On the defect of compactness for the Strichartz estimates of the Schrödinger equations

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Abstract

In this paper, we prove that every sequence of solutions to the linear Schrödinger equation, with bounded data in $\dot{H}^1(\mathbb{R}^d)$, $d \geq 3$, can be written, up to subsequence, as an almost orthogonal sum of sequences of the type $h_n^{-\frac{d-2}{2}} V(\frac{t-t_n}{h_n^2}, \frac{x-x_n}{h_n})$, where V is a solution of the linear Schrödinger equation, with a small remainder term in Strichartz norms.
Using this decomposition, we prove a similar one for the defocusing H^1-critical nonlinear Schrödinger equation, assuming that the initial data belong to a ball in the energy space where the equation is solvable. This implies in particular the existence of an *a priori* estimate of the Strichartz norms in terms of the energy.

Résumé

On décrit le défaut de compacité des estimations de Strichartz pour l'équation de Schrödinger linéaire avec données dans $\dot{H}^1(\mathbb{R}^d)$, $d \geq 3$, en montrant que toute suite de solutions bornée dans l'espace d'énergie peut s'écrire, à une sous-suite prés, comme une somme presque orthogonale de suites de translatées-dilatées de solutions fixes, et d'un reste tendant vers 0 en normes de Strichartz.
A l'aide de cette décomposition, on montre que les suites de solutions de l'équation de Schrödinger non linéaire H^1-critique défocalisante ayant des données initiales dans une boule d'énergie où l'équation est résoluble, admettent une décomposition analogue. On en déduit, en particulier, l'existence d'une estimation *a priori* des normes de Strichartz en fonction de l'énergie.

Keywords. Schrödinger equations, Strichartz estimates, compactness, asymptotic analysis, a priori estimates.
Classification MSC. 35B40, 35Q55, 35J10, 35B35.

1 Introduction and results

To study the Cauchy problem for the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t v + \Delta v = F(v) \\ v(0,x) = \varphi(x) \end{cases} \tag{1.1}$$

one uses a family of space-time estimates for the solutions of the associated free problem

$$\begin{cases} i\partial_t v + \Delta v = 0 \\ v(0,x) = \varphi(x) \end{cases} \tag{1.2}$$

classically called Strichartz inequalities .
The first inequality of this type was established by Segal [S] in the context of the wave equation. This work is then generalized by Strichartz in his famous paper [St] in connection with the work by Tomas on restriction of Fourier transforms [T].
A large amount of work has been devoted to this subject and much progress was made in several directions. In [GV] Ginibre and Velo introduced another point of view which completely exploited by Yajima in [Y] to provide a large set of inequalities for the linear Schrödinger equation (1.2).
Strichartz estimates play an important role in the study of nonlinear equations; for example it was the fundamental tool used by Kato (see [K]) to establish some results of wellposedness for the sub-critical Schrödinger equation.
As can be seen easily, the Strichartz estimates for the Schrödinger equation 1.2 are not compact, due to the invariance of this equation under some transformations. In this paper we will study this loss of compactness.
Finally, let us mention that for sake of simplicity, we consider the case of three space dimensions. Higher dimensions can be handled similarly.

Here, we list several basic notations which will be used throughout the paper.

$\partial_t = \partial/\partial t$
$\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$
$\Delta = \sum_1^3 \partial_{x_i}^2$
$\dot{W}^{1,s}(\mathbb{R}^3) = \{u \in \mathcal{S}' : \hat{u}(\xi) = \frac{f(\xi)}{|\xi|^s}, \quad f \in L^2(\mathbb{R}^3)\}, \qquad \dot{H}^1(\mathbb{R}^3) = \dot{W}^{1,2}(\mathbb{R}^3)$
$L^q(I, L^r(\mathbb{R}^3)) = \{u(t,x) : (\int_{t\in I}(\int_{\mathbb{R}^d} |u(t,x)|^r dx)^{\frac{q}{r}} dt)^{\frac{1}{r}} < \infty\}$
$\dot{B}^0_{2,\infty}(\mathbb{R}^3) = \{u = u(x) : \|u\|^2_{\dot{B}^0_{2,\infty}} = \sup_{k\in\mathbb{Z}} \int_{2^k \leq |\xi| \leq 2^{k+1}} |\hat{u}(\xi)|^2 d\xi < +\infty\}$

Here $\hat{u}$ is the Fourier transform of f, defined by

$$\hat{u}(\xi) = \int_{\mathbb{R}^3} e^{-ix\xi} f(x) dx.$$

For $\varphi \in \dot{H}^1(\mathbb{R}^3)$ the solution of (1.2) belongs to $(\mathcal{C} \cap L^\infty)(\mathbb{R}_t, \dot{H}^1(\mathbb{R}^3_x))$ and we have the conservation law

$$E_0(v)(t) := \frac{1}{2}\|\nabla v(t,.)\|^2 = E_0(\varphi) \tag{1.3}$$

Other notations will be introduced when required.

First, we recall the main results related to Strichartz estimates in the context of Schrödinger equation (see [CW], [Y], [GV] for more details).

Definition 1.1 **1)** *A pair (q,r) is L^2-admissible, if $r \in [2,6[$ and q satisfy*

$$\frac{2}{q}+\frac{3}{r}=\frac{3}{2}. \tag{1.4}$$

2) *A pair (q,r) is H^1-admissible; if $r<\infty$ and q satisfy*

$$\frac{2}{q}+\frac{3}{r}=\frac{1}{2}. \tag{1.5}$$

The main result proved in [GV] and [Y] is the following.

Proposition 1.2 *Let (q,r) be a L^2-admissible pair. There exists $C=C(r)$ such that*

$$\|e^{it\Delta}\varphi; L^q(\mathbb{R}_t, L^r(\mathbb{R}^3_x)\| \leq C_r\|\varphi\|_{L^2(\mathbb{R}^3)} \tag{1.6}$$

for every $\varphi \in L^2(\mathbb{R}^3)$.

A direct consequence of Proposition 1.2, via Sobolev's inequality, is the following.

Proposition 1.3 *Let (q,r) be a H^1-admissible pair. There exists $C=C(r)$ such that*

$$\|e^{it\Delta}\varphi; L^q(\mathbb{R}_t, L^r(\mathbb{R}^3_x))\| \leq C_r\|\nabla\varphi\|_{L^2(\mathbb{R}^3)} \tag{1.7}$$

for every $\varphi \in \dot{H}^1(\mathbb{R}^3)$.

In the sequel, we will need also the following compactness result, which is an easy consequence of the local smoothing effect ([S], [CS] and [V]).

Proposition 1.4 *The operator $\varphi \longmapsto e^{it\Delta}\varphi$ is compact from $L^2(\mathbb{R}^3)$ to $L^2_{\text{loc}}(\mathbb{R}^4)$.*

Given $x_0 \in \mathbb{R}^3$, $t_0 \in \mathbb{R}$ and $h \in \mathbb{R}^*_+$ we denote by τ_{x_0}, S_h and R_{t_0} the operators defined by

$$\tau_{x_0}\varphi(x)=\varphi(x-x_0), \qquad S_h\varphi(x)=\frac{1}{\sqrt{h}}\varphi(\frac{x}{h}), \qquad R_{t_0}\varphi=e^{it_0\Delta}\varphi(x). \tag{1.8}$$

Let $(x_n)_{n\geq 0}$ be a sequence of $\mathbb{R}^3$ and $(t_n)_{n\geq 0}$ be a sequence of $\mathbb{R}$ both going to infinity. Also, let $(h_n)_{n\geq 0}$ be a sequence of $\in \mathbb{R}^*_+$ going to 0. It is clear that for every fixed function $\varphi \in \dot{H}^1(\mathbb{R}^3)$ the sequences $(\tau_{x_n}\varphi)_{n\geq 0}$, $(S_{h_n}\varphi)_{n\geq 0}$ and $(R_{t_n}\varphi)_{n\geq 0}$ converge weakly to 0. However, for every L^2-admissible pair (q,r) the L^qL^r norms of these sequences are equal to $\|e^{it\Delta}\varphi, L^q(L^r)\|$ for every n. These sequences are not relatively compact in $L^q(L^r)$. The invariance of the $\dot{H}^1(\mathbb{R}^3)$ norm by τ_{x_0}, S_{h_n} and R_{t_0} generate a defect of compactness in the estimates (1.7).
Our main motivation in the first part of the present paper is to give some description of this defect of compactness. Before stating our main result, we introduce some definitions.

Definition 1.5 *1. We call scale, every sequence* $\mathbf{h} = (h_n)_{n\geq o}$ *of positive numbers and core, every sequence* $\mathbf{z} = (z_n)_{n\geq 0} = (t_n, x_n)_{n\geq 0}$ *in* $\mathbb{R}\times\mathbb{R}^3$.

2. We say that two pairs $(\mathbf{h}, \mathbf{z})$ *and* $(\mathbf{h}', \mathbf{z}')$ *are orthogonal if*

$$\frac{h_n}{h_n'} + \frac{h_n'}{h_n} + |\frac{t_n - t_n'}{(h_n)^2}| + |\frac{x_n - x_n'}{h_n}| \underset{n\to\infty}{\longrightarrow} +\infty. \tag{1.9}$$

Now, we are able to state the first result.

Theorem 1.6 *Let* $(\varphi_n)_{n\geq 0}$ *be a bounded sequence in* $\dot{H}^1(\mathbb{R}^3)$. *Let* $(v_n)_{n\geq 0}$ *be the sequence of solutions of* (1.2) *with initial data* $v_n(0,x) = \varphi_n(x)$. *Then there exist a subsequence* (v_n') *of* (v_n), *a sequence* $(\mathbf{h}^j)_{j\geq 1}$ *of scales, a sequence* $(\mathbf{z}^j)_{j\geq 1}$ *of cores and a sequence* $(V^j)_{j\geq 1}$ *of solutions to* (1.2) *such that*
i) the pairs $(\mathbf{h}^j, \mathbf{z}^j)$ *are pairwise orthogonal.*
ii) for every $l \geq 1$,

$$v_n'(t,x) = \sum_{j=1}^{l} \frac{1}{\sqrt{h_n^j}} V^j\left(\frac{t - t_n^j}{(h_n^j)^2}, \frac{x - x_n^j}{h_n^j}\right) + w_n^l(t,x) \tag{1.10}$$

with

$$\limsup_{n\to\infty} \|w_n^l\|_{L^q(L^r)} \underset{l\to\infty}{\longrightarrow} 0 \tag{1.11}$$

for every H^1*-admissible pair* (q,r), *and, for every* $l \geq 1$,

$$E_0(v_n') = \sum_{j=1}^{l} E_0(V^j) + E_0(w_n^l) + o(1) \quad \text{as} \quad n \to \infty \tag{1.12}$$

A few remarks are in order.

Remark 1.7 A similar result has been proved for wave equation by Bahouri and P.Gérard [BG].
Using some refined Strichartz estimates, Merle and Vega [MV] proved a similar result for the Schrödinger equation in two spatial dimensions for L^2 data. Observe that, due to the Galilean invariance, $\limsup_{l\to\infty}\limsup_{n\to\infty} \|\nabla w_n^l\|_{L^q(L^r)} \neq 0$ for the L^2-admissible pairs (q,r). The corresponding refined estimate is not available in in three spatial dimension. However, some refinement of Sobolev's inequalities, stated in [GMO], allows us to deduce sharper versions of (1.7).

Remark 1.8 In the study of the nonlinear H^1-critical Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = |u|^4 u \\ u(0,x) = u_0 \in \dot{H}^1(\mathbb{R}^3) \end{cases}$$

the $L^{10}_{t,x}$ norm plays a fundamental role. Indeed, if the lifespan T^* of the solution is finite, then $\|u\|_{L^{10}([0,T^*[\times\mathbb{R}^3)}$ is infinite (see [CW]). So, in order to establish global wellposedness for this Cauchy problem, it suffices to get a uniform bound for $\|u\|_{L^{10}(I\times\mathbb{R}^3)}$ where I is a unit time interval. This fact is completely exploited by Bourgain to establish global wellposedness for this Cauchy problem in the radial case (see [B]).

The following Corollary is a consequence of Theorem 1.6.

Corollary 1.9 *Let $(\varphi_n)_{n\geq 0}$ be a sequence of $\dot{H}^1(\mathbb{R}^3)$ satisfying*

$$\|\varphi_n\|_{\dot{H}^1(\mathbb{R}^3)} \leq M \quad \text{and} \quad \|e^{it\Delta}\varphi_n\|_{L^{10}(\mathbb{R}^4)} \geq A, \tag{1.13}$$

then, there exists a sequence $(x_n, t_n, h_n)_{n\geq 0}$ of $\mathbb{R}^3\times\mathbb{R}\times\mathbb{R}^*_+$ such that, up to subsequence,

$$\sqrt{h_n}e^{i(.h_n^2+t_n)\Delta}\varphi(.h_n+x_n) \rightharpoonup U, \qquad \text{weakly} \tag{1.14}$$

with $\|U\|_{L^{10}(\mathbb{R}^4)} \geq C(M,A)$.

