On the L^p stabilization of the double integrator with bounded controls

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Résumé:

Soit Σ $\dot{x}(t) = f(x(t), u(t))$ un système avec commandes admettant un feedback k qui rende $\dot{x} = f(x, k(x))$ globalement asymtotiquement stable. On suppose que, pour (h, p, q) avec h une application sortie et $1 \leq p \leq q \leq \infty$, il existe une fonction α de classe $\mathcal{K}_\infty$ telle que $\|x_u\|_q \leq \alpha(\|u\|_p)$ où x_u est la solution maximale de Σ correspondant à u et à la condition initiale $x(0) = 0$. Alors la fonction gain $G_{(h,p,q)}$ de (h, p, q) donnée par

$$G_{(H,p,q)}(X) \stackrel{def}{=} \sup_{\|u\|_p = X} \|x_u\|_q,$$

est bien définie. Nous appelons profil de k pour (h, p, q), la fonction $\beta_{(h,p,q)}$ de classe $\mathcal{K}_\infty$ équivalente à $G_{(h,p,q)}$. Pour le double integrateur avec commande saturatée et stabilisé par $-(11)^T x$, on détermine les profils correspondant aux principales applications sorties et nous donnons une famille de feedbacks qui ont un profil linéaire pour la sortie $h(x, u) = x$.

Mots-Clefs: système avec commandes, $iISS$ et ISS stabilité, L^p stabilisation, commande saturée, fonction de Lyapunov, fonction gain.

Abstract:

We consider a finite-dimensional control system Σ $\dot{x}(t) = f(x(t), u(t))$, such that there exists a feedback stabilizer k that renders $\dot{x} = f(x, k(x))$ globally asymptotically stable. Moreover we assume that, for (h, p, q) with h an output map, $1 \leq p \leq q \leq \infty$, there exists a $\mathcal{K}_\infty$ function α such that $\|x_u\|_q \leq \alpha(\|u\|_p)$ where x_u is the maximal solution of Σ corresponding to u and to the initial condition $x(0) = 0$. Then the gain function $G_{(h,p,q)}$ of (h, p, q) given by

$$G_{(H,p,q)}(X) \stackrel{def}{=} \sup_{\|u\|_p = X} \|x_u\|_q,$$

is well-defined and we call profile of k for (h, p, q) a $\mathcal{K}_\infty$ function $\beta_{(h,p,q)}$ equivalent to $G_{(h,p,q)}$. For the double integrator subject to input saturation and stabilized by $-(11)^T x$, we determine the profiles corresponding to the main output maps and we provide a class of feedback stabilizers that have a linear profile for $h(x, u) = x$.

Key words: nonlinear control systems, L^p stabilization, $iISS$ and ISS, input saturation, Lyapunov functions, gain function.

MSC: 93D15, 93D21, 93D30.

1 Introduction

Let (Σ_1) be the finite dimensional control system of the form:

$$(\Sigma_1) \quad \dot{x}(t) = f(x(t), u(t)), \tag{1}$$

where states $x(t) \in I\!R^n$ for all $t \geq 0$ (the integer n is the dimension of (Σ_1)), inputs $u : [0, \infty) \to I\!R^m$ are measurable locally essentially bounded maps (the integer m is the dimension of the input space), and f is locally Lipschitz in (x, u) with $f(0, 0) = 0$. Assume that (Σ_1) admits a feedback stabilizer i.e. a map $k : I\!R^n \to I\!R^m$ locally bounded such that $k(0) = 0$ and the closed loop system given by

$$\dot{x} = f(x, k(x)), \tag{2}$$

is globally asymptotically stable (GAS for short) with respect to the origin (for basic terminology on control theory cf. [8]). We are interested in the robustness of the feedback k i.e. in the stability of k with respect to measurement and actuator noise. We first recall the following definition: the class of $\mathcal{K}_\infty$ functions consists of all $\alpha : I\!R^+ \to I\!R^+$ which are continuous, strictly increasing, unbounded, and satisfy $\alpha(0) = 0$. We also assume for simplicity of the exposition that k is locally Lipschitz and we consider the control system $(\Sigma_1)_k$ given by

$$(\Sigma_1)_k \quad \dot{x}(t) = f(x(t), k(x(t)) + u(t)). \tag{3}$$

We use x_u to denote the maximal solution of (3) corresponding to u and to the initial condition $x(0) = 0$. For $p, q \in [1, \infty]$, $p \leq q$, and an output map $H : I\!R^n \times I\!R^m \to I\!R^l$ locally Lipschitz with $H(0, 0) = 0$, we then ask if $(\Sigma_1)_k$ verifies the following properties:

$(P^1_{(H,p,q)})$ the assignment $u \mapsto H(x_u, u)$ defines a map $F_{(H,p,q)} : L^p(0, \infty) \to L^q(0, \infty)$;

$(P^2_{(H,p,q)})$ there exists $\alpha_{(H,p,q)} \in \mathcal{K}_\infty$ such that

$$\|F_{(H,p,q)}(u)\|_q \leq \alpha_{(H,p,q)}\Big(\|u\|_p\Big). \tag{4}$$

(Indices such as (H, p, q) are dropped when the context is clear.) The main output maps H are x, $f(x, u)$, (the integer l is then equal to n), $h^T x$, $h^T f(x, u)$ where h is a fixed vector of $I\!R^n$ (the integer l is then equal to 1). We also have local versions of $(P^2_{(H,p,q)})$ when (4) only holds in a neighborhood of 0 or a neighborhood of ∞. If (H, p, q) satisfies $(P^2_{(H,p,q)})$ (in a neighborhood of 0 or ∞ respectively) with $\alpha \in \mathcal{K}_\infty$ given by (4), then k is said to be *of shape* α *for* (H, p, q) *(at* 0 or *at* ∞ respectively) or α *is a shape of* k *for* (H, p, q) *(at* 0 or *at* ∞ respectively). Remark that if there exists a shape α of k for (H, p, q), we can define the *gain function* $G_{(H,p,q)} : I\!R^+ \to I\!R^+$ associated to (H, p, q) by

$$G_{(H,p,q)}(X) \stackrel{def}{=} \sup_{\|u\|_p = X} \|F_{(H,p,q)}(u)\|_q. \tag{5}$$

Then G is an increasing function and $G(0) = 0$. We say that k is of *profile* α *for* (H, p, q) *at* 0 *(at* ∞ respectively) if α is a shape of k for (H, p, q) at 0 (at ∞ respectively) and $G(X)/\alpha(X)$ is bounded below by a positive constant for $X > 0$ small enough (for X large enough respectively). As soon as the gain function G is defined, such a profile exists at 0 and if in addition G is unbounded then it also exists at ∞ (see proof in Lemma 1 below). Moreover a profile is uniquely determined up to the multiplication by a positive bounded function. A profile of k for (H, p, q) both at 0 and at ∞ is simply called a profile of k for (H, p, q).

A natural way to quantify the robustness of a feedback stabilizer k satisfying $(P^2_{(H,p,q)})$ for some (H,p,q) is to determine (when possible) the asymptotic behaviors of a corresponding profile α when $\|u\|_p$ tends firstly to 0 and secondly to ∞. Moreover, given two feedback stabilizers k_1 and k_2 for (Σ_1) and (H,p,q) as introduced previously, we can compare k_1 and k_2 by comparing the asymptotic behaviors at 0 and ∞ of their respective profiles (when they exist). In particular we say that k_1 is better than k_2 for (H,p,q) if we have for their corresponding profiles α_1, α_2 (when they exist) that $\limsup_{x\to 0}\frac{\alpha_1(x)}{\alpha_2(x)}$ and $\limsup_{x\to\infty}\frac{\alpha_1(x)}{\alpha_2(x)}$ are both finite and at least one of them is equal to 0. Therefore, the notion of profile is a generalization of the concept of finite-gain stability (though weaker) as it is defined in input/output studies and related works on computation of norms for nonlinear systems in state-space form (cf. [1], [14] and the references given therein). In addition, the notion of profile is closely linked to the concept of input-to-state stability (ISS) and related notions (e.g. integral input-to-state stability etc.) see [9]. For instance, if the output map $H(x,u)$ is equal to x and $p=q=\infty$, having a shape for (x,∞,∞) is equivalent to have the (ISS) property (cf. theorem H in [9]). To illustrate the previous definitions, we consider a simple case namely the linear one. We assume that (Σ_1) is a linear system $\dot{x}=Ax+Bu$ such that the uncontrollable modes have negative real parts. Then, there exists a linear feedback k such that for every $p,q\in[1,\infty]$, $p\le q$ and main output map H, k is of *linear profile* (i.e. the corresponding α is a linear function for (H,p,q)). We can see with the previous example that that the notion of linear profile is less precise than the notion of finite gain.

General questions such as under which conditions on f, k and on (H,p,q), the control system $(\Sigma_1)_k$ satisfies the properties $(P^1)_{(H,p,q)}$, $(P^2)_{(H,p,q)}$. Moreover, one can ask about the possible relationships between the two previous properties and (in the spirit of ISS results) the links between the fact of admitting a profile and admitting a control Lyapunov function satisfying an appropriate dissipative inequality. However in this paper, we focus on a particular control system (SI_2) (defined below) and we address through that example the two following issues: suppose that a control system admits a feedback stabilizer k. Firstly find its profiles for the main output maps and secondly, given (H,p,q) with H a main output map, find a feedback stabilizer that is better than any other for (H,p,q).

The control system (SI_2) belongs to a specific class of control systems, namely the linear ones subject to input saturation

$$\dot{x}=Ax+B\sigma(u). \tag{6}$$

Here A is an $n\times n$ matrix, B an $n\times m$ matrix and σ is of "saturation" type. For instance, arctan, tanh and the standard saturation function $\sigma_0(s)=\frac{s}{\max(1,|s|)}$ are 1-dimensional examples of saturation functions. For this class of systems, examples of feedback stabilizers are well-known under the necessary and sufficient condition that the controllable modes are of non positive real part and the uncontrollable ones of negative real part (cf. [7], [10], [11], [13], [12], and references therein). The interesting problems for the control systems under consideration occur when A admits at least one eigenvalue λ with zero real part and more particularly when the multiplicity of λ is larger than one (we refer to that case as the critical one). The feedback stabilizer is necessarily nonlinear in general (cf. [2], [13] where it is shown that if (A,B) is controllable, $m=1$ and A is equal to the nth-Jordan matrix, then the corresponding control system (SI_n) called the nth-integrator with saturated input cannot be globally stabilized by means of linear feedback if $n\ge 3$). In [4], the authors consider the case when (A,B) is controllable and A is just neutrally stable, (i.e. the eigenvalues of A have non positive real parts and there are no nontrivial Jordan blocks). It is shown there that for the linear feedback stabilizer given by $k(x)\stackrel{def}{=}-B^Tx$, we have for every $p\le q\in[1,\infty)$ or $p=q=\infty$ and $H(x,u)=x$, a linear profile for (H,p,q). Moreover, for $(H,p,q)=(x,p,\infty)$ with p finite, the profile of k is linear at 0 and equal to $X\mapsto X^{p/p+1}$ at ∞. Finally, as a first result regarding the critical case, it is shown in [4] that for (SI_2), the linear feedback stabilizer $k_L(x)\stackrel{def}{=}-(1\ 1)^Tx$ is not of linear profile at ∞ for

(x,p,p) with $p \in [1,\infty]$. For related work, in [5], the authors show that any linear system subject to input saturation with poles of non positive real parts can be semi-globally (that is, on compact sets) stabilized by means of linear feedback. In addition, they show in [6] that for every $D > 0$, there exists a linear feedback stabilizer which is of linear profile (depending on D) under the restriction that all the inputs considered are essentially bounded by D.

In the present paper, we consider the control system (SI_2) given by

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \sigma(u). \end{cases}$$

In a first part we determine the profiles of the linear feedback stabilizer k_L for the main output maps. We show that at 0 all the profiles considered next are linear and at ∞ they are of the type $X \mapsto X^r$, where r depends on (H,p,q) (cf. Theorem 1 below). In a second part, we exhibit a set of feedback stabilizers of (SI_2) each of them having a linear profile for the output map $H(x,u) = x$, which is the best possible for this specific output map (cf. Theorem 3 below). We also provide additional results regarding nonzero initial states and the perturbed control system $(PSI_2)_k$:

$$(PSI_2)_k \begin{cases} \dot{x}_1 &= x_2 + v_1, \\ \dot{x}_2 &= \sigma(k(x)+u) + v_2, \end{cases}$$

where the input $v = (v_1, v_2)$ with $v_1, v_2 \in L^{p'}(0,\infty)$, $p' \geq 1$ and $\|v\|_\infty \leq C_0$ for some constant C_0 depending only on (SI_2) (cf. Theorems 2 and 4 below). We conclude this introduction with a remark regarding the techniques used to obtain the results. In [4] and [6], the passivity technique is essentially generalized: for $p \in [1,\infty]$, the main argument of the proofs consists of finding a suitable 'storage function' V_p and establishing for it a 'dissipation inequality' of the form

$$\frac{dV_p(x_u(t))}{dt} \leq -\|x_u(t)\|^p + \lambda_p \|u(t)\|^p, \tag{7}$$

for some constant $\lambda_p > 0$. We recall that in [4] a nonsmooth V_p was needed. For more discussion on passivity, see for instance [3] and [14]. In our situation, we are not able to find such storage functions. We just have at our disposal a control Lyapunov function V (in the case of the feedback k_L, the saturation function σ has to be increasing and V is not even $(iISS)$-Lyapunov -see [9]-). Therefore the 'dissipation inequality' satisfied by V does not lead directly to the desired estimates. However, on a time interval $I = [t_0, t_1]$ where the variation of V is positive and 'large', we find subsets of I of 'large' size where the input u is 'large'. We can then control integrals on I of various quantities in terms of $\int_I |u|^p$ and get global results.

2 Notations and Statement of the Results

In this section we define the class of saturation functions to be considered and state the main results we obtain for (SI_2). In addition, we provide simple remarks.

