Reflection and coalescence between independent one-dimensional Brownian paths

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Abstract

Take two independent one-dimensional brownian motions $(B_t, t \in [0,1])$ and $(\beta_t, t \in [0,1])$ with $B_0 = 0$ and $\beta_0 = 0$ (β can be seen as running backwards in time). Define $(\gamma_t, t \in [0,1])$ as being the function that is obtained by reflecting B on β. Then γ is still a Brownian motion. Similar and more general results (with families of coalescing Brownian motions) are also derived. They enable to give a precise definition (in terms of reflection) of the joint realisation of finite families of coalescing/reflecting brownian motions.

Key words: Brownian motion, coalescence, reflection.
MSC Class.: 60J65

1 Introduction

The main goal of this paper is to derive some facts concerning reflection and coalescence between independent one-dimensional Brownian motions.

Several papers in recent years studied and used families of coalescing one-dimensional random walks or Brownian motions. These families and their main properties have been initially (to our knowledge) studied by Richard Arratia [1, 2] with applications to the voter model and stochastic flows. More recently, they have received attention for various reasons: coalescing random walks are the local time lines of certain self-interacting walks or processes ([7] and the references therein) and families of coalescing Brownian motions enable to construct natural continuous "self-repelling" processes (see [8]); they provide examples of "non-Brownian" filtrations [9].

As already pointed out by Arratia [1, 2], families of coalescing random walks and families of coalescing Brownian motions have a natural "duality property" (we very briefly recall this in the appendix). To each family of "forward" (running from left to right i.e. forward in time) coalescing random walks in $\mathbb{Z}$, one can associate a family of "backward" coalescing random walks (i.e. running backward in time) as shown on Figure 7 in the appendix (see also Harris [4] for the corresponding statement for stochastic flows). A natural question is how forward and backward lines interact. Clearly the definition of the backward lines show that forward and backward lines never cross; on the other hand, another quick look at the picture (see Figure 7 in the Appendix) leads to the following loose observation: the local behaviour of a forward line and a backward line are independent when they do not touch each other. Our aim is to derive the corresponding results for families of Brownian motions (i.e. in the scaling limit).

It turns out that the correct formulation in the continuous setting is that backward lines are reflected (in the sense of Skorokhod) on the forward lines (or vice-versa). This leads to a natural and simple way to construct finite families of coalescing/reflecting Brownian motions running in both directions, with the constraint that when two Brownian motions running in the same direction meet, then they coalesce, whereas when two Brownian motions running into opposite directions meet, then they reflect on each other (so that they do not cross).

In [8] (see also [1, 2]), the 'dual' family of coalescing/reflecting Brownian motions was contructed in a different way. In particular, in order to construct the dual family, we used all (i.e. a countable family) the forward coalescing Brownian motions, and the reflection property was not apparent.

The results that we will derive are in fact simple striking statements concerning linear Brownian motion, and interesting on their own (not only because of the link with the families of coalescing brownian motions discussed above). Actually, we are mostly going to focus on these statements and then say a few words on their applications to families of coalescing Brownian motion.

In order to avoid messy notation in the introduction, let us detail a very particular case of our results: Suppose that $(B(t),\ t \in [0,1])$ is a linear Brownian motion started from $B(0) = 0$ defined on the time-interval $[0,1]$. Suppose that $(\beta(t),\ t \in [0,1])$ is an independent linear Brownian motion running backwards in time started from $\beta(1) = 0$ (in other words, the law of $(\beta(1-t), t \in [0,1])$ is identical to that of B). Define the reflection $(C(t), t \geq 0)$ of the function B on the path $(\beta(t),\ t \in [0,1])$. More precisely,

$$C(t) \quad = \quad \begin{cases} B(t) + \sup_{s \leq t}(B(s) - \beta(s))_- \text{ if } \beta(0) < 0 \\ B(t) - \sup_{s \leq t}(B(s) - \beta(s))_+ \text{ if } \beta(0) > 0. \end{cases}$$

where $x_+ = x 1_{x>0}$ and $x_- = -x 1_{x<0}$. In other words, C behaves locally exactly as B, but it is pushed each time it hits the function β just enough in such a way that C never crosses β. Figure 1 below shows a realization of β (in grey) and C (in black).

An important observation is that this reflection is not a symmetric procedure. Here, the function β has priority whereas B is changed into C.

In a similar symmetric way, one can define a process γ obtained by reflecting β on B, but "backward in time". More precisely,

$$\gamma(t) \quad = \quad \begin{cases} \beta(t) + \sup_{s \in [t,1]}(\beta(s) - B(s))_- \text{ if } B(0) < 0 \\ \beta(t) - \sup_{s \in [t,1]}(\beta(s) - B(s))_+ \text{ if } B(0) > 0. \end{cases}$$

Note that the laws of $(\gamma(1-t), t \in [0,1])$ and $(C(t), t \in [0,1])$ are identical. In order to define (B, γ), we decided that B had priority and γ is obtained by reflecting β "backwards" on B.

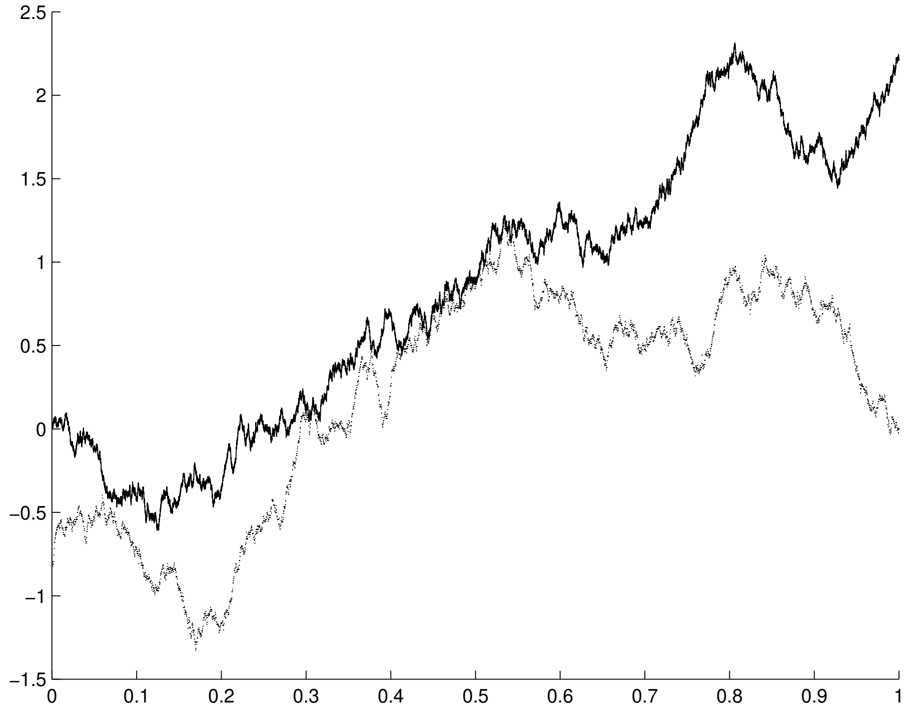

Figure 1: A joint realization of C (in black) and β (in grey)

Theorem 3 states that in fact the two couples of processes (C, β) and (B, γ) are identical in law. In particular, it implies that the law of C is again that of a Brownian motion started from 0. This result may seem surprising as C is obtained from B by pushing the path of B in one direction.

More general statements hold involving finite families of coalescing/reflecting Brownian motions; see Theorems 8 and 11.

Our proofs are based on discrete approximations (we first derive the corresponding results for simple random walks using simple combinatorial arguments) and invariance principles. We want to stress that it seems (at least to us) difficult to derive directly these results for Brownian motions (i.e. without using discrete approximations) using for instance stochastic calculus methods because of the fact that filtrations get messed up (C is constructed using both the forward running Brownian motion B and the backward running Brownian motion β). One might wish to compare this with Tsirelson's results relating filtration generated by Brownian coalescence to "Black noise" [9].

The paper is structured as follows. After recalling some relevant facts concerning Skorohod reflection, we derive the "two-component" version of the result (corresponding to Figure 1 above).

In Section 3, we state and prove the more complicated results concerning finite families of coalescing/reflected Brownian motions. In both sections, the proofs are based on discrete approximations and invariance principles. Finally in section 4, we say a few words on generalizations and consequences of these statements.

2 Two Brownian motions

2.1 Reflection

We first derive some easy facts concerning reflection. Let us recall the following lemma (see e.g. [6]) often referred to as Skorohod's reflection lemma:

Lemma 1 *Suppose that f is a continuous function defined on the interval $[0, T]$ such that $f(0) > 0$. Then, there exists a unique function f_0 defined on $[0, T]$ such that*

- *The function $f_0 - f$ is non-decreasing and continuous.*
- *The function f_0 is non-negative.*
- *The function $f_0 - f$ increases only when f_0 is equal to 0, i.e.*

$$\int_0^T 1_{\{f_0(t) \neq 0\}} d(f(t) - f_0(t)) = 0.$$

Moreover,

$$f_0(t) = f(t) + \sup_{s \leq t} (f(s))_-$$

where $x_- = -x 1_{\{x<0\}}$.

This Lemma is for instance useful when studying the local time at 0 of a linear Brownian motion: In the case where $f = B$ is a linear Brownian motion, it is easy to see that the law of f_0 is that of a reflected Brownian motion (i.e. the same law as $|B|$) (see e.g. [6]) and $f_0 - f$ is its local time at 0.

Describing the application $f \mapsto f_0$ as "reflection" is in fact rather misleading. It is in fact a "pushing" but we will use the usual terminology ("reflection" is also used for multi-dimensional Brownian motion pushed on the boundary of a domain etc).

As we shall now see it is easy to generalize this lemma in the following way:

Lemma 2 *Suppose that f and g are two continuous functions defined on the interval $[0, T]$ such that $f(0) > g(0)$. There exists a unique function f_g defined on $[0, T]$ such that*

- *The function $f_g - f$ is non-decreasing and continuous.*

- *The function $f_g - g$ is non-negative.*
- *The function $f_g - f$ increases only when $f_g = g$, i.e.*

$$\int_0^T 1_{\{f_g(t)\neq g(t)\}} d(f_g(t) - f(t)) = 0.$$

Moreover, for any $t \in [0,T]$,

$$f_g(t) = f(t) + \sup_{s\leq t}(f(s) - g(s))_-.$$

Proof. Note that this Lemma can be viewed as a consequence of the previous one as in fact,

$$f_g = g + (f - g)_0.$$

Instead of using this observation, we prefer to give here the self-contained proof that goes along the same line as that of Lemma 1 (see [6]), as we will want to generalize it later in this paper.