The second part of the paper deals with the defocusing H^1-critical nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = |u|^4 u \\ u(0,x) = \varphi, \quad \varphi \in \dot{H}^1(\mathbb{R}^3). \end{cases} \tag{1.15}$$

Before going to the main subject, we summarize basic properties of the Cauchy problem (1.15)(see, e.g, [CW])

i. For every $\varphi \in \dot{H}^1(\mathbb{R}^3)$, the IVP (1.15) has a unique maximal solution

$$u \in (\mathcal{C}\cup L^\infty)(]T_*,T^*[,\dot{H}^1(\mathbb{R}^3)) \cap L^q_{\text{loc}}(]T_*,T^*[,\dot{W}^{1,r}(\mathbb{R}^3)),$$

for every (q,r) pair L^2-admissible.

ii. The solution satisfies the following conservation low of energy

$$E(u(t)) = \frac{1}{2}\|\nabla u(t,.)\|^2_{L^2} + \frac{1}{6}\|\ u(t,.)\|^6_{L^6} = E(\varphi). \tag{1.16}$$

iii. If T_* or T^* is finite, then we have $\|\nabla u\|_{L^q(]T_*,T^*[,L^r(\mathbb{R}^3))} = \infty$, for every L^2-admissible pair (q,r) with $r>2$.

Furthermore, there exists $\lambda > 0$ such that if

$$\|\varphi\|_{\dot{H}^1(\mathbb{R}^3)} \leq \lambda, \tag{1.17}$$

there exists a unique global solution to (1.15)

$$u \in (\mathcal{C}\cup L^\infty)(\mathbb{R},\dot{H}^1(\mathbb{R}^3)) \cap L^{10/3}(\mathbb{R},\dot{W}^{1,10/3}(\mathbb{R}^3)). \tag{1.18}$$

Definition 1.10 *We define λ_0 as the supremum of λ in* (1.17) *such that the global existence in $L^{10/3}(\mathbb{R}, \dot{W}^{1,10/3}(\mathbb{R}^3))$ holds.*

In the ball $\|\varphi\|_{L^2(\mathbb{R}^3)} < \lambda_0$ the evolution problem (1.15) admits a complete scattering theory with respect to linear problem (1.2).
It is an open problem to prove that $\lambda_0 = \infty$, i.e. global wellposedness of the IVP (1.15) for any data in $\dot{H}^1(\mathbb{R}^3)$. Recently, Bourgain solved this problem in the particular case of radially symmetric data [B].
The second main result of the present paper uses Theorem 1.6 to provide a decomposition similar to (1.10) for solution of (1.15). We begin by introducing the following definition.

Definition 1.11 *We say that U is the nonlinear profile associated to $\mathcal{W} = (V, \mathbf{h}, \mathbf{z})$ if U is solution to the equation* (1.15) *satisfying*

$$\|\nabla(U - V)(-\frac{t_n}{h_n^2})\|_{L^2} \underset{n\to\infty}{\longrightarrow} 0 \, . \tag{1.19}$$

Let us consider now a sequence (u_n) of solutions to (1.15) such that $\|u_n(0,.)\|_{\dot{H}^1(\mathbb{R}^3)} < \lambda_0$. The following theorem provides a decomposition of (u_n).

Theorem 1.12 *Let $(v_n^{'})$ be the subsequence of (v_n) given by Theorem 1.6. Let $(u_n^{'})$ be the sequence of solutions of* (1.15), *with the same Cauchy data at $t = 0$ as $v_n^{'}$, then*

$$u_n^{'}(t,x) = \sum_{j=1}^{l} \frac{1}{\sqrt{h_n^j}} U^j\left(\frac{t - t_n^j}{(h_n^j)^2}, \frac{x - x_n^j}{h_n^j}\right) + w_n^l(t,x) + r_n^l(t,x) \tag{1.20}$$

with

$$\limsup_{n\to\infty}(\sup_{t\in\mathbb{R}} E_0(r_n^l(t,.)) + \|r_n^l\|_{L^{10}(\mathbb{R}^4)} + \|\nabla r_n^l\|_{L^{10/3}(\mathbb{R}^4)}) \underset{l\to\infty}{\longrightarrow} 0 \tag{1.21}$$

where $V^j, \mathbf{h}^j, \mathbf{z}^j, w_n^l$ are as in (1.10) *and U^j is the nonlinear profile associated to $(V^j, \mathbf{h}^j, \mathbf{z}^j)$.*

Remark 1.13 This theorem remains true for the focusing H^1-critical equation with small data. Indeed, the local theory is the same. However there is a phenomenon of blow-up for some initial data.

Remark 1.14 Using the decomposition of solutions to 2D linear Schrödinger equation proved by Merle and Vega in [MV] we may prove a similar decomposition as Theorem 1.12 for the solution of the Cauchy problem

$$\begin{cases} i\partial_t u + \Delta u = |u|^2 u & x \in \mathbb{R}^2,\, t \in \mathbb{R} \\ u(0,x) = \varphi \in L^2(\mathbb{R}^2). \end{cases}$$

Finally, we obtain an additional result as a consequence of Theorem 1.12

Corollary 1.15 *There exists a nondecreasing function $M : [0,\infty[\to [0,\infty[$ such that , for every solution u to* (1.15) *with $\|u(0,.)\|_{\dot{H}^1(\mathbb{R}^3)} < \lambda_0$, we have*

$$\|\nabla u\|_{L^{10/3}} + \|u\|_{L^{10}} \leq A(E_0(u)). \tag{1.22}$$

The rest of the paper is organized as follows. In section 2 we prove Theorem 1.6 and Corollary 1.9. Section 3 is devoted to the proofs of Theorem 1.12 and Corollary1.15.

2 Proof of Theorem 1.6

The proof is divided into three steps.
Step 1: We start with some general results of decomposition of bounded sequences in L^2. We recall some definitions and results (see [BG] and [Ge] for more details).

Definition 2.1 *Let* $\mathbf{f} = (f_n)_{n\in\mathbb{N}}$ *be a bounded sequence of* $L^2(\mathbb{R}^3)$ *and* $\mathbf{h} = (h_n)_{n\geq 0}$ *a scale.*

1. *We say that* $\mathbf{f}$ *is* $\mathbf{h}$*-oscillatory if*

$$\limsup_{n\to\infty}\Big(\int_{h_n|\xi|\leq\frac{1}{R}}|\hat{f}_n(\xi)|^2d\xi+\int_{h_n|\xi|\geq R}|\hat{f}_n(\xi)|^2d\xi\Big)\underset{R\to\infty}{\longrightarrow}0 \tag{2.1}$$

2. *We say that* $\mathbf{f}$ *is* $\mathbf{h}$*-singular if, for every* $b>a>0$*, we have*

$$\int_{a\leq h_n|\xi|\leq b}|\hat{f}(\xi)|^2d\xi\underset{n\to\infty}{\longrightarrow}0. \tag{2.2}$$

Definition 2.2 *We say that two scales* $\mathbf{h} = (h_n)$ *and* $\tilde{\mathbf{h}} = (\tilde{h}_n)$ *are orthogonal (we note* $\mathbf{h}\perp\tilde{\mathbf{h}}$*) if*

$$\frac{h_n}{\tilde{h}_n}+\frac{\tilde{h}_n}{h_n}\underset{n\to\infty}{\longrightarrow}+\infty\ . \tag{2.3}$$

The following remark will be useful.

Remark 2.3 Let $\mathbf{h} = (h_n)$ be a scale. Let $\mathbf{f}$ and $\mathbf{g}$ be two bounded sequences in L^2 such that $\mathbf{f}$ is $\mathbf{h}$-oscillatory and $\mathbf{g}$ is $\mathbf{h}$-singular. Then, via Plancherel's formula and Cauchy-Schwarz inequality, they are decoupled in infinity, in the sense

$$\int_{\mathbb{R}^3}f_n(x)g_n(x)dx\underset{n\to\infty}{\longrightarrow}0. \tag{2.4}$$

From (2.4) we get

$$\|f_n+g_n\|^2_{L^2(\mathbb{R}^3)}=\|g_n\|^2_{L^2(\mathbb{R}^3)}+\|f_n\|^2_{L^2(\mathbb{R}^3)}+o(1)\quad\text{as}\quad n\to+\infty. \tag{2.5}$$

The main result related to the above notations is the following

Proposition 2.4 *Let* $\mathbf{f}$ *be a bounded sequence in* $L^2(\mathbb{R}^3)$*. there exists a subsequence* $\mathbf{f}'$ *of* $\mathbf{f}$*, a family* $(\mathbf{h}^j)_{j\geq 1}$ *of pairwise orthogonal scales and a family of sequences* $\mathbf{g}^j$ *bounded in* $L^2(\mathbb{R}^3)$ *, such that,*

i) *For every j, $\mathbf{g}^j$ is $\mathbf{h}^j$-oscillatory.*

ii) *For every $l \geq 1$ and $x \in \mathbb{R}^3$,*

$$f'_n(x) = \sum_{j=1}^{l} g_n^j(x) + r_n^l(x), \tag{2.6}$$

where $\mathbf{r}^l$ is $\mathbf{h}^j$-singular for every $j \in \{1, ..., l\}$, and

$$\limsup_{n\to\infty} \|r_n^l\|_{\dot{B}^0_{2,\infty}} \underset{l\to\infty}{\longrightarrow} 0 \ . \tag{2.7}$$

and

$$\|f'_n\|^2_{L^2(\mathbb{R}^3)} = \sum_{j=1}^{l} \|g_n^j\|^2_{L^2(\mathbb{R}^3)} + \|r_n^l\|^2_{L^2(\mathbb{R}^3)} + o(1) \quad \text{as} \quad n \to +\infty. \tag{2.8}$$

Applying this decomposition to the sequence $(\nabla\varphi_n)_{n\geq 0}$, we obtain a family of scales $(\mathbf{h}^j)_{j\geq 1}$ and a family of sequences $(\varphi^j)_{j\geq 1}$ bounded in $\dot{H}^1(\mathbb{R}^3)$, such that

$$\varphi_n^{'} = \sum_{j=1}^{l} \varphi_n^j + \Phi_n^l \tag{2.9}$$

$(\nabla\varphi_n^j)$ is $\mathbf{h}^j$-oscillatory for every $j \geq 1$, $(\nabla\Phi_n^l)$ is $\mathbf{h}^j$-singular for all $1 \leq j \leq l$, and

$$\limsup_{n\to\infty} \|\nabla\Phi_n^l\|_{\dot{B}^0_{2,\infty}} \underset{l\to\infty}{\longrightarrow} 0 \ . \tag{2.10}$$

With the following almost orthogonality's identity

$$E_0(\varphi_n^{'}) = \sum_{j=1}^{l} E_0(\varphi_n^j) + E_0(\Phi_n^l) + o(1), \quad \text{as} \quad n \to \infty. \tag{2.11}$$

To (2.9) corresponds a decomposition of $(v_n^{'})$ solutions of (1.2)

$$v_n^{'} = \sum_{j=1}^{l} p_n^j + q_n^l. \tag{2.12}$$

The energy conservation law and (2.11) imply

$$E_0(v_n^{'}) = \sum_{j=1}^{l} E_0(p_n^j) + E_0(q_n^l) + o(1), \quad \text{as} \quad n \to \infty, \tag{2.13}$$

for every $l \geq 1$.

Step 2:

In order to show that the remainder term q_n^l converges to 0 as n goes at infinity, in Strichartz norm, we need a refinement of the estimate for the norm $L_t^\infty L_x^6$ provided by the following proposition which is a particular case of a more general family of refined Sobolev inequalities (see [GMO] for more details).

Proposition 2.5 *For every $f \in \dot{H}^1(\mathbb{R}^3)$, we have*

$$\|f\|_{L^6(\mathbb{R}^3)} \leq C\|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{3}}\|\nabla f\|_{\dot{B}^0_{2,\infty}}^{\frac{2}{3}}. \tag{2.14}$$

If q is a solution of (1.2) then $\sigma(D)q$ is also solution to the same equation, where $\sigma(\xi) = \mathbf{1}_{2^k \leq |\xi| \leq 2^{k+1}}(\xi)$. The conservation law reads

$$\|\nabla q(t)\|_{\dot{B}^0_{2,\infty}} = \|\nabla q(0)\|_{\dot{B}^0_{2,\infty}}. \tag{2.15}$$

Applying (2.15) to q_n^l, we obtain

$$\|\nabla q_n^l\|_{L^\infty(\mathbb{R},\dot{B}^0_{2,\infty})} = \|\nabla \Phi_n^l\|_{\dot{B}^0_{2,\infty}}. \tag{2.16}$$

From (2.10) and (2.16) one has that

$$\limsup_{n\to\infty} \|\nabla q_n^l\|_{L^\infty(\mathbb{R},\dot{B}^0_{2,\infty})} \underset{l\to\infty}{\longrightarrow} 0\,. \tag{2.17}$$

On the other hand, by (2.14), we get

$$\limsup_{n\to\infty} \|q_n^l\|_{L^\infty(\mathbb{R},L^6)} \leq C \limsup_{n\to\infty} E_0(q_n^l)^{\frac{1}{6}} \limsup_{n\to\infty} \|\nabla q_n^l\|_{L^\infty(\mathbb{R},\dot{B}^0_{2,\infty})}^{\frac{1}{3}}. \tag{2.18}$$

Moreover, from(2.13), it follows that

$$\limsup_{n\to\infty} E_0(q_n^l) \leq \limsup_{n\to\infty} E_0(v_n^{'}), \tag{2.19}$$

for every $l \geq 1$. Combining (2.18) and (2.19), we infer

$$\limsup_{n\to\infty} \|q_n^l\|_{L^\infty(\mathbb{R},L^6)} \underset{l\to\infty}{\longrightarrow} 0\,. \tag{2.20}$$

Let (q,s) be a H^1-admissible pair. By Hölder's inequality we obtain

$$\|q_n^l\|_{L^s(\mathbb{R},L^r)} \leq \|q_n^l\|_{L^\infty(\mathbb{R},L^6)}^{\alpha}\|q_n^l\|_{L^{\tilde{s}}(\mathbb{R},L^{\tilde{r}})}^{\beta} \tag{2.21}$$

where

$$\tilde{s} = \frac{s(r-3)}{r}, \quad \tilde{r} = (2r-6), \quad \beta = (1-\frac{3}{r}), \quad \alpha = \frac{3}{r}$$

It is easy to see that the pair $(\tilde{s},\tilde{r})$ is H^1-admissible. Therefore, via Strichartz estimate (1.7), it follows

$$\limsup_{n\to\infty} \|q_n^l\|_{L^{\tilde{s}}(\mathbb{R},L^{\tilde{r}})} \leq C \limsup_{n\to\infty} E_0(q_n^l)^{1/2} \leq C \limsup_{n\to\infty} E_0(v_n)^{1/2} \leq C. \tag{2.22}$$

And finally

$$\limsup_{n\to\infty} \|q_n^l\|_{L^s(\mathbb{R},L^r)} \underset{l\to\infty}{\longrightarrow} 0, \tag{2.23}$$

for every H^1-admissible pair (s,r).

