^d} (E - H_\Lambda)^{-1}$$

exists by using spectral theory;

$$\lim_{\Lambda\nearrow\mathbf{Z}^d}\langle (E-H_\Lambda)^{-1}(\mu,\nu)\rangle_g=\lim_{\Lambda\nearrow\mathbf{Z}^d}\langle G_\Lambda(E;\mu,\nu)\rangle_g=\langle G(E;\mu,\nu)\rangle_g$$

and is independent of the boundary conditions on Λ, as mentioned in sect. IV. In fact for all given μ, ν, let μ, $\nu\ni\Lambda\subset\mathbf{Z}^d$ be a cube of side length N, then

$$|G_\Lambda(E;\mu,\nu)-G_\Lambda(E;\mu,\nu)|=e^{-\mathbf{O}(N)}\to 0$$

as $N\to\infty$.

For the computations in this section, it is more convenient to revert back to the point of view of associating $(\mathbf{R}^2)^\Lambda$ to Λ. The indices i,j,$\mu\cdots$ etc. below are once again in Λ and $x_i,\cdots\in\mathbf{R}^2$.

For the cubes mentioned above, using (3.22), we have

(7.1)

$$\begin{aligned}\langle G_\Lambda(E;\mu,\nu)\rangle_g=&\sum_{\alpha=1,2}(e^{-\phi}dx_\mu^{(\alpha)},(\Delta_\phi^{(1)})^{-1}e^{-\phi}dx_\nu^{(\alpha)})\\=&\sum_{\alpha=1,2}(e^{-\phi}dx_\mu^{(\alpha)},E_\Lambda e^{-\phi}dx_\nu^{(\alpha)})+(e^{-\phi}dx_\mu^{(\alpha)},(E_{+,\Lambda}E_{-+,\Lambda}^{-1}E_{-,\Lambda})e^{-\phi}dx_\nu^{(\alpha)}).\end{aligned}$$

From Theorem 5.1, $\|\rho_\Lambda E_\Lambda\rho_\Lambda^{-1}\|_{\mathcal{L}(L^2,L^2)}=\mathbf{O}(1)$ uniformly in Λ, where as before, ρ_Λ is the restriction of ρ to Λ and ρ satisfies (6.10) (6.11) with

$$R=E+3(E-2d)-\frac{1}{\mathbf{O}(1)}.$$

We therefore have

$$|(e^{-\phi}dx_\mu^{(\alpha)},E_\Lambda e^{-\phi}dx_\nu^{(\alpha)})|=\mathbf{O}(1)e^{-d_r(\mu,\nu)}$$

uniformly in Λ, where

$$r=R-\frac{1}{\mathbf{O}(1)}=E+3(E-2d)-\frac{1}{\mathbf{O}(1)}. \tag{7.2}$$

(Recall that this is approximately twice the decay rate that we expect from $\langle G_\Lambda(E;\mu,\nu)\rangle_g$.) As a consequence, we obtain that

$$\overline{\lim}_{\Lambda\nearrow\mathbf{Z}^d}(e^{-\phi}dx_\mu^{(\alpha)},E_\Lambda e^{-\phi}dx_\nu^{(\alpha)})=\mathbf{O}(1)e^{-d_r(\mu,\nu)}. \tag{7.3}$$

To study the second term of (7.1), we use Theorem 5.1, (5.15) and (5.16) we obtain

$$\begin{aligned}&(e^{-\phi}dx_\mu^{(\alpha)},(E_{+,\Lambda}E_{-+,\Lambda}^{-1}E_{-,\Lambda})(e^{-\phi}dx_\nu^{(\alpha)}))\\=&(R_{+,\Lambda}(e^{-\phi}dx_\mu^{(\alpha)}),E_{-+,\Lambda}^{-1}R_{+,\Lambda}(e^{-\phi}dx_\nu^{(\alpha)}))(1+\mathbf{O}(\gamma^4))\\=&(1+\mathbf{O}(\gamma^4))(\Lambda_+^1(E_{-+,\Lambda}^{-1})\Lambda_+^1)(\mu,\nu),\end{aligned} \tag{7.4}$$