Definition 1 *We call $\sigma : I\!\!R \to I\!\!R$ a saturation function (or an S-function) if there exist two real numbers $0 < a \leq K_\sigma$ such that for all $t, t' \in I\!\!R$*

(i) $|\sigma(t) - \sigma(t')| \leq K_\sigma \inf(1, |t-t'|)$*;*

(ii) $|\sigma(t) - at| \leq K_\sigma t\sigma(t)$*.*

A constant K_σ defined as above is called an S-bound for σ and the existence of finite S-bounds for S-functions is shown in [4]. When the context is clear we simply use K to denote an S-bound. Note that (i) is equivalent to the fact that σ is bounded and globally Lipschitz and (ii) is equivalent to

$$t\sigma(t) > 0 \text{ for } t \neq 0,\ \liminf_{|t|\to\infty} |\sigma(t)| > 0,\ \limsup_{t\to 0} \frac{\sigma(t) - \sigma'(0)t}{t^2} < \infty.$$

We say that σ is an iS-function if σ is an increasing S-function. If m is a positive integer, σ is an $\mathbb{R}^m$-valued S-function if $\sigma = (\sigma_1, \ldots, \sigma_m)$, where each component σ_i is an S-function and

$$\sigma(t) = (\sigma_1(t), \ldots, \sigma_m(t)),$$

for $x = (x_1, \ldots, x_m)^T$. Here we use $(\cdots)^T$ to denote the transpose of $(\cdots)$. Throughout this paper, if $x \in \mathbb{R}^n$, we use $\|x\| \overset{def}{=} (\Sigma_{i=1}^n x_i^2)^{1/2}$ to denote the usual Euclidean norm. If E is a measurable set of $\mathbb{R}$, we use $|E|$ to denote its Lebesgue measure. If S is a finite set, we use $S^\#$ to denote its cardinality. For $p \in [1, \infty]$ and $T > 0$, we use L^p to denote $L^p(0, \infty)$ and we let $\|x\|_p$ ($\|x\|_{p,[0,T]}$ respectively) denote the L^p-norm:

$$\|x\|_p \overset{def}{=} \Big(\int_0^\infty \|x(t)\|^p dt\Big)^{1/p}, (\|x\|_{p,[0,T]} \overset{def}{=} \Big(\int_0^T \|x(t)\|^p dt\Big)^{1/p} \text{ respectively}),$$

and

$$\|x\|_\infty \overset{def}{=} \text{ess} \sup_{0\le t<\infty} \|x(t)\|, (\|x\|_{\infty,[0,T]} \overset{def}{=} \text{ess} \sup_{0\le t\le T} \|x(t)\| \text{ respectively}).$$

Various constants depending only on (SI_2) and (H, p, q) will be considered. We will use the symbol *const.* to denote those that depend on (SI_2) and (H, p, q). Sometimes, a constant term may depend on an extra parameter R. In this case, instead of *const.*, we use $const._R$. For $a \in \mathbb{R}$, let $a^+ \overset{def}{=} \sup(a, 0)$ and we use sgn a to denote the sign of a ($\frac{a}{|a|}$ if $a \neq 0$ and 0 otherwise). For $a, b > 0$, the symbol $a << b$ means that b is much larger than a. If $a, b \in \mathbb{R}$, we use $\Delta f\Big|_a^b$ or $\Delta f\Big|_I$ to denote $f(b) - f(a)$ where f is a function on $\mathbb{R}$ and I is the interval between a and b.

Let us consider a control system of the form (Σ_1) together with a feedback stabilizer k that gives rise to a control system of the form $(\Sigma_1)_k$. For (H, p, q) with H a main output map and $1 \le p \le q \le \infty$, assume that $(P^1_{(H,p,q)})$ holds. Suppose that there exists $X_0 > 0$ such that $\sup_{\|u\|_p = X_0} \|F_{(H,p,q)}(u)\|_q$ is infinite. We say in this case that k has *infinite gain for* (H, p, q). Otherwise, the gain function $G_{(H,p,q)} : \mathbb{R}^+ \to \mathbb{R}^+$ is well-defined and we say that k has *finite gain* G *for* (H, p, q). The function G is not necessarily continuous but we have

Lemma 1 *Let $(\Sigma_1)_k$ and (H, p, q) be defined as above. Then $G_{(H,p,q)}$ is a well-defined map from $\mathbb{R}^+$ to $\mathbb{R}^+$ if and only if $(P^2_{(H,p,q)})$ holds. In addition, the shape α given by (4) can be taken as a profile of k for (H, p, q) if, for $p < \infty$ $G_{(H,p,q)}$ is unbounded or if, for $p = \infty$, $G_{(H,p,q)}$ is unbounded and $\|f(0, u_n)\| > 0$ for some sequence $(u_n)_{n\ge 0}$, $u_n \in \mathbb{R}^m$, such that* $\lim_{n\to\infty} u_n = 0$.

Proof: We have two alternatives; either $u \mapsto f(0, u)$ is the zero function or there exists $X_0 \ge 0$ such that $G(X) > 0$ if $X > X_0$ and $G(X_0) = 0$. In the first case, $x_u \equiv 0$ for every locally essentially bounded measurable u. Any $\alpha \in \mathcal{K}_\infty$ is a shape of k for (H, p, q). In the second case, let $g \overset{def}{=} G(X_0 + 1)$. The argument that follows is standard. Since $G(X_0) = 0$ and G is increasing, we can find a sequence of times $(t_n)_{n\ge 0}$ such that $t_0 = X_0 + 1$, $G(t_n) = g/2^n$ and for $n \ge 0$, $G(t_{n+1}) \le G(t) \le G(t_n)$ for $t \in [t_{n+1}, t_n]$. Then $(t_n)_{n\ge 0}$ is a decreasing sequence to X_0. Let α be the piecewise linear function defined on

$[X_0, X_0+1]$ such that for $n \geq 0$ and $t \in [t_{n+1}, t_n]$, α is the line segment between $(t_{n+1}, 2G(t_{n+1}))$ and $(t_n, 2G(t_n))$. Then α is continuous, increasing, $\alpha(X_0) = 0$ and for all $t \in [X_0, X_0+1]$ we have $\frac{\alpha(t)}{4} \leq G(t) \leq \alpha(t)$. If $p < \infty$, then $X_0 = 0$ since there exists $u \in L^p$ with arbitrary small L^p-norm and arbitrary large L^∞-norm. If $p = \infty$, then $\bar{\alpha}(t) \stackrel{def}{=} t + \alpha(t)$ is a shape of k for (H,p,q). When G is unbounded, we can take $X_0 = 0$ (for p finite, this is the case and for $p = \infty$ it results from the extra hypothesis). Then the same procedure as above holds exactly by considering a sequence of times $(s_n)_{n\geq 0}$ such that $s_0 = 1$, $G(s_n) = 2^n g$ and for $n \geq 0$, $G(s_n) \leq G(t) \leq G(s_{n+1})$ for $t \in [s_n, s_{n+1}]$. It is clear that the function α previously defined is a $\mathcal{K}_\infty$ function and is a profile of k for (H,p,q). If G is bounded, there exists n_0 such that $2^{n_0} g \leq \sup_{t \in \mathbb{R}^+} G(t) < 2^{n_0+1} g$. Then for $t \geq s_{n_0+1}$, we take for α a half-line of positive slope starting at $(s(t_{n_0+1}), 2G(s_{n_0+1}))$. ■

We give next a criterion for a shape to be a profile. This result will be used repeatedly in the sequel.

Lemma 2 *Let $(\Sigma_1)_k$ and (H,p,q) be defined as above. Suppose that α is a shape of k for (H,p,q) such that there exists $K > 1$ for which*

(a) *there exist $r > 0, X_0 > 1$ and $A \geq 1$ such that we have $\alpha(X/K) \geq \alpha(X)/AK^r$ for $X \leq 1/X_0$ ($\alpha(KX) \leq AK^r \alpha(X)$ for $X \geq X_0$ respectively)*

(b) *there exist $a > 0$ and a sequence $(u_n)_{n\geq 0}$ ($(v_n)_{n\geq 0}$ respectively) such that $\|u_n\|_p \to 0$, $\|u_{n+1}\|_p \geq \|u_n\|_p/K$ ($\|v_n\|_p \to \infty$, $\|v_{n+1}\|_p \leq K\|v_n\|_p$) and*

$$\frac{\|F(u_n)\|_q}{\alpha\Big(\|u_n\|_p\Big)} \geq a \quad \Big(\frac{\|F(v_n)\|_q}{\alpha\Big(\|v_n\|_p\Big)} \geq a \text{ respectively}\Big).$$

Then α is a profile of k for (H,p,q) at 0 (at ∞ respectively).

Condition (a) simply says that if $\beta(X) \stackrel{def}{=} \frac{\alpha(X)}{X^r}$, then $\beta(KX)/\beta(X)$ is bounded below and above for X in a neighborhood of 0 (of ∞ respectively). This is obvious if β is constant. Then in order to show that the profile of k for (H,p,q) is a function $\alpha \in \mathcal{K}$, in a first step we establish the inequality (4) and in a second step, we exhibit a sequence of inputs such that the hypothesis of the previous lemma are satisfied.

Proof: We just prove the lemma in the neighborhood of ∞. We may also assume that $(v_n)_{n\geq 0}$ is increasing. For X large enough, there exists n_0 such that $\|v_{n_0}\|_p \leq X < \|v_{n_0+1}\|_p$. Then, we get

$$\frac{G(X)}{\alpha(X)} \geq \frac{G(\|v_{n_0}\|_p)}{\alpha(\|v_{n_0}\|_p)} \frac{\alpha(\|v_{n_0}\|_p)}{\alpha(X)} \geq \frac{1}{AK^r} \frac{F(\|v_{n_0}\|_p)}{\alpha(\|v_{n_0}\|_p)} \frac{AK^r \alpha(\|v_{n_0}\|_p)}{\alpha(\|v_{n_0+1}\|_p)} \geq \frac{a}{AK^r}. \ ■$$

Remark 1 *Let $(\Sigma_1)_k$ and (H,p,q) be defined as above. Suppose that $(x,u) \mapsto f(x,u)$ and $x \mapsto k(x)$ are differentiable at $(0,0)$ and k has finite gain for (H,p,q). If $\dot{x} = Ax + Bu$ is the linearized control system at $(0,0)$, it is clear that A is Hurwitz. Then a profile of k at 0 is linear and if $H \neq 0$ is a main output map and (p,q) are both finite, there exist $\gamma > 0$ and a sequence $(u_n)_{n\geq 0}$ with $\|u_n\|_p \to \infty$ such that*

$$\frac{\|F(u_n)\|_q}{\|u_n\|_p} \geq \gamma. \tag{8}$$

(Simply consider inputs of small L^∞-norm but large L^p-norm. Then the corresponding trajectories have L^∞-norm.) Therefore, if $(x,u) \mapsto f(x,u)$ and $x \mapsto k(x)$ are differentiable at $(0,0)$ and k has finite gain G for (H,p,q), the real issue is to determine the profile of k for (H,p,q) at ∞.

We now state the main results of the paper. In section 3, we study the linear feedback $k_L = -(x_1 + x_2)$ for (SI_2) and get the following two theorems:

Theorem 1 *Let consider the control system (SI_2) where σ is an (iS)-function together with the linear feedback stabilizer $k_L(x)$ and (H, p, q), H a main output map and $p \leq q \in [1, \infty]$. Then, the profile of k_L at 0 is linear and at ∞ is as follows:*

(1) *for $H(x, u) = x$ or x_1, $X \mapsto X^{r_1(p,q)}$ with $r_1(p, q) = \frac{2p(q+1)}{q(p+1)}$;*

(2) *for $H(x, u) = x_2$, $X \mapsto X^{r_2(p,q)}$ with $r_2(p, q) = \frac{p(q+2)}{q(p+1)}$;*

(3) *for $H(x, u) = \sigma(-x_1 - x_2 + u) = \dot{x}_2$, $X \mapsto X^{p/q}$ with $q < \infty$.*

In the previous results, we have

$$r_i(p, \infty) \stackrel{def}{=} \lim_{q \to \infty} r_i(p, q) \text{ if } p \text{ finite and } r_i(\infty, \infty) \stackrel{def}{=} \lim_{p \to \infty} r_i(p, \infty).$$

In order to state results relative to $(PSI_2)_{k_L}$, we consider the following Lyapunov function for (SI_2) (introduced in [13]):

$$V(x_1, x_2) = x_2^2 + \int_0^{x_1} \sigma(s) ds + \int_0^{x_1 + x_2} \sigma(s) ds. \tag{9}$$

If σ is an (iS)-function, then V is a strict Lyapunov function for the closed loop system (2) associated to (SI_2) and k_L. In addition, V is not $(iISS)$-Lyapunov function for the system $(SI_2)_{k_L}$. Note though that $V^{\frac{1}{2}}$ is $(iISS)$ Lyapunov (but not (ISS)) for the system $(SI_2)_{k_L}$. We have

Theorem 2 *Let consider $(PSI_2)_{k_L}$. There exists $\varepsilon_0 > 0$ such that for $p \in [1, \infty]$, $p_1 \leq p$, $u \in L^p$, $v \in L^{p_1}$ and $\|v\|_\infty \leq \varepsilon_0$, $\bar{x} \in \mathbb{R}^2$, if (x_1, x_2) is the trajectory of $(PSI_2)_{k_L}$ corresponding to (u, v) starting at $\bar{x}$, we get*

(1) *for $p_1 \in [1, 2)$,*

$$\|x_1\|_p \leq const.\Big(L(v, u, \bar{x}) + \|u\|_p^2 + \|v\|_{p_1}^{\frac{p+1}{p} c_{p_1}} + V(\bar{x})^{\frac{p+1}{p}}\Big), \tag{10}$$

$$\|x_2\|_p \leq const.\Big(L(v, u, \bar{x}) + \|u\|_p^{\frac{p+2}{p+1}} + \|v\|_{p_1}^{\frac{p+2}{2p} c_{p_1}} + V(\bar{x})^{\frac{p+2}{2p}}\Big), \tag{11}$$

$$\sup(\|x_2\|_\infty, \|x_2\|_\infty^2) \leq const.\|x_1\|_\infty \leq const.\Big(L(v, u, \bar{x}) + \|u\|_p^{\frac{2p}{p+1}} + \|v\|_{p_1}^{c_{p_1}} + V(\bar{x})\Big), \tag{12}$$

$$\|\dot{x}_2\|_p \leq const.\Big(L(v, u, \bar{x}) + \|v\|_{p_1}^{\frac{c_{p_1}}{2p}} + V(\bar{x})^{\frac{p+2}{2p}}\Big), \tag{13}$$

with $L(v, u, \bar{x}) \stackrel{def}{=} \|v\|_{p_1} + \|u\|_p + V(\bar{x})^{1/2}$ and $c_{p_1} = \frac{2p_1}{2-p_1}$;

(2) *for $p_1 = 2$, set $G(v, u, \bar{x}) \stackrel{def}{=} \Big(\|u\|_p^{\frac{2p}{p+1}} + V(\bar{x}) + 1\Big) e^{const.\|v\|_2^2} - 1$. We have*

$$\|x_1\|_p \leq const.\Big(L(v, u, \bar{x}) + \|u\|_p^2 + V(\bar{x})^{\frac{p+1}{p}} + G(v, u, \bar{x})^{\frac{p+1}{p}} \|v\|_2^{\frac{2}{p}}\Big), \tag{14}$$

$$\|x_2\|_p \leq const.\Big(L(v, u, \bar{x}) + \|u\|_p^{\frac{p+2}{p+1}} + V(\bar{x})^{\frac{p+2}{2p}} + G(v, u, \bar{x})^{\frac{p+2}{2p}} \|v\|_2^{\frac{2}{p}}\Big), \tag{15}$$

$$\sup(\|x_2\|_\infty, \|x_2\|_\infty^2) \leq const.\|x_1\|_\infty \leq const.\Big(L(v, u, \bar{x}) + G(v, u, \bar{x})\Big), \tag{16}$$

$$\|\dot{x}_2\|_p \leq const.\Big(L(v, u, \bar{x}) + V(\bar{x})^{\frac{p+1}{2p}} + (G(v, u, \bar{x})\|v\|_2)^{\frac{2}{p}}\Big); \tag{17}$$

(3) *for $p_1 > 2$ and $\varepsilon > 0$, there exist u, v with u continuous and compactly supported, $v \in L^{p_1}$ and $\|v\|_\infty \leq \varepsilon$, such that the trajectory of $(PSI_2)_{k_L}$ corresponding to (u, v) starting at $(0,0)$ is unbounded.*

Theorem 2 holds for $p = \infty$ as we simply take the limits in the exponents when p tends to ∞.