Step 3:

In this step we deal with the family of sequences (p_n^j), $j \geq 1$. We denote by $\mathbf{1}$ the scale with all terms are equal to 1.

Proposition 2.6 *Let* $\mathbf{P} = (P_n)_{n\geq 0}$ *be a sequence of solutions to (1.2) such that* $(\nabla P_n(0,.))_{n\geq 0}$ *is bounded and* **1***-oscillatory. Then, there exist a subsequence* $(P_n^{'})$ *of* (P_n)*, a family* $(Z^\alpha)_{\alpha\geq 1}$ *of cores and a sequence* $(V^\alpha)_{\alpha\geq 1}$ *of solutions to (1.2) , such that ,*

i) *for every* $\alpha \neq \beta$; $\quad |z_n^\alpha - z_n^\beta| \underset{n\to\infty}{\longrightarrow} +\infty,$

ii) *for every* $A \geq 1$*, every* $x \in \mathbb{R}^3$ *and* $t \in \mathbb{R}$*, we have*

$$P_n^{'}(t,x) = \sum_{\alpha=1}^{A} V_n^\alpha(t - t_n^\alpha, x - x_n^\alpha) + P_n^A(t,x), \tag{2.24}$$

with

$$\limsup_{n\to\infty} \|P_n^A\|_{L^s(L^r)} \underset{A\to\infty}{\longrightarrow} 0 \tag{2.25}$$

for every H^1*- admissible pair* (s,r)*, and*

$$E_0(P_n^{'}) = \sum_{\alpha=1}^{A} E_0(V^\alpha) + E_0(P_n^A) + o(1), \quad n \to \infty \tag{2.26}$$

Proof. Let $\mathcal{V}(\mathbf{P})$ be the set of solutions to (1.2) obtained as weak limits in $L^\infty(\mathbb{R}, \dot{H}^1(\mathbb{R}^3))$ of subsequences of translated $(P_n(.+s_n, .+y_n))$ of $\mathbf{P}$. We denote

$$\eta(\mathbf{P}) = \sup\{E_0(V)^{\frac{1}{2}} : V \in \mathcal{V}(\mathbf{P})\} \tag{2.27}$$

Clearly

$$\eta(\mathbf{P}) \leq \limsup_{n\to\infty} E_0(P_n)^{\frac{1}{2}}.$$

We shall prove that for every sequence $\mathbf{P}$ there exist a sequence $(V^\alpha)_{\alpha\geq 1}$ of $\mathcal{V}(\mathbf{P})$ and a family (s_n^α, y_n^α) of sequences of $\mathbb{R}\times\mathbb{R}^3$ such that

$$\alpha \neq \beta \Longrightarrow |s_n^\alpha - s_n^\beta| + |y_n^\alpha - y_n^\beta| \underset{n\to\infty}{\longrightarrow} \infty, \tag{2.28}$$

and, up to extracting a subsequence, the sequence $(P_n)_{n\geq 1}$ can be written as

$$P_n(s,y) = \sum_{\alpha=1}^{A} V^\alpha(s - s_n^\alpha, y - y_n^\alpha) + P_n^A(t,x), \quad \eta(\mathbf{P}^A) \underset{A\to\infty}{\longrightarrow} 0 \tag{2.29}$$

with the almost orthogonality

$$E_0(P_n) = \sum_{\alpha=1}^{A} E_0(V^\alpha) + E_0(P_n^A) + o(1), \quad n \to \infty \tag{2.30}$$

If $\eta(\mathbf{P})=0$, we can take $V^\alpha\equiv 0$ for all α; otherwise we choose $V^1\in\mathcal{V}(\mathbf{P})$ such that

$$E_0(V^1)^{\frac{1}{2}}\geq\frac{1}{2}\eta(\mathbf{P})>0. \tag{2.31}$$

By definition, there exists some sequence (s_n^1,y_n^1) of $\mathbb{R}\times\mathbb{R}^3$ such that, up to extracting a subsequence, we have

$$P_n(.+s_n^1,.+y_n^1)\rightharpoonup V^1(s,y). \tag{2.32}$$

We set

$$P_n^1(s,y)=P_n(s,y)-V^1(s-s_n^1,y-y_n^1). \tag{2.33}$$

Since $P_n^1(.+s_n^1,.+y_n^1)\rightharpoonup 0$, we get

$$E_0(P_n)=E_0(V^1)+E_0(P_n^1)+o(1),\quad n\to\infty. \tag{2.34}$$

Now, we replace $\mathbf{P}$ by $\mathbf{P}^1$ and repeat the same process . If $\eta(\mathbf{P}^1)>0$ we get V^2, (s_n^2,y_n^2) and $\mathbf{P}^2$. Moreover

$$|s_n^1-s_n^2|+|y_n^1-y_n^2|\longrightarrow\infty,\quad\text{as}\quad n\to\infty$$

otherwise, up to extracting of subsequence, $P_n^1(.+s_n^2,.+y_n^2)$ converge weakly to 0, which implies that $V^2=0$, therefore $\eta(\mathbf{P}^1)=0$, a contradiction. An easy argument of iteration and orthogonal extraction achieve the proof of claims (2.28), (2.29) and (2.30). To complete the proof of Proposition 2.6 we just need that the family $(\mathbf{P}^A)$ satisfies

$$\limsup_{n\to\infty}\|P_n^A\|_{L^s(L^r)}\underset{A\to\infty}{\longrightarrow}0$$

for every H^1-admissible pair (s,r). Firstly, let us remark that if we apply the operator $\sigma_R(D)$, where $\sigma_R=\mathbf{1}_{\{|\xi|\leq\frac{1}{R}\}\cup\{|\xi|\geq R\}}$, to the both side of (2.33) we get

$$E_0(\sigma_R(D)P_n)=E_0(\sigma_R(D)V^1)+E_0(\sigma_R(D)P_n^1)+o(1),\quad n\to\infty. \tag{2.35}$$

By iteration we get

$$E_0(\sigma_R(D)P_n)=\sum_{\alpha=1}^{A}E_0(\sigma_R(D)V^\alpha)+E_0(\sigma_R(D)P_n^A)+o(1),\quad n\to\infty. \tag{2.36}$$

From (2.36) it follows that

$$\limsup_{n\to\infty}\int_{\{|\xi|\leq\frac{1}{R}\}\cup\{|\xi|\geq R\}}|\xi|^2|\hat{P}_n^A(0,\xi)|^2d\xi\leq\limsup_{n\to\infty}\int_{\{|\xi|\leq\frac{1}{R}\}\cup\{|\xi|\geq R\}}|\xi|^2|\hat{P}_n(0,\xi)|^2d\xi, \tag{2.37}$$

for every $A\geq 1$. In particular, $(\mathbf{P}^A)$ is $\mathbf{1}$-oscillatory , for every $A\geq 1$.
Let us summarize the properties of the family $(\mathbf{P}^A)_{A\geq 1}$.

i) for every $A \geq 1$, $\mathbf{P}^A$ is uniformly bounded energy solutions to (1.2).

ii) For every $A \geq 1$

$$\limsup_{n\to\infty} \int_{\{|\xi|\leq\frac{1}{R}\}\cup\{|\xi|\geq R\}} |\xi|^2 |\hat{P}_n^A(0,\xi)|^2 d\xi \leq \limsup_{n\to\infty} \int_{\{|\xi|\leq\frac{1}{R}\}\cup\{|\xi|\geq R\}} |\xi|^2 |\hat{P}_n(0,\xi)|^2 d\xi \tag{2.38}$$

iii)

$$\eta(\mathbf{P}^A) \underset{A\to\infty}{\longrightarrow} 0. \tag{2.39}$$

Using these properties, we shall prove that

$$\limsup_{n\to\infty} \|P_n^A\|_{L^\infty(L^6(\mathbb{R}^3))} \underset{A\to\infty}{\longrightarrow} 0,$$

and by Hölder's inequality we get, for every H^1-admissible pair (r,s),

$$\limsup_{n\to\infty} \|P_n^A\|_{L^s(L^r)} \underset{A\to\infty}{\longrightarrow} 0\ .$$

We start by introducing a family of functions $\chi_R(t,x) = \chi_R^1(x).\chi_R^2(t)$ in $\mathcal{C}_0^\infty$ satisfying the following properties

$$\begin{cases} 0 \leq |\hat{\chi}_R^1| + |\tilde{\chi}_R^2| \leq 2 \quad . \\ \mathrm{Supp}(\hat{\chi}_R^1) \subset \{\frac{1}{3R} \leq |\xi| \leq 3R\}. \\ \hat{\chi}_R^1 \equiv 1 \quad \text{for} \quad \frac{1}{2R} \leq |\xi| \leq 2R. \\ \tilde{\chi}_R^2(-|\xi|^2) = 1 \quad \text{on } \mathrm{Supp}(\hat{\chi}_R^1). \end{cases} \tag{2.40}$$

Here $\hat{}$ denotes the partial Fourier transform in x and $\tilde{}$ denotes the partial Fourier transform in t. One has

$$\|P_n^A\|_{L_t^\infty(L_x^6)} \leq \|\chi_R * P_n^A\|_{L_t^\infty(L_x^6)} + \|(\delta - \chi_R) * P_n^A\|_{L_t^\infty(L_x^6)}, \tag{2.41}$$

where $*$ denotes the convolution in (t,x).
<u>Control of $\|\chi_R * P_n^A\|_{L_t^\infty(L_x^6)}$.</u> We have

$$\|\chi_R * P_n^A\|_{L_t^\infty(L_x^6)} \leq \|\chi_R * P_n^A\|_{L_t^\infty(L_x^2)}^{\frac{1}{3}} \|\chi_R * P_n^A\|_{L_{t,x}^\infty(\mathbb{R}^4)}^{\frac{2}{3}}. \tag{2.42}$$

Since $\chi_R * P_n^A$ is solution to (1.2), the L_x^2 conservation law gives

$$\|\chi_R * P_n^A\|_{L_t^\infty(L_x^2)}^2 = \|(\chi_R * P_n^A)(0,.)\|_{L_x^2} = \frac{1}{(2\pi)^3} \|(\widehat{\chi_R * P_n^A(0,.)})(\xi)\|_{L_\xi^2}^2. \tag{2.43}$$

On the other hand

$$\chi_R * P_n^A(0,x) = \int_{\mathbb{R}_s} \chi_R^2(-s) \int_{\mathbb{R}_y^3} \chi_R^1(x-y) P_n^A(s,y)\,dy\,ds. \tag{2.44}$$

By Plancherel, we get

$$\chi_R * P_n^A(0,x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}_s}\chi_R^2(-s)\int_{\mathbb{R}^3_\xi}\hat{\chi}_R^1(-\xi)\widehat{P_n^A(s,.)}(-\xi)e^{-ix\xi}d\xi ds. \tag{2.45}$$

So that

$$\chi_R * P_n^A(0,x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}_s}\chi_R^2(-s)\int_{\mathbb{R}^3_\xi}\hat{\chi}_R^1(-\xi)e^{-is|\xi|^2}\widehat{P_n^A(0,.)}(-\xi)e^{-ix\xi}d\xi ds. \tag{2.46}$$

thereby

$$\chi_R * P_n^A(0,x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3_\xi}\tilde{\chi}_R^2(-|\xi|^2)\hat{\chi}_R^1(\xi)\widehat{P_n^A(0,.)}(\xi)e^{ix\xi}d\xi ds. \tag{2.47}$$

Hence

$$\chi_R * P_n^A(0,x) = \mathcal{F}^{-1}_{\xi\to x}[\tilde{\chi}_R^2(-|\xi|^2)\hat{\chi}_R^1(\xi)\widehat{P_n^A(0,.)}(\xi)](x). \tag{2.48}$$

Consequently,

$$(\widehat{\chi_R * P}_n^A(0,.))(\xi) = \tilde{\chi}_R^2(-|\xi|^2)\hat{\chi}_R^1(\xi)\widehat{P_n^A(0,.)}(\xi). \tag{2.49}$$