Suppose first that there exist two functions f^1 and f^2 satisfying the required conditions. Then for any $t \in [0,T]$,

$$f^1(t) - f^2(t) = f^1(t) - f(t) + f(t) - f^2(t)$$

is of bounded variation (it is the difference between two non-decreasing functions) and for all $t \in [0,T]$,

$$\begin{aligned}
&(f^2(t) - f^1(t))^2 \\
&= 2\int_0^t (f^2(s) - f^1(s))d(f^2(s) - f^1(s)) \\
&= -2\int_0^t (f^2(s) - g(s))d(f^1(s) - g(s)) - 2\int_0^t (f^1(s) - g(s))d(f^2(s) - g(s)) \\
&\leq 0
\end{aligned}$$

so that $f^1 = f^2$.

Then, it suffices to check that if we define

$$f_g(t) = f(t) + \sup_{s\leq t}(f(s) - g(s))_-$$

then f_g meets the required conditions. This is straightforward: $f_g - f$ is clearly continuous non-decreasing, and

$$f_g(t) - g(t) = f(t) - g(t) + \sup_{s\leq t}(f(s) - g(s))_- \geq 0.$$

Moreover, $f_g - f$ can increase only when $f(t) - g(t) = \sup_{s\leq t}(f(s) - g(s))_-$ and in that case $f_g(t) = g(t)$.

Define the set $C_T = C([0,T])^2$ of pairs of real-valued continuous functions endowed with the uniform distance

$$d((f,g),(\tilde{f},\tilde{g})) = \max\left(\sup_{t\le T}|f(t)-\tilde{f}(t)|\ ,\ \sup_{t\le T}|g(t)-\tilde{g}(t)|\right).$$

Note that the mapping $(f,g)\mapsto(f_g,g)$ defined on C_T, is clearly continuous on the set

$$C_T^+ = \{(f,g)\in C_T\ :\ f(0)>g(0)\}$$

as for any $(f,g)\in C_T^+$, and for any $t\le T$,

$$\begin{aligned}|f_g(t)-\tilde{f}_{\tilde{g}}(t)| &= \left|\sup_{s\le t}(f(s)-g(s))_- - \sup_{s\le t}(\tilde{f}(s)-\tilde{g}(s))_-\right| \\ &\le 2d((f,g),(\tilde{f},\tilde{g})).\end{aligned}$$

Similarly, it is easy to define the reflection f_g in the case where $f(0)<g(0)$. In that case, f_g is obtained by pushing f downwards each time it hits g. More precisely, if f and g are two continuous functions defined on $[0,T]$ with $f(0)<g(0)$ we define for any $t\in[0,T]$,

$$f_g(t) = f(t) - \sup_{s\le t}(f(s)-g(s))_+$$

and an analogous statement to Lemma 2 holds. The mapping $(f,g)\mapsto(f_g,g)$ is then continuous at any point in the set

$$\{(f,g)\in C_T\ :\ f(0)\ne g(0)\}.$$

Finally, suppose now that we look at the functions f and g backwards in time. In other words, they start at time T and run backwards until time 0. In this case, we wish to define the backwards reflection of f on g. For the sake of clarity, we will call this function f^g (and omit the dependance in T). Then, if $f(T)>g(T)$, we define for all $t\in[0,T]$,

$$f^g(t) = f(t) + \sup_{t\le s\le T}(f(s)-g(s))_-$$

and in the case where $f(T)<g(T)$,

$$f^g(t) = f(t) - \sup_{t\le s\le T}(f(s)-g(s))_+.$$

The mapping $(f,g)\mapsto(f^g,g)$ is continous on the set

$$\{(f,g)\in C_T\ :\ f(T)\ne g(T)\}.$$

Note that we have not defined f_g when $f(0)=g(0)$ and that f^g isn't defined when $f(T)=g(T)$. In these cases, there is the choice between “pushing downwards” or “pushing upwards”.

2.2 Brownian invariance property

2.2.1 Statement

We are now ready to state the first result:

Theorem 3 *Suppose that $T > 0$, $a \in \mathbb{R}$ and $a' \in \mathbb{R}$ are fixed. Let $(B(t), t \in [0,T])$ and $(\beta(t), t \in [0,T])$ denote two independent Brownian motions with*

$$B(0) = a' \text{ and } \beta(T) = a$$

(β should be understood as a Brownian motion running backwards in time: The law of $(\beta(T-t), t \in [0,T])$ is that of a Brownian motion started at a). Then the two pairs of continuous processes,

$$(B_\beta, \beta) \text{ and } (B, \beta^B) \text{ are identical in law.}$$

In particular, B_β is a Brownian motion. This can seem somewhat surprising, as B_β is obtained by a one-sided pushing of B. However, this can be an 'upwards' pushing or a 'backwards' pushing depending on the value of $\beta(0)$. Figure 1 (in the introduction) shows the case where $a = a' = 0$ and $T = 1$.

Note that B_β and β^B are almost surely well-defined as almost surely,

$$B(T) \neq a \text{ and } \beta(0) \neq a'.$$

By translation invariance, we can assume that $a' = 0$, and the Brownian scaling property shows that we can restrict ourselves to the case $T = 1$.

Let us first derive a simple technical lemma:

Lemma 4 *Almost surely, $B_\beta(1) \neq a$ and $\beta^B(0) \neq 0$.*

Proof of Lemma 4. By symmetry, it suffices to show that almost surely, $B_\beta(1) \neq a$. Note that almost surely, $\beta(0) \neq 0$. Suppose now for a moment that $\beta(0) < 0$. In this case, B_β is defined by

$$B_\beta(t) = B(t) + \sup_{s \leq t}(B(s) - \beta(s))_-$$

and $B_\beta \geq \beta$. Hence, if $B_\beta(1) = a = \beta(1)$ then

$$-B(1) + \beta(1) = \sup_{s \leq 1}(\beta(s) - B(s))_+.$$

This implies that the one-dimensional Brownian motion

$$\left(\frac{\beta(t) - B(t) - \beta(0)}{\sqrt{2}},\ t \in [0,1]\right)$$

hits its maximum at time 1; this is almost surely not the case.

Similary, almost surely, if $\beta(0) > 0$ then $B_\beta(1) < a$ so that finally, we get that almost surely, $B_\beta(1) \neq a$.

We now turn our attention towards the proof of Theorem 3. For symmetry reasons, and using Lemma 4, it is sufficient to prove that (for any fixed a)

$$1_{\{B_\beta(0)>\beta(0),B_\beta(1)>\beta(1)\}}(B_\beta,\beta) \overset{\text{(in law)}}{=} 1_{\{B(0)>\beta^B(0),B(1)>\beta^B(1)\}}(B,\beta^B). \tag{1}$$

In this identity (and we shall use a similar notation in the rest of the paper) we say that when F is a function defined on a set S, then

$$1_{\{S\}}F(x) \quad = \quad \begin{cases} 0 \text{ if } x \notin S \\ F(x) \text{ if } x \in S \end{cases}$$

The idea of the proof is the following. We shall first observe that a corresponding statement for random walks holds, and we then use Donsker's invariance principle and the continuity properties of the mappings $(f,g) \mapsto (f_g,g)$ and $(f,g) \mapsto (f,g^f)$.

2.2.2 The discrete picture

Suppose that N is an even positive integer and that A is an even integer. Suppose that $(S(n), n \in [0,N])$ is a simple random walk started from 0 and that $(R(n), n \in [0,N])$ is an independent simple random walk running backwards in time with $R(N) = A$.

We define $S(t)$ and $R(t)$ for any real $t \in [0,N]$ by linear interpolation.

Note that as N is even, the probability that $S(N) = A$ is positive when $|A| \le N$ and that $P(S(N) = A) = P(R(0) = 0)$; in particular, the reflected functions S_R and R^S are not always well-defined.

Suppose now that $R(0) < 0$. In this case, the reflected function S_R is well-defined and is obtained by pushing S "upwards" when it hits R. Let $(x_0, \cdots, x_N)$ and $(y_0, \cdots, y_N)$ denote a pair in $\mathbb{Z}^{N+1}$ such that $x_0 = 0 > y_0$, $y_N = A < x_N$, and for any $j \in \{0, \cdots, N-1\}$,

$$|x_{j+1} - x_j| = 1, \ |y_{j+1} - y_j| = 1, \ x_j \ge y_j.$$

In other words, $x = (x_0, \cdots, x_N)$ and $y = (y_0, \cdots, y_N)$ is a possible realization of

$$S_R = (S_R(0), \cdots, S_R(N)) \text{ and } R = (R(0), \cdots, R(N))$$

with $R(0) < 0$.

Suppose that x and y are as above; define the set $C(x,y)$ of indices corresponding to common "upward" edges of the two paths x and y as follows:

$$C(x,y) = \{j \in \{0, \cdots, N-1\} \ : \ (x_j, x_{j+1}) = (y_j, y_{j+1}) \text{ and } x_{j+1} - x_j = +1\}.$$

Define

$$c(x,y) = \#C(x,y).$$

It is easy to notice that out of the $(2^N)^2 = 4^N$ possible configurations of S and R, there are $2^{c(x,y)}$ configurations such that

$$R = y \text{ and } S_R = x :$$

The condition $R = y$ gives just one single possible configuration for R. The condition $S_R = x$ implies that

$$S(j+1) - S(j) = x_{j+1} - x_j$$

for any $j \notin C(x,y)$, but there is no condition on $S(j+1) - S(j)$ when $j \in C(x,y)$ corresponds a common "upward" edge.

Similarly, suppose that x and y are defined just as above (note that $x_N > A$ so that if $S = x$, then R^S is well-defined), then, out of the possible 4^N possible configurations of S and R, there are $2^{c(x,y)}$ configurations such that

$$S = x \text{ and } R^S = y.$$

Hence, the two processes

$$1_{\{S(N)>A \text{ and } R^S(0)<0\}}(S, R^S)$$

and

$$1_{\{R(0)<0 \text{ and } S_R(N)>A\}}(S^R, R)$$

are identically distributed.

2.2.3 Conclusion of the proof of Theorem 3

We now simply have to apply Donsker's invariance principle carefully. For each fixed even integer $N = 2m$, define the two simple random walks S and R as above with $A = A(N)$ being an even integer such that $|A(N) - b\sqrt{N}| \leq 1$. Then, define for any $t \in [0,1]$,

$$B^N(t) = \frac{1}{\sqrt{N}} S(Nt)$$

and

$$\beta^N(t) = \frac{1}{\sqrt{N}} R(Nt).$$

We use the notation B^N and β^N for simplicity. This shouldn't be confused with the notation we used for reflection.

Donsker's invariance principle asserts that the law of the pair (B^N, β^N) converges weakly towards that of (B, β) when $N \to \infty$ with the topology of uniform convergence on $[0,1]$.