In section 4, we consider nonlinear feedbacks for (SI_2). For that purpose, we define the class of $\mathcal{F}$ functions as follows:

Definition 2 *A function $F : I\!R \to I\!R$ is an $\mathcal{F}$-function if F is C^1, odd, $F'(0) = 0$ and there exist $r \geq 1$ such that for $|x_2| \geq 1$ we have*

$$F'(x_2) \geq \sup\Big(3|x_2|, r\frac{F(x_2)}{x_2}\Big). \tag{18}$$

The next theorems describe the performances of the feedback k_F defined below:

Theorem 3 *Let consider the control system (SI_2) with σ an (S)-function, and the feedback $k_F(x)$ given by*

$$k_F(x) \stackrel{def}{=} -\Big(x_1 + x_2 + F(x_2)\Big), \tag{19}$$

where F an $\mathcal{F}$ function. Then for $p \in [1, \infty]$, k_F is a feedback stabilizer for (SI_2) and is of linear profile for (x, p, p).

For instance $-(x_1 + x_2 + 3x_2|x_2|)$ and $-(x_1 + x_2 + x_2^3)$ are examples of k_F-feedbacks. The function $\bar{F} : I\!R \to I\!R$ defined by $\bar{F}(x_2) \stackrel{def}{=} x_2 + F(x_2)$ is a C^1-diffeomorphism and let $K : I\!R \to I\!R$ be defined by $K(y) \stackrel{def}{=} \bar{F}^{-1}(y)$. In order to state the results for the perturbed system $(PSI_2)_{k_F}$, we need to consider a Lyapunov function V_F defined in (97) below for which there exist $a, b > 0$ such that for all $(x_1, x_2) \in I\!R^2$ we have

$$a\Big(x_1^2 + \bar{F}(x_2)^2\Big) \leq V_F(x_1, x_2) \leq b\Big(x_1^2 + \bar{F}(x_2)^2\Big). \tag{20}$$

We get the following theorem (compare to theorem 2 of [4])

Theorem 4 *Let consider $(PSI_2)_{k_F}$. There exists $\varepsilon_0 > 0$ such that for $p \in [1, \infty]$, $u, v \in L^p$ and $\|v\|_\infty \leq \varepsilon_0$, $\bar{x} \in I\!R^2$, if (x_1, x_2) is the trajectory of $(PSI_2)_{k_F}$ corresponding to (u, v) starting at $\bar{x}$, the following estimate holds*

$$\|x\|_p \leq const.\Big(\|v\|_p + \|u\|_p + L_p(V_F(\bar{x})^{1/2})\Big), \tag{21}$$

where

$$L_p(X) \stackrel{def}{=} \Big(\int_0^X \frac{s^p}{K(s)} ds\Big)^{1/p} \text{ for } p \text{ finite and } L_\infty(X) \stackrel{def}{=} X. \tag{22}$$

3 The linear feedback k_L

We rewrite $(SI_2)_{k_L}$ as follows

$$\begin{cases} \dot{z} &= y - \sigma(z + u), \\ \dot{y} &= -\sigma(z + u), \end{cases}$$

where $z = x_1 + x_2$, $y = x_2$, $\sigma(\cdot)$ ($u(\cdot)$ respectively) stands for $-\sigma(-\cdot)$ ($-u(-\cdot)$ respectively). Moreover, up to a time reparameterization and a linear change of variables, we can assume that $\sigma'(0) = 1$. Let us define $\sigma_+ = \lim_{+\infty} \sigma$, $\sigma_- = -\lim_{-\infty} \sigma$ and for $r > 0$

$$K_r \stackrel{def}{=} \inf\{|\sigma(z)|,\ |z| > r\} \text{ and } \delta_r \stackrel{def}{=} \sup_{|z|,|z'|>\frac{2}{3}r, zz'>0} |\sigma(z) - \sigma(z')|.$$

Let K be an S-bound for σ and $\sigma_m \stackrel{def}{=} \min(\sigma_+, \sigma_-)$, $\sigma_M \stackrel{def}{=} \max(\sigma_+, \sigma_-)$. Then we have

$$K \geq \max(1, \sigma_M, \|\sigma'\|_\infty).$$

Fix $C, \rho > 0$ and ρ_p for p finite such that

$$\begin{array}{rcl} \rho & << & 1 << C, \\ \sigma(\varepsilon C) & > & \frac{3}{4}\sigma_\varepsilon \text{ if } \varepsilon = \pm, \\ 2KC\rho_p & < & \frac{1}{2}\min(1, \frac{K_\rho^p}{2}), \\ \rho_p & < & \frac{\rho}{2}, \\ \delta_C & << & \min(1, K_C, \frac{3}{16}\sigma_m,). \end{array}$$

In addition, let $\theta \stackrel{def}{=} \frac{C}{2(C+K)} \in (0, 1/2)$. The Lyapunov function defined in (9) is written now

$$V(z, y) = y^2 + \int_0^{z-y} \sigma(s)ds + \int_0^z \sigma(s)ds. \tag{23}$$

Set for $r > 0$ $V_r \stackrel{def}{=} \max\{V(z, y),\ \|(z, y)\| \leq r\}$. We use $\dot{V}$ to denote the time derivative of V along trajectories of $(SI_2)_{k_L}$ and we have

$$\dot{V} = -2y\Delta\sigma\Big|_z^{z+u} + y\Delta\sigma\Big|_z^{z-y} - \sigma(z)\sigma(z+u). \tag{24}$$

We sometimes consider a classical Lyapunov function for (SI_2) given by

$$W(z, y) \stackrel{def}{=} \frac{y^2}{2} + \int_0^z \sigma(s)ds. \tag{25}$$

Remark 2 *It is easy to see that V cannot give rise to a dissipation inequality of the kind (7) when $yz < 0$, $z(z+u) \leq 0$ and $|y| >> \max(|z|, |u|) >> 1$ and then $\dot{V} \geq \frac{K}{2}|y|$. However in this case $|\dot{z}| \geq |y| - K$ which implies that the previous situation firstly cannot last more than a time interval of length 1 and secondly is preceded and followed by time intervals of length of magnitude $|y|$, where $|z| \geq |y|$. Therefore, by integrating (24) on appropriate time intervals, one hopes to achieve the desired results.*

3.1 Proof of Theorem 1

We need the next proposition

Proposition 1 *Let $p \in [1, \infty)$, $T > 0$ and $u \in L^p$. Then we have*

(a) $\|y\|^2_{\infty,[0,T]} \leq const.\|z\|_{\infty,[0,T]}$;

(b) *either* $\|y\|^2_{\infty,[0,T]} \geq \frac{\sigma_m}{2}\|z\|_{\infty,[0,T]}$ *or*

$$\|z\|_{\infty,[0,T]} \leq const.\min(\|u\|^2_{\infty,[0,T]}, \|u\|^{\frac{2p}{2p+1}}_{p,[0,T]}).$$

Proof of Proposition 1: We use y_{max} and z_{max} to denote $\|y\|_{\infty,[0,T]}$ and $\|z\|_{\infty,[0,T]}$ respectively. For (a), we pick $[t_0,t_1] \in [0,T]$ such that we have for $t \in [t_0,t_1]$

$$y(t_0) = \frac{y_{max}}{2} \leq y(t) \leq y(t_1) = y_{max}.$$

Let $L \overset{def}{=} t_1 - t_0$. Then $\frac{y_{max}}{2} = \Delta y\Big|_{t_0}^{t_1} = -\int_{t_0}^{t_1} \sigma(z+u) \leq KL$. In addition, we have

$$2z_{max} \geq \Delta z\Big|_{t_0}^{t_1} = \int_{t_0}^{t_1} y + \Delta y\Big|_{t_0}^{t_1} \geq \frac{y_{max}}{2}(L+1).$$

Then the two previous equations imply that $z_{max} \geq \frac{y^2_{max}}{8K}$.

As for item (b), we pick $[t_0,t_1] \in [0,T]$ such that we have for $t \in [t_0,t_1]$

$$W(t_0) = \frac{W_{max}}{2} \leq W(t) \leq W(t_1) = W_{max},$$

where W is given by (25) and $W_{max} = \|W\|_{\infty,[0,T]}$. Note that

$$\dot{W} = -y\Delta\sigma\Big|_z^{z+u} - \sigma(z)\sigma(z+u) \text{ and } \sigma_m \leq \frac{W_{max}}{z_{max}} \leq const.. \tag{26}$$

Consider the two following subsets of $[t_0,t_1]$ given by

$$E_1 = \{t \in [t_0,t_1],\ |u(t)| < \frac{|z(t)|}{3}\},\ E_2 = \{t \in [t_0,t_1],\ |u(t)| \geq \frac{|z(t)|}{3}\}. \tag{27}$$

We may assume that for every $t \in [t_0,t_1]$, $|y(t)| < \sqrt{\frac{W_{max}}{2}}$ and $W_{max} \geq C$. This clearly implies that for every $t \in [t_0,t_1]$

$$|z(t)| \geq \frac{W_{max}}{4\sigma_M},$$

and then z has a constant sign. Writing (26) as

$$\dot{W} = -\dot{z}\Delta\sigma\Big|_z^{z+u} - \sigma^2(z+u), \tag{28}$$

and integrating it, we get

$$\sigma_m^2(L-|E_2|) + \frac{W_{max}}{2} \leq (L-|E_2|)\delta_C(\sqrt{\frac{W_{max}}{2}}+K) + (\sqrt{\frac{W_{max}}{2}}+K)2K|E_2|,$$

which can be written as

$$\sigma_m^2 L + \frac{W_{max}}{2} \leq L\delta_C\sqrt{2W_{max}} + 3K|E_2|(\sqrt{\frac{W_{max}}{2}}+K). \tag{29}$$

On the other hand, we have

$$\Delta y\Big|_{t_0}^{t_1} = -\int_{t_0}^{t_1}\sigma(z) + \int_{t_0}^{t_1}\Delta\sigma\Big|_z^{z+u} \tag{30}$$

and then

$$\sigma_m L \leq 2y_{max} + (L - |E_2|)\delta_C + 2K|E_2|. \tag{31}$$

We thus obtain

$$L \leq \frac{4}{\sigma_m}\sqrt{\frac{W_{max}}{2}} + \frac{6K}{\sigma_m}|E_2| \tag{32}$$

Equations (29) and (32) give $\sqrt{W_{max}} \leq const.|E_2|$ which in turn implies that

$$\|u\|^p_{p,[0,T]} \geq \int_{E_2} |u|^p \geq W_{max}^{p+\frac{1}{2}}.$$

The proof of Proposition 1 is finished. ■

We now prove (4) for $p = \infty$. Taking into account the previous proposition, it is enough to show that

Proposition 2 *For $u \in L^\infty$ and $\|u\|_\infty \geq C$ we have*

$$\|z\|_\infty \leq const.\|u\|^2_\infty.$$

Proof of Proposition 2: let $M = \|u\|_\infty \geq C$. Let $T > 0$ such that we can pick $[t_0, t_1] \in [0, T]$ for which we have for $t \in [t_0, t_1]$

$$W(t_0) = \frac{\lambda}{2}M^2 \leq W(t) \leq W(t_1) = \lambda M^2,$$

with $\lambda > 2$. Thanks to Proposition 1, we can suppose that

$$const.\lambda M^2 \leq y^2_{max} \leq const.z_{max} \leq const.\lambda M^2, \tag{33}$$

where y_{max}, z_{max} are defined as before. We first show that

Lemma 3 *With the previous notations, there exists $[t_2, t_3] \in [t_0, t_1]$ such that $|z(t_2)| = |z(t_3)| = M+C$ and $\Delta W\Big|_{t_2}^{t_3} \geq \frac{\lambda}{8}M^2$.*

Proof of Lemma 3: let assume that $|z(t_0)| > M + C$ and $|z(t_1)| > M + C$. In addition suppose that

$$\text{for all } t \in [t_0, t_1],\ |z(t)| \geq M + C. \tag{34}$$

Then $|\dot{y}| \geq \frac{3}{4}\sigma_m$ which implies that $L = t_1 - t_0 \leq \frac{y_{max}}{2\sigma_m}$. Then by integrating (28) we get

$$\frac{\lambda}{2}M^2 \leq \delta_C \frac{\lambda M^2}{8\sigma_m} - \sigma^2_m L,$$

which is impossible. Then (34) cannot hold. Therefore, let $t_2 \in [t_0, t_1]$ such that $|z(t_2)| = M + C$ and $|z| \geq M + C$ on $[t_0, t_2]$. Then $\Delta W\Big|_{t_2}^{t_1} \geq \frac{\lambda}{4}M^2$. Then by reproducing the previous argument on $[t_2, t_1]$, we can find $t_3 \in [t_2, t_1]$ such that $|z(t_3)| = M + C$ and $\Delta W\Big|_{t_2}^{t_3} \geq \frac{\lambda}{8}M^2$.

We assume now that $|z(t_0)| < M + C$ and $|z(t_1)| < M + C$. In addition suppose that

$$\text{for all } t \in [t_0, t_1],\ |z(t)| \geq M + C. \tag{35}$$

Then for all $t \in [t_0, t_1]$ we have $|y(t)| \geq \frac{\sqrt{\lambda}}{4}M$. Hence, $2(M+C) \geq |\Delta z\Big|_{t_0}^{t_1}| \geq L\frac{\sqrt{\lambda}}{8}M$. On the other hand, we also have $\frac{\lambda}{2}M^2 = \Delta W\Big|_{t_0}^{t_1} \leq const.\sqrt{\lambda}ML$. We finally get that M is bounded by a constant independent of u, which is impossible. Therefore, (35) cannot hold and there exists $t_0' \in [t_0, t_1]$ such that $|z(t_0')| > M + C$. Then if t_2 is the first time after t_0 such that $|z(t_2)| = M + C$ and t_3 is the first time before t_1 such that $|z(t_3)| = M + C$, we have that $\Delta W\Big|_{t_2}^{t_3} \geq \frac{\lambda}{8}M^2$.