where $\mathbf{O}(\gamma^4)$ is uniform in Λ. Using (6.34), (6.23) we have

$$(\Lambda^1_+(E^{-1}_{-+,\Lambda})\Lambda^1_+)(\mu,\nu) = (\langle 2\phi''_\Lambda\rangle + \mathbf{O}(\gamma^4))^{-1}(\mu,\nu) \tag{7.5}$$

where $\mathbf{O}(\gamma^4)$ (uniformly in Λ) is in the sense of $\mathcal{L}(\ell^2_{\rho_\Lambda}(\bar\Lambda);\ell^2_{\rho_\Lambda}(\bar\Lambda))$ and ρ_Λ is the restriction of ρ to Λ with R lowered to

$$R = E + 2(E-2d) - \frac{1}{\mathbf{O}(1)}.$$

To simplify things further, we use

Lemma 7.1.

$$\langle\phi''_\Lambda\rangle = (E-\Delta_\Lambda - diag\ \{\gamma^2\{\langle v^2\rangle_g[(E-\Delta_\Lambda)^{-1}(j,j)] + \gamma^3\langle v^3\rangle_g[(E-\Delta_\Lambda)^{-1}(j,j)]^2\})\otimes\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \mathbf{O}(\gamma^4),$$

where the operator in the parenthesis acts on $\ell^2(\Lambda)$, $\mathbf{O}(\gamma^4)$ (uniformly in Λ) is in the sense of $\mathcal{L}(\ell^2_{\rho_\Lambda}(\bar\Lambda);\ell^2_{\rho_\Lambda}(\bar\Lambda))$ with R as in

$$R = E + 2(E-2d) - \frac{1}{\mathbf{O}(1)}.$$

Proof. We use (2.13) to estimate (6.6). We estimate term by term. To compute the integral, it is convenient to go back to the supersymmetric representation. We have

(7.6)

$$\begin{aligned}\langle k'(\gamma x_i\cdot x_i)\rangle &= \int k'(\gamma x_i\cdot x_i)e^{-2\phi}\prod_{j\in\Lambda}d^2x_j\\ &= \int k'(\gamma x_i\cdot x_i)e^{-(\sum_{j\in\Lambda}EX_j\cdot X_j - \sum_{j,k\in\Lambda,|j-k|_1=1}X_j\cdot X_k - \sum_{j\in\Lambda}k(\gamma X_j\cdot X_j))}\prod_{j\in\Lambda}d^2X_j.\end{aligned}$$

Since

(7.7)

$$\int k'(\gamma X_i\cdot X_i)e^{-(\sum_{j\in\Lambda}EX_j\cdot X_j - \sum_{j,k\in\Lambda,|j-k|_1=1}X_j\cdot X_k - \sum_{j\in\Lambda}k(\gamma X_j\cdot X_j))}\prod_{j\in\Lambda}d^2X_j = k'(0) = 0$$

by using (2.13), and

$$k'(\gamma X_i\cdot X_i) = k'(\gamma x_i\cdot x_i) + \xi_i\eta_i\gamma k''(\gamma x_i\cdot x_i); \tag{7.8}$$

we have

(7.9)

$$\begin{aligned}\langle k'(\gamma x_i\cdot x_i)\rangle &= \gamma\int \eta_i\xi_i k''(\gamma x_i\cdot x_i)e^{-(\sum_{j\in\Lambda}EX_j\cdot X_j - \sum_{j,k\in\Lambda,|j-k|_1=1}X_j\cdot X_k - \sum_{j\in\Lambda}k(\gamma X_j\cdot X_j))}\prod_{j\in\Lambda}d\xi_j d\eta_j\prod_{j\in\Lambda}d\\ &= \gamma\int k''(\gamma x_i\cdot x_i)(E-\Delta_\Lambda - \gamma\,\mathrm{diag}\,k'(\gamma x_j\cdot x_j))^{-1}_{ii}e^{-2\phi}\prod_{j\in\Lambda}d^2x_j,\end{aligned}$$

where we used (2.6) and integrated out the anticommutative variables. Hence the average of the third and the fifth term in the RHS of (6.6) cancel.