Using the properties of χ_R described in (2.40), we get

$$\|\chi_R * P_n^A\|_{L^\infty(L^2)} \leq \frac{3R}{(2\pi)^{3/2}}\|\xi\widehat{P_n^A(0,.)}\|_{L^2} \leq C_1(R)E_0(P_n^A)^{\frac{1}{2}}, \tag{2.50}$$

where $C_1(R)$ is an R-dependent constant.
Now, observe that

$$\limsup_{n\to\infty}\|\chi_R * P_n^A\|_{L^\infty} = \sup_{(s_n,y_n)}\limsup_{n\to\infty}|\chi_R * P_n^A(s_n,y_n)|. \tag{2.51}$$

Therefore, using the definition of $\mathcal{V}(\mathbf{P^A})$, we have

$$\|\chi_R * P_n^A\|_{L^\infty} \leq \sup\{|\int_{\mathbb{R}}\int_{\mathbb{R}^3}\chi_R(t,x)V(t,x)dxdt|, \quad V\in\mathcal{V}(\mathbf{P^A})\}. \tag{2.52}$$

So that, by Hölder's inequality,

$$\|\chi_R * P_n^A\|_{L^\infty} \leq C_2(R)\sup\{\|V\|_{L^\infty(L^6)}, \quad V\in\mathcal{V}(\mathbf{P}^A)\}. \tag{2.53}$$

Here $C_2(R) = \|\chi_R, L^1_t(L_x^{\frac{6}{5}})\|$ quantity which dependent only on R.
Since, for every $V\in\mathcal{V}(\mathbf{P}^A)$,

$$\|V\|_{L^\infty(L^6)} \leq E_0(V)^{\frac{1}{2}} \leq \eta(\mathbf{P}^A) \tag{2.54}$$

we infer that

$$\|\chi_R * P_n^A\|_{L^\infty_{t,x}(\mathbb{R}^4)} \leq C_2(R)\eta(\mathbf{P}^A). \tag{2.55}$$

Combining (2.42), (2.50) and (2.55), we obtain

$$\|\chi_R * P_n^A\|_{L^\infty(L^6)} \leq C(R)\eta(\mathbf{P}^A)^{\frac{2}{3}} E_0(P_n^A)^{\frac{1}{6}}. \tag{2.56}$$

<u>Control of $\|(\delta-\chi_R) * P_n^A\|_{L^\infty(L^6)}$.</u> The function $(\delta-\chi_R) * P^A$ is a solution to (1.2). We have, by Sobolev's injection,

$$\|(\delta-\chi_R) * P_n^A\|^2_{L^\infty(L^6)} \leq C E_0((\delta-\chi_R) * P_n^A). \tag{2.57}$$

By Plancherel and (2.49), we get

$$E_0((\delta-\chi_R) * P_n^A) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3_\xi} |\xi|^2 |\widehat{P_n^A(0,.)}(\xi)[1-\tilde{\chi}_R^2(-|\xi|^2)\hat{\chi}_R^1(\xi)]|^2 d\xi. \tag{2.58}$$

Observe that, by (2.40) the quantity $[1-\tilde{\chi}_R^2(-|\xi|^2)\hat{\chi}_R^1(\xi)]$ is equal to zero for $\frac{1}{R} \leq |\xi| \leq R$ and uniformly bounded by 2. Consequently

$$\limsup_{n\to\infty} \|(\delta-\chi_R) * P_n^A\|^2_{L^\infty(L^6)} \leq 2\limsup_{n\to\infty}\int_{\{|\xi|\leq\frac{1}{R}\}\cup\{|\xi|\geq R\}} ||\xi|^2 \hat{P}_n^A(0,\xi)|^2 d\xi. \tag{2.59}$$

Hence, by (2.38)

$$\limsup_{n\to\infty} \|(\delta-\chi_R) * P_n^A\|_{L^\infty(L^6)} \leq 2(\limsup_{n\to\infty}\int_{\{|\xi|\leq\frac{1}{R}\}\cup\{|\xi|\geq R\}} |\xi|^2 \hat{P}_n(0,\xi)|^2 d\xi)^{\frac{1}{2}}. \tag{2.60}$$

<u>Conclusion.</u> From estimates (2.56) and (2.60), we obtain

$$\|P_n^A\|_{L^\infty(L^6)} \leq 2(\limsup_{n\to\infty}\int_{\{|\xi|\leq\frac{1}{R}\}\cup\{|\xi|\geq R\}} |\xi|^2 \hat{P}_n(0,\xi)|^2 d\xi)^{\frac{1}{2}} + C(R)\eta(\mathbf{P}^A)^{\frac{2}{3}} E_0(P_n^A)^{\frac{1}{6}}. \tag{2.61}$$

Now we let A go to infinity , then R go to infinity and we use the fact that $\eta(\mathbf{P}^A) \underset{A\to\infty}{\longrightarrow} 0$ and that $(\nabla P_n(0,.))$ is $\mathbf{1}$-oscillatory, we obtain

$$\limsup_{n\to\infty} \|P_n^A\|_{L^\infty(L^6)} \underset{A\to\infty}{\longrightarrow} 0 \tag{2.62}$$

and this completes the proof of Proposition 2.6. •

Step 4: End of the proof of Theorem 1.6.
Let (φ_n) be a bounded sequence in $\dot{H}^1(\mathbb{R}^3)$ and let $(v_n)_{n\geq 0}$ be the sequence of solutions to (1.2) with initial data $v_n(0,x) = \varphi_n(x)$. In the step 1,2 we proved the existence of some subsequence $(v_n^{'})$ of (v_n), a sequence $(\mathbf{h}^j)$ of scales and a family $(\mathbf{p}_n^j)$ of solutions to (1.2) such that

1) $(\nabla p_n^j(0,.))$ is $\mathbf{h}^j$-oscillatory for every $j \geq 1$.

2)

$$v_n^{'} = \sum_{j=1}^{l} p_n^j + q_n^l. \tag{2.63}$$

3) For all H^1-admissible pair (s,r), we have

$$\|q_n^l\|_{L^s(\mathbb{R},L^r)} \underset{n\to\infty}{\longrightarrow} 0 \ . \tag{2.64}$$

Denote

$$P_n^j(s,y) = \sqrt{h_n^j}\, p_n^j((h_n^j)^2 s, h_n^j y). \tag{2.65}$$

It is clear that, for every $j \geq 1$, the sequence $(P_n^j(0,.))_{n\geq 0}$ is bounded and $\mathbf{1}$-oscillatory. Applying Proposition 2.6 to these sequences, we get , for every $j \geq 1$,

$$P_n^j(t,x) = \sum_{\alpha=1}^{A} V^{(\alpha,j)}(t - t_n^{(\alpha,j)}, x - x_n^{(\alpha,j)}) + P_n^{(j,A)}(t,x). \tag{2.66}$$

Thereby

$$p_n^l = \sum_{\alpha=1}^{A_j} \frac{1}{\sqrt{h_n^j}} V^{(\alpha,j)}(\frac{t - t_n^{(\alpha,j)}}{(h_n^j)^2}, \frac{x - x_n^{(\alpha,j)}}{h_n^j}) + w_n^{(j,A)}(t,x), \tag{2.67}$$

where

$$x_n^{(\alpha,j)} = h_n^j y_n^\alpha, \quad t_n^{(\alpha,j)} = (h_n^j)^2 s_n^\alpha, \quad w_n^{(j,A)}(t,x) = \frac{1}{\sqrt{h_n^j}} P_n^{(j,A)}(\frac{t}{(h_n^j)^2}, \frac{x}{h_n^j}). \tag{2.68}$$

Note that the $L^s(\mathbb{R}, L^r)$ norms are invariant by the last scaling.
From (2.63), (2.65) and (2.67) it follows that

$$v_n^{'}(t,x) = \sum_{j=1}^{l} (\sum_{\alpha=1}^{A_j} \frac{1}{\sqrt{h_n^j}} V^{(\alpha,j)}(\frac{t - t_n^{(\alpha,j)}}{(h_n^j)^2}, \frac{x - x_n^{(\alpha,j)}}{h_n^j}) + w_n^{(j,A)}(t,x)) + q_n^l(t,x), \tag{2.69}$$

which can be written as

$$v_n^{'}(t,x) = \sum_{j=1}^{l} (\sum_{\alpha=1}^{A_j} \frac{1}{\sqrt{h_n^j}} V^{(\alpha,j)}(\frac{t - t_n^{(\alpha,j)}}{(h_n^j)^2}, \frac{x - x_n^{(\alpha,j)}}{h_n^j})) + w_n^{(l,A_1,\ldots,A_l)} \tag{2.70}$$

where

$$w_n^{(l,A_1,\dots,A_l)}(t,x) = \sum_{j=1}^{l} w_n^{(j,A_j)}(t,x) + q_n^l(t,x). \tag{2.71}$$

We enumerate the pairs (α, j) by ω satisfying

$$\omega(j,\alpha) < \omega(k,\beta) \quad \text{if} \quad j+\alpha < k+\beta \quad \text{or} \quad j+\alpha = k+\beta \quad \text{and} \quad j > k. \tag{2.72}$$

We have three points to investigate.
i) The family $(\mathbf{h}^j, \mathbf{z}^j)$ are pairwise orthogonal. indeed, we have two possibilities

- The two pairs $(\mathbf{h}^j, \mathbf{z}^j)$ and $(\mathbf{h}^k, \mathbf{z}^k)$ are in the form $(\mathbf{h}^j, \mathbf{z}^j) = (\mathbf{h}^s, \mathbf{z}^{(\alpha,s)})$, $(\mathbf{h}^k, \mathbf{z}^k) = (\mathbf{h}^m, \mathbf{z}^{(\beta,m)})$ with $s \neq m$. In this case the orthogonality of the two scales is equivalent to

$$|\frac{h_n^s}{h_n^m}| + |\frac{h_n^m}{h_n^s}| \underset{n\to\infty}{\longrightarrow} +\infty \tag{2.73}$$

  which is trivial, because $(\mathbf{h}^j)_{j\geq 1}$ are pairwise orthogonal.

- The two pairs $(\mathbf{h}^j, \mathbf{z}^j)$ and $(\mathbf{h}^k, \mathbf{z}^k)$ are in the form $(\mathbf{h}^j, \mathbf{z}^j) = (\mathbf{h}^m, \mathbf{z}^{(\alpha,m)})$, $(\mathbf{h}^k, \mathbf{z}^k) = (\mathbf{h}^m, \mathbf{z}^{(\beta,m)})$ with $\alpha \neq \beta$. In this case the orthogonality reads

$$|\frac{t_n^\alpha - t_n^\beta}{(h_n^m)^2}| + |\frac{x_n^\alpha - x_n^\beta}{h_n^m}| \underset{n\to\infty}{\longrightarrow} +\infty \tag{2.74}$$

  From (2.68), (2.74) is equivalent to

$$|s_n^\alpha - s_n^\beta| + |y_n^\alpha - y_n^\beta| \underset{n\to\infty}{\longrightarrow} +\infty \tag{2.75}$$

  which is $i)$ in Proposition 2.6.

ii) The almost orthogonlity'identity (1.12)) is satisfied. Indeed, combining (2.13) and (2.26), we obtain

$$E_0(v_n^{'}) = \sum_{j=1}^{l} (\sum_{\alpha=1}^{A_j} E_0(V^{(\alpha,j)}) + E_0(w_n^{(A_j)})) + E_0(q_n^l) + o(1),\ n \to \infty \tag{2.76}$$

On the other hand, recall that (q_n^l) is singular to $\mathbf{h}^j$ for every $1 \leq j \leq l$. Furthermore, the sequence $(w_n^{(A_j)})$ is $\mathbf{h}^j$-oscillatory (because $(P_n^{(A_j)})$ is $\mathbf{1}$-oscillatory). Thus, (see Remark 2.3) we have the following almost orhogonality's identity

$$E_0(w_n^{(l,A_1,\dots,A_l)}) = \sum_{j=1}^{l} E_0(w_n^{(A_j)}) + E_0(q_n^l) + o(1) \quad n \to \infty. \tag{2.77}$$

Combined with (2.76), (2.77) gives

$$E_0(v_n') = \sum_{j=1}^{l}\sum_{\alpha=1}^{A_j} E_0(V^{(\alpha,j)}) + E_0(w_n^{(l,A_1,\dots,A_l)}) + o(1) \quad n\to\infty \tag{2.78}$$

which is to prove.
iii) The last point is the convergence of the remainder to zero in the Strichartz norms. We begin by proving that

$$\limsup_{n\to\infty} \|w_n^{(l,A_1,\dots,A_l)}\|_{L^{10}(\mathbb{R}^4)} \longrightarrow 0 \quad \text{as} \quad \inf_{1\le j\le l}\{l, j+A_j\}\to\infty. \tag{2.79}$$

Let $\epsilon>0$ be a small arbitrary number. Take l_0 such that, for every $l\ge l_0$

$$\limsup_{n\to\infty}\|q_n^l\|_{L^{10}(\mathbb{R}^4)} \le \frac{\epsilon}{3}. \tag{2.80}$$

For every $l\ge l_0$, there exists B_l such that $A_j\ge B_l$, for every $j\in\{1,\dots,l\}$

$$\limsup_{n\to\infty}\|w_n^{A_j}\|_{L^{10}(\mathbb{R}^4)} \le \frac{\epsilon}{3l}. \tag{2.81}$$

The remainder can be written in the following form

$$w_n^{(l,A_1,\dots,A_l)} = q_n^l + \sum_{1\le j\le l} w_n^{(j,A_j\bigvee B_l)} + S_n^{(l,A_1,\dots,A_l)}, \tag{2.82}$$

where

$$S_n^{(l,A_1,\dots,A_l)} = \sum_{\substack{1\le j\le l\\ A_j<B_l}} (w_n^{(j,A_j)} - w_n^{(j,B_l)}). \tag{2.83}$$

However

$$w_n^{(j,A_j)} - w_n^{(j,B_l)} = \sum_{A_j<\alpha\le B_l} \frac{1}{\sqrt{h^j}} V^{(\alpha,j)}\Big(\frac{t-t_n^{(\alpha,j)}}{(h_n^j)^2}, \frac{x-x_n^{(\alpha,j)}}{h_n^j}\Big)$$

hence

$$S_n^{(l,A_1,\dots,A_l)} = \sum_{\substack{1\le j\le l\\ A_j<B_l}} \sum_{A_j<\alpha\le B_l} \frac{1}{\sqrt{h^j}} V^{(\alpha,j)}\Big(\frac{t-t_n^{(\alpha,j)}}{(h_n^j)^2}, \frac{x-x_n^{(\alpha,j)}}{h_n^j}\Big). \tag{2.84}$$

From (2.80) and (2.81) it follows that

$$\limsup_{n\to\infty}\|w_n^{(l,A_1,\dots,A_l)}\|_{L^{10}(\mathbb{R}^4)} \le \frac{2\epsilon}{3} + \limsup_{n\to\infty}\|S_n^{(l,A_1,\dots,A_l)}\|_{L^{10}(\mathbb{R}^4)}. \tag{2.85}$$

Now, we need the following lemma.