All the other cases come down to those studied above and Lemma 3 is thus proved. ∎

Note that at $t \in [t_0, t_1]$ such that $|z(t)| = M + C$ we have $|\dot{z}(t)| \geq \frac{\sqrt{\lambda}}{4}M$. We then have on $[t_2, t_3]$ a sequence (t^l) of zeros of $|z| - (M + C)$ with $t^1 = t_2$, $t^{2N+1} = t_3$. For $l = 1, \cdots, N$, we can assume that $|z| \geq M + C$ on $I_{2l-1} = [t^{2l-1}, t^{2l}]$ and $|z| \leq M + C$ on $I_{2l} = [t^{2l}, t^{2l+1}]$. For $j = 1, \cdots, 2N$ define $\Delta W_j \stackrel{def}{=} \Delta W\Big|_{t_j}^{t_{j+1}}$. We show that

Lemma 4 *With the previous notations, we have for* $1, \cdots, N$,

$$\Delta W_{2l-1} + \Delta W_{2l} \leq -const.\sqrt{\lambda}M + const.M. \tag{36}$$

Proof of Lemma 4: On I_{2l-1}, $|\dot{y}| \geq \frac{3}{4}\sigma_m$ and therefore there exist $t^i < t^f \in I_{2l-1}$ such that $|y| > \sigma_M$ on $[t^{2l-1}, t^i[$ and $]t^f, t^{2l-1}]$ and $|y| \leq \sigma_M$ on $[t^i, t^f]$. We get $|\dot{z}| > 0$ on $[t^{2l-1}, t^i[$ and $]t^f, t^{2l-1}]$. We may assume that $z > 0$ on I_{2l-1} and then

$$\begin{aligned}
\Delta W\Big|_{t^{2l-1}}^{t^i} &= \int_{t^{2l-1}}^{t^i} -\Big(\dot{z}\Delta\sigma\Big|_z^{z+u} + \sigma^2(z+u)\Big)dt \\
&= -\int_{z(t^{2l-1})}^{z(t^i)} \Delta\sigma\Big|_z^{z+u} dz - \int_{t^{2l-1}}^{t^i} \sigma^2(z+u)dt \\
&\leq \int_{z(t^{2l-1})}^{z(t^i)} \Delta\sigma\Big|_z^{z+M} dz - \int_{t^{2l-1}}^{t^i} \sigma^2(z-M)dt \\
&\leq \iint_{\substack{M+C\leq z\leq z(t^i)\\ z\leq s\leq z+M}} \sigma'(s)ds\, dz - \frac{\sigma_m^2}{2}(t^i - t^{2l-1}).
\end{aligned}$$

We use J to denote the double integral in the last equation above. By Fubini's theorem, we can permute the order of integration in J and we obtain

$$J \leq M \iint_{\substack{s\leq z\leq s+M\\ C\leq s\leq const.\lambda M^2}} \sigma'(s)ds\, dz \leq \int_C^{const.\lambda M^2} \sigma'(s) \int_s^{s+M} dzds \leq M\delta_C.$$

Then we get $\Delta W\Big|_{t^{2l-1}}^{t^i} \leq M\delta_C - \frac{\sigma_m^2}{2}(t^i - t^{2l-1})$. In addition, we have

$$\sigma_M(t^i - t^{2l-1}) \geq |\Delta y\Big|_{t^{2l-1}}^{t^i}| \geq const.\sqrt{\lambda}M,$$

and then

$$\Delta W\Big|_{t^{2l-1}}^{t^i} \leq M\delta_C - const.\sqrt{\lambda}M \leq -const.\sqrt{\lambda}M.$$

Moreover, since $\sigma_M(t^f - t^i) \geq |\Delta y\Big|_{t^i}^{t^f}| = 2\sigma_M$, we have

$$\Delta W\Big|_{t^i}^{t^f} \leq \sigma_M\delta_C - (\frac{3\sigma_m}{4})^2(t^f - t^i) \leq 0.$$

Then we obtain $\Delta W_{2l-1} \leq -const.\sqrt{\lambda}M$. On I_{2l}, $|y| \geq const.\sqrt{\lambda}M$ and then

$$\Delta W_{2l} \leq const.\sqrt{\lambda}M(t^{2l+1} - t^{2l}) \leq const.|\int_{I_{2l}} y| \leq const.|\Delta z\Big|_{t^{2l}}^{t^{2l+1}}| \leq const.M.$$

Then adding the appropriate inequalities, we get the conclusion of Lemma 4. ■

Finally, by adding all the inequalities for $l = 1, \cdots, N$, we obtain that

$$\frac{\lambda}{8}M^2 \leq \Delta W\Big|_{t_2}^{t_3} \leq NM(-const.\sqrt{\lambda} + const.).$$

Then λ has to be bounded above by some *const.* and the proof of Proposition 2 is finished. ■

We now consider (3) of Theorem 1 in the case where $p = q < \infty$. The first step consists of showing the next proposition

Proposition 3 *Let $p \in [1, \infty)$, $u \in L^p$ and $x_u = (z, y)$ the trajectory of $(SI_2)_{k_L}$ corresponding to u and starting at 0. Then we have*

$$\|\sigma(z + u)\|_p \leq const.\|u\|_p. \tag{37}$$

Proof of Proposition 3: Equation (37) will be obtained from the integration of (24) on two by two disjoint measurable sets E_i whose union is $I\!R^+$. We will distinguish several cases such that every E_i falls in one of these cases and for each case we look for the inequality

$$\int_{E_i} (\dot{V} + const.|\sigma(z + u)|^p \leq const. \int_{E_i} |u|^p, \tag{38}$$

either by the corresponding differential inequality $\dot{V} + const.|\sigma(z + u)|^p \leq const.|u|^p$ or by a direct computation. For the rest of the proof, we assume that $\|u\|_p \geq C$. We will consider the following cases:

(A) $|y| \leq C$; we have two sub-cases defined by

- $(A1)$ $|z| \geq \rho$:
- $(A2)$ $|z| < \rho$;

(B) $|y| > C$.

If $(A1)$ holds, we have clearly have

$$\dot{V} + const.|\sigma(z + u)|^p \leq const.|u|^p, \tag{39}$$

whether $|u| \geq \rho_p$ or not.

If $(A2)$ holds, we consider the set $S_0 = \{t \geq 0 \mid |y(t)| \leq C,\ V(z(t), y(t)) < 4\rho_p^2\}$. Then S_0 is open and is included in $S_1 = \{t \geq 0 \mid |y(t)| \leq C,\ |z(t)| < \rho\}$. On S_0, we have

$$\dot{V} \leq const.|u|\sqrt{V} - const.V. \tag{40}$$

Set $W_p \overset{def}{=} V^{p/2}$. Then using (40) we get

$$\dot{W}_p \leq const.W_p^{\frac{p-1}{p}} - const.W_p. \tag{41}$$

On each finite interval (t,t') of S_0, we have $W_p(t) = W_p(t')$ which implies that $\Delta V\Big|_t^{t'} = 0$. By integrating (41) (and applying Holder inequality fro $p>1$) we get

$$\Delta V\Big|_t^{t'} + const. \int_t^{t'} |\sigma(z+u)|^p \leq const. \int_t^{t'} |u|^p. \tag{42}$$

If now $(t,\infty) \subset S_0$, we have

$$\int_t^\infty W_p^p \leq W_p^p(t) + const. \int_t^\infty |u|^p,$$

which leads to (42) with $t'=\infty$. On S_1/S_0, we clearly have (39) since the dominant term in $\dot V$ is $\frac{1}{2}y\sigma(-y)$ if $|u| \leq \rho$.

We assume that (B) $|y|>C$ holds. Since $u \in L^p$, the open set $S_2 = \{t \geq 0| \, |y(t)|>C\}$ is a union (at most countable) of intervals of finite length. To see that, consider $I=(t_0,t_1)$ $(t_1 \leq \infty)$ such that $y>C$ on I. Then z is increasing and $z \geq C$ for some $t \geq t'$. By considering (27) and (30), equation (31) becomes

$$\Delta y\Big|_{t'}^{t} \leq -\frac{3}{4}\sigma_m(t-t') + (t-t'-|E_2|)\delta_C + 2K|E_2|. \tag{43}$$

As a consequence of Tchebysheff inequality, $|E_2| \leq \|u\|_p^p$. Then y becomes eventually negative if $t-t'$ is large.

We now focus on an interval $I=[t_0,t_1]$ where $y(t_0)=y(t_1)=C$ and $y>C$ on (t_0,t_1). We will prove (38) for $E_i = I$. Let $T \stackrel{def}{=} t_1 - t_0$. We assume that $T \geq \frac{10C}{\sigma_m}$ otherwise we are back to case (A). We say that condition (R) is satisfied at $t \in I$ if $y(t)+C \leq |z(t)|$. Moreover z is strictly increasing from I to $[z_0,z_1]$ where $z_0=z(t_0)$ and $z_1=z(t_1)$. Then z is equal to 0 at most once. We have the following situation

Lemma 5 *With the notations above, one of the two next cases occurs:*

$(B1)$ *z has a constant sign on I and (R) is satisfied on I except (maybe) on an interval (t_0',t_1') such that either $t_1'=t_1$ and there is equality in (R) at t_0' if $z \leq 0$ or either $t_0'=t_0$ and there is equality in (R) at t_1' if $z \geq 0$;*

$(B2)$ *z has a unique zero at $\bar t$ and (R) is satisfied on I except on an interval $I'=(t_0',t_1')$ such that $\bar t \in I'$ and there is equality in (R) at t_0' and t_1'.*

Proof of Lemma 5: Consider the function f defined on I by

$$f(t) \stackrel{def}{=} |z(t)| - y(t) - C = \text{sgn } z(t)z(t) - y(t) - C.$$

For all $t \in I$ except maybe for at most one point we have

$$(\text{sgn } z)\dot f = y(t) - (\text{sgn } z(t)+1)\sigma(z(t)+u(t)) > 0.$$

Depending on the sign of z, $f(t_0)$ and $f(t_1)$, we easily get the conclusion. ∎

In each above case, we will establish (38) first on $E_i = I'$ and second on $E_i = I/I'$. We also need to rewrite (24) as

$$\dot V = -2\dot z\Delta\sigma\Big|_{z-\theta y}^{z+u} - \dot z\Delta\sigma\Big|_{z}^{z-\theta y} - \dot z\Delta\sigma\Big|_{z-y}^{z-\theta y} - \sigma(z+u)\Delta\sigma\Big|_{z-y}^{z+u} - \sigma^2(z+u). \tag{44}$$

We get

$$\Delta V\Big|_{t_0}^{t_1} = J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 &= -\int_{t_0}^{t_1} \dot{z}\Delta\sigma\Big|_{z}^{z-\theta y} - \int_{t_0}^{t_1} \dot{z}\Delta\sigma\Big|_{z-y}^{z-\theta y}, & (45)\\ J_2 &= -2\int_{t_0}^{t_1} \Big(\dot{z}\Delta\sigma\Big|_{z-\theta y}^{z+u} + \sigma(z-y)\Delta\sigma\Big|_{z-y}^{z+u}\Big), & (46)\\ J_3 &= -\int_{t_0}^{t_1} \Big([\Delta\sigma\Big|_{z-y}^{z+u}]^2 + \sigma^2(z+u)\Big) \leq 0. & (47) \end{aligned}$$

We now suppose that $(B1)$ holds. We will only treat the case when $z \geq 0$ since the other one is entirely similar. Then we assume $I' = [t_0, t_1')$. Let $T' = t_1' - t_0$. We have

$$T' \leq \frac{2}{1-\frac{2K}{C}},\ |\Delta V\Big|_{I'}|,\ \Delta z\Big|_{I'} \leq CT' \text{ and } \Delta y\Big|_{I'} \leq T',$$

by simply taking into account that $\dot{z} \geq C - K$ and $\dot{y} \geq K$. Let us consider the set E given by

$$E = \{t \in I',\ |u(t)| \geq \frac{z(t)}{3}\}. \tag{48}$$

Then we have

Lemma 6 *With the previous notations, $|E| \geq const.T'$ with const. independent of C.*

Proof of Lemma 6: From (30) written on I' and the fact that $\Delta y\Big|_{I'} \geq 0$, we get

$$0 \leq -\sigma(\frac{C}{3})(T' - |E| - |F|) + 2K|E|,$$

where $F = \{t \in I',\ z \leq 1\}$. We clearly have $|F| \leq \frac{\frac{C}{3}}{C+\frac{2}{1-\frac{2K}{C}}} \leq \frac{1}{3}$ and if F is not empty, we have

$$T' - |F| \geq \frac{z(t_1') - \frac{C}{3}}{C + \frac{2}{1-\frac{2K}{C}}} \geq \frac{5}{4}. \tag{49}$$

We conclude easily. ∎

In the case where F is not empty, E contains a subset E' of E of length $\frac{|E|}{2} \geq const.T'$ such that $z \geq const.C$ on E'. Then $|u|$ will be larger than $const.C$ on E' of E and we have

$$\int_{I'} |u|^p \geq \int_{E'} |u|^p \geq const.C^pT' \geq |\Delta V\Big|_{I'}| + const. \int_{I'} |\sigma(z+u)|^p.$$

In the case where F is empty, we simply take $E' = E$ and now

$$\int_{I'} |u|^p \geq \int_{E} z \geq const.\frac{C}{2}T' \geq const.C(|\Delta V\Big|_{I'}| + const. \int_{I'} |\sigma(z+u)|^p).$$

Then (38) is obtained.

We next establish (38) on $I/I' = [t_1', t_1]$. This is a consequence of the following lemma:

Lemma 7 *Let consider J_1, J_2 and J_3 be defined in (45), (46) and (47) respectively. Then we have*

(J1) $J_1 \leq const. \sup(y(t_1), C)\delta_C \leq const.\delta_C T$*;*

(J2) $J_2 \leq const. \int_{I/I'} |u|^p$*;*

(J3) $J_3 \leq -const.T + const. \int_{I/I'} |u|^p$*;*

where const. is independent of C.

Proof of Lemma 7: We start with $(J2)$. Notice that if $u + y \geq 0$ then $\Delta\sigma\Big|_{z-\theta y}^{z+u} \geq 0$ and $\Delta\sigma\Big|_{z-y}^{z+u} \geq 0$ since σ is increasing. Now if $u + y < 0$ then $|u| > y$ in which case we have

$$|\dot{z}\Delta\sigma\Big|_{z-\theta y}^{z+u} + \sigma(z-y)\Delta\sigma\Big|_{z-y}^{z+u}| \leq const.|u|^p.$$

Then $(J2)$ is proved. For $(J3)$, we have

$$J_3 \leq -\frac{\sigma_m^2}{4}|I/I'| + K^2|E'|,$$

where $E' = \{t \in I/I',\ |u(t)| \geq \frac{z(t)}{3}\}$. Since $z \geq 1$ on I/I', we get $(J3)$.