Using similar arguments as in (7.6)–(7.9), we have

(7.10)
$$\langle x_i^{(\alpha)} x_i^{(\beta)} k''(\gamma x_i \cdot x_i)\rangle = \frac{1}{2}\delta_{\alpha\beta}\int \eta_i\xi_i k''(\gamma x_i \cdot x_i) e^{-(\sum_{j\in\Lambda} EX_j\cdot X_j - \sum_{j,k\in\Lambda,|j-k|_1=1} X_j\cdot X_k - \sum_{j\in\Lambda} k(\gamma X_j\cdot X_j))}\prod_{j\in\Lambda}$$
$$+\frac{1}{2}\delta_{\alpha\beta}\gamma\langle x_i^{(\alpha)} x_i^{(\beta)} k'''(\gamma x_i\cdot x_i)(E-\Delta_\Lambda-\gamma \mathrm{diag}\, k'(\gamma x_j\cdot x_j))_{ii}^{-1}\rangle.$$

After multiplying by $-4\gamma^2$, the second term in the RHS of (7.10) cancelles the average of the sixth term. So we only need to estimate the first term in the RHS of (7.10). After integrating out the anticommutative variables, we have

(7.11)
$$\int \eta_i\xi_i k''(\gamma x_i\cdot x_i) e^{-(\sum_{j\in\Lambda} EX_j\cdot X_j - \sum_{j,k\in\Lambda,|j-k|_1=1} X_j\cdot X_k - \sum_{j\in\Lambda} k(\gamma X_j\cdot X_j))}\prod_{j\in\Lambda} d^2X_j$$
$$= \int k''(\gamma x_i\cdot x_i)(E-\Delta_\Lambda-\gamma\mathrm{diag}\, k'(\gamma x_j\cdot x_j))_{ii}^{-1} e^{-2\phi}\prod_{j\in\Lambda} d^2x_j$$
$$= \int k''(\gamma x_i\cdot x_i)\{(E-\Delta_\Lambda)_{ii}^{-1} + \gamma[(E-\Delta_\Lambda)^{-1}(\mathrm{diag}\, k')(E-\Delta_\Lambda)^{-1}]_{ii}\} e^{-2\phi}\prod_{j\in\Lambda} d^2x_j + \mathbf{O}(\gamma^2)$$
$$= k''(0)(E-\Delta_\Lambda)_{ii}^{-1} + k'''(0)[(E-\Delta_\Lambda)_{ii}^{-1}]^2 + \mathbf{O}(\gamma^2),$$

where we used (2.13) and resolvent series for

$$(E-\Delta_\Lambda-\gamma\mathrm{diag}\, k'(\gamma x_j\cdot x_j))_{ii}^{-1}.$$

Since the last term of the RHS of (6.6) is of order $\mathbf{O}(\gamma^4)$ by using a similar argument as in (7.10), (7.11), we obtain the Lemma by using (2.2,2.3).

□

Remark. We note the rather remarkable cancellations that occured in the computation of $\langle\phi''\rangle$. We suspect that this continues to higher orders.

Let $\widetilde{E} = E - \gamma^2\langle v^2\rangle_g[(E-\Delta)^{-1}(0,0)] - \gamma^3\langle v^3\rangle_g[(E-\Delta)^{-1}(0,0)]^2$; I be as defined in (1.3), we then have the principal result of the paper:

Theorem 7.2. *Assume g satisfies (H1,2). There exists $\gamma_0 > 0$, such that for all $0 < \gamma < \gamma_0$, all $E \in I$, all μ, $\nu \in \mathbf{Z}^d$,*

$$\begin{aligned}\log\langle G(E,\mu,\nu)\rangle =& \log\left((E-\Delta-\gamma^2\langle v^2\rangle[(E-\Delta)^{-1}(0,0)]-\gamma^3\langle v^3\rangle[(E-\Delta)^{-1}(0,0)]^2\right)^{-1}(\mu-\nu,0)\\ &+\mathbf{O}(\gamma^4)(|\mu-\nu|+1)\\ =& \log\left(\widetilde{E}-\Delta\right)^{-1}(\mu-\nu,0)+\mathbf{O}(\gamma^4)(|\mu-\nu|+1),\end{aligned}$$

where $|\widetilde{E}| > 2d$.

Proof. Since all the estimates are uniform in Λ and the limit $\Lambda \nearrow \mathbf{Z}^d$ exists, combinning (7.1)–(7.5), Lemma 7.1, (6.23) and taking the limit $\Lambda \nearrow \mathbf{Z}^d$, we obtain the Theorem.

□

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