Let us put $U_N = (B^N, \beta^N)$, $U = (B, \beta)$, $\Phi(f,g) = (f_g, g)$ and $\Phi'(f,g) = (f, g^f)$. Define the two open sets D^1 and D^2 in C_T as follows:

$$D^1 = \{(f,g) \in C_T \ : \ f(0) > g(0) \text{ and } f_g(1) > g(1)\}$$

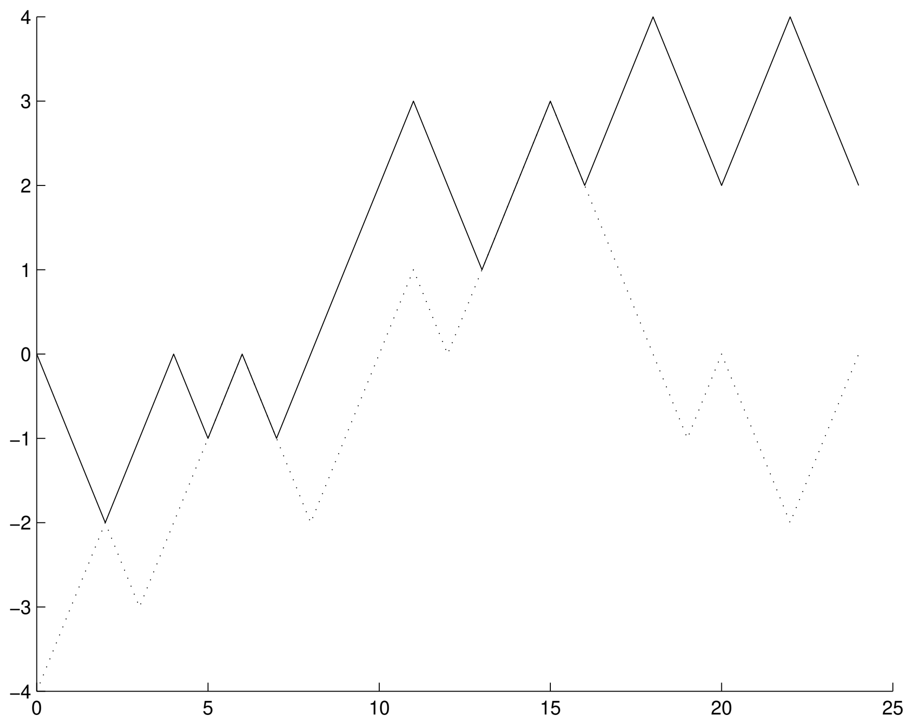

Figure 2: A joint realization of S (in black) and R^S (in dotted line). Here $N = 24$, $A = 0$ and $c = 3$ (corresponding to the three common upward edge)

and

$$D^2 = \{(f, g) \in C_1 \;:\; f(1) > g(1) \text{ and } g^f(0) < f(0)\}$$

and let ∂D^1 and ∂D^2 denote their frontiers with respect to the topology of uniform convergence on $[0, 1]$.

The result derived in the previous subsection can be reformulated as follows:

$$1_{\{U_N \in D^1\}} \Phi(U_N) \stackrel{\text{(in law)}}{=} 1_{\{U_N \in D^2\}} \Phi'(U_N).$$

Moreover, we have also seen that the two functions Φ and Φ' are continuous in D^1 and D^2. Lemma 4 shows that almost surely,

$$U = (B, \beta) \notin \partial D^1 \cup \partial D^2.$$

The law of U_N converges weakly towards the law of U when $N \to \infty$ and the law of U does charge neither ∂D^1 nor ∂D^2. Hence, when $N/2 = m \to \infty$,

$$1_{\{U_N \in D^1\}} \Phi(U_N) \stackrel{\text{in law}}{\Longrightarrow} 1_{\{U \in D^1\}} \Phi(U)$$

and

$$1_{\{U_N \in D^2\}} \Phi'(U_N) \stackrel{\text{in law}}{\Longrightarrow} 1_{\{U \in D^2\}} \Phi'(U)$$

so that we eventually see that

$$1_{\{U \in D^1\}} \Phi(U) \stackrel{\text{(in law)}}{=} 1_{\{U \in D^2\}} \Phi'(U)$$

i.e. precisely (1); this concludes the proof of Theorem 3.

3 More than two Brownian motions

We now turn our attention to the case where we are considering more than two Brownian motions and more precisely systems of coalescing and reflecting Brownian motions. The proofs go along similar lines as the 'two-component' case but there are several additional difficulties.

We will use the following terminology: When v is a real-valued function defined on an interval I, we say that

- The function v is of constant sign on I if $v(I) \subset \mathbb{R}_-$ or $v(I) \subset \mathbb{R}_+$.
- The sign of v oscillates on I if v is not of constant sign on I.
- We say that the sign of v oscillates just after t if there exists $\varepsilon_0 > 0$ such that $[t, t+\varepsilon_0] \subset I$ and if for all $\varepsilon \in (0, \varepsilon_0)$, the sign of v oscilates on $[t, t+\varepsilon]$.

3.1 Reflection/Coalescence operator

3.1.1 Coalescing/reflecting families

From now on, we will consider continuous real functions f that are defined on a half-line. In other words, there exists a real number $T(f)$ such that f is just a continuous function on $(-\infty, T(f)]$ or on $[T(f), +\infty)$. In the first case, we put $\varepsilon(f) = -$ and in the latter one, we put $\varepsilon(f) = +$. One should think of f as a function running forward with time when $\varepsilon(f) = +$ and backwards in time when $\varepsilon(f) = -$. Let C denote the class of such functions. Note that when we say that we choose $f \in C$, then we choose also its direction $\varepsilon(f)$ and its starting time $T(f)$. $I(f)$ denotes the interval on which f is defined and $J(f)$ denotes its interior.

Suppose now that $g_1, \cdots, g_p$ are p functions in C such that $\varepsilon(g_1) = \cdots = \varepsilon(g_p) = +$. We say that $(g_1, \ldots, g_p)$ is a set of coalescing forward functions if the following property is satisfied: For any $i \neq j \in \{1, \cdots, p\}$, for any $s \geq \max(T(g_i), T(g_j))$, if $g_i(s) = g_j(s)$ then $g_i = g_j$ on the whole half-line $[s, +\infty)$.

Similarly, we define families of coalescing backward functions as follows: If $\varepsilon(g_1) = \cdots = \varepsilon(g_p) = -$ and if for any $i \neq j \in \{1, \cdots, p\}$ and any $s \leq \min(T(g_i), T(g_j))$,

$$g_i(s) = g_j(s) \Rightarrow g_i = g_j \text{ on } (-\infty, s]$$

then $(g_1, \cdots, g_p)$ is a coalescing family of backward functions.

Suppose now that $g_1, \cdots, g_p$ are p functions in C but with no condition on the $\varepsilon(g_j)$'s. For sake of simplicity, we put

$$\varepsilon_i = \varepsilon(g_i) \text{ and } T_i = T(g_i).$$

We say that $(g_1, \cdots, g_p)$ is a perfectly coalescing/reflecting family if the following statements hold for any $i \neq j$ in $\{1, \cdots, p\}$:

- If $\varepsilon_i = +$, $\varepsilon_j = -$, and if $T_i < T_j$, then

  $$g_i(T_i) \neq g_j(T_i) \text{ and } g_i(T_j) \neq g_j(T_j).$$

  Moreover,

  $$g_i - g_j \text{ is of constant sign on } [T_i, T_j];$$

  let us call this sign $S(i,j) = S(j,i) \in \{+, -\}$. When $\varepsilon_i = \varepsilon_j$ or if $J_i \cap J_j = \emptyset$, we put $S(i,j) = 0$.

- If $\varepsilon_i = \varepsilon_{i'} = +$ and if there exists $s \geq \max(T_i, T_{i'})$ such that $g_i(s) = g_{i'}(s)$ then $g_i = g_{i'}$ on $[s, \infty)$.

- If $\varepsilon_i = \varepsilon_{i'} = -$ and if there exists $s \leq \max(T_i, T_{i'})$ such that $g_i(s) = g_{i'}(s)$ then $g_i = g_{i'}$ on $(-\infty, s]$.

The last two conditions are the "coalescing conditions" for functions running in the same direction and the first one implies that two functions running in opposite direction can never cross.

Note that we do not allow two forward lines to meet without coalescing even if they meet on a backward line and stay on two different sides of the backward line (we will come back to this later).

Note also that this definition indeed extends the definitions of coalescing forward functions and coalescing backward functions.

3.1.2 Coalescence

Suppose now that $g_1, \cdots, g_p$ is a coalescing foward family. Take $f \in C$ with $\varepsilon(f) = +$ and $T(f) = T$. Define the function $\hat{f} = C(f; g_1, \cdots, g_p)$ (in plain words: $\hat{f}$ is f coalesced with $(g_1, \cdots, g_p)$), as follows: Let

$$\tau = \inf\{t \geq T \ : \ \exists j \in \{1, \cdots, p\}, \ f(t) = g_j(t)\},$$

and then define $\hat{f} = f$ on the interval $[T, \tau)$ and when $\tau < \infty$, $\hat{f} = g_j$ on $[\tau, \infty)$ where j is chosen in such a way that $f(\tau) = g_j(\tau)$.

Note that when $g_1, \cdots, g_p$ and T are fixed, the mapping $f \mapsto \hat{f}$ is not continuous everywhere on the space $C[T, \infty)$ (i.e. the space of continuous functions on $[T, \infty)$ with the topology of uniform convergence on compact intervals). But it is easy to check the following lemma:

Lemma 5 *If f^* is such that $\tau < \infty$ and the sign of $(f^* - \hat{f}^*)$ oscillates just after τ, then the mapping $f \mapsto C(f; g_1, \cdots, g_p)$ is continuous at f^*.*

Note that clearly, for any permutation σ of $\{1, \cdots, p\}$,

$$C(f; g_1, \cdots, g_p) = C(f; g_{\sigma(1)}, \cdots, g_{\sigma(p)}). \tag{2}$$

When $f \in C$ with $\varepsilon(f) = -$ and $(g_1, \cdots, g_p)$ is a coalescing backward family, then we define $C(f; g_1, \cdots, g_p)$ in a similar (symmetric) way.

3.1.3 Reflection

Suppose now that $(g_1, \cdots, g_p)$ is a coalescing backward family. Suppose again that $f \in C$ with $\varepsilon(f) = +$ and $T(f) = T$.

We are now going to define a function $\tilde{f}$ on the interval $[T, \infty)$ that is loosely speaking "f reflected on the family $g_1, \cdots, g_p$". We shall denote $\tilde{f}$ by $R(f; g_1, \cdots, g_p)$.

Let us be more precise. We assume the following important condition:

(H0) For any $j \in \{1, \cdots, p\}$ such that $T < T_j$, we have $f(T) \neq g_j(T)$.

Without any restriction, we can assume that for all $j \in \{1, \cdots, p\}$, $T < T_j$ (otherwise, just drop those g_j's for which $T_j \leq T$).