It remains to show $(J1)$. Set $z_1' = z(t_1')$, $z_1 = z(t_1)$. Since $\dot{z} \geq C$ on I/I', we rewrite (45) as

$$\begin{aligned} J_1 &= -\int_{z_1'}^{z_1} \Delta\sigma\Big|_{z}^{z-\theta y} dz - \int_{z_1'}^{z_1} \Delta\sigma\Big|_{z-y}^{z-\theta y} dz, \\ &= \iint_{\substack{z_1' \leq z \leq z_1 \\ z-\theta y(z) \leq s \leq z}} \sigma'(s)ds\ dz - \iint_{\substack{z_1' \leq z \leq z_1 \\ z-y(z) \leq s \leq z-\theta y(z)}} \sigma'(s)ds\ dz. \end{aligned} \tag{50}$$

Note that $z - \theta y$ and $z - y$ as functions of $z \in [z_0 - y(z_0), z_1]$ verify

$$\frac{1}{2} \leq \frac{d(z-\theta y)}{dz},\ \frac{d(z-y)}{dz} \leq 2.$$

Therefore, we can define for $s \in [z_0 - y(z_0), z_1]$ two functions $Z(s)$ and $\bar{Z}(s)$ as follows

$$\begin{aligned} Z(s) = z_0 &\quad \text{for} \quad s \in [z_0 - y(z_0), z_0], \\ Z(s) - \theta y(Z(s)) = s &\quad \text{for} \quad s \in [z_0, z_1 - \theta y(z_1)], \\ Z(s) = z_1 &\quad \text{for} \quad s \in [z_1 - \theta y(z_1), z_1]. \end{aligned}$$

and

$$\begin{aligned} \bar{Z}(s) - y(\bar{Z}(s)) = s &\quad \text{for} \quad s \in [z_0 - y(z_0), z_1 - y(z_1)], \\ \bar{Z}(s) = z_1 &\quad \text{for} \quad s \in [z_1 - y(z_1), z_1]. \end{aligned}$$

By Fubini's theorem, we can permute the order of integration in J and we obtain

$$J_1 = J_1' + J_2' + J_3' + J_4',$$

where

$$
\begin{aligned}
J_1' &= \int_{z_0}^{z_1-y(z_1)} \Big(2Z(s)-s-\bar Z(s)\Big)\sigma'(s)ds,\\
J_2' &= \int_{z_0-y(z_0)}^{z_0-\theta y(z_0)} \Big(Z(s)-\bar Z(s)\Big)\sigma'(s)ds,\\
J_3' &= \int_{z_0-\theta y(z_0)}^{z_0} \Big(2Z(s)-z_0-\bar Z(s)\Big)\sigma'(s)ds,\\
J_4' &= \int_{z_1-y(z_1)}^{z_1} \Big(Z(s)-s\Big)\sigma'(s)ds.
\end{aligned}
$$

Since $Z(s)\leq \bar Z(s)$ and $Z(s)\leq s$ on $s\in[z_0-y(z_0),z_0]$, we have $J_2'\leq 0$ and $J_3'\leq 0$. In addition we have

$$
J_4' = \int_{z_1-y(z_1)}^{z_1} \Big(\sigma(z)-\sigma(z-\theta y)\Big)dz \leq 2\sup(y(t_1),C)\delta_C, \tag{51}
$$

since $z>z-\theta y\geq 2C$ for $s\in[z_1-y(z_1),z_1]$. As for J_1', we can write it as follows

$$
J_1' = \int_{z_0}^{z_1-y(z_1)} \Big(2\theta y(Z(s))-y(\bar Z(s))\Big)\sigma'(s)ds.
$$

Then we have $J_1'\leq 0$ as a consequence of the next lemma

Lemma 8 *For $s\in[z_0,z_1-y(z_1)]$, we have $2\theta y(Z(s))-y(\bar Z(s))\leq 0$.*

Proof of Lemma 8: We have

$$
2\theta y(Z(s))-y(\bar Z(s)) = 2\theta\Delta y\Big|_{Z(s)}^{\bar Z(s)} - (1-2\theta)y(\bar Z(s) \leq -(1-2\theta)C + 2\theta\Delta y\Big|_{Z(s)}^{\bar Z(s)}.
$$

On the other hand, we have

$$
|\Delta y^2\Big|_{Z(s)}^{\bar Z(s)}| \leq 2|\int_{Z(s)}^{\bar Z(s)} \frac{ydy}{dz}dz| \leq 2\int_{Z(s)}^{\bar Z(s)} \frac{|y\sigma(z+u)|}{y-\sigma(z+u)}dz \leq \frac{2K}{1-\frac{K}{C}}|\Delta z\Big|_{Z(s)}^{\bar Z(s)}| \leq \frac{2CK}{C-K}y(\bar Z(s)).
$$

We thus obtain

$$
|\Delta y\Big|_{Z(s)}^{\bar Z(s)}| \leq \frac{2CK}{C-K}\frac{y(\bar Z(s))}{y(Z(s))+y(\bar Z(s))} \leq \frac{2CK}{C-K}.
$$

Then $2\theta y(Z(s))-y(\bar Z(s))\leq -(1-2\theta)C+\frac{2CK}{C-K}=0$. ■

Then Lemma 7 is proved, (38) is obtained and case $(B1)$ is ended.

We now suppose that case $(B2)$ holds. It is enough to establish (38) on the interval $[\bar t,t_1]$ where z remains positive. The interval $[\bar t,t_1]$ is the union of $I'=[\bar t,t_1')$ and $[t_1',t_1]$. It is easy to see that the procedure that lead to (38) on I/I' above (the inequalities (45), (46) and (47) together with Lemmas 7 and 8 will also prove (38) on $[t_1',t_1]$. Therefore, the last part of the proof of Proposition 3 consists of getting (38) on I'.

We can suppose that $y(\bar t)\geq 8C$ as a consequence of the fact that $t_1-\bar t$ can be taken larger than $\frac{10C}{\sigma_m}$. Indeed, if $y(\bar t)\leq 8C$, we reproduce the argument involving the set E defined in (49) and Lemma 6 with $[\bar t,t_1]$ now instead of I' and obtain (38). Then for the rest of the section, we assume that $y(\bar t)\geq 8C$.

We have that $z(\bar t)=0$ and $z(t_1')=y(t_1')+C$. Let $T''\overset{def}{=}t_1'-\bar t$ and $\bar y\overset{def}{=}\frac{\int_{\bar t}^{t_1'}y}{T''}$. Since

$$
-K\leq \dot y\leq K \text{ and } T''C\leq T''\bar y = \int_{\bar t}^{t_1'} y = C+y(\bar t),
$$

we get $T' \leq 1 + \frac{y(\bar{t})}{C}$ and $y(\bar{t}) - KT'' \leq \bar{y} \leq y(\bar{t}) + KT''$. Then we get

$$y(\bar{t})(1 - \frac{2K}{C}) \leq \bar{y} \leq y(\bar{t})(1 + \frac{2K}{C}).$$

We finally obtain the following estimate for T''

$$\frac{6}{7} \leq 1 - \frac{2K}{C} \leq \frac{1 + \frac{C}{y(\bar{t})}}{1 + \frac{2K}{C}} \leq T'' \leq \frac{1 + \frac{C}{y(\bar{t})}}{1 - \frac{2K}{C}} \leq \frac{9}{8}(1 + \frac{4K}{C}). \tag{52}$$

Define $t_0'' \in [\bar{t}, t_1']$ such that $z(t_0'') = y(t_0'')$. We have

$$t_1' - t_0'' \leq \frac{\Delta(z-y)\Big|_{t_0''}^{t_1'}}{y(\bar{t}) - KT''} \leq \frac{1}{7},$$

Then $T_1 \stackrel{def}{=} t_0'' - \bar{t} \geq 1 - \frac{2K}{C} - \frac{1}{7} \geq \frac{5}{7}$. Using (24), we have

$$\Delta V\Big|_{\bar{t}}^{t_1'} + \int_{\bar{t}}^{t_1'} |\sigma(z+u)|^p \leq J + J' - \int_{\bar{t}}^{t_1'} \sigma(z)\sigma(z+u), \tag{53}$$

where $J \stackrel{def}{=} -2\int_{\bar{t}}^{t_1'} y\Big(\sigma(z+u) - \sigma(z)\Big)$ and $J' \stackrel{def}{=} \int_{\bar{t}}^{t_0''} y\Big(\sigma(z-y) - \sigma(z)\Big)$. We first have

$$-\int_{\bar{t}}^{t_1'} \sigma(z)\sigma(z+u) \leq 2K^2,$$

and

$$J' \leq -\int_{\bar{t}}^{t_0''} y\sigma(z) \leq -\frac{\sigma(1)}{2} T_1 y(\bar{t}).$$

Suppose that $|J| \leq |J'|$. Then (38) follows readily. If $|J| > |J'|$, by taking into account the next equation,

$$|J| \leq 3(T'' - |E| - |F|)y(\bar{t})\delta_C + 5Ky(\bar{t})|E|,$$

with $E = \{t \in [\bar{t}, t_1'],\ |u| > \frac{z}{3}\}$ and $F = \{t \in [\bar{t}, t_1'],\ |z| \leq 1\}$, we obtain as in Lemma 6

$$|E| \geq const.T'',$$

which implies that (38) holds on $[\bar{t}, t_1']$. Then case (B) is finished which ends the proof of Proposition 3. ■

The second step of (3) of Theorem 1 in the case where $p = q < \infty$ consists of providing an appropriate sequence of inputs $u_n \in L^p$ satisfying the conditions of Lemma 2. This is done in the appendix. We now prove (1) and (2) of Theorem 1 in the case where $p = q < \infty$. The first step consists in the next proposition

Proposition 4 *Let $p \in [1, \infty)$, $u \in L^p$ and $x_u = (z, y)$ the trajectory of $(SI_2)_{k_L}$ corresponding to u and starting at 0. Then there exists const. > 0 such that for $\|u\|_p \geq$ const. we have*

$$\|z\|_p \leq const.\|u\|_p^2, \tag{54}$$

$$\|y\|_p \leq const.\|u\|_p^{\frac{p+2}{p+1}}, \tag{55}$$

$$\|y\|_\infty^2 \leq const.\|z\|_\infty \leq const.\|u\|_p^{\frac{2p}{p+1}}. \tag{56}$$

Before starting the proof of the proposition above, we give three lemmas.

Lemma 9 *Let $p \in [1, \infty)$. If $\gamma_p > 0$ is the L^p-gain of the linear system given by $\dot{x} = Ax + bu$ with*

$$A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } b = (1\ 1)^T,$$

then for every $u \in L^p$ with $x_u = (z, y)$ the trajectory of $(SI_2)_{k_L}$ corresponding to u and starting at 0, we have

$$\|x_u\|_p \leq \gamma_p \|\bar{u}\|_p, \tag{57}$$

where $\bar{u} = z - \sigma(z + u)$.

Proof of Lemma 9: This is simply a consequence of the rewriting of (SI_2) as

$$\dot{x}_u = Ax_u + b\bar{u}.$$

Note that for all $t \in \mathbb{R}^+$,

$$|\bar{u}(t)| \leq |u(t)| + K(z(t) + u(t))\sigma(z(t) + u(t)) \leq const.|u(t)| + Kz(t)\sigma\Big(z(t) + u(t)\Big).$$

■

Lemma 10 *Let $p \in [1, \infty)$, $u \in L^p$ and $x_u = (z, y)$ the trajectory of $(SI_2)_{k_L}$ corresponding to u and starting at 0. Then we have*

$$\|y^2\|_p \leq const.\|z\sigma(z + u)\|_p. \tag{58}$$

Consider the function

$$Q(z, y) \stackrel{def}{=} -zy|y|^{2p-2} + \frac{|y|^{2p}}{p}.$$

If we use $\dot{Q}$ to denote the time derivative of Q along x_u, we get

$$\dot{Q} = -\int_0^\infty |y|^{2p} + (2p - 1)\int_0^\infty |y|^{2p-2} z\sigma(z + u).$$

By applying Holder inequality (for $p > 1$) we get the result. ■

Let $u \in L^p$ and $x_u = (z, y)$ the trajectory of $(SI_2)_{k_L}$ corresponding to u and starting at 0. For $R > 0$, consider the set E_R given by

$$E_R = \{t \in \mathbb{R}^+,\ V(t) > R\}.$$

Lemma 11 *With the notations above, we either have $\|V\|_p \leq const._R \|u\|_p$ or*

$$\|V\|_p^p \leq const._R \int_{E_R} V^p. \tag{59}$$

Proof of Lemma 11: The two previous lemmas clearly imply that

$$\|V\|_p \leq const.\|V\sigma(z + u)\|_p.$$

Therefore we have by using Proposition 3

$$\begin{aligned}\int_0^\infty V^p &\leq const. \int_0^\infty V^p|\sigma(z+u)|^p \leq const.\Big(\int_{E_R} V^p|\sigma(z+u)|^p + \int_{\mathbb{R}^+/E_R} V^p|\sigma(z+u)|^p\Big)\\ &\leq const. \int_{E_R} V^p|\sigma(z+u)|^p + const.R^p\|\sigma(z+u)\|_p^p \leq const. \int_{E_R} V^p + const.R^p\|u\|_p^p.\end{aligned}$$

From the last equation, we conclude. ■

Proof of Proposition 4: For the rest of this proof, we choose $R \stackrel{def}{=} C^2$. For $T > 0$, we will "slice" $[0,T] \cup E_R$ into a finite number of disjoint measurable sets L_k defined below and prove for each L_k the estimates

$$\int_{L_k} |z|^p \leq const.\Big(\int_{L_k} |u|^p\Big)^2 \tag{60}$$

$$\int_{L_k} |y|^p \leq const.\Big(\int_{L_k} |u|^p\Big)^{\frac{p+2}{p+1}}, \tag{61}$$

with of course $const.$ independent of k and T.

Let $V_m \stackrel{def}{=} V_{\infty,[0,T]}$ and $k_m \stackrel{def}{=} \Big[\frac{\ln \frac{V_m}{R}}{\ln 2}\Big]$. For $k = 0, \cdots, k_m$ we define L_k as

$$L_k \stackrel{def}{=} \{t \in [0,T] \cup E_R, \quad \frac{V_m}{2^{k+1}} < V(t) \leq \frac{V_m}{2^k}\}. \tag{62}$$

Then $\displaystyle\int_{[0,T]\cup E_R} V^p \leq \sum_{k=0}^{k_m} \int_{L_k} V^p$. Therefore, estimates (54) and (55) of Proposition 4 follow readily from (60) and (61).

In the sequel, we fix $k \in \{0, \cdots, k_m\}$ and use L and $\bar{V}$ to denote L_k and $\frac{V_m}{2^k}$ respectively. It is easy to see that L is the countable union of intervals $I_j = (t^j, t'^j]$, $j \in \Omega$, with two by two disjoint interiors such that for every $t \in (t^j, t'^j)$ we have

$$\frac{\bar{V}}{2} < V(t) < \bar{V}.$$

Moreover, each I_j falls into one of the three following cases

$(c1)$ $V(t^j) = V(t'^j)$ $(= \frac{\bar{V}}{2}$ or $\bar{V})$;

$(c2)$ $V(t^j) = \frac{\bar{V}}{2}$ and $V(t'^j) = \bar{V}$;

$(c3)$ $V(t^j) = \bar{V}$ and $V(t'^j) = \frac{\bar{V}}{2}$.

For $l = 1, 2, 3$ and $j \in \Omega$, we use Ω_l to denote the set of indices $j \in \Omega$ such that I_j verifies (cl). Let $T^{(j)} \stackrel{def}{=} t'^j - t^j$.

Remark 3 *For each interval I_j, $j \in \Omega_3$, there exists $j_0 < j$ with $j_0 \in \Omega_2$ and for every $j' = j_0 + 1, \cdots, j-1$, $j' \in \Omega_1$. In addition, Ω_2 is non empty. If $(V^{\frac{1}{2}})^{\cdot}$ is the time derivative of $V^{\frac{1}{2}}$ along x_u, it is bounded by $5K$. We then have that*

$$\Omega_3^\# \leq \Omega_2^\# \leq const.\frac{T}{\bar{V}^{\frac{1}{2}}}. \tag{63}$$

Moreover, for every $j \in \Omega$ we have on I_j that $|\dot V| \leq 8\sigma_M \bar V^{\frac12}$ and

$$\int_{I_j} |z|^p \leq const.T^{(j)}\bar V^p \text{ and } \int_{I_j} |y|^p \leq const.T^{(j)}\bar V^{\frac{p}{2}}. \tag{64}$$

We fix some $j \in \Omega$ and we will get an estimate of $\Delta V\Big|_{I_j}$. We first assume that z has at least two zeros on I_j. Since $|\dot z| \geq const.\bar V^{\frac12}$ at a zero of z, then z has a finite number N_j of zeros on I_j and $N_j \leq const.\frac{T_j}{\bar V^{\frac12}}$. Let $t_0 < \cdots < t_{N_j}$ the sequence of zeros of z on I_j. Set $T_i \stackrel{def}{=} t_{i+1} - t_i \geq const.\bar V^{\frac12}$ and $\Delta V_i \stackrel{def}{=} \Delta V\Big|_{t_i}^{t_{i+1}}$. We define for $i = 0, \cdots, N_j$

$$\begin{aligned}
E_i &= \{t \in [t_i, t_{i+1}],\ |u| > \frac{|z|}{3}\}; &(65)\\
E'_i &= \{t \in [t_i, t_{i+1}],\ |u| \leq \frac{|z|}{3}\}; &(66)\\
F_i &= \{t \in [t_i, t_{i+1}],\ |y| \leq |z| - C\}; &(67)\\
F'_i &= \{t \in F_i,\ |u| > \frac{|y|}{2},\ \ uy < 0\} &(68)\\
G_i &= \{t \in F'_i,\ |u| \geq \rho \bar V^{\frac12}\}. &(69)
\end{aligned}$$

Set $\Theta = \{i \in \{0, \cdots, N_j\},\ |E_i| > \rho\}$. Note that $\Theta^\# \leq \frac{|E|}{\rho}$. Note also that for $i \in \{0, \cdots, N_j - 1\}$, $y(t_i)y(t_{i+1}) < 0$.