Lemma 2.7 *Let $(\mathbf{h}^j, \mathbf{z}^j)$ a family of pairwise orthogonal scales-cores and (V^j) a family of functions in $L^{10}(\mathbb{R}^4)$. Then, for every $l \geq 1$, we have*

$$\limsup_{n\to\infty} \| \sum_{j=1}^{l} \frac{1}{\sqrt{h_n^j}} V^j\left(\frac{\cdot - t_n^j}{(h_n^j)^2}, \frac{\cdot - x_n^j}{h_n^j}\right)\|_{L^{10}(\mathbb{R}^4)}^{10} = \sum_{j=1}^{l} \|V^j\|_{L^{10}(\mathbb{R}^4)}^{10}. \tag{2.86}$$

Proof. Denote

$$V_n^j(t,x) = \frac{1}{\sqrt{h_n^j}} V^j\left(\frac{t - t_n^j}{(h_n^j)^2}, \frac{x - x_n^j}{h_n^j}\right) \tag{2.87}$$

We have to prove that

$$I_n = \int_{\mathbb{R}^4} V_n^{j_1} V_n^{j_2} V_n^{j_3} V_n^{j_4} V_n^{j_5} V_n^{j_6} V_n^{j_7} V_n^{j_8} V_n^{j_9} V_n^{j_{10}}\, dxdt \underset{n\to\infty}{\longrightarrow} 0. \tag{2.88}$$

Where $1 \leq j_k \leq l$ and at least two j_k's different. Assume, for example that $j_1 \neq j_2$. By Hölder's inequality, we estimate

$$|I_n| \leq C \|V_n^{j_1} V_n^{j_2}\|_{L^5(\mathbb{R}^4)} \tag{2.89}$$

Where $C = \prod_{k=3}^{10} \|V^{j_k}\|_{L^{10}(\mathbb{R}^4)}$. Let us compute the right hand-side of (2.89),

$$\|V_n^{j_1} V_n^{j_2}\|_{L^5}^5 =$$
$$\int_{\mathbb{R}} \frac{1}{(h_n^{j_1} h_n^{j_2})^{5/2}} \int_{\mathbb{R}^3} |V^{j_1}|^5\left(\frac{t - t_n^{j_1}}{(h_n^{j_1})^2}, \frac{x - x_n^{j_1}}{h_n^{j_1}}\right) |V^{j_2}|^5\left(\frac{t - t_n^{j_2}}{(h_n^{j_2})^2}, \frac{x - x_n^{j_2}}{h_n^{j_2}}\right) dxdt. \tag{2.90}$$

The orthogonality of $(\mathbf{h}^{j_1}, \mathbf{z}^{j_1})$, $(\mathbf{h}^{j_2}, \mathbf{z}^{j_2})$ means that

$$\text{either} \quad \frac{h_n^{j_1}}{h_n^{j_2}} + \frac{h_n^{j_2}}{h_n^{j_1}} \underset{n\to\infty}{\longrightarrow} +\infty \quad \text{or} \quad \mathbf{h^{j_1}} = \mathbf{h^{j_2}} \quad \text{and} \quad \left|\frac{t_n^{j_1} - t_n^{j_2}}{(h_n^{j_1})^2}\right| + \left|\frac{x_n^{j_1} - x_n^{j_2}}{h_n^{j_1}}\right| \underset{n\to\infty}{\longrightarrow} +\infty. \tag{2.91}$$

Without loss of generality we can assume that V^{j_1}, V^{j_2} are continuous and compactly supported.

- If $(\frac{h_n^{j_1}}{h_n^{j_2}} + \frac{h_n^{j_2}}{h_n^{j_1}}) \underset{n\to\infty}{\longrightarrow} +\infty$, then either $\frac{h_n^{j_1}}{h_n^{j_2}} \underset{n\to\infty}{\longrightarrow} +\infty$ or $\frac{h_n^{j_2}}{h_n^{j_1}} \underset{n\to\infty}{\longrightarrow} +\infty$. We assume that $\frac{h_n^{j_1}}{h_n^{j_2}} \underset{n\to\infty}{\longrightarrow} +\infty$ (the other case is symmetric). A change of variables $x = h_n^{j_1} y + x_n^{j_1}, t = (h_n^{j_1})^2 s + t_n^{j_1}$ gives

$$\|V_n^{j_1} V_n^{j_2}\|_{L^5}^5 =$$
$$= \int_{\mathbb{R}} \left(\frac{h_n^{j_1}}{h_n^{j_2}}\right)^{5/2} \int_{\mathbb{R}^3} |V^{j_1}|^5(s,y) |V^{j_2}|^5\left(\frac{(h_n^{j_1})^2}{(h_n^{j_2})^2} s + \frac{t_n^{j_1} - t_n^{j_2}}{(h_n^{j_2})^2}, \frac{h_n^{j_1}}{h_n^{j_2}} y + \frac{x_n^{j_1} - x_n^{j_2}}{h_n^{j_2}}\right) dxdt. \tag{2.92}$$

So that

$$\|V_n^{j_1} V_n^{j_2}\|_{L^5}^5 = O\left(\left(\frac{h_n^{j_1}}{h_n^{j_2}}\right)^{5/2}\right) \underset{n\to\infty}{\longrightarrow} 0. \tag{2.93}$$

• If $\mathbf{h^{j_1}} = \mathbf{h^{j_2}}$, and by same change of variables as above, we get

$$\|V_n^{j_1}V_n^{j_2}\|_{L^5}^5 = \\ = \int_{\mathbb{R}}\int_{\mathbb{R}^3} |V^{j_1}|^5(s,y)|V^{j_2}|^5(s+\frac{t_n^{j_1}-t_n^{j_2}}{(h_n^{j_2})^2}, y+\frac{x_n^{j_1}-x_n^{j_2}}{h_n^{j_2}})dxdt. \tag{2.94}$$

which goes to 0 as n goes to infinity.

•

Remark 2.8 Using the elementary inequality

$$||\sum_{j=1}^{l} a_j|^{10/3} - \sum_{j=1}^{l}|a_j|^{10/3}| \leq C_l \sum_{j\neq k}|a_j||a_k|^{7/3} \tag{2.95}$$

and arguing in the same way as in the proof of the previous lemma, we prove that

$$\limsup_{n\to\infty}\|\nabla(\sum_{j=1}^{l}\frac{1}{\sqrt{h_n^j}}V^j(\frac{\cdot-t_n^j}{(h_n^j)^2},\frac{\cdot-x_n^j}{h_n^j}))\|_{L^{10/3}(\mathbb{R}^4)}^{10/3} = \sum_{j=1}^{l}\|\nabla V^j\|_{L^{10/3}(\mathbb{R}^4)}^{10/3}. \tag{2.96}$$

Returning to the proof of (2.79). From Lemma 2.7, we infer

$$\limsup_{n\to\infty}\|S_n^{(l,A_1,\ldots,A_l)}\|_{L^{10}(\mathbb{R}^4)}^{10} = \sum_{\substack{1\leq j\leq l\\ A_j<B_l}}\sum_{A_j<\alpha\leq B_l}\|V^{(\alpha,j)}\|_{L^{10}}^{10}. \tag{2.97}$$

Moreover, by Proposition 1.3, we have

$$\sum_{\substack{1\leq j\leq l\\ A_j<B_l}}\sum_{A_j<\alpha\leq B_l}\|V^{(\alpha,j)}\|_{L^{10}}^{10} \leq \sum_{\substack{1\leq j\leq l\\ A_j<B_l}}\sum_{A_j<\alpha\leq B_l}E_0(V^{(\alpha,j)})^5 \leq \sum_{(j,\alpha)}E_0(V^{(\alpha,j)})^5. \tag{2.98}$$

Using the formula (2.76), we obtain

$$\|S_n^{(l,A_1,\ldots,A_l)}\|_{L^{10}(\mathbb{R}^4)}^{10} \leq \limsup_{n\to\infty}E_0(v_n)^5 < \infty. \tag{2.99}$$

Therefore

$$(\sum_{\substack{(j,\alpha)\\ \alpha>A_j,1\leq j\leq l}}\sum_{A_j<\alpha\leq B_l}\|V^{(\alpha,j)}\|_{L^{10}}^{10})^{\frac{1}{10}} \leq \frac{\epsilon}{3}. \tag{2.100}$$

for $\inf\{l, j+A_j, 1\leq j\leq l\}$ large enough. Combined with (2.85), (2.100) gives

$$\limsup_{n\to\infty}\|w_n^{(l,A_1,\ldots,A_l)}\|_{L^{10}(\mathbb{R}^4)} \leq \epsilon. \tag{2.101}$$

for $\inf\{l, j+A_j, 1\leq j\leq l\}$ large enough. Then, we obtain finally

$$\limsup_{n\to\infty}\|w_n^{(l,A_1,\dots,A_l)}\|_{L^{10}(\mathbb{R}^4)} \underset{l\to\infty}{\longrightarrow} 0. \tag{2.102}$$

Now it remains to prove the same result to (2.102) with the $L^q(\mathbb{R}, L^r(\mathbb{R}^3))$ norms for every H^1-admissible pair (q,r). For this it is enough to prove

$$\limsup_{n\to\infty}\|w_n^{(l,A_1,\dots,A_l)}\|_{L^\infty(L^6)} \underset{l\to\infty}{\longrightarrow} 0 \tag{2.103}$$

and complete by interpolation argument. This will follow from the following lemma

Lemma 2.9 *For every solution v to the linear Schrödinger equation we have*

$$\|V\|_{L^\infty(L^6)} \leq \|V\|_{L^{10}(\mathbb{R}^4)}^{\frac{2}{3}}\|\nabla V\|_{L^{\frac{10}{3}}(\mathbb{R}^4)}^{\frac{1}{3}} \tag{2.104}$$

Proof. Take V a solution to (1.2). Since

$$\int_{\mathbb{R}^3}|V(t,x)|^6dx = -\int_t^\infty\int_{\mathbb{R}^3}6|V(t,x)|^4Re(\overline{V}(t,x)\partial tV(t,x))dxdt, \tag{2.105}$$

it follows that

$$\int_{\mathbb{R}^3}|V(t,x)|^6dx = Im(\int_t^\infty\int_{\mathbb{R}^{32}}6|V(t,x)|^4\overline{V}(t,x)\Delta V(t,x)dxdt). \tag{2.106}$$

An integration by part in the variable x gives

$$\int_{\mathbb{R}^3}|V(t,x)|^6dx = 6Im(\int_t^\infty\int_{\mathbb{R}^3}\nabla(|V|^4)\overline{V}\nabla Vdxdt) = 12Im(\int_t^\infty\int_{\mathbb{R}^3}V\overline{V}^3(\nabla V)^2dxdt) \tag{2.107}$$

hence

$$\int_{\mathbb{R}^3}|V(t,x)|^6dx \leq \int_{-\infty}^{+\infty}\int_{\mathbb{R}^3}|V|^4|\nabla V|^2dxdt \tag{2.108}$$

By Hölder's inequality we get

$$\|V\|_{L^\infty(L^6)} \leq C\|V\|_{L^{10}(\mathbb{R}^4)}^{\frac{2}{3}}\|\nabla V\|_{L^{\frac{10}{3}}(\mathbb{R}^4)}^{\frac{1}{3}} \tag{2.109}$$

which yields the lemma. •

By Lemma 2.9, we have

$$\limsup_{n\to\infty}\|w_n^{(l,A_1,\dots,A_l)}\|_{L^\infty(L^6)} \leq C\limsup_{n\to\infty}\|w_n^{(l,A_1,\dots,A_l)}\|_{L^{10}(\mathbb{R}^4)}^{\frac{2}{3}}\limsup_{n\to\infty}E_0(w_n^{(l,A_1,\dots,A_l)})^{\frac{1}{6}}. \tag{2.110}$$

From (2.78) we get

$$E_0(w_n^{(l,A_1,\dots,A_l)}) \le \limsup_{n\to\infty} E_0(v_n) \le C \tag{2.111}$$

(2.103) follows from (2.102) and (2.111).

Proof of Corollary 1.9.