Lemma 6 *There exists a unique continuous function $\tilde{f} : [T, \infty) \to \mathbb{R}$ such that:*

- *$\tilde{f}(T) = f(T)$.*
- *For any $j \in \{1, \cdots, p\}$, the function $\tilde{f} - g_j$ is of constant sign on $[T, T_j]$.*
- *there exist p continuous functions $v_1, \cdots, v_p$ defined on $[T, \infty)$ such that v_j is non-decreasing if $g_j(T) < f(T)$ and non-increasing if $g_j(T) > f(T)$, and anyway constant on $[T_j, \infty)$, such that*
$$\tilde{f}(t) = f(t) + \sum_{j=1}^{p} v_j(t)$$
and
$$\int_T^{T_j} 1_{\{g_j(t) \neq \tilde{f}(t)\}} dv_j(t) = 0.$$

Note that the case $p = 1$ is precisely Lemma 2 (and then, define the function $\tilde{f}$ on $[T_1, \infty)$ by $\tilde{f}(t) = f(t) + \tilde{f}(T_1) - f(T_1)$).

Proof. As the case $p = 1$ has already been proved, we suppose that $p \geq 2$. The unicity part is very similar to that of the proof of Lemmas 1 and 2: Suppose that f^1 and f^2 both meet the conditions required for $\tilde{f}$, with the corresponding functions $v_1^1, \cdots v_p^1, v_1^2, \cdots, v_p^2$. Then for any $t \geq T$,

$$\begin{aligned}
&(f^1(t) - f^2(t))^2 \\
&= 2\int_T^t (f^1(s) - f^2(s))d(f^1(s) - f^2(s)) \\
&= 2\sum_{j=1}^{p} \int_T^{\min(t,T_j)} (f^1(s) - g_j(s) + g_j(s) - f^2(s))d(v_j^2 - v_j^1)(s) \\
&= 2\sum_{j=1}^{p} \int_T^{\min(t,T_j)} (f^1(s) - g_j(s))dv^2(s) + (f^2(s) - g_j(s))dv^1(s) \\
&\leq 0,
\end{aligned}$$

so that $f^1 = f^2$.

To prove existence of $\tilde{f}$, things are little more complicated than in Lemma 2 due to the fact that there is no explicit formula for $\tilde{f}$ in terms of f and g_j's. However, here is a simple outline for how to construct $\tilde{f}$. We will need some further notation. For any $t \in \mathbb{R}$, define

$$G(t) = \{g_j(t) \ : \ 1 \leq j \leq p \text{ and } t \leq T_j\}.$$

When $t > \max_j(T_j)$, then $G(t) = \emptyset$. When $a \in G(t)$, we define the two indices $j^+(a,t)$ and $j^-(a,t)$ in such a way that loosely speaking g_{j^-} (resp. g_{j^+}) is the lowest (resp. the highest) of the curves that go through the point (t,a). More precisely, for any j such that $g_j(t) = a$, either $g_j(T_j) \leq g_{j^+}(T_j)$ or $g_j(T_{j^+}) < g_{j^+}(T_{j^+})$ (and the similar definition for j^-).

We are now ready to construct $\tilde{f}$. Define

$$\sigma_1 = \inf\{t > T \ : \ f(t) \in G(t)\}.$$

It is the first time at which f touches the backward system. Because of $(H0)$, $\sigma_1 > T$. We now define

$$\tilde{f} = f \text{ on } [T, \sigma_1).$$

In the case where $\sigma_1 < \infty$, then let $a_1 = f(\sigma_1)$. Suppose for instance that f hits the backward system at σ_1 "from above"; rigorously speaking, suppose that

$$f(T) > g_{j^+(a_1,\sigma_1)}(T).$$

Then, f will be pushed upwards by the backward system near σ_1, and more precisely by the function g_{j_1} where

$$j_1 = j^+(a_1, \sigma_1).$$

So, we define

$$\tilde{f}_1(t) = R(f; g_{j_1})(t)$$

for all $t \geq \sigma_1$. We define

$$\sigma_2 = \inf\{t > \sigma_1 \ : \ \tilde{f}_1(t) \in G(t) \setminus \{g_{j_1}(t)\}\}$$

and we define

$$\tilde{f} = \tilde{f}_1 \text{ on } [T, \sigma_2).$$

We then proceed by induction: For any $n \geq 2$, if $\sigma_n < \infty$, we define $a_n = \tilde{f}_{n-1}(\sigma_n)$,

$$\varepsilon_n = + \text{ or } - \text{ according whether this collision at time } \sigma_n \text{ is from below or above}$$

$$j_n = j^{\epsilon_n}(a_n, \sigma_n)$$

and

$$\begin{aligned} h_n(t) &= f(t) - f(\sigma_n) + \tilde{f}_{n-1}(\sigma_n), \ t \geq \sigma_n \\ \tilde{f}_n(t) &= R(h_n; g_{j_n})(t), \ t \geq \sigma_n \\ \sigma_{n+1} &= \inf\{t > \sigma_n \ : \ \tilde{f}_n(t) \in G(t) \setminus \{g_{j_n}(t)\}\} \\ \tilde{f} &= \tilde{f}_n \text{ on } [\sigma_n, \sigma_{n+1}). \end{aligned}$$

We leave the details to the reader. Note that uniform continuity of the continuous function f on $[T, \max_j T_j]$ ensure that after for some finite n_0, $\sigma_{n_0} = \infty$, and that this procedure indeed defines a function $\tilde{f}$ on the whole interval $[T, \infty)$.

We now make a list of some simple remarks concerning this definition of $\tilde{f} = R(f; g_1, \cdots, g_p)$:

- Note that the explicit construction of $\tilde{f}$ implies that if $\tilde{f}(t) = g_j(t)$ for some t, then t is (the time of) a local one-sided extremum of $f - g_j$. More precisely, there exists α such that on $[t - \alpha, t]$ the function
$$s \mapsto f(s) - g_j(s) - (f(t) - g_j(t))$$
is of constant sign. We shall use this observation later.
- Just as for the reflection on one function, the mapping $f \mapsto R(f; g_1, \cdots, g_p)$ is not well-defined when $(H0)$ is not satisfied.
- It is simple to check the following lemma:

  Lemma 7 *For any fixed family of backward coalescing functions $(g_1, \cdots, g_p)$, the mapping $f \mapsto 1_{(H0)}\tilde{f}$ defined on $C[T, \infty)$ is continuous at the point f_0 as soon as*
  $$f_0 \in \{f \in C[T, \infty) \ : \ (H0) \text{ is satisfied}\}.$$
- Finally note the following obvious statement: For any permutation σ of $\{1, \cdots, p\}$,
$$R(f; g_1, \cdots, g_p) = R(f; g_{\sigma(1)}, \cdots, g_{\sigma(p)}). \tag{3}$$

3.1.4 Coalescence/reflection

We are now going to combine coalescence and reflection. As we shall see, things become more complicated.

Suppose now that $(g_1, \cdots, g_p)$ is a perfectly coalescing/reflecting system (with no conditions on ε_j's). Define $i(1), \cdots, i(l)$ and $j(1), \cdots, j(k)$ in such a way that i and j are increasing, $l+k=p$ and that for all n, $\varepsilon_{i(n)} = +$ and $\varepsilon_{j(n)} = -$. In other words, $g_{i(1)}, \cdots, g_{i(l)}$ are the forward functions and $g_{j(1)}, \cdots, g_{j(k)}$ are the backward ones.

Suppose that f is a forward function defined on $[T, \infty)$ and that for any $u \leq k$ such that $T_{j(u)} > T$, one has $f(T) \neq g_{j(u)}(T)$. Then, define

$$CR(f; g_1, \cdots, g_p) = C\left(R(f; g_{j(1)}, \cdots, g_{j(k)}); g_{i(1)}, \cdots, g_{i(l)}\right).$$

In other words, we first construct the reflected function $R(f; g_{j(1)}, \cdots, g_{j(k)})$ and then let it coalesce with $(g_{i(1)}, \cdots, g_{i(l)})$.

Note that for fixed $(g_1, \cdots, g_p)$, and T, the mapping $f \mapsto CR(f; g_1, \cdots, g_p)$ is not well-defined and not continuous everywhere on $C[T, \infty)$ but Lemma 5 and Lemma 7 give conditions that ensure continuity at f_0 when f_0 belongs to a large class of functions.

Note also that (2) and (3) ensure that for any permutation σ of $\{1, \cdots, p\}$,

$$CR(f; g_1, \cdots, g_p) = CR(f; g_{\sigma(1)}, \cdots, g_{\sigma(p)}).$$

Similarly, when f is such that $\varepsilon(f) = -$, define $CR(f; g_1, \cdots, g_p)$ in an analogous way. We first let f reflect on $(g_{i(1)}, \cdots, g_{i(l)})$ and then let the obtained function coalesce with $(g_{j(1)}, \cdots, g_{j(l)})$. Then, the (symmetric) statements corresponding to those derived so far hold.

Let us stress at this point the following problem that is going to cause trouble later on in our proofs: The system

$$(g_1, \cdots, g_p, CR(f; g_1, \cdots, g_p))$$

is not necessarily a perfectly coalescing/reflecting system anymore. It might for instance happen that

$$R(f; g_{j(1)}, \cdots, g_{j(k)})$$

coalesces with $g_{i(1)}$ precisely on one of the backward functions, say $g_{j(1)}$. Then, the coalesced function $CR(f; g_1, \cdots, g_p)$ could cross the backward line $g_{j(1)}$ at that point.

Suppose now that $f_1, \cdots, f_p$ is a family of p functions in C (with no conditions on the "directions"). Then, we wish to define the coalesced/reflected system

$$CR(f_1, f_2, \cdots, f_p) = (g_1, \cdots, g_p)$$

as follows: $g_1 = f_1$ and for any $i \in \{2, \cdots, p\}$,

$$g_i = CR(f_i; g_1, g_2, \cdots, g_{i-1}).$$

Note that this time, $(g_1, \cdots, g_p)$ depends in a crucial way on the ordering of $f_1, \cdots, f_p$. For instance, $g_1 = f_1$ whereas it can happen that $f_p \neq g_p$. Such a definition is indeed possible by induction provided that at each step $i \in \{2, \cdots, p\}$,

- For all $i' \in \{1, \cdots, i-1\}$ such that $J_i \cap J_{i'} \neq \emptyset$ and $\varepsilon_i \neq \varepsilon_{i'}$, one has $g_{i'}(T_i) \neq g_i(T_i)$.
- The system $(g_1, \cdots, g_i)$ is a perfectly coalescing/reflecting system.

When these conditions are satisfied (and therefore $CR(f_1, \cdots, f_p)$ is well-defined), we say that

$$(f_1, \cdots, f_p) \in W.$$

Finally, we define

$$\Psi(f_1, \cdots, f_p) = 1_{\{(f_1, \cdots, f_p) \in W\}} CR(f_1, \cdots, f_p).$$

3.1.5 Main result

We are now ready to state the multi-component version of our main result: For any family of functions $f = (f_1, \cdots, f_p)$, we define $f^{(\sigma)} = (f_{\sigma(1)}, \cdots, f_{\sigma(p)})$ (for any permutation σ of $\{1, \cdots, p\}$).