Lemma 12 *With the above notations, we have*

$$\Delta V_i + \frac{\sigma_m^2}{2}T_i + \frac{\sigma_m^2}{8}\bar V^{\frac12} \leq const.\bar V^{\frac12}\Big(|E_i| + |G_i|\Big). \tag{70}$$

Proof of Lemma 12: The set F_i is an interval $[t', t'']$ such that $\frac12 \leq t' - t_i \leq \frac32$ and $\frac12 \leq t_{i+1} - t'' \leq \frac32$. Set $S_i = [t_i, t_{i+1}]/F_i$. We have $1 \leq |S_i| \leq 3$ and $|y| \geq \frac{\bar V^{\frac12}}{2}$ on S_i. Then

$$\begin{aligned}
\int_{S_i} \dot V &= \int_{S_i \cap E_i} \dot V + \int_{S_i \cap E'_i} \dot V\\
&\leq const.|S_i \cap E_i|\bar V^{\frac12} - |S_i \cap E'_i|\frac{\sigma_m^2 + \bar V^{\frac12}}{2}\\
&\leq -\tfrac14 \bar V^{\frac12} + const.\bar V^{\frac12}|S_i \cap E_i|.
\end{aligned}$$

On F_i, using the estimates obtained in Lemma 7 of case $(B1)$, we get

$$\int_{F_i} \dot V \leq -\frac{\sigma_m^2}{4}\bar V^{\frac12} + const.|F_i \cap E_i| + 2\int_{F'_i} |\dot z||\Delta\sigma\Big|_{z+u}^{z-\theta y}|.$$

We also have

$$\int_{F'_i/G_i} |\dot z||\Delta\sigma\Big|_{z+u}^{z-\theta y}| \leq \int_{F'_i} |\dot z|\Delta\sigma\Big|_{z-\rho\bar V^{\frac12}}^{z} \leq \int_{F_i} |\dot z|\Delta\sigma\Big|_{z-\rho\bar V^{\frac12}}^{z}.$$

We need the next lemma to complete the proof of Lemma 12

Lemma 13 *With the previous notations, we have*

$$\int_{F_i} |\dot z|\Delta\sigma\Big|_{z-\rho\bar V^{\frac12}}^{z} \leq const.\delta_{\bar V^{\frac12}}\Big(|E_i| + \bar V^{\frac12}\Big).$$

Proof of Lemma 13: Let $L_i = [t', \bar{t}'] \cup [\bar{t}'', t'']$ and $M_i = (\bar{t}', \bar{t}'')$ such that $|y| > \sigma_M + 1$ on L_i and $|y(\bar{t}')| = |y(\bar{t}'')| = \sigma_M + 1$. Assume first that

$$|M_i \cap E_i| < \frac{\sigma_m}{4K}|L_i|. \tag{71}$$

Then $|M_i| \leq const.$ and then $|y|$ remains bounded. Indeed, this is a consequence of the following equation

$$\Delta y\Big|_{\bar{t}'}^{\bar{t}''} \leq -\int_{M_i} \sigma(z) + \int_{M_i} |\Delta\sigma\Big|_{z+u}^{z}|,$$

which implies that

$$\sigma_m|M_i| \leq 2(\sigma_M + 1) + \delta_C|M_i| + 2K|M_i \cap E_i|.$$

Then using (71), we show that $|M_i|$ is bounded. Therefore, we have

$$\int_{M_i} |\dot{z}|\Delta\sigma\Big|_{z-\rho\bar{V}^{\frac{1}{2}}}^{z} \leq const.\delta_{\bar{V}^{\frac{1}{2}}}$$

Assume now that $|M_i \cap E_i| \geq \frac{\sigma_m}{4K}|L_i|$. Then $|M_i| \leq const.|E_i|$, in which case we obtain

$$\int_{M_i} |\dot{z}|\Delta\sigma\Big|_{z-\rho\bar{V}^{\frac{1}{2}}}^{z} \leq const.\delta_{\bar{V}^{\frac{1}{2}}}|E_i|.$$

Finally, we have

$$\begin{aligned}\int_{L_i} |\dot{z}|\Delta\sigma\Big|_{z-\rho\bar{V}^{\frac{1}{2}}}^{z} &\leq 3\int_{\bar{t}'}^{\bar{t}''} \Delta\sigma\Big|_{z-\rho\bar{V}^{\frac{1}{2}}}^{z} \leq \iint_{\substack{z(\bar{t}') \leq z \leq z(\bar{t}'') \\ z-\rho\bar{V}^{\frac{1}{2}} \leq s \leq z}} \sigma'(s)dsdz \\ &\leq \rho\bar{V}^{\frac{1}{2}}\Delta\sigma\Big|_{z(\bar{t}')-\rho\bar{V}^{\frac{1}{2}}}^{z(\bar{t}'')} \leq const.\delta_{\bar{V}^{\frac{1}{2}}}\bar{V}^{\frac{1}{2}}. \qquad \blacksquare\end{aligned}$$

We also have $\int_{G_i} |\dot{z}||\Delta\sigma\Big|_{z+u}^{z-\theta y}| \leq const.|G_i|\bar{V}^{\frac{1}{2}}$. Adding up all the inequalities we get (70). $\blacksquare$

Lemma 14 *With the above notations, we have for $i \in \Omega$*

$$\int_{E_i} |z|^p \geq const.|E_i|\bar{V}^{\frac{p}{2}}. \tag{72}$$

Define $U_i = \{t \in E_i, \quad |z| \geq \frac{\bar{V}^{\frac{1}{2}}}{2\sigma_M}\}$. Then we clearly have

$$\int_{U_i} |z|^p \geq const.|U_i|\bar{V}^{\frac{p}{2}}.$$

We may assume that $|U_i| < \rho|E_i|$ otherwise we are done. Moreover since E_i/U_i is included in S_i, $|E_i|$ is bounded. Since $|E_i/U_i| \geq \frac{\rho}{2}$ and $|y| \geq \bar{V}^{\frac{1}{2}}$ on E_i/U_i, the set E_i/U_i contains an interval of length $\frac{\rho}{4}$ in S_i and then

$$\int_{E_i} |z|^p \geq const.\int_0^{\frac{\rho}{4}} (\frac{\bar{V}}{2}s)^{\frac{p}{2}}ds \leq const.\bar{V}^{\frac{p}{2}} \leq const.|E_i|\bar{V}^{\frac{p}{2}}. \blacksquare$$

Since $[t_i, t_{i+1}]$ contains E_i and G_i, we get by Lemma 12 and Lemma 14 that

$$\int_{t_i}^{t_{i+1}} |u|^p \geq const.\bar{V}^{\frac{p-1}{2}}\Big(\Delta V_i + \frac{\sigma_m^2}{2}T_i + \frac{\sigma_m^2}{8}\bar{V}^{\frac{1}{2}}\Big).$$

By adding up the previous inequality for $i = 0, \cdots, N_j$, we get

$$\int_{t_0}^{t_{N_j}} |u|^p \geq const.\bar{V}^{\frac{p-1}{2}}\Big(\Delta V\Big|_{t_0}^{t_{N_j}} + \frac{\sigma_m^2}{2}(t_{N_j} - t_0) + N_j\frac{\sigma_m^2}{8}\bar{V}^{\frac{1}{2}}\Big).$$

In addition, on the intervals $[t^j, t_0]$ and $[t_{N_j}, t'_j]$, we have on each interval

$$\Delta V + \frac{\sigma_m^2}{4}\Delta t \leq const.\bar{V}^{\frac{1}{2}}\Big(|E^1| + |G^1|\Big),$$

where ΔV and Δt are the variations of V and t respectively, E^1 and G^1 correspond to E_i and G_i defined in (65) and (69) for $[t^j, t_0]$ and $[t_{N_j}, t'_j]$. We hence obtain for every $j \in \Omega$

$$\bar{V}^{\frac{p-1}{2}}\Big(\Delta V\Big|_{I_j} + \frac{\sigma_m^2}{4}T^{(j)} + N_j\frac{\sigma_m^2}{8}\bar{V}^{\frac{1}{2}}\Big) \leq const.\bar{V}^{\frac{p-1}{2}}\Big(|E^{(j)}| + |G^{(j)}|\Big) \leq \int_{I_j} |u|^p, \tag{73}$$

where

$$\begin{aligned} E^{(j)} &= \{t \in I_j, \ \ |u| > \tfrac{|z|}{3}\}, \\ G^{(j)} &= \{t \in I_j, \ |u| > \sup(\rho\bar{V}^{\frac{1}{2}}, \tfrac{|y|}{2}), \ uy < 0, \ |y| \leq |z| - C\}. \end{aligned}$$

Note that (73) remains valid in the case where z has at most one zero on I_j.

For $j \in \Omega_1$, equation (73) implies that

$$\int_{I_j} |u|^p \geq const.T^{(j)}\bar{V}^{\frac{p-1}{2}}. \tag{74}$$

For $j \in \Omega_2$, equation (73) implies that

$$\int_{I_j} |u|^p \geq const.\Big(T^{(j)} + \bar{V}^{\frac{1}{2}}\Big)\bar{V}^{\frac{p-1}{2}}. \tag{75}$$

As for $j \in \Omega_3$, two possibilities occur whether $T^{(j)} \leq \frac{2\bar{V}}{\sigma_m^2}$ (case $(c31)$) or not (case $(c32)$). Thanks to Remark 3, if $j \in \Omega_3$ verifies case $(c32)$ and $j_0 \in \Omega_2$ (with $j_0 < j$), we have

$$\int_{I_j \cup I_{j'}} |u|^p \geq const.\Big(T^{(j)} + \bar{V}\Big)\bar{V}^{\frac{p-1}{2}} \geq const.\Big(T^{(j)} + T^{(j')} + \frac{\bar{V}}{4}\Big)\bar{V}^{\frac{p-1}{2}}.$$

If $j \in \Omega_3$ verifies case $(c31)$, then

$$\frac{\sigma_m^2}{4}T^{(j)} \leq \Big(\Delta V\Big|_{I_j} + \frac{\sigma_m^2}{4}T^{(j)}\Big),$$

which means that this case is equivalent to case $(c1)$. From now on, Ω_1 is the set of indices j for which case $(c1)$ or $(c32)$ holds and Ω_2 is the set of indices j for which case $(c2)$ or $(c31)$ holds.

Set $T_{1,\bar{V}} \overset{def}{=} \sum_{j\in\Omega_1} T^{(j)}$ and $T_{2,\bar{V}} \overset{def}{=} \sum_{j\in\Omega_2}\Big(T^{(j)} + \frac{\bar{V}}{4}\Big)$. Note that $T_{2,\bar{V}} \geq \frac{\bar{V}}{4}$. Then we have

$$\int_L |u|^p \geq const.\bar{V}^{\frac{p-1}{2}}(T_{1,\bar{V}} + T_{2,\bar{V}}), \tag{76}$$

$$\int_L |z|^p \leq const.\bar{V}^p(T_{1,\bar{V}} + T_{2,\bar{V}}), \tag{77}$$

and

$$\int_L |y|^p \leq const.\bar{V}^{\frac{p}{2}}(T_{1,\bar{V}} + T_{2,\bar{V}}), \tag{78}$$

Using (76) and (77), we get

$$\begin{aligned}\Big(\int_L |u|^p\Big)^2 &\geq const\bar{V}^{p-1}(T_{1,\bar{V}} + T_{2,\bar{V}})^2 \\ &\geq const.\bar{V}^{\frac{p}{2}}(T_{1,\bar{V}} + T_{2,\bar{V}})\frac{T_{1,\bar{V}} + T_{2,\bar{V}}}{\bar{V}} \geq const.\int_L |z|^p,\end{aligned}$$

and

$$\begin{aligned}\Big(\int_L |u|^p\Big)^{\frac{p+2}{p+1}} &\geq const.\bar{V}^{\frac{(p-1)(p+2)}{2(p+1)}}(T_{1,\bar{V}} + T_{2,\bar{V}})^{\frac{p+2}{p+1}} \\ &\geq const.\bar{V}^{\frac{p}{2}}(T_{1,\bar{V}} + T_{2,\bar{V}})\Big(\frac{T_{1,\bar{V}} + T_{2,\bar{V}}}{\bar{V}}\Big)^{\frac{1}{p+1}} \geq const.\int_L |y|^p.\end{aligned}$$

Therefore the proof of (54) and (55) is finished. As for (56), it is obtain by considering (76) on L_0,

$$\int_{L_0} |u|^p \geq const.V_m^{\frac{p-1}{2}}(T_{1,V_m} + T_{2,V_m}) \geq const.V_m^{\frac{p+1}{2}}.$$

Then the proof of Proposition 4 is completed. ∎

Proposition 5 *Let $p \leq q \in [1, \infty)$, $u \in L^p$ and $x_u = (z, y)$ the trajectory of $(SI_2)_{k_L}$ corresponding to u and starting at 0. Then there exists const. > 0 such that for $\|u\|_p \geq$ const. we have*

$$\begin{aligned}\|z\|_q &\leq const.\|u\|_p^{r_1(p,q)}, &(79)\\ \|y\|_q &\leq const.\|u\|_p^{r_2(p,q)}, &(80)\\ \|\dot{y}\|_q &\leq const.\|u\|_p^{\frac{p}{q}}, &(81)\end{aligned}$$

where $r_1(p,q) = \frac{2p(q+1)}{q(p+1)}$ and $r_2(p,q) = \frac{p(q+2)}{q(p+1)}$.

Proof of Proposition 5: By interpolating q between p and ∞, and if $\xi = z, y$ or $\dot{x}$ we have

$$\|\xi\|_q \leq \|\xi\|_\infty^{\frac{q-p}{q}} \|\xi\|_p^{\frac{p}{q}}.$$

Using Propositions 4 and 3, the conclusions of the above proposition are reached. ∎

In order to finish the proof of Theorem 1, we have to show that the several shapes obtained in Propositions 3 and 4 are in fact the profiles announced in Theorem 1. For that purpose, it remains to exhibit sequences of inputs verifying the hypotheses of Lemma 2. This is done in the Appendix where we construct two such sequences (u_n^1) and (u_n^2). It turns out that (u_n^1) works to get (1) and (2) of Theorem 1 and (u_n^2) to get (3) of Theorem 1.