Let $(\varphi_n)_{n\ge 0}$ be a sequence of $\dot{H}^1(\mathbb{R}^3)$ satisfying (1.13). By Theorem1.6 the sequence $(e^{it\Delta}\varphi_n)_{n\ge 0}$ can be written, up a subsequence, as

$$e^{it\Delta}\varphi_n(t,x) = \sum_{j=1}^{l} \frac{1}{\sqrt{h_n^j}} V^j\Big(\frac{t-t_n^j}{(h_n^j)^2}, \frac{x-x_n^j}{h_n^j}\Big) + w_n^l(t,x) \tag{2.112}$$

with

$$\limsup_{n\to\infty} \|w_n^l\|_{L^q(L^r)} \underset{l\to\infty}{\longrightarrow} 0 \tag{2.113}$$

for every H^1-admissible pair (q,r), and, for every $l \ge 1$,

$$E_0(\varphi_n) = \sum_{j=1}^{l} E_0(V^j) + E_0(w_n^l) + o(1) \quad \text{as} \quad n\to\infty. \tag{2.114}$$

From Lemma 2.7 we have, for every $l \ge 1$

$$\lim_{n\to\infty} \|\sum_{j=1}^{l} V_n^j\|_{L^{10}(\mathbb{R}^4)}^{10} = \sum_{j=1}^{l} \|V^j\|_{L^{10}(\mathbb{R}^4)}^{10}. \tag{2.115}$$

However,

$$\limsup_{n\to\infty} \|\sum_{j=1}^{l} V_n^j - e^{it\Delta}\varphi_n\|_{L^{10}(\mathbb{R}^4)} \le \limsup_{n\to\infty} \|w_n^l\|_{L^{10}(\mathbb{R}^4)} \underset{l\to\infty}{\longrightarrow} 0. \tag{2.116}$$

Thereby

$$\|e^{it\Delta}\varphi_n\|_{L^{10}(\mathbb{R}^4)} \underset{n\to\infty}{\longrightarrow} \sum_{j=1}^{\infty} \|V^j\|_{L^{10}(\mathbb{R}^4)}^{10}. \tag{2.117}$$

On the other hand, via Stricharz estimates, we have

$$\sum_{j\ge 1} \|V^j\|_{L^{10}(\mathbb{R}^4)}^{10} \le C \sup\{\|V^j\|_{L^{10}(\mathbb{R}^4)}^{8}, j\ge 1\} \sum_{j\ge 1} E_0(V^j). \tag{2.118}$$

However, from (2.113), we have

$$\sum_{j\ge 1} E_0(V^j) \le \limsup_{n\to\infty} E_0(\varphi_n) \le M.^2 \tag{2.119}$$

Therefore

$$\sup\{\|V^j\|^8_{L^{10}(\mathbb{R}^4)}, j \geq 1\} \geq \frac{A^{10}}{M^2}. \tag{2.120}$$

In particular, there exists j_0 such that

$$\|V^{j_0}\|^8_{L^{10}(\mathbb{R}^4)} \geq \frac{A^{10}}{2M^2}. \tag{2.121}$$

A change of variables $x = h_n^{j_0} y + x_n^{j_0}, t = (h_n^{j_0})^2 s + t_n^{j_0}$ gives

$$\sqrt{h_n^{j_0}} e^{i((h_n^{j_0})^2 t + t_n^{j_0})\Delta} \varphi(x h_n^{j_0} + x_n^{j_0}) = V^{j_0}(t,x) + \sum_{\substack{1 \leq j \leq l \\ j \neq j_0}} \tilde{V}^j(t,x) + \\ + \tilde{w}_n^l(t,x), \tag{2.122}$$

where

$$\begin{aligned} \tilde{V}_n^j(t,x) &= \sqrt{\frac{h_n^{j_0}}{h_n^j}} V^j\left(\frac{(h_n^{j_0})^2}{(h_n^j)^2} t + \frac{t_n^{j_0} - t_n^j}{(h_n^j)^2}, \frac{h_n^{j_0}}{h_n^j} x + \frac{x_n^{j_0} - x_n^j}{h_n^j}\right) \\ \tilde{w}_n^l(t,x) &= \sqrt{h_n^{j_0}} w_n^l((h_n^{j_0})^2 t + t_n^{j_0}, x h_n^{j_0} + x_n^{j_0}). \end{aligned} \tag{2.123}$$

As in the proof of Lemma 2.7, we prove

$$\tilde{V}_n^j \rightharpoonup 0 \quad \text{weakly}, \tag{2.124}$$

for every $j \neq j_0$. Then,

$$\sqrt{h_n^{j_0}} e^{i((h_n^{j_0})^2 . + t_n^{j_0})\Delta} \varphi(h_n^{j_0} . + x_n^{j_0}) \rightharpoonup V^{j_0} + \tilde{w}^l, \tag{2.125}$$

where $\tilde{w}^l$ denote the weak limit of $(\tilde{w}_n^l)_{n \geq 0}$. However, we have

$$\|\tilde{w}^l\|_{L^{10}(\mathbb{R}^4)} \leq \limsup_{n\to\infty} \|\tilde{w}_n^l\|_{L^{10}(\mathbb{R}^4)} \underset{l\to\infty}{\longrightarrow} 0 \ . \tag{2.126}$$

Thereby, by uniqueness of weak limit, we get

$$\tilde{w}^l = 0 \tag{2.127}$$

for every $l \geq j_0$. So that

$$\sqrt{h_n^{j_0}} e^{i((h_n^{j_0})^2 . + t_n^{j_0})\Delta} \varphi(h_n^{j_0} . + x_n^{j_0}) \rightharpoonup V^{j_0}. \tag{2.128}$$

The function V^{j_0} fulfills our conditions. •

3 Proof of Theorem 1.12

The following notations will be used

$$\left\{\begin{array}{rcl} p(z) & = & |z|^4 z \\ V_n^j(x,t) & = & \frac{1}{(h_n^j)^{\frac{1}{2}}} V^j\left(\frac{t-t_n^j}{(h_n^j)^2}, \frac{x-x_n^j}{h_n^j}\right) \\ U_n^j(x,t) & = & \frac{1}{(h_n^j)^{\frac{1}{2}}} U^j\left(\frac{t-t_n^j}{(h_n^j)^2}, \frac{x-x_n^j}{h_n^j}\right) \\ W_n^l & = & \sum_{j=1}^l U_n^j \\ f_n^l & = & \sum_{j=1}^l p(U_n^j) - p(\sum_{j=1}^l U_n^j + w_n^l + r_n^l) \end{array}\right. \tag{3.1}$$

The function r_n^l satisfies the following equation

$$\left\{\begin{array}{l} i\partial_t r_n^l + \Delta r_n^l = f_n^l \\ r_n^l(0,x) = \sum_{j=1}^l (V_n^l - U_n^l)(0,x). \end{array}\right. \tag{3.2}$$

We also introduce the norm

$$\|\!|g\|\!|_I = \|\nabla g\|_{L^{10/3}(I\times\mathbb{R}^3)} + \|g\|_{L^{10}(I\times\mathbb{R}^3)} \tag{3.3}$$

Note that, by Strichartz estimates (1.6) and (1.7), we get

$$\|\!|e^{it\Delta}\varphi\|\!|_I \leq \|\nabla\varphi\|_{L^2(\mathbb{R}^3)}, \tag{3.4}$$

for very $\varphi \in \dot{H}^1(\mathbb{R}^3)$.
Recall that our purpose is to prove that

$$\limsup_{n\to\infty}(\sup_{t\in\mathbb{R}} E_0(r_n^l(t,.)) + \|\!|r_n^l\|\!|_{\mathbb{R}}) \underset{l\to\infty}{\longrightarrow} 0\ . \tag{3.5}$$

The following lemma is combination of Strichartz estimates and energy estimates.

Lemma 3.1 *Let v be a solution on $I=[a,b]$ of the inhomogeneous Schrödinger equation*

$$i\partial_t v + \Delta v = f. \tag{3.6}$$

Then, we have

$$\|\!|v\|\!|_I + \sup_{t\in I} E_0^{1/2}(v(t)) \leq C(E_0^{1/2}(v(a)) + \|\nabla f_n^l\|_{L^{10/7}(\mathbb{R}^4)}). \tag{3.7}$$

Applying Lemma 3.1 to r_n^l on $I=[a,b]$ we obtain

$$\|\!|r_n^l\|\!|_I + \sup_{t\in I} E_0^{1/2}(r_n^l(t)) \leq C(\|\nabla r_n^l(a,.)\|_{L^2(\mathbb{R}^3)} + \|\nabla f_n^l\|_{L^{10/7}(\mathbb{R}^4)}. \tag{3.8}$$

On the other hand, we estimate

$$
\begin{aligned}
\|\nabla f_n^l\|_{L^{10/7}(\mathbb{R}^4)} &\leq \|\nabla(\sum_{j=1}^{l} p(U_n^j) - p(W_n^l))\|_{L^{10/7}(\mathbb{R}^4)} + \\
&+ \|\nabla(p(W_n^l + w_n^l) - p(W_n^l))\|_{L^{10/7}(\mathbb{R}^4)} + \\
&+ \|\nabla(p(W_n^l + w_n^l + r_n^l) - p(W_n^l + w_n^l))\|_{L^{10/7}(\mathbb{R}^4)}. \qquad (3.9)
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\|\nabla(p(W_n^l + w_n^l + r_n^l) - p(W_n^l + w_n^l))\|_{L^{10/7}(\mathbb{R}^4)} &\leq \|W_n^l + w_n^l\|_{\mathbb{R}}^3 \|w_n^l + w_n^l\|_{L^{10}(I\times\mathbb{R}^3)} \|r_n^l\|_I + \\
&+ \sum_{\alpha=2}^{\alpha=5} \|W_n^l + w_n^l\|_{\mathbb{R}}^{(5-\alpha)} \|r_n^l\|_I^{\alpha}. \qquad (3.10)
\end{aligned}
$$

Denote

$$
\delta_n^l = \|\nabla(p(W_n^l + w_n^l) - p(W_n^l))\|_{L^{10/7}(\mathbb{R}^4)} + \|\nabla(\sum_{j=1}^{l} p(U_n^j) - p(\sum_{j=1}^{l} U_n^j))\|_{L^{10/7}(\mathbb{R}^4)} \qquad (3.11)
$$

Combining (3.8), (3.9) and (3.10), we obtain

$$
\begin{aligned}
\|r_n^l\|_I + \sup_{t\in I} E_0^{1/2}(r_n^l(t)) &\leq C(\gamma_n^l(a) + \delta_n^l + \\
&+ \|W_n^l + w_n^l\|_{\mathbb{R}}^3 \|w_n^l + w_n^l\|_{L^{10}(I\times\mathbb{R}^3)} \|r_n^l\|_I \\
&+ \sum_{\alpha=2}^{\alpha=5} \|W_n^l + w_n^l\|_{\mathbb{R}}^{(5-\alpha)} \|r_n^l\|_I^{\alpha} +). \qquad (3.12)
\end{aligned}
$$

We prepare several propositions. The first one gives the bound on $\|W_n^l + w_n^l\|_{\mathbb{R}}$.

Proposition 3.2 *There exists $C > 0$ such that, for every $l \geq 1$,*

$$
\limsup_{n\to\infty} \|W_n^l\|_{\mathbb{R}} + \|w_n^l\|_{\mathbb{R}} \leq C. \qquad (3.13)
$$

In view of (3.12) and Proposition 3.2, we get for every $l \geq 1$ and $n \geq N(l)$,

$$
\begin{aligned}
\|r_n^l\|_I + \sup_{t\in I} E_0^{1/2}(r_n^l(t)) &\leq C_1(\gamma_n^l(a) + \delta_n^l + \\
&+ \|W_n^l + w_n^l\|_{L^{10}(I\times\mathbb{R}^3)} \|r_n^l\|_I + \sum_{\alpha=2}^{5} \|r_n^l\|_I^{\alpha} +). \qquad (3.14)
\end{aligned}
$$

The second proposition shows that under a suitable finite partition of $\mathbb{R}$, one can absorb the linear term in r_n^l in the right-hand side of (3.14).

Proposition 3.3 *For every ϵ strictly positive number, there exists an n-dependent partition of* $\mathbb{R}_+$

$$\mathbb{R}_+ = \underbrace{[0, a_n^1]}_{I_n^1} \cup \underbrace{[a_n^1, a_n^2]}_{I_n^2} \cup ... \cup \underbrace{[a_n^{p-1}, +\infty[}_{I_n^p}. \tag{3.15}$$

and $l_0 \geq 1$, such that, for every $1 \leq i \leq p$, every $l \geq l_0$, one has

$$\limsup_{n\to\infty} \|W_n^l + w_n^l\|_{L^{10}(I_n^i \times \mathbb{R}^3)} \leq \epsilon. \tag{3.16}$$

The third proposition proves the smallness of δ_n^l.