Theorem 8 *Suppose that $B^1, \cdots, B^p$ are p (forward or backward) Brownian motions respectively started at times $t_1, \cdots, t_p$, levels $a_1, \cdots, a_p$ and running in the directions $\varepsilon(1), \cdots, \varepsilon(p)$. Suppose that for any $i \neq j$, $(t_i, a_i, \varepsilon(i)) \neq (t_j, a_j, \varepsilon(j))$ and let σ denote any permutation of $\{1, \cdots, p\}$.*

Then almost surely

$$B = (B^1, \cdots, B^p) \in W \text{ and } B^{(\sigma)} = (B^{\sigma(1)}, \cdots, B^{\sigma(p)}) \in W$$

and

$$\Psi(B)^{(\sigma)} \overset{(in\ law)}{=} \Psi(B^{(\sigma)}).$$

In other words, the order with which the coalescence/reflection rule has been used does not affect the law of the outcome. In particular, it shows for instance that if

$$(Y^1, \cdots, Y^p) = CR(B^1, \cdots, B^p)$$

then Y^p and B^p have the same law (i.e. Y^p is a Brownian motion).

Figure 3 shows a picture of $CR(B^1, B^2, B^3, B^4)$ obtained from two forward Brownian motions and two backward Brownian motions.

In the case where $p = 2$, $\varepsilon(1) = +$ and $\varepsilon(2) = -$, we simply recover Theorem 3. Note also that when all Brownian motions run in the same direction i.e. when $\varepsilon(1) = \cdots = \varepsilon(p)$ then the Theorem is an easy consequence of the strong Markov property (see for instance [8]).

Hence, the case $p = 2$ has already been proved. We are in fact going to prove Theorem 8 using an induction over p (this is not absolutely necessary but it will simplify some technical details). We are also first going to derive it in the case where $t_1, \cdots, t_p$ are all rational numbers.

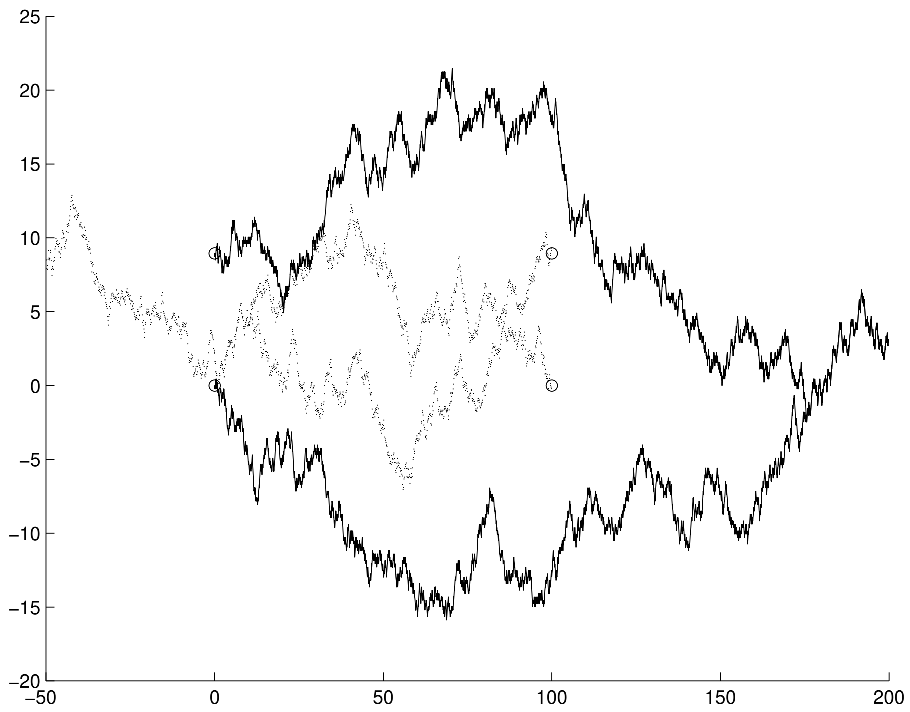

Figure 3: A joint realization of $CR(B^1, B^2, B^3, B^4)$ with four prescribed starting points. The forward lines are in black and the backward lines in grey. The starting points are circled.

3.2 The discrete picture

Take $p+1$ (forward or backward) running simple random walks $S^1, \cdots, S^{p+1}$ started respectively from the even integer times $T_1, \cdots, T_{p+1}$ at the even integer levels

$$S^1(T_1) = A_1, \cdots, S^{p+1}(T_{p+1}) = A_{p+1}$$

and in the directions $\varepsilon(1), \cdots, \varepsilon(p+1)$. Define also the corresponding intervals $I_1, J_1, \cdots, I_{p+1}, J_{p+1}$.

Suppose that σ is a permutation of $\{1, \cdots, p+1\}$. We put

$$S = (S^1, \cdots, S^{p+1}) \text{ and } S^{(\sigma)} = (S^{\sigma(1)}, \cdots, S^{\sigma(p+1)}).$$

Then, the following identity in law holds:

$$\Psi(S)^{(\sigma)} \overset{\text{(in law)}}{=} \Psi(S^{(\sigma)}) \tag{4}$$

This is easy to see just by counting the possible configurations corresponding to the times in the interval $[T_-, T_+] = [\min_{i\le p}(T_i), \max_{i\le p}(T_i)]$.

Define the set of edges

$$E = \cup_{i\in[T_-,T_+-1]} \cup_{j\in\mathbb{Z}} \{((i,j),(i+1,j+1)),((i,j),(i+1,j-1))\}$$

We say that an edge of the type $((i,j),(i+1,j+1))$ is an upward edge and that $((i,j),(i+1,j-1))$ is a downward edge.

Define

$$(U^1,\cdots,U^{p+1}) = \Psi(S^1,\cdots,S^{p+1}).$$

We declare that the edge $((i,j),(i+1,j'))$ is occupied by $(U^1,\cdots,U^{p+1})$ if for some $l\le p+1$,

$$U^l(i) = j \text{ and } U^l(i+1) = j'.$$

It is a forward (respectively backward) occupied edge if we add the condition $\varepsilon_l = +$ (resp. $\varepsilon_l = -$).

Note that if $(S^1,\cdots,S^{p+1}) \in W$ and if a given edge e is occupied simulateously by forward edges and backward edges of $(U^1,\cdots,U^{p+1})$, the fact that $(S^1,\cdots,S^{p+1})$ belongs to W ensures that for any forward path U^i that occupies e and any backward path that occupies e, $S(i,j)$ is well-defined and its value is actually independent of the particular choice of i and j. We therefore put $S(e) = S(i,j)$. In other words, $S(e) = +$ when e is occupied simultaneously by forward and backward paths and when the corresponding forward paths come from "above" the backward paths.

We define j^+ the number of such edges that are also upward edges. We define j^- as the number of downward edges with $S(e) = -$. Loosely speaking, those are the edges where one path has been forced to move up or down.

The probability of a given configuration (in the time-interval $[T_-,T_+]$) is $1/2^q$ where $q = f + b - (j^+ + j^-)$ and

- f is the number of forward occupied edges
- b is the number of backward occupied edges

and this holds independently of the order of the coalescing/reflection procedure. Hence, one indeed gets (4).

3.3 Part I of the proof

3.3.1 Continuity of CR

We will want to derive Theorem 8 via an invariance principle argument. In order to achieve this, we have to define as set of paths such that almost surely $(B^1,\cdots,B^p)$ belongs to this set, and the mapping Ψ defined on $\prod_{j=1}^p \mathcal{C}(I_j)$ by

$$\Psi \ : \ (f^1,\cdots,f^p) \mapsto 1_{(f^1,\cdots,f^p)\in W} CR(f^1,\cdots,f^p)$$

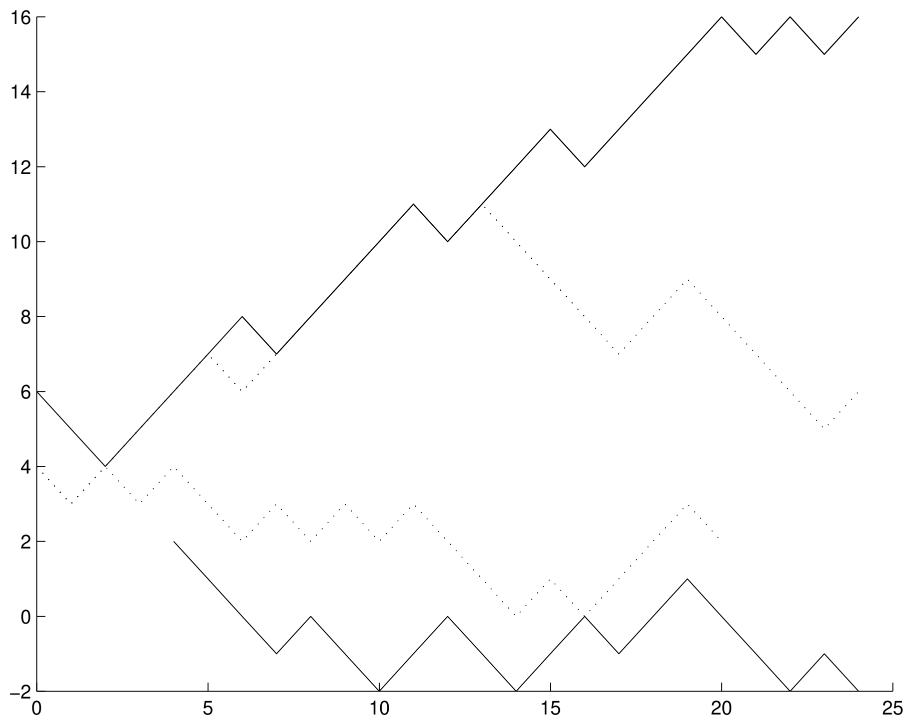

Figure 4: A joint realization of $CR(S^1, \cdots, S^4)$ with four prescribed starting points. The forward lines are in black and the backward lines in grey. Here $j^+ = 9$ and $j^- = 0$

is continuous at any point of this set.

W is not quite the correct choice for this set because continuity of Ψ fails. We therefore define $W' \subset W$ as the set of points in W at which Ψ is continuous.