3.2 Proof of Theorem 2

First recall that $(SI_2)_{k_L}$ is controllable. In addition, we have the following proposition

Proposition 6 *For $p \in [1, \infty]$, there exists C_p such that for every $(z, y) \in I\!R^2$ and $\|(z, y)\| \geq C_p$ there exists $u_{(z,y)} \in L^p$ such that*

$$const.\|u_{(z,y)}\|_p \leq V(z,y)^{\frac{p+1}{2p}} \leq const.\|u_{(z,y)}\|_p. \tag{82}$$

Proof of Proposition 6: Let $(z,y) \in I\!R^2$ and $\|(z,y)\| \geq C_p$. It is enough to show that we can reach $(0,\bar{y})$ with $\bar{y} = V(z,y)^{\frac{1}{2}}$. This can be deduced from the construction of the sequence (u_n^1) in the Appendix. ■

Therefore, any trajectory of $(PSI_2)_{k_L}$ starting at $(z,y) \in I\!R^2$ and corresponding to $u \in L^p$ and $v \in L^{p_1}$ (with $p_1 \in [1,\infty])$ can be seen as the trajectory of $(PSI_2)_{k_L}$ starting at 0 and corresponding to $\tilde{u} \in L^p$ and $v \in L^{p_1}$ (with $p_1 \in [1,\infty]$) where $\tilde{u}$ is the concatenation of $u_{(z,y)}$ defined in Proposition 6 and u. For the rest of the section, we will only consider trajectories of $(PSI_2)_{k_L}$ starting at 0 and use the estimates given below to get Theorem 2: for $p \in [1,\infty]$, for every $(z,y) \in I\!R^2$ and $\|(z,y)\| \geq C_p$, for every $u \in L^p$, we have

$$const.(V(z,y)^{\frac{p+1}{2p}} + \|u\|_p) \leq \|\tilde{u}\|_p \leq const.(V(z,y)^{\frac{p+1}{2p}} + \|u\|_p). \tag{83}$$

Remark 4 *The previous paragraph indicates how to get (ISS)-type estimates. It is easy to see by taking into account (83) that for $p \in [1,\infty]$, there exist $C_0, C_1, C_2 > 0$ such that for $(z,y) \in I\!R^2$ and $u \in L^p$, if x is the trajectory of $(PSI_2)_{k_L}$ starting at (z,y) and corresponding to $u \in L^p$, the following inequality holds for $t \geq 0$*

$$\begin{aligned} \|x(t)\| &\leq const.V(x(t)) \leq const.\Big(\sup(\|x\|,\|x\|^2)e^{-C_2 t}(1 - [sgn\ (\sup(\|x\|,\|x\|^2) - C_0)]^+) \\ &+ (\sup(\|x\|,\|x\|^2) - C_1 t)[sgn\ (\sup(\|x\|,\|x\|^2) - C_0)]^+\Big) + \|u\|_{p,[0,t]}^{\frac{2p}{p+1}}. \end{aligned} \tag{84}$$

Let $p \in [1,\infty]$ and $u \in L^p$, $p_1 \in [1,\infty)$ with $p_1 \leq p$ and $v = (v_1, v_2)) \in L^{p_1}$. Set $\varepsilon_0 = \frac{\sigma_m}{4\gamma_\infty}$ where γ_∞ is the L^∞-gain of the linear control system $(L2)$ $\dot{x} = A_1 x + bu$ with

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Let $x = (z,y)$ be the trajectory of $(PSI_2)_{k_L}$ starting at 0 and corresponding to u, v and $x_1 = (z_1, y_1)$ be the trajectory of $(L2)$ starting at 0 and corresponding to v. Then $(Z,Y) = (z - z_1, y - y_1)$ is the trajectory of $(PSI_2)_{k_L}$ starting at 0 and corresponding to $\bar{u} = z_1 + u$ and $\bar{v} = (0, z_1 + y_1)$. We suppose that Note then that $\|v\|_\infty \leq \varepsilon_0$. Then $\bar{v}$ is a locally absolutely continuous (l.a.s. for short) input such that

$$\bar{v} \in L^{p_1}, \quad \|\bar{v}\|_\infty \leq \frac{\sigma_m}{2}. \tag{85}$$

We first prove part (3) of Theorem 3 and suppose hence that $p_1 > 2$. For every $\varepsilon > 0$, we construct a curve x which is l.a.s., unbounded and can be seen as the trajectory of $(PSI_2)_{k_L}$ starting at 0 and corresponding to some continuous and compactly supported input u and to some $v \in L^{p_1}$ such that $\|\bar{v}\|_\infty \leq \varepsilon$. We will suppose that σ is equal to the standard saturation function σ_0. Let $V_0 > (\frac{5}{\varepsilon})^{\frac{2p}{p_1-1}}$. There exists a continuous input u which steers 0 to $(0, V_0^{\frac{1}{2}})$ in time $T > 0$ along $(SI_2)_{k_L}$. Then x is defined on $[0,T]$ as the resulting trajectory. For $t > T$, we take $u = 0$ and v as follows: for $k \geq 0$,

$$v(t) \stackrel{def}{=} c_k \frac{y}{(1+y^2)^{\frac{p_1+1}{2p_1}}},$$

Here, c_k is piecewise constant and equal to $\frac{5}{V_k^{\frac{p_1-1}{2p_1}}}$ on $(t_k, t_{k+1}]$ where (t_k) is the sequence of zeros of z $(t_0 = T)$ and the sequence (V_k) is equal to $\Big(V(z(t_k), y(t_k))\Big)$. We prove that the above sequences are well-defined by induction using the next equation: for $k \geq 0$, we have

$$V_k^{\frac{1}{2}} \leq t_{k+1} - t_k \leq 2V_k^{\frac{1}{2}}, \quad 2V_k^{\frac{1}{2}} \leq V_{k+1} - V_k \leq 6V_k^{\frac{1}{2}}.$$

We also obtain that for $k \geq 0$, $const.k^2 + V_0 \leq V_k \leq const.k^2 + V_0$. Taking into account the next equation: for $k \geq 1$,

$$\int_{t_k}^{t_{k+1}} |v|^{p_1} \leq const.\frac{\ln k}{k^{p_1-1}},$$

we are done. It is easy to adapt the above construction to an arbitrary iS-function.

From now on, we only consider trajectories of $(PSI_2)_{k_L}$ starting at 0 and corresponding to $u, \bar{v}$ with $u \in L^p$, $\bar{v} = (0, v))$ with v l.a.s. and satisfying (85). Moreover, we assume that $p_1 \leq 2$. The key estimates to get Proposition 2 ($p = \infty$) and Proposition 4 (p finite) respectively are equation (4) and equation (12) respectively. Considering $(PSI_2)_{k_L}$, the two previous equations are still valid, with however the addition on their respective right-hand side of the term $\int_I |yv|$ where I is the ad hoc interval. In the sequel, we will only treat the case where p is finite since the argument when $p = \infty$ follows the same lines.

In the present situation, equation (73) becomes

$$\Delta V\Big|_{I_j} + const.(T^{(j)} + \bar{V}^{\frac{1}{2}}) \leq const.\bar{V}^{\frac{1}{2}}\Big(|E^{(j)}| + |G^{(j)}|\Big) + const.\int_{I^{(j)}} |yv|. \tag{86}$$

Setting $T_{\bar{V}} = T_{1,\bar{V}} + T_{2,\bar{V}} \geq const.\bar{V}$, equation (76) becomes

$$\bar{V}^{\frac{p-1}{2}} T_{\bar{V}} \leq const.\Big(\int_L |u|^p + \bar{V}^{\frac{p}{2}} \int_L |v|\Big). \tag{87}$$

We have to consider two sub-cases. First we suppose that $p_1 < 2$. Then by applying Holder inequality in (87), we get

$$\bar{V}^{\frac{p+1}{2}} \leq const.\bar{V}^{\frac{p-1}{2}} T_{\bar{V}} \leq const.\Big(\int_L |u|^p + \bar{V}^{\frac{p}{2}} T_{\bar{V}}^{1-\frac{1}{p_1}} \|v\|_{p_1}\Big). \tag{88}$$

Therefore, we obtain

$$\begin{aligned} T_{\bar{V}} &\leq const.\Big(\int_L |u|^p + (\int_L |v|^{p_1})^{\frac{2}{2-p_1}}\Big) \\ \bar{V}^{\frac{p+1}{2}} &\leq const.\Big(\int_L |u|^p + (\int_L |v|^{p_1})^{\frac{2}{2-p_1}}\Big). \end{aligned}$$

Recall that

$$\begin{aligned} \int_L |z|^p &\leq const.\bar{V}^p T_{\bar{V}}, \\ \int_L |y|^p &\leq const.\bar{V}^{\frac{p}{2}} T_{\bar{V}}, \\ \int_L |\sigma(z+u)|^p &\leq const.T_{\bar{V}}. \end{aligned}$$

Next we only prove (10). If $\int_L |u|^p \geq \bar{V}^{\frac{p}{2}} T_{\bar{V}}^{1-\frac{1}{p_1}} \|v\|_{p_1}$, then we get (54). Otherwise, we get

$$\int_L |z|^p \leq const.\bar{V}^{p+\frac{p_1}{2}} \|v\|_{p_1}^{p_1} \leq \|v\|_{p_1}^{p_1+(p+\frac{p_1}{2})p_1}.$$

and we conclude.

Next, we suppose that $p_1 = 2$. This case is more delicate than the previous one since (86) does not provide directly any interesting estimate. We will rewrite it in two different ways: first we easily get that

$$\Delta V\Big|_{I_j} + const.(T^{(j)} + \bar{V}^{\frac{1}{2}}) \leq const.\int_{I_j} \frac{|u|^p}{\bar{V}^{\frac{p-1}{2}}} + \int_{I_j} |yv|, \tag{89}$$

or if we divide (86) by $\bar{V}$, we can write it as

$$\int_{I_j} \frac{\dot{V}}{V} + const. \int_{I_j} \frac{dt}{V} \leq const. \int_{I_j} \frac{|u|^p}{\bar{V}^{\frac{p+1}{2}}} + const. \int_{I_j} \frac{|v|}{V^{\frac{1}{2}}}. \tag{90}$$

This leads to rewrite (87) as

$$T_{\bar{V}} \leq const. \int_L \frac{|u|^p}{\bar{V}^{\frac{p-1}{2}}} + \int_L |yv|, \tag{91}$$

or

$$1 + \int_L \frac{dt}{V} \leq const. \int_L \frac{|u|^p}{\bar{V}^{\frac{p+1}{2}}} + const. \int_L \frac{|v|}{V^{\frac{1}{2}}}. \tag{92}$$

By applying Cauchy-Scharwz inequality above we get

$$1 + \int_L \frac{dt}{V} \leq const. \int_L \frac{|u|^p}{\bar{V}^{\frac{p+1}{2}}} + const. \int_L |v|^2. \tag{93}$$

We next divide the set of integers $\{0, \cdots, k_m\}$ in two disjoint subsets $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $k \in \mathcal{K}_1$ if

$$\int_{L_k} |u|^p \geq \bar{V}^{\frac{p-1}{2}} \int_{L_k} |yv|,$$

and $k \in \mathcal{K}_2$ if the opposite inequality holds. Let $k_2 = \mathcal{K}_2^{\#}$ and $k_1 = \inf_{\mathcal{K}_1} k$. Then $k_2 \geq k_1$. By applying (88) to k_1, we have

$$V_{k_1} \leq const. \|u\|^{\frac{2p}{p+1}},$$

and by summing up (92) for $k \in \mathcal{K}_2$, we obtain

$$k_2 \leq const \|v\|_2^2.$$

Since $V_m = 2^{k_1} V_{k_1} \leq 2^{k_2} V_{k_1}$, we get $V_m \leq const \|u\|^{\frac{2p}{p+1}} e^{const \|v\|_2^2}$. Then (16) follows. The other inequalities can be then deduced as done before and the proof of Theorem 2 is completed. ■

4 The feedback k_F

From now on, σ is an S-function and F is an $\mathcal{F}$-function. Note that K is odd, C^1 and increasing. In addition, we have the following obvious estimates

Lemma 15 *The function $y \mapsto \frac{y}{K(y)}$ is even, increasing on $\mathbb{R}^+$ and for every $y \in \mathbb{R}$,*

$$const. \inf(|y|, 1) \leq |K(y)| \leq const. \inf(|y|, |y|^{\frac{1}{2}}, |y|^{\frac{1}{r}}) \text{ and } K'(y) \geq \frac{1}{1 + |K(y)|}.$$

Proof of Lemma 15: This easily follows from (18). ■

As a consequence of Lemma 15, we have that for every $\lambda \in (0, 1)$ and $y \geq 0$

$$\frac{K(\lambda y)}{K(y)} \geq \lambda. \tag{94}$$

Up to a time reparameterization and linear change of variable in (SI_2), we may assume that $|\sigma(t)| \geq 2$ for $|t| \geq 1$. We use the C^1 change of variable given by

$$\begin{aligned} z &\stackrel{def}{=} x_1 + \bar{F}(x_2), \\ y &\stackrel{def}{=} \bar{F}(x_2); \end{aligned} \tag{95}$$

to rewrite $(SI_2)_{k_F}$ as

$$\left\{ \begin{array}{lcl} \dot{z} &=& K(y) - \frac{\sigma(z+u)}{K'(y)}, \\ \dot{y} &=& -\frac{\sigma(z+u)}{K'(y)}. \end{array} \right.$$

For $y \in I\!\!R$, we define the odd function G by

$$G(y) \stackrel{def}{=} y - \text{sgn }(y)K(y)^2 - \frac{1}{4}K(y), \tag{96}$$

and we have $\frac{2}{3}|y|\frac{1+|y|}{1+3|y|} \leq |G(y)| \leq |y|$.

We define the Lyapunov function V_F by

$$V_F(z,y) = \frac{1}{2}z^2 - zG(y) + \int_0^y G(s)ds. \tag{97}$$

There exist $const. > 0$ such that for every $z, y \in I\!\!R^2$

$$const.(z^2 + y^2) \leq V_F(z,y) \leq const.(z^2 + y^2),$$

which implies (20). Moreover, V_F is a strict Lyapunov function for $(SI_2)_{k_F}$ as shown by the dissipation inequality given below

Lemma 16 *If we use $\dot{V}_F$ to denote the time derivative of V_F along the trajectories of $(SI_2)_{k_F}$, we have*

$$\dot{V}_F + \frac{1+|K(y)|}{2}V_F^{\frac{1}{2}} \inf(V_F^{\frac{1}{2}}, 1)+ \leq 4(1+|K(y)|)|z|\sigma(3|u|)M(z,y,u), \tag{98}$$

where $M(z,y,u) \stackrel{def}{=} 1$ if $|u| > \frac{|z|}{2} > \inf(V_F^{\frac{1}{2}}, 1)\frac{V_F^{\frac{1}{2}}}{12}$ and 0 otherwise.

Proof of Lemma 16: We compute $\dot{V}_F$ and obtain

$$\dot{V}_F = zK(y) - z\sigma(z+u)(2|K(y)| + \frac{1}{4}) - K(y)G(y).$$

By considering the sub-cases defined whether $\frac{|z|}{2} < |u|$ or not and $V_F^{\frac{1}{2}} > 1$ or not, we conclude. ∎

As a consequence, we can see that V_F is an (ISS)-Lyapunov function.