Proposition 3.4

$$\limsup_{n\to\infty} \delta_n^l \underset{l\to\infty}{\longrightarrow} 0 \tag{3.17}$$

We postpone the proofs of Proposition 3.2, 3.3 and 3.4 for a while.
Now, we are in position to achieve the proof of Theorem 1.12. Applying (3.14) on an interval I_n^i, defined by Proposition 3.3, we get

$$\|r_n^l\|_{I_n^i} + \sup_{t\in I_n^i} E_0^{1/2}(r_n^l(t)) \leq C(\gamma_n^l(a_n^i) + \delta_n^l + \sum_{\alpha=2}^{5} \|r_n^l\|_{I_n^i}^{\alpha} \tag{3.18}$$

for every $l \geq l_0$ and $n \geq N(l)$. Of course, the smallness of ϵ is only used to absorb the linear term in r_n^l in the right-side hand of the inequality (3.14).
On I_n^1, we have

$$\|r_n^l\|_{I_n^1} + \sup_{t\in I_n^1} E_0^{1/2}(r_n^l(t)) \leq C(\gamma_n^l(0) + \delta_n^l + \sum_{\alpha=2}^{5} \|r_n^l\|_{I_n^1}^{\alpha}). \tag{3.19}$$

Recall that by definition of nonlinear profiles we have, for every $l \geq 1$,

$$\gamma_n^l(0) \leq \sum_{j=1}^{l} E_0^{1/2}(U^j - V^j)(-\frac{t_n^j}{(h_n^j)^2}) \underset{n\to\infty}{\longrightarrow} 0. \tag{3.20}$$

Now, we need the following lemma

Lemma 3.5 *Let $M = M(t)$ be a positive continuous function on $[0,T]$ such that,$M(0) = 0$ and, for every $t \in [0,T]$, we have*

$$M(t) \leq C(a + \sum_{\alpha=2}^{5} M(t)^{\alpha}), \tag{3.21}$$

with $0 < a < a(C)$. Then, we have, for every $t \in [0,T]$

$$M(t) \leq 2Ca. \tag{3.22}$$

From (3.17) and (3.20), it follows that for every l large enough there exists $n(l)$ such that, for every $n \geq n(l)$, we have

$$\gamma_n^l(0) + \delta_n^l \leq a(C). \tag{3.23}$$

We denote by M_n^l the function defined on $I_n^1 = [0, a_n^1]$ by

$$M_n^l(s) = \|r_n^l\|_{[0,s]} + \frac{s}{a_n^1} \sup_{t \in I_n^1} E_0^{1/2}(r_n^l(t)). \tag{3.24}$$

It is clear that for every $n \geq n(l)$, M_n^l satisfies the conditions of Lemma 3.5. Hence it follows that

$$M_n^l(a_n^1) \leq 2C(\gamma_n^l(0) + \delta_n^l), \tag{3.25}$$

namely

$$\|r_n^l\|_{I_n^1} + \sup_{t \in I} E_0^{1/2}(r_n^l(t)) \leq C(\gamma_n^l(0) + \delta_n^l). \tag{3.26}$$

From (3.26) one concludes

$$\limsup_{n \to \infty} (\|r_n^l\|_{I_n^1} + \sup_{t \in I_n^1} E_0^{1/2}(r_n^l(t))) \underset{l \to \infty}{\longrightarrow} 0. \tag{3.27}$$

On the other hand, note that

$$\gamma_n^l(a_n^1) \leq \sup_{t \in I_n^2} E_0^{1/2}(r_n^l(t)), \tag{3.28}$$

which gives

$$\limsup_{n \to \infty} \gamma_n^l(a_n^1) \underset{l \to \infty}{\longrightarrow} 0. \tag{3.29}$$

This fact allows us to repeat the same argument on $I_n^2 = [a_n^1, a_n^2]$. We get

$$\|r_n^l\|_{I_n^2} + \sup_{t \in I_n^2} E_0^{1/2}(r_n^l(t)) \leq C(\gamma_n^l(a_n^1) + \delta_n^l). \tag{3.30}$$

So that

$$\limsup_{n \to \infty} (\|r_n^l\|_{I_n^2} + \sup_{t \in I_n^2} E_0^{1/2}(r_n^l(t))) \underset{l \to \infty}{\longrightarrow} 0. \tag{3.31}$$

By iterating this process, we prove that, for every $1 \leq i \leq p$,

$$\limsup_{n \to \infty} (\|r_n^l\|_{I_n^i} + \sup_{t \in I_n^2} E_0^{1/2}(r_n^l(t))) \underset{l \to \infty}{\longrightarrow} 0\ . \tag{3.32}$$

On the other hand, we have

$$\limsup_{n\to\infty}(\|r_n^l\|_{\mathbb{R}_+}\sup_{t\in\mathbb{R}_+}E_0^{1/2}(r_n^l(t))\leq\limsup_{n\to\infty}\sum_{i=1}^{p}(\|r_n^l\|_{I_n^i}+\sup_{t\in I_n^i}E_0^{1/2}(r_n^l(t))). \tag{3.33}$$

Hence

$$\limsup_{n\to\infty}(\|r_n^l\|_{\mathbb{R}_+}+\sup_{t\in\mathbb{R}_+}E_0^{1/2}(r_n^l(t)))\underset{l\to\infty}{\longrightarrow}0. \tag{3.34}$$

A similar decomposition to (3.58) for $\mathbb{R}_-$ can be provided. Arguing as above we prove

$$\limsup_{n\to\infty}(\|r_n^l\|_{\mathbb{R}_-}+\sup_{t\in\mathbb{R}_-}E_0^{1/2}(r_n^l(t))\underset{l\to\infty}{\longrightarrow}0 \tag{3.35}$$

which gives finally

$$\limsup_{n\to\infty}[\|r_n^l\|_{\mathbb{R}}+\sup_{t\in\mathbb{R}}E_0^{1/2}(r_n^l(t))]\underset{l\to\infty}{\longrightarrow}0. \tag{3.36}$$

This concludes the proof of Theorem 1.12.

Proof of Proposition 3.2. Firstly, observe that from (3.4) it follows that

$$\limsup_{n\to\infty}\|w_n^l\|\leq C\limsup_{n\to\infty}E^{1/2}(w_n^l)\leq C\limsup_{n\to\infty}E^{1/2}(v_n^{'})\leq C. \tag{3.37}$$

Thereby, it suffices to prove that

$$\limsup_{n\to\infty}\|W_n^l\|\leq C, \tag{3.38}$$

for every $l\geq 1$. Secondly, by Lemma 2.7 and Remark 2.8 we have, for every l,

$$\limsup_{n\to\infty}\|W_n^l\|_{L^{10}(\mathbb{R}^4)}^{10}=\sum_{j=1}^{l}\|U^j\|_{L^{10}(\mathbb{R}^4)}^{10} \tag{3.39}$$

and

$$\limsup_{n\to\infty}\|\nabla W_n^l\|_{L^{10/3}(\mathbb{R}^4)}^{10/3}=\sum_{j=1}^{l}\|\nabla U^j\|_{L^{10/3}(\mathbb{R}^4)}^{10/3}. \tag{3.40}$$

Hence, if we prove that the series $\sum_{j=1}^{l}\|U^j\|_{L^{10}(\mathbb{R}^4)}^{10}$ and $\sum_{j=1}^{l}\|\nabla U^j\|_{L^{10/3}(\mathbb{R}^4)}^{10/3}$ are convergent, the proof of (3.38) is done. However, via (1.6), (1.7) and (1.12), we have

$$\sum_{l\geq 1}\|\nabla V^j\|_{L^{10/3}(\mathbb{R}^4)}^{10/3}+\sum_{l\geq 1}\|V^j\|_{L^{10}(\mathbb{R}^4)}^{10}\leq C(\sum_{l\geq 1}E_0(V^j)^{10/6}+\sum_{l\geq 1}E_0(V^j)^5)\leq C. \tag{3.41}$$

Then, it suffices to prove that, for j large enough,

$$\|U^j\|_{\mathbb{R}}\leq C\|V^j\|_{\mathbb{R}}. \tag{3.42}$$

The fact (3.42) will follow from the following lemma

Lemma 3.6 *There exists $\delta_0 > 0$, such that, if v is solution of equation (1.2) with*

$$\|v\|_{\mathbb{R}} \leq \delta_0, \tag{3.43}$$

then the solution u of equation (1.15) *with $v(T,x) = u(T,x)$ for some $T \in [-\infty, +\infty]$ satisfies*

$$\|u\|_{\mathbb{R}} \leq 2\|v\|_{\mathbb{R}}. \tag{3.44}$$

Proof. We assume that $\|\nabla(u-v)(x,-\infty)\|_{L^2(\mathbb{R}^3)} = 0$ (the others cases can be handled similarly). Let $(T_n)_{n\geq 0}$ be a sequence of numbers converging to $+\infty$ as $n \longrightarrow +\infty$. The following notations will be used only in this proof,

$$J_n = [-T_n, T_n] \tag{3.45}$$

Denote

$$w = u - v$$

satisfying the difference equation

$$\begin{cases} i\partial_t w + \Delta w = |w+v|^4(w+v) \\ w(t=-T_n, x) = (u-v)(-T_n, x) \end{cases} \tag{3.46}$$

From (3.7) it follows that

$$\|w\|_{J_n} \leq C(\|\nabla(u-v)(x,T_n)\|_{L^2(\mathbb{R}^3)} + \|v\|_{J_n}^5 + \|w\|_{J_n}^5). \tag{3.47}$$

Using the fact that $\|\nabla(u-v)(x,T_n)\|_{L^2(\mathbb{R}^3)} \longrightarrow 0$ as n goes to infinity and Lemma 3.5, we get, for n large,

$$\|w\|_{J_n} \leq 2C(\|\nabla(u-v)(x,T_n)\|_{L^2(\mathbb{R}^3)} + \|v\|_{J_n}^5). \tag{3.48}$$

Letting n goes to ∞, we infer

$$\|w\|_{\mathbb{R}} \leq 2C\|v\|_{\mathbb{R}}^5. \tag{3.49}$$

Hence

$$\|u\|_{J_n} \leq 2C\|v\|_{J_n}^5 + \|v\|_{J_n} \leq (2C\|v\|_{J_n}^4 + 1)\|v\|_{J_n}. \tag{3.50}$$

The choice of δ_0 such that $2C\delta_0 \leq 1$ conclude the proof of Lemma 3.6. •

Returning to the proof of Proposition 3.2. Observe that if j is taken large enough, V^j and U^j satisfy the conditions of Lemma 3.6. Indeed, the inequality (3.41) implies that, for j large enough,

$$\|V^j\|_{\mathbb{R}} \leq \delta_0. \tag{3.51}$$

Moreover, by definition of U^j given in Definition 1.11 we have

$$\|\nabla(U^j - V^j)(T_j)\|_{L^2(\mathbb{R}^3)} = 0, \tag{3.52}$$

where $T^j = \lim_{n\to\infty} -\frac{t_n^j}{(h_n^j)^2}$. Consequently, there exists some j_0 such that

$$\|U^j\|_{\mathbb{R}} \leq 2\|V^j\|_{\mathbb{R}} \tag{3.53}$$

for every $j \geq j_0$, as desired. •

Proof of Proposition 3.3. First, recall that

$$\limsup_{n\to\infty} \|w_n^l\|_{L^{10}(\mathbb{R}^4)} \underset{l\to\infty}{\longrightarrow} 0. \tag{3.54}$$

Then, there exists $l_1 \geq 1$ such that

$$\limsup_{n\to\infty} \|w_n^l\|_{L^{10}(\mathbb{R}^4)} \leq \epsilon/2. \tag{3.55}$$

for every $l \geq l_1$. On the other hand, by Lemma 2.7, we have, for every $l \geq l_2 \geq 1$,

$$\limsup_{n\to\infty} \|W_n^l\|^{10}_{L^{10}(\mathbb{R}^4)} \leq \limsup_{n\to\infty} \|W_n^{l_2}\|^{10}_{L^{10}(\mathbb{R}^4)} + \sum_{j=l_2+1}^{l} \|U^j\|^{10}_{L^{10}(\mathbb{R}^4)}. \tag{3.56}$$

Recall also that the series $\sum_{j=1}^{\infty} \|U^j\|^{10}_{L^{10}(\mathbb{R}^4)}$ is convergent, then we can choose l_2 such that

$$\sum_{j\geq l_2} \|U^j\|_{L^{10}(\mathbb{R}^4)} \leq \frac{\epsilon}{4}. \tag{3.57}$$

View of (3.56) , it is sufficient to prove the proposition for $1 \leq l \leq l_0 = \sup(l_1, l_2)$. Note the partition is needed for n large, then in the next constructions we take n large enough. For $l = 1$, we have three possibilities

i) If $\frac{t_n^1}{(h_n^1)^2} \underset{n\to\infty}{\longrightarrow} +\infty$, then

$$\begin{aligned}\mathbb{R}_+ = {} & [0, t_n^1 - T^p(h_n^1)] \cup [t_n^1 - T^p(h_n^1)^2, t_n^1 - T^{p-1}(h_n^1)^2] \cup ... \cup [t_n^1 - T^1(h_n^1)^2, t_n^1] \cup \\ & \cup [t_n^1, t_n^1 + T^1(h_n^1)^2] \cup [t_n^1 + T^1(h_n^1)^2, t_n^1 + T^2(h_n^1)^2] \cup .... \cup [t_n^1 + T^p(h_n^1)^2, +\infty[,\end{aligned} \tag{3.58}$$

where $T^0 = 0 < T^1 < ... < T^p < T^{p+1} = +\infty$ satisfying $\|U^1\|_{L^{10}(T^i \leq |t| \leq T^{i+1}, L^{10/3})} \leq \frac{\epsilon}{4l_0}$ for every $0 \leq i \leq p$.

ii) If $\frac{t_n^1}{(h_n^1)^2} \underset{n\to\infty}{\longrightarrow} \beta$, then

$$\mathbb{R}_+ = [0, T^1(h_n^1)^2] \cup [T^1(h_n^1)^2, T^2(h_n^1)^2] \cup ... \cup [T^p(h_n^1)^2, +\infty[. \tag{3.59}$$

where $T^0 = 0 < T^1 < ... < T^p < T^{p+1} = +\infty$ satisfying $\|U^1\|_{L^{10}([T^i - \beta, T^{i+1} - \beta], L^{10/3})} \leq \frac{\epsilon}{4l_0}$ for every $0 \leq i \leq p$.

iii)If $\frac{t_n^1}{(h_n^1)^2} \underset{n\to\infty}{\longrightarrow} -\infty$, then

$$\mathbb{R}_+ = [0, +\infty[. \tag{3.60}$$

This gives the decomposition for $l=1$. Arguing similarly we construct a partial decomposition for every $2 \leq l \leq l_0$. Finally, the global decomposition is obtained by intersecting all the partial ones. •

Proof of Proposition 3.4. The proof is divided in two parts. In the first part, we prove that, for every $l \geq 1$ we have

$$\|\nabla(\sum_{j=1}^{l} p(U_n^j) - p(\sum_{j=1}^{l} U_n^j))\|_{L^{10/7}} \underset{n\to\infty}{\longrightarrow} 0 \,. \tag{3.61}$$

In the second part, we prove

$$\limsup_{n\to\infty} \|\nabla(p(W_n^l + w_n^l) - p(W_n^l))\|_{L^{10/7}(\mathbb{R}^4)} \underset{l\to\infty}{\longrightarrow} 0. \tag{3.62}$$

(Part 1). The left hand-side of (3.61) is bounded by a sum of quantities as

$$D_n = \|\nabla(U_n^{j_1} U_n^{j_2} U_n^{j_3} U_n^{j_4} U_n^{j_5})\|_{L^{10/7}(\mathbb{R}^4)}, \tag{3.63}$$

when not all j_k are equal. Arguing in the same way at the proof of Lemma 2.7, we get easily

$$D_n \underset{n\to\infty}{\longrightarrow} 0, \tag{3.64}$$

and concludes the proof of (3.61).