We now briefly describe a condition that will ensure continuity of Ψ. Let W'' be the subset of W with functions that oscillate just after coalescence. More precisely, for any $(f_1, \cdots, f_p) \in W$ we define

$$(g_1, \cdots, g_p) = CR(f_1, \cdots, f_p)$$

and for any $j \leq p$,

$$h_j = R(f_j; g_{i(1)}, \cdots, g_{i(u)})$$

where

$$\{i(1), \cdots, i(u)\} = \{i < j \ : \ \varepsilon(i) \neq \varepsilon(j)\}.$$

Note that

$$g_j = C(h_j; g_{l(1)}, \cdots, g_{l(v)})$$

where

$$\{l(1),\cdots,l(v)\} = \{1,\cdots,j-1\}\setminus\{i(1),\cdots,i(u)\}.$$

Let τ_j denote the coalescence time of h_j i.e. the first time at which h_j meets one of the other functions $g_{l(1)},\cdots,g_{l(v)}$. Then, W'' is the set of functions such that for any $\alpha>0$ and for any $j\in\{1,\cdots,p\}$,

$$\text{if } v\neq 0 \text{ then } \tau_j\in\mathbb{R} \text{ and } g_j-h_j \text{ is not of constant sign on } (\tau_j,\tau_j+\alpha\varepsilon_j).$$

Lemma 9 *For any fixed p, and $I_1,\cdots,I_p$. The function Ψ (defined on $\prod_{j=1}^p \mathcal{C}(I_j)$) is continuous at any $(f_1,\cdots,f_p)\in W''$.*

3.3.2 The induction

We now concentrate again on the continuous setting. In this subsection, we will assume that the starting times are rational numbers and we will prove Theorem 8 by induction over p.

Let us first state clearly the induction hypothesis. For any $p\geq 2$, we define the set

$$\begin{aligned}\mathcal{A}_p &= \{(t_1,\cdots,t_p,a_1,\cdots,a_p,\varepsilon_1,\cdots,\varepsilon_p)\in\mathbb{Q}^p\times\mathbb{R}^p\times\{+,-\}^p \ : \\ &\quad \forall i\neq j \text{ in } \{1,\cdots,p\},\ (t_i,a_i,\varepsilon_i)\neq(t_j,a_j,\varepsilon_j)\}.\end{aligned}$$

We say that $B^1,\cdots,B^p$ is a set of independent Brownian motions corresponding to $(t_1,\cdots,\varepsilon_p)\in\mathcal{A}_p$ when $B^1,\cdots,B^p$ are independent, and for any $i\in\{1,\cdots,p\}$, B^i is a Brownian motion with ε_i-direction and started at time t_i from level a_i.

Finally, let I_i denote the time-interval on which B^i is defined (i.e. $[t_i,\infty)$ if $\varepsilon_i=+$ and $(-\infty,t_i]$ if $\varepsilon_i=-$) and J_i its interior.

The induction hypothesis $\mathcal{P}_p$ is the following: For any $(t_1,\cdots,\varepsilon_p)\in\mathcal{A}_p$, if $B^1,\cdots,B^p$ is a set of independent Brownian motions corresponding to $(t_1,\cdots,\varepsilon_p)$, then

- (1.p) Almost surely, $(B^1,\cdots,B^p)\in W$.
- (2.p) Almost surely, $(B^1,\cdots,B^p)\in W'$.
- (3.p) For any permutation σ of the set $\{1,\cdots,p\}$,

$$(Y^{\sigma(1)},\cdots,Y^{\sigma(p)})\overset{\text{(in law)}}{=}CR(B^{\sigma(1)},\cdots,B^{\sigma(p)}).$$

Note that (3.p) is the statement corresponding to Theorem 8. The other two statements (1.p) and (2.p) ensure that $(Y^1,\cdots,Y^p)=CR(B^1,\cdots,B^p)$ is well-defined, and they will also be useful to derive (3.p) using the invariance principle. Note that (2.p) in fact contains (1.p) as $W'\subset W$, but we will prove them separately.

We split the proof into several short steps. Note first that $\mathcal{P}_2$ holds. So, we are going to assume that $\mathcal{P}_p$ holds for some fixed $p\geq 2$, and we want to prove that $\mathcal{P}_{p+1}$ holds as well. Let us fix

$$(t_1,\cdots t_{p+1},a_1,\cdots,a_{p+1},\varepsilon_1,\cdots,\varepsilon_{p+1})\in\mathcal{A}_{p+1}$$

and let $(B^1, \cdots, B^{p+1})$ denote $p+1$ independent Brownian motions started from $(t_1, \cdots, \varepsilon_{p+1})$. We can assume by symmetry that $\varepsilon(p+1) = +$.

3.3.3 Reduction

Suppose that $\varepsilon_1 = \cdots = \varepsilon_p = \varepsilon_{p+1} = +$ (i.e. all the Brownian motions run forward). In that case, the coalescence/reflection is simply a coalescence, and the statement (1.p+1) is in this case straightforward as almost surely for any $i \neq j$ with $t_i < t_j$, $B^i(t_j) \neq a_j$, and as a.s. $U^i(t_j) = B^{i'}(t_j)$ for some i'. Statements (2.p+1) and (3.p+1) are straightforward consequences of the strong Markov property. Hence, we will from now on assume that for some $j \in \{1, \cdots, p\}$, $\varepsilon_j = -$.

Define whenever it is possible

$$(Y^1, \cdots, Y^{p+1}) = CR(B^1, \cdots, B^{p+1}).$$

Note that statements (1.p+1), (2.p+1) and (3.p+1) are statements concerning laws. As we assumed $\mathcal{P}_p$, and as

$$(Y^1, \cdots, Y^p) = CR(B^1, \cdots, B^p)$$

we can change the order with which we perform the coalescence rule for the first p Brownian paths without affecting the law of $(Y^1, \cdots, Y^{p+1})$. Hence, we can in fact assume that there exists $l \in \{1, \cdots, p\}$ such that

$$\varepsilon_1 = \cdots = \varepsilon_l = - \text{ and } \varepsilon_{l+1} = \cdots = \varepsilon_{p+1} = +.$$

3.3.4 Proof of (1.p+1)

Note that

$$(Y^1, \cdots, Y^p) = CR(B^1, \cdots, B^p)$$

and that we assumed $\mathcal{P}_p$ so that $(Y^1, \cdots, Y^p)$ are a.s. well-defined and (1.p) holds. Moreover, for each $j \leq p$, the law of Y^j is that of a Brownian motion started from time t_j at level a_j and in the ε_j direction. This implies immediately that almost surely, for all $j \in \{1, \cdots, p\}$

$$Y^j(t_{p+1}) \neq a_{p+1} \text{ if } t_{p+1} \in J_j.$$

In particular, this shows that Y^{p+1} is a.s. well-defined. We therefore only have to check two facts: Almost surely,

1. Y^{p+1} does not hit the starting points of $Y^1, \cdots, Y^l$.

2. Y^{p+1} does not cross a backward path.

These two facts recall the “fine topological structure properties of the system of forward and backward lines” derived in [8].

(1) Recall that the explicit definition of the reflected function $\tilde{f} = R(f; g_1, \cdots, g_p)$ shows that if $\tilde{f}(t) = g_j(t)$ for some t, then t is a local one-sided maximum or minimum (depending on whether $\tilde{f}$ is below or above g_j) of the function $f - g_j$. More precisely, there exists $\alpha > 0$ such that either $f(s) - g_j(s) \leq f(t) - g_j(t)$ for any $s \in [t-\alpha, t]$, or $f(s) - g_j(s) \geq f(t) - g_j(t)$ for any $s \in [t-\alpha, t]$. In particular, if Y^{p+1} hits the starting point of Y^1, then T_1 is a time of a local one-sided maximum (or minimum) of $B^j - B^1$ (for some $j \in \{l+1, \cdots, p+1\}$ such that Y^{p+1} is locally following the reflection of a translate of B^j just before T_1). As the time T_1 is deterministic, we know that this is almost surely not the case.

(2) This second fact is of a slightly different nature. If Y^{p+1} crosses a backward path (say Y^1), then it means that Y^{p+1} has coalesced with another forward path (say Y^p) on Y^1 in such a way that Y^p and Y^{p+1} are coming from two different "sides" of Y^1.

Then, at the coalescence time σ, two events occur simultaneously: for some $j \neq j' > l$: $B^j - B^1$ is at a local one-sided maximum and $B^{j'} - B^1$ is at a local one-sided minimum (j and j' are the two brownian motions used to describe the evolution of Y^p and Y^{p+1} just before σ).

Now define

$$\gamma^1 = \frac{B^j - B^1}{\sqrt{2}}$$

and

$$\gamma^2 = \frac{B^j + B^1 - 2B^{j'}}{\sqrt{6}}.$$

It is easy to see that the conditions for $B^1, B^j, B^{j'}$ near σ imply that σ is a (local time of a) one-sided cone point of angle θ_0 (with $\theta_0 < \pi/2$) for the two-dimensional Brownian motion $\tilde{\gamma}(t) = (\gamma^1(t) - \gamma^1(t_0), \gamma^2(t) - \gamma^2(t_0))$ (for some rational $t_0 < t$). We know (see e.g. [5]) that such points almost surely never exist. This proves the second statement.

3.3.5 Proof of (2.p+1)

As we assumed $\mathcal{P}_p$, we know that $(B^1, \cdots, B^p)$ is almost surely a point of continuity for Ψ. Hence, it will be sufficient to show that almost surely, the mapping

$$f_{p+1} \mapsto CR(f_{p+1}; Y^1, \cdots, Y^p)$$

is continuous at B^{p+1}. Recall that we assumed that $\varepsilon(1) = \cdots = \varepsilon(l) = -$ and that $\varepsilon(l+1) = \cdots = \varepsilon(p+1) = +$.

We are again going to treat the cases $l = p$ and $l < p$ separately.

Case 1. We assume that $l = p$. In that case, for any $f \in C[a, \infty)$,

$$CR(f; Y^1, \cdots, Y^p) = R(f; Y^1, \cdots, Y^p).$$

Hence Lemma 7 shows that

$$f \mapsto CR(f; Y^1, \cdots, Y^p)$$

is a.s. continuous at $f = B^{p+1}$.

Case 2. We now assume that $l < p$ (remember that $l \geq 1$). Then,

$$(Y^{l+1}, \cdots, Y^{p+1})$$

is obtained by coalescence of the reflected paths $(Z^{l+1}, \cdots, Z^{p+1})$ where

$$Z^j = R(B^j; Y^1, \cdots, Y^l).$$

Because of (1.p+1) we know that the reflection operation is almost surely continuous. It therefore remains to check that if $i \neq j$ in $\{l+1, \cdots, p+1\}$, and if

$$T = \inf\{t > \max(t_i, t_j) \ : \ Z^i(t) = Z^j(t)\}$$

then the sign of $Z^i - Z^j$ osillates just after T. Define the σ-field

$$\mathcal{G} = \sigma\left(Y^1, \ldots, Y^l, (Z^j(t), t \leq T), (Z^i(t), t \leq T)\right).$$

Define the random variable U as follows:

- $U = 2$ if there exists $\alpha > 0$ such that $Z^j = Z^i$ on $[T, T + \alpha]$.
- $U = 1$ if there exists $\alpha > 0$ such that $Z^j \geq Z^i$ on $[T, T + \alpha]$ (and if $U \neq 2$).
- $U = -1$ if there exists $\alpha > 0$ such that $Z^j \leq Z^i$ on $[T, T + \alpha]$ (and if $U \neq 2$).
- $U = 0$ in all other cases.