(Part 2). Let us recall that our purpose is to prove that

$$\limsup_{n\to\infty} \|\nabla(p(W_n^l + w_n^l) - p(W_n^l))\|_{L^{10/7}(\mathbb{R}^4)} \underset{l\to\infty}{\longrightarrow} 0. \tag{3.65}$$

It is easy to check that

$$\begin{aligned} \|\nabla(p(W_n^l + w_n^l) - p(W_n^l))\|_{L^{10/7}(\mathbb{R}^4)} &\leq C\|w_n^l\|_{L^{10}(\mathbb{R}^4)}(1 + \|W_n^l + w_n^l\|_{\mathbb{R}})^4 + \\ &+ \|W_n^l\|_{\mathbb{R}}^3 \|W_n^l \nabla(w_l^n)\|_{L^{5/2}(\mathbb{R}^4)}. \end{aligned} \tag{3.66}$$

But, from Proposition 3.2 we have

$$\limsup_{n\to\infty} (\|W_n^l\|_{\mathbb{R}} + \|w_n^l\|_{\mathbb{R}}) \leq C \tag{3.67}$$

and

$$\limsup_{n\to\infty} \|w_n^l\|_{L^{10}(\mathbb{R}^4)} \underset{l\to\infty}{\longrightarrow} 0. \tag{3.68}$$

Then it suffices to show that

$$\limsup_{n\to\infty} \|W_n^l \nabla(w_n^l)\|_{L^{5/2}(\mathbb{R}^4)} \underset{l\to\infty}{\longrightarrow} 0. \tag{3.69}$$

On the other hand, the series $\sum_{j\geq 1}\|U^j\|^{10}_{L^{10}(\mathbb{R}^4)}$ is convergent. Then, for every $\epsilon>0$, there exists $l(\epsilon)$ such that

$$\sum_{j\geq l(\epsilon)}\|U^j\|^{10}_{L^{10}(\mathbb{R}^4)}\leq\epsilon. \tag{3.70}$$

In particular

$$\begin{aligned}\limsup_{n\to\infty}\|(\sum_{j=l(\epsilon)}^{l}U^j_n)\nabla(w^n_l)\|^{10}_{L^{5/2}(\mathbb{R}^4)} &\leq (\sum_{j\geq l(\epsilon)}\|U^j\|^{10}_{L^{10}(\mathbb{R}^4)})\limsup_{n\to\infty}\|\nabla W^l_n\|^{10}_{L^{10/3}(\mathbb{R}^4)}\\ &\leq C\epsilon. \end{aligned}\tag{3.71}$$

Since, for l larger than $l(\epsilon)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\|W^l_n\nabla(w^l_n)\|_{L^{5/2}(\mathbb{R}^4)} &\leq \limsup_{n\to\infty}\|W^{l(\epsilon)}_n\nabla(w^l_n)\|_{L^{5/2}(\mathbb{R}^4)}\\ &+ \limsup_{n\to\infty}\|(\sum_{j=l(\epsilon)}^{l}U^j_n)\nabla(w^l_n)\|_{L^{5/2}(\mathbb{R}^4)}, \end{aligned}\tag{3.72}$$

it follows that

$$\limsup_{n\to\infty}\|W^l_n\nabla(w^n_l)\|_{L^{5/2}(\mathbb{R}^4)}\leq\limsup_{n\to\infty}\|W^{l(\epsilon)}_n\nabla(w^l_n)\|_{L^{5/2}(\mathbb{R}^4)}+C\epsilon. \tag{3.73}$$

Hence, it suffices to prove that

$$\limsup_{n\to\infty}\|W^{l_0}_n\nabla(w^l_n)\|_{L^{5/2}(\mathbb{R}^4)}\underset{l\to\infty}{\longrightarrow}0. \tag{3.74}$$

for every $l_0\geq 1$. We shall prove that, for every $j\geq 1$,

$$\limsup_{n\to\infty}\|U^j_n\nabla(w^l_n)\|_{L^{5/2}(\mathbb{R}^4)}\underset{l\to\infty}{\longrightarrow}0, \tag{3.75}$$

which implies (3.74) and so (3.65) . A change of variables $x=h^j_ny+x_n, t=(h^j_n)^2s+t_n$ gives

$$\|U^j_n\nabla(w^l_n)\|_{L^{5/2}(\mathbb{R}^4)}=\|U^j\nabla(\tilde{w}^l_n)\|_{L^{5/2}(\mathbb{R}^4)} \tag{3.76}$$

where

$$\tilde{w}^n_l(s,y)=w^l_n((h^j_n)^2s+t_n,h^j_ny+x_n) \tag{3.77}$$

Observe that

$$\|w^l_n\|_{L^{10}(\mathbb{R}^4)}=\|\tilde{w}^l_n\|_{L^{10}(\mathbb{R}^4)}\quad\text{and}\quad\|\nabla w^l_n\|_{L^{10/3}(\mathbb{R}^4)}=\|\nabla\tilde{w}^l_n\|_{L^{10/3}(\mathbb{R}^4)}. \tag{3.78}$$

By density, we can take $U^j\in C^\infty_0$. Using Hölder's inequality, it is enough to prove

$$\limsup_{n\to\infty}\|\nabla w^l_n\|_{L^2(\mathcal{B})}\underset{l\to\infty}{\longrightarrow}0. \tag{3.79}$$

where $\mathcal{B}$ denote a fixed compact of $\mathbb{R}_t\times\mathbb{R}^3_x$. This fact will follow from the following lemma.

Lemma 3.7 *Let $\mathcal{B}$ be a compact set of $\mathbb{R}_t \times \mathbb{R}^3_x$. Then, for every $\epsilon > 0$, there exists a constant C_ϵ such that*

$$\|\nabla v\|_{L^2(\mathcal{B})} \leq C_\epsilon \|v\|_{L^{10}(\mathbb{R}^4)} + \epsilon \|\nabla v(0,.)\|_{L^2(\mathbb{R}^3)}. \tag{3.80}$$

for every v solution to linear Schrödinger equation.

Proof. Assume that (3.80) does not hold. Then there exists an $\epsilon > 0$ and a sequence $(v^p)_{p\geq 0}$ of solutions to linear Schrödinger equation such that,

$$\|\nabla v^p\|_{L^2(\mathcal{B})} > p\|v\|_{L^{10}(\mathbb{R}^4)} + \epsilon \|\nabla v^p(0,.)\|_{L^2(\mathbb{R}^3)}. \tag{3.81}$$

We replace v^p by $\frac{v_p}{\|\nabla v^p\|_{L^2(\mathcal{B})}}$ and we preserve the same notation. By Hölder's inequality and Strichartz estimates, we infer

$$\begin{aligned} \|\nabla v^p\|_{L^2(\mathcal{B})} \leq \|\chi_\mathcal{B}\|_{L^5(\mathbb{R}^4)} \|\nabla v^p\|_{L^{10/3}(\mathbb{R}^4)} &\leq C\|\chi_\mathcal{B}\|_{L^5(\mathbb{R}^4)} \|\nabla v^p(0,.)\|_{L^2(\mathbb{R}^3)} \\ &\leq B^* \|\nabla v^p(0,.)\|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{3.82}$$

Hence, the sequence $(v^p)_{p\geq 1}$ satisfies

$$1 \geq p\|v^p\|_{L^{10}(\mathbb{R}^4)} + \epsilon B^* \tag{3.83}$$

From (3.83) it follows that

$$\|v^p\|_{L^{10}(\mathbb{R}^4)} \underset{p\to\infty}{\longrightarrow} 0. \tag{3.84}$$

Then, there exists a subsequence of $(v^p)_{p\geq 0}$, also denoted $(v^p)_{p\geq 0}$, such that

$$\nabla v^p \rightharpoonup 0 \quad \text{weakly in } L^{10/3}(\mathbb{R}^4). \tag{3.85}$$

We need the following lemma (The proof is easy, see Lemma 3.23 in [MV] for two spatial dimensions)

Lemma 3.8 *Let $(\varphi_p)_{n\geq 1}$ and φ be in $L^2(\mathbb{R}^3)$. The following statements are equivalent*
i) $\varphi_p \rightharpoonup \varphi$ weakly in $L^2(\mathbb{R}^3)$.
ii) $e^{it\Delta}\varphi_p \rightharpoonup e^{it\Delta}\varphi$ weakly in $L^{10/3}(\mathbb{R}^4)$.

We set $w^p = \nabla v^p(0,.)$. From (3.85) and the previous lemma it follows that, up a subsequence,

$$\|e^{it\Delta} w^p\|_{L^2(\mathcal{B})} = 1 \quad \text{and} \quad w^p \rightharpoonup 0 \quad \text{weakly in } L^2(\mathbb{R}^3). \tag{3.86}$$

The fact (3.86) is contrary to the compactness of the operator $\varphi \longmapsto U(t)\varphi$ from $L^2(\mathbb{R}^3)$ to $L^2_{\text{loc}}(\mathbb{R}^4)$ (see Proposition 1.4). Then (3.80) holds. •
Returning to the proof of Proposition 3.4. From (3.68) and Lemma 3.7, we get

$$\limsup_{l\to\infty} \limsup_{n\to\infty} \|\nabla w_n^l\|_{L^2(\mathcal{B})} \leq C\epsilon \tag{3.87}$$

Because ϵ is arbitrary, we get

$$\limsup_{l\to\infty}\limsup_{n\to\infty}\|\nabla w_n^l\|_{L^2(\mathcal{B})}=0, \tag{3.88}$$

which is (3.79). •

Proof of Corollary 1.15. Assume that the *a priori* estimate (1.22) fails, then there exists some sequence (u_n) of solution to (1.15) such that

$$\sup_n E_0(u_n)<\lambda_0, \qquad \|\|u_n\|\|_{\mathbb{R}} \underset{n\to\infty}{\longrightarrow} \infty. \tag{3.89}$$

In view of Theorem 1.12, there exists a subsequence $(u_n^{'})$ of (u_n) such that

$$u_n^{'}=\sum_{j=1}^{l}\frac{1}{\sqrt{h_n^j}}U^j(\frac{t-t_n^j}{(h_n^j)^2},\frac{x-x_n^j}{h_n^j})+w_n^l(t,x)+r_n^l(t,x) \tag{3.90}$$

with

$$\limsup_{n\to\infty}\|\|w_n^l+r_n^l\|\|_{\mathbb{R}}\le C \tag{3.91}$$

for every $l\geq 1$. Hence,

$$\limsup_{n\to\infty}\|\|u_n^{'}\|\|_{\mathbb{R}}\le\limsup_{n\to\infty}\|\|w_n^l+r_n^l\|\|_{\mathbb{R}}+\sum_{j=1}^{l}\|\|U^j\|\|_{\mathbb{R}}<+\infty. \tag{3.92}$$

This implies that $(u_n^{'})$ is bounded in $\|\|\|\|_{\mathbb{R}}$ norm, which contradicts (3.89) and proves the existence of some function A satisfying (1.22).

Remark 3.9 Let us observe that in the statement of Corollary 1.15 we can replace $\|\|u\|\|_{\mathbb{R}}$ by $\|u\|_{L^{10}}$ without any loss. Indeed, it is easy to show that the *a priori* estimate of $L^{10}(\mathbb{R}^4)$ norm in terms of the energy implies the one for $\dot{W}^{1,10/3}(\mathbb{R}^4)$ norm.

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ACKNOWLEDGMENT

Thanks to my Ph.D. advisor Professor Patrick Gérard for introducing me into the subject and the support during the preparation of the work.

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