Note that $U = 0$ implies that $Z^j - Z^i$ oscillates just after T.

It is very easy to check that almost surely, $U \neq 2$. Conditionaly on $\mathcal{G}$ and for all $\alpha > 0$, the random variable U is determined by the knowledge of

$$(B^j(T+t) - B^j(T), 0 \leq t \leq \alpha) \text{ and } (B^i(T+t) - B^i(T), 0 \leq t \leq \alpha).$$

These two Brownian motions are independent of $\mathcal{G}$ because of the strong Markov property. Moreover, a simple symmetry argument shows that

$$P(U = 1) = P(U = -1).$$

Hence, the 0-1 law implies that $U = 0$ almost surely.

3.3.6 Proof of (3.p+1)

We are now ready to show that $\mathcal{P}_{p+1}$ indeed holds by deriving (3.p+1). As $t_1, \cdots, t_{p+1}$ are rational numbers, there exists m_0 such that $t_1 m_0, \cdots, t_p m_0$ are all even integers. Suppose for a while that $N \geq 1$ is fixed and define

$$T_1 = m_0 t_1 N, \cdots, T_{p+1} = m_0 t_{p+1} N,$$

and choose even integers $A_1, \cdots, A_{p+1}$ such that $|A_j - a_j\sqrt{m_0 N}| \leq 1$. Define the $p+1$ independent simple random walks $S^1, \cdots, S^{p+1}$ respectively started at time $T_1, \cdots, T_{p+1}$, levels $A_1, \cdots, A_{p+1}$ and running in the $\varepsilon_1, \cdots, \varepsilon_{p+1}$ directions. To mark the dependence in N, we put

$$S_N = (S^1, \cdots, S^{p+1}) \text{ and } S_N^{(\sigma)} = (S^{\sigma(1)}, \cdots, S^{\sigma(p+1)}).$$

Then, as observed in (4),

$$\Psi(S_N)^{(\sigma)} \overset{\text{(in law)}}{=} \Psi(S_N^{(\sigma)}).$$

Now, define for $j \in \{1, \cdots, p+1\}$,

$$\begin{aligned} B_N^j(t) &= \frac{1}{\sqrt{Nm_0}} S^j(Nm_0 t) \\ B_N &= (B_N^1, \cdots, B_N^{p+1}) \end{aligned}$$

Then, by scaling

$$\Psi(B_N)^{(\sigma)} = \Psi(B_N^{(\sigma)}).$$

We know that:

- The law of B_N converges weakly to the law of $B = (B^1, \cdots, B^{p+1})$ (seen of the set $\prod_{i=1}^{p+1} C(I_i)$).
- Ψ is almost surely continuous at B.

Hence, the law of $\Psi(B_N)$ converges to that of $\Psi(B)$. Similarly, the law of $\Psi(B_N^{(\sigma)})$ converges to that of $\Psi(B^{(\sigma)})$. Hence, we indeed obtain the identity in law (3.p+1) so that $\mathcal{P}_{p+1}$ holds.

3.4 Conclusion of the proof of Theorem 8

It now remains to remove the asumption that $t_1, \cdots, t_p$ are rational numbers. Define the p Brownian motions $B^1, \cdots, B^p$ started at times $t_1, \cdots, t_p$ from levels $a_1, \cdots, a_p$ and in directions $\varepsilon_1, \cdots, \varepsilon_p$. Define for each $i \in \{1, \cdots, p\}$ a sequence $(t_i^n)_{n \geq 1}$ of rational numbers such that

- t_i^n converges to t_i when $n \to \infty$.
- t_i^n is decreasing if $\varepsilon_i = +$ and increasing if $\varepsilon_i = -$.

Then, define also a Brownian motion W_i^n started at time t_i^n from level a_i and in the ε_i direction. Finally, define

$$B_n^i = C(W_i^n; B^i).$$

Note that in the case where $\varepsilon_i = +$, for any $\alpha > 0$,

$$P\left(B_n^i = B^i \text{ on } [t_i + \alpha, \infty)\right) \overset{n\to\infty}{\longrightarrow} 1.$$

But we know that Theorem 8 holds for $(B_n^1, \cdots, B_n^p)$. Letting $n \to \infty$ leads to Theorem 8 for $(B^1, \cdots, B^n)$.

4 Generalizations and consequences

As opposed to the rest of this paper, we will not go into details in this section.

4.1 Stationary true self-repelling motion

A consequence of Theorem 8 is the following: Let E denote the set of finite subsets of $\mathbb{R}^2$. For any $A = \{(x_1, y_1), \cdots, (x_p, y_p)\}$ define the law P_A of families of forward and backward coalescing brownian motions started at these points. More precisely, let $B^1, \cdots, B^p$ denote p independent forward brownian motions started at times $x_1, \cdots, x_p$ from levels $y_1, \cdots, y_p$ respectively. Let $\gamma^1, \cdots, \gamma^p$ denote p independent backward Brownian motions started at the same points. Then define

$$(U^1, V^1, U^2, V^2, \cdots, U^p, V^p) = CR(B^1, \gamma^1, \cdots, B^p, \gamma^p)$$

(it is almost surely well-defined). And finally, define for any $j \in \{1, \cdots, p\}$ and $x \in \mathbb{R}$,

$$W^j(x) = W^{(x_j, y_j)}(x) = 1_{\{x < x_j\}} V^j(x) + 1_{\{x \geq x_j\}} U^j(x).$$

We define P_A to be the law of $(W^1, \cdots, W^p)$. Note that the perfectly coalescing/reflecting property of $(U^1, V^1, \cdots, U^p, V^p)$ implies that for any $i \neq j$, the two functions W^i and W^j never cross and are different (because $W^i(x_j) \neq W^j(x_j)$).

Theorem 8 shows that this law is independent of the order in which the coalescence/reflection rule has been used. Moreover, it shows that the families of laws $(P_A, A \in E)$ is a compatible family of probability measures.

In particular, it is therefore possible to define P_A where $A = \mathbb{Q}^2$. Then for any $(x, y) \in \mathbb{Q}^2$, define

$$T(x, y) = \int_{\mathbb{R}} (W^{(x,y)}(u) - W^{(0,0)}(u)) du.$$

As seen above, almost surely, the (random) mapping $(x, y) \mapsto T(x, y)$ (defined on $\mathbb{Q}^2$) is injective. Define $\mathcal{T} = T(\mathbb{Q}^2)$. For any $\tau \in \mathcal{T}$, define $(x(\tau), y(\tau))$ as the unique $(x, y) \in \mathbb{Q}^2$ such that $T(x, y) = \tau$.

Proposition 10 *Almost surely, $\overline{\mathcal{T}} = \mathbb{R}$. Moreover, almost surely, the mapping $(x(\cdot), y(\cdot))$ defined on $\mathcal{T}$ has a unique continuous extension $(X(\cdot), H(\cdot))$ to $\mathbb{R}$. This defines a continuous random process $(X_t, t \in \mathbb{R})$.*

This process has been defined in [8] and is referred to as true self-repelling motion with stationary increments. For a proof of this result and properties of this process, see [8].

4.2 Coalescence/reflection on deterministic curves

We now briefly discuss the case, where some of the forward (or backward lines) are deterministic. Our aim in the present section is not to give the "maximal" result but just to show the ideas of what is going on in this case.

Suppose that $f_1, \cdots, f_q$ is a (fixed) coalescing/reflecting system such that for each j, f_j is uniformly Lipschitz on any compact interval in I_j. Suppose also (for ease) that for any j, a Brownian motion running in the same direction as f_j will almost surely hit f_j (independently of where the Brownian motion starts); this is for instance the case if $|f_j|$ does not increase faster than $\sqrt{(2-\epsilon)|t|\log\log|t|}$ for large $|t|$). We the say that $(f_1, \cdots, f_q)$ is a nice CR family.

Theorem 11 *Suppose that $(f_1, \cdots, f_q)$ is a nice CR family and that $B^1, \cdots, B^p$ are p (forward or backward) Brownian motions started respectively at times $t_1, \cdots, t_p$ from the levels $a_1, \cdots, a_p$ and in the directions $\varepsilon_1, \cdots, \varepsilon_p$.*

Suppose that σ is a permutation of the set $\{1, \cdots, p\}$. Define

$$\begin{aligned} U^1 &= CR(B^1; f_1, \cdots, f_q) \\ V^{\sigma(1)} &= CR(B^{\sigma(1)}; f_1, \cdots, f_q) \end{aligned}$$

and for $i \in \{2, \cdots, p\}$,

$$\begin{aligned} U^i &= CR(B^i; f_1, \cdots f_q, U^1, \cdots, U^{i-1}) \\ V^{\sigma(i)} &= CR(B^{\sigma(i)}; f_1, \cdots, f_q, V^{\sigma(1)}, \cdots, V^{\sigma(i-1)}). \end{aligned}$$

Then,

$$(U^1, \cdots, U^p) \overset{\textit{(in law)}}{=} (V^1, \cdots, V^p).$$

In other words, the law of $U^1, \cdots, U^p$ does not depend on the order used to construct the paths and to apply the reflection/coalescence rule. The proof of this Theorem is almost identical to that of Theorem 8. Use a discretization of the functions $f_1, \cdots, f_q$, approximate the Brownian motions by simple random walks, and use Donsker's invariance principle carefully. The main difference lies in the proof of the two facts corresponding to (1.p+1) in the induction procedure. We leave this to the interested reader.

In order to stress that -in some way- the proof of the two facts corresponding to (1.p+1) is the crucial part of the proof, let us very briefly describe an example of a (non-lipschitz) function f where things go wrong:

Take the function $f(t) = -t^{1/3}$ defined on $\mathbb{R}_+$, and let $(B^n, n \geq 0)$ denote a family of independent forward Brownian motions started at time 0 respectively from the levels $B^n(0) = 1/n$. Take also another independent backward Brownian motion β started at time 1 from level $\beta(1) = 0$. As f is Lipschitz on any compact subinterval of $(0, \infty)$, it is easy to see that for any $n \geq 1$, if we put

$$U_n^1 = C(B^n; f),\ U_n^2 = R(\beta; f, U_n^1)$$

and

$$V_n^1 = V^1 = R(\beta; f),\ V_n^2 = CR(B^n; V^1, f)$$

then

$$(U_n^1, U_n^2) \overset{\text{(in law)}}{=} (V_n^2, V_n^1)$$

and in particular,

$$U_n^2 \overset{\text{(in law)}}{=} V^1.$$

But, if B is a Brownian motion started at time 0 from level 0,

$$p := P(\forall t \in (0,1]\ ,\ B(t) > -t^{1/3} \text{ and } B(1) > 0)$$

is strictly positive. Hence, by comparison, for all $n \geq 1$,

$$P(\forall t \in [0,1]\ ,\ U_n^1(t) > -t^{1/3} \text{ and } U_n^1(1) > 0) \geq p > 0.$$

But if this event is true, then necessarily, $U_n^2(0) \in [0, 1/n]$. Hence, we finally see that for any $n \geq 1$,

$$P(V^1(0) \in [0,1/n]) = P(U_n^2(0) \in [0,1/n]) \geq p$$

so that

$$P(R(\beta; f)(0) = 0) \geq p > 0.$$

In other words, reflection on the curve f forces the Brownian motion β to hit the point $(0,0)$ with positive probability (see Figure 5).

As a consequence, consider for instance the two functions

$$\begin{aligned} f_1(t) &= -t^{1/3} \\ f_2(t) &= -|t|^{1/3} \end{aligned}$$

defined on $\mathbb{R}^+$ and $\mathbb{R}^-$ respectively (f_1 is a forward function and f_2 a backward function). Consider the two Brownian motions B^1 and B^2 such that

$$\varepsilon_1 = +,\ t_1 = -1,\ a_1 = 0,\ \varepsilon_2 = -,\ t_2 = 1,\ a_2 = 0.$$

Then, with positive probability, both coalesced/reflected Brownian motions go through the point $(0,0)$ and coalesce with f_1 and f_2 at this point. Hence, the two curves U^1 and U^2 cross with positive probability (see Figure 6).

4.3 True self-repelling motion

Just in the same way than stationary TSRM has been constructed using Theorem 8, one can actually construct TSRM using Theorem 11.

Just apply Theorem 11 in the case where $q = 2$ and

- f_1 is defined on $[0, \infty)$ (it is running forward) and $f_1(t) = 0$ for all $t \geq 0$.

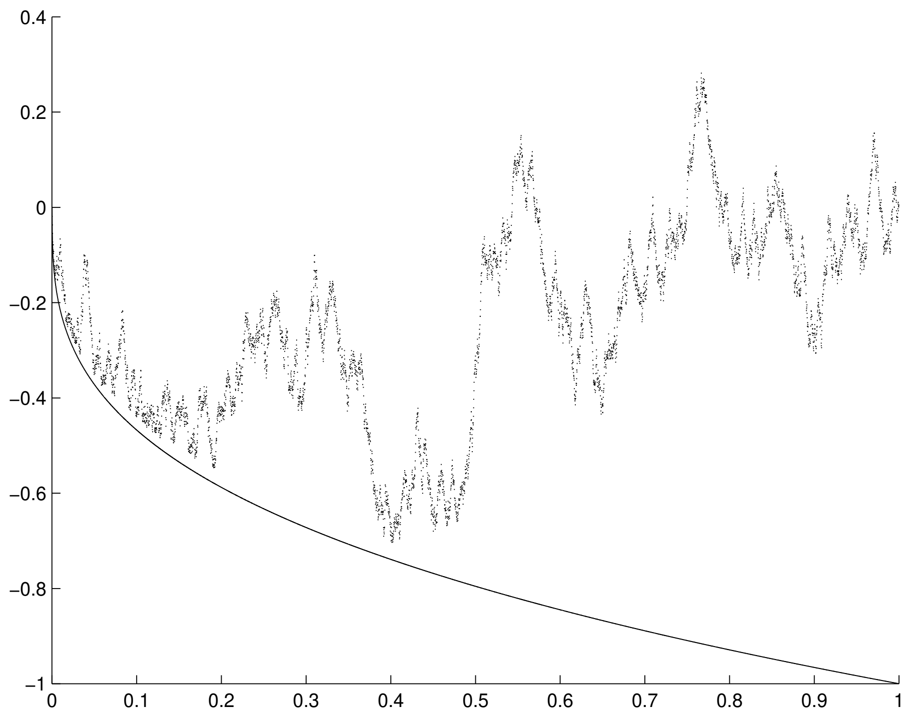

Figure 5: The reflected backward Brownian motion β (in grey) started from the point $(1,0)$ hits the point $(0,0)$

- f_2 is defined on $(-\infty, 0]$ (it is running backwards) and $f_2(t) = 0$ for all $t \leq 0$.
- $p = 2m$ is even, and for any $j \in \{0, \cdots, m-1\}$, B^{2j+1} and B^{2j+2} are two Brownian motions running in opposite directions, started at the same time t_j and from the same level $a_j \geq 0$.

Let us conclude briefly with the following remark: One might think that TSRM (and its stationary version) can be interpreted as a continuous Markov process on a space of rooted continuous functions (see [8]). Our results (and the remarks at the end of the previous subsection) show that this Markov process is not well-defined on the whole space of continuous functions: It is well defined on the support of Wiener measure, on the space of functions obtained by coalescing Brownian motions with deterministic nice locally Lipschitz functions, but its definition can not easily be extended to the whole space of continuous functions. For instance, it is not possible to start with the initial condition

$$X_0 = 0, \ f(x) = -|x|^{1/3}.$$

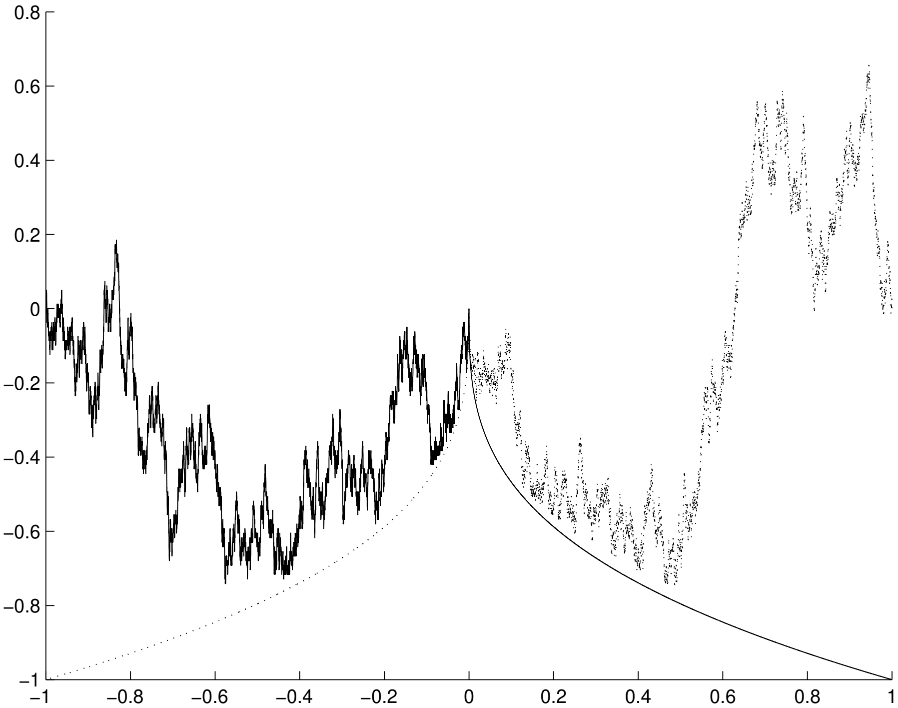

Figure 6: The two lines cross

5 Appendix

Consider a system of coalescing simple random walks in $\mathbb{Z}$ started from any point (x, y) (i.e. time x and level y) in $\mathbb{Z}^2$ such that $x+y$ is even. Such a system can be constructed easily; At each such site (x, y) (such that $x+y$ is even), with probability $1/2$ draw an upward edge (to $(x+1, y+1)$) or a downward edge (to $(x+1, y-1)$). The dual system is then simply obtained as follows (see [1, 2]): In the case where the edge $((x, y), (x+1, y+1))$ is in the original system, then the edge $((x+1, y), (x, y-1))$ is in the dual system (and if $((x, y), (x+1, y-1))$ is in the original system, then $((x+1, y), (x, y+1))$ is in the dual system). Hence, the backward edge (in the dual system) starting at $(x+1, y)$ has a probability $1/2$ to go upwards and $1/2$ to go downawards so that the backward system is also a system of coalescing simple random walks (running backwards).

It is easy to see directly in this setting that the backward lines are reflected on ‘tubes’ around the forward lines. This leads naturally to conjecture and understand intuitively Theorem 8 (in the scaling limit), but unfortunately, it does not provide a simpler proof than the one presented in this paper.

Note that a backward line can be viewed as the “upper envelope” of the family of all coalescing

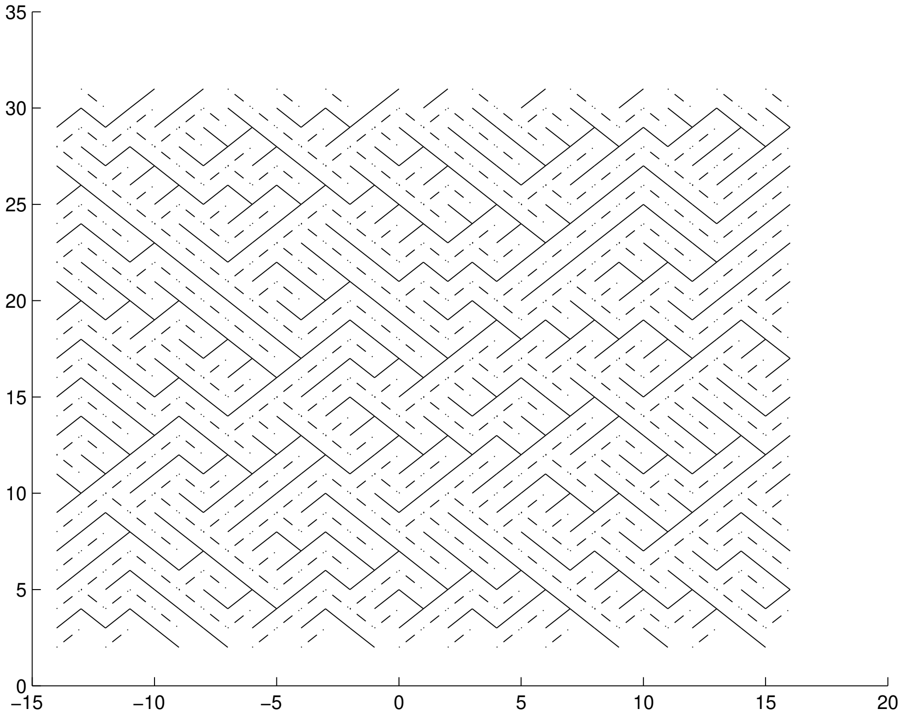

Figure 7: The forward system (in black) and the dual backward line (in dashed)

forward lines running through a point; this is the observation used in [1, 2, 8] in order to define the dual family in the continuous setting.

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