4.1 Proof of Theorem 3

We first assume that $p = \infty$ and $\|u\|_\infty \geq C_0$. Then by the previous lemma, we have

$$\dot{V}_F \leq -(1+|K(y)|)\inf(1, V_F^{\frac{1}{2}})\Big(V_F^{\frac{1}{2}} - 24K\|u\|_\infty\Big).$$

Therefore, $\|V_F^{\frac{1}{2}}\|_\infty \leq 24K\|u\|_\infty$.

We have a similar result to Lemma 10 for $(SI_2)_{k_F}$

Lemma 17 *Let $p \in [1,\infty)$, $u \in L^p$ and $x_u = (z,y)$ the trajectory of $(SI_2)_{k_F}$ corresponding to u and starting at 0. Then we have*

$$\|K(y)^2\|_p \leq const.\|z\sigma(z+u)\|_p, \tag{99}$$

with equality if $p = 1$.

Consider the function

$$Q_F(z,y) \stackrel{def}{=} -zK(y)|K(y)|^{2p-2} - \int_0^y K(s)|K(s)|^{2p-2}ds.$$

Let $\dot{Q}_F$ be the time derivative of Q_F along x_u. By computing $\dot{Q}_F$, integrating it between 0 and ∞ and applying Holder inequality (for $p > 1$) we get the result. ∎

As a corollary, we prove Theorem 3 for $p = 1$. By the previous lemma, we have

$$\|z\sigma(z+u)\|_1 = \|K(y)^2\|_1 \leq const.\|y\|_1.$$

Moreover, dividing (98) by $V_F^{\frac{1}{2}}$ implies that

$$\Big(L_1(V_0^{\frac{1}{2}})\Big)^{\cdot} + V_F^{\frac{1}{2}}\frac{1+|K(y)|}{2K(V_F^{\frac{1}{2}})}\inf(V_F^{\frac{1}{2}},1) \leq 4\frac{1+|K(y)|}{K(V_F^{\frac{1}{2}})}|z|\sigma(3|u|)M(z,y,u),$$

where L_1 was defined in (22). Since $y \mapsto \frac{y}{K(y)}$ is increasing for $y \geq 0$, we have $\frac{V_0^{\frac{1}{2}}}{K(V_0^{\frac{1}{2}})} \geq const.\frac{|y|}{|K(y)|}$ and we get from the equation above

$$\Big(L_1(V_0^{\frac{1}{2}})\Big)^{\cdot} + \frac{|y|}{2} \leq 12K|u|,$$

After integration, we obtain $\|y\|_1 \leq const.\|u\|_1$. A trajectory of $(SI_2)_{k_F}$ starting at 0 and corresponding to the input u is also the trajectory of the linear system given by $\dot{x} = A_1x + b_1w$ with

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \text{ and } b_1 = (0\ 1)^T,$$

with $w = \Big(z+u-\sigma(z+u)\Big) - u - y + K(y)$. By (ii) of Definition 1, we get $|w| \leq const.(|z\sigma(z+u)| + |u| + |y|)$. Since A_1 is Hurwitz, we conclude.

In the sequel, we assume that $p \in (1,\infty)$. For $X \geq 0$, let $R(X) \stackrel{def}{=} L_p(X^{\frac{1}{2}})^p$, where L_p was defined in (22). In the sequel we use R and $\dot{R}$ respectively to denote the Lyapunov function $R\Big(V_F^{\frac{1}{2}}(z,y)\Big)$ and its time derivative respectively. Multiplying (98) by $\frac{V_F^{\frac{p-1}{2}}}{K(V_F^{\frac{1}{2}})}$, we get

$$\dot{R} + \frac{1+|K(y)|}{2\sup(K(V_F^{\frac{1}{2}}),1)}V_F^{\frac{p}{2}} \leq 4\frac{1+|K(y)|}{K(V_F^{\frac{1}{2}})}|z|V_F^{\frac{p-1}{2}}\sigma(3|u|)M(z,y,u). \tag{100}$$

Then we have

$$\dot{R} + V_F^{\frac{p}{2}} \leq 8V_F^{\frac{p-1}{2}}\sigma(3|u|)M(z,y,u), \text{ if } V_F \leq 1 \tag{101}$$

$$\dot{R} + \frac{1+|K(y)|}{2K(V_F^{\frac{1}{2}})}V_F^{\frac{p}{2}} \leq 4\frac{1+|K(y)|}{K(V_F^{\frac{1}{2}})}|z|V_F^{\frac{p-1}{2}}\sigma(3|u|)M(z,y,u) \text{ if } V_F > 1. \tag{102}$$

We deduce that

$$\dot{R} \leq const.|u|^p,$$

which implies that $L_p(\|V_F^{\frac{1}{2}}\|_\infty) \leq const.\|u\|_p$. Moreover since x_u is bounded, $\dot{x}_u$ is also bounded and by (101) and (102) we also have

$$const._{\|u\|_p} \int_0^\infty V_F^{\frac{p}{2}} \leq \int_0^\infty \frac{V_F^{\frac{p}{2}}}{\sup(K(V_F^{\frac{1}{2}}),1)} \leq const.\|u\|_p^p.$$

Therefore, by a classical result, we get that the trajectory converges to 0 at infinity. From now on, we assume that $R_m \stackrel{def}{=} \|R\|_\infty, \|V_F^{\frac{1}{2}}\|_\infty, \|u\|_p \geq C_0^2$. In order to prove Theorem 3, we look for establishing the inequality

$$\int_0^\infty V_0^{\frac{p}{2}} \leq const. \int_0^\infty V_0^{\frac{p-1}{2}}|u|, \tag{103}$$

which provides the result together with Holder inequality.

Set $k_m \stackrel{def}{=} \left[\frac{\ln \frac{R_m}{C_0}}{\ln 2}\right]$. We consider

$$\begin{aligned} S_0 &= \{t \geq 0, \ \ R\Big(z(t), y(t)\Big) > C_0\}, \\ S_1 &= \{t \geq 0, \ \ R\Big(z(t), y(t)\Big) \leq C_0\}. \end{aligned}$$

On each interval I' of S_1, we look for

$$\Delta R\Big|_{I'} + const. \int_{I'} V_F^{\frac{p}{2}} \leq const. \int_{I'} V_F^{\frac{p-1}{2}}|u|. \tag{104}$$

This last equation is a direct consequence of the fact that (100) can be simply written on S_1 as

$$\dot{R} + const.V_1^{\frac{p}{2}} \leq const.V_1^{\frac{p-1}{2}}|u|.$$

Similarly to the proof of Proposition 4, we "slice" up S_0 according to (62) (with the obvious modifications) and on each interval I of L_k falling into one of the (ci) cases, we look for

$$\Delta R\Big|_{I} + const. \int_{I} V_F^{\frac{p}{2}} \leq const. \int_{I} V_F^{\frac{p-1}{2}}|u|. \tag{105}$$

Equation (103) is finally obtained by summing up (105) on S_0 and (104) on S_1.

Therefore, it remains to show (105) to finish the proof of Theorem 3. Let $I = (t_0, t_1)$ an interval of some L_k such that $\Delta R\Big|_I = \mu\frac{\bar{R}}{2}$ where $\mu = \pm 1, 0$ and $\bar{R} = \frac{R_m}{2^k}$. Let $T \stackrel{def}{=} t_1 - t_0$ and $\bar{V} = R^{-1}(\bar{R})$. We easily get that $R^{-1}(\frac{\bar{R}}{2}) = \lambda\bar{V}$ with

$$(\frac{1}{2})^{\frac{1}{p}} \leq \lambda \leq (\frac{1}{2})^{\frac{1}{p+1}}.$$

Therefore $\Delta V_F\Big|I = (1-\lambda)\bar{V}$ and we have for every $t \in I$ $\lambda\bar{V} \leq V_F(t) \leq \bar{V}$. In addition, we get

$$\frac{1}{\sqrt{2}}T\bar{V}^{\frac{p}{2}} \leq \int_I V_F^{\frac{p}{2}} \leq T\bar{V}^{\frac{p}{2}}.$$

Consider the following subsets of I

$$\begin{aligned} E^1 &= \{t \in I, \quad |y| \geq \rho V_F^{\frac{1}{2}}\}, \\ E^2 &= \{t \in I, \quad |y| < \rho V_F^{\frac{1}{2}}\}, \\ E &= \{t \in E_2, \quad |u| > \tfrac{|z|}{2}\}. \end{aligned}$$

We first assume that $|E_1| \geq \rho T$. Then $\frac{1+|K(y)|}{2K(V_F^{\frac{1}{2}})} \geq \frac{\rho}{2}$ on E_1 and then

$$\int_I \frac{1+|K(y)|}{2K(V_F^{\frac{1}{2}})} V_F^{\frac{p}{2}} \geq const.\rho \int_{E_1} V_F^{\frac{p}{2}} \geq const.\rho T \bar{V}^{\frac{p}{2}},$$

and we get (105) by integration of (102).

We next assume that $|E_1| < \rho T$. Then $|z| > (1-\rho)V_F^{\frac{1}{2}}$ on E_2. Let $k_0 \stackrel{def}{=} \sup_{E_2} |K(y)|$. We have $k_0 \leq K(\rho V_F^{\frac{1}{2}})$ and the following lemma

Lemma 18 *With the above notations, let $U = \{t \in E_2, \quad |K(y)| \geq \frac{k_0}{2}\}$. Then either $E_2 = I$ or $|U| + |E| \geq const.T$.*

Proof of Lemma 18: If E_1 is not empty, then $k_0 = K(\rho V_F^{\frac{1}{2}})$. Notice that E_2 is open and then it is the countable union of open intervals. For every such an interval $J = (t, t')$ of E_2 let $U_J = U \cap J$ and $E_J = E \cap J$. We have $|K(y(t))|, |K(y(t'))| \leq k_0$ and there is equality at t or at t'. Similarly to (30), we get

$$\Big|\sigma(C)(t'-t) - const.|E_J|\Big| \leq 2k_0, \tag{106}$$

and since the time derivative of $K(y)$ is bounded (by K_σ), we have $k_0 \leq const.|U_J|$. Therefore $|U_J| + |E_J| \geq const.(t'-t)$. Adding up, we conclude. ∎

According to the previous lemma, three possibilities may occur: E_1 is empty, $|E| \geq const.T$ or $|U| \geq const.T$. Suppose first that $|E| \geq const.T$. Then $|u| > \frac{|z|}{2} > const.V_F^{\frac{1}{2}}$ on E and we get

$$\int_I V_F^{\frac{p-1}{2}} |u| \geq const. \int_I V_F^{\frac{p}{2}}.$$

This implies (105) except in the case where $\Delta R\Big|_I = \frac{\bar{R}}{2}$. But

$$\bar{R} \leq const.\frac{V_F^{\frac{p+1}{2}}}{K(V_F^{\frac{1}{2}})} \text{ and } \bar{V} \leq const.\Delta V_F\Big|_I \leq const.TK(\bar{V}^{\frac{1}{2}})\bar{V}^{\frac{1}{2}}.$$

Then $\Delta R\Big|_I \leq const. \int_I V_F^{\frac{p-1}{2}} |u|$ and we conclude this case.

Suppose now that $|U| \geq const.T$. We may also assume that $k_0 = K(\rho V_F^{\frac{1}{2}})$. Then

$$\int_U \frac{1+|K(y)|}{2K(V_F^{\frac{1}{2}})} V_F^{\frac{p}{2}} \geq const.\frac{K(\rho \bar{V}^{\frac{1}{2}})}{K(\bar{V}^{\frac{1}{2}})}(1-\lambda)T\bar{V}^{\frac{p}{2}} \geq T\bar{V}^{\frac{p}{2}}.$$

We easily conclude here.

Finally, suppose that E_1 is empty. Then z has a constant sign on I, let say $z > (1-\rho)V_F^{\frac{1}{2}}$ on I. Equation (106) becomes

$$\left|\sigma(C)T - const.|E|\right| \leq 2k_0 \leq const.K(\rho\bar{V}^{\frac{1}{2}}). \tag{107}$$

If $|\Delta R\Big|_I| = \frac{\bar{R}}{2}$, then $|\Delta V_F\Big|_I| \geq const.\bar{V}$ which implies that $T \geq const.\frac{\bar{V}^{\frac{1}{2}}}{K(\rho\bar{V}^{\frac{1}{2}})}$ with the last $const.$ independent of ρ. Since $K(\rho\bar{V}^{\frac{1}{2}})^2 \leq const.\rho\bar{V}^{\frac{1}{2}}$, we obtain from (107) that $|E| \geq const.T$. That situation was already treated. Therefore the proof of Theorem 3 is finished.

4.2 Proof of Theorem 4

Using the change of variables given by (95), we rewrite $(PSI_2)_{k_F}$ as

$$\begin{cases} \dot{z} &= K(y) + \frac{-\sigma(z+u)+v_2}{K'(y)} + v_1, \\ \dot{y} &= \frac{-\sigma(z+u)+v_2}{K'(y)}. \end{cases}$$

We choose $\varepsilon_0 = \frac{\sigma(1)}{4}$. The time derivative of V_F along trajectories of $(PSI_2)_{k_F}$ becomes

$$\begin{aligned} \dot{V}_F + \tfrac{1+|K(y)|}{2} V_F^{\frac{1}{2}} \inf(V_F^{\frac{1}{2}}, 1) &\leq 8(1+|K(y)|)|z|\sigma\Big(const.(|u|+|v_1|+|v_2|)\Big)M(z,y,u) \\ &+ const.|v_1||y|N(z,y), \end{aligned}$$

where $N(z,y) = 1$ if $V_F < 1$ and 0 otherwise. From the previous equation, we can follow exactly the lines of the proof of Theorem 3 only changing $|u|$ by $|u|+|v_1|+|v_2|$ and checking at each step that the extra term $|v_1||y|N(z,y)$ does not alter the computations. The term $L_p(V_F(\bar{x})^{1/2})$ appearing in (21) comes from the integration of (100).

5 Appendix

We first notice that (SI_2) is controllable. We construct the required sequences assuming that σ is the standard saturation function σ_0. The difference with an arbitrary iS-function simply consists in a different set of *const.* appearing below.

5.1 Construction of $(u_n^{(1)})$

We construct an l.a.c. curve $c : [0,\infty) \to I\!R^2$ such that c is a trajectory of $(SI_2)_{k_L}$ corresponding to an piecewise C^1 input u. Then each input of the sequence $(u_n^{(1)})$ will be the concatenation of u_n, the restriction of u to an finite time interval $[0, T_n]$, and the zero input for $t > T_n$. There exists a continuous input $\bar{u}$ which steers 0 to $(0, V_0^{\frac{1}{2}})$ with $V_0^{\frac{1}{2}} \geq C$ in time $T_0 > 0$ along $(SI_2)_{k_L}$. Then c is defined on $[0, T_0]$ as the resulting trajectory. We take $u = C - z$ until $z = y + 2$ and then $u = 0$ until $z = 0$ again. By symmetry with respect to $(0,0)$, we do the same from $(0, -V')$ and we end up at $(0, V_1)$ for a time T_1. The above construction is then repeated indefinitely. We thus define two sequences (V_k) and (T_k) such that we have *const.* for which

$$const.V_k^{\frac{1}{2}} \leq T_{k+1} - T_k \leq const.V_k^{\frac{1}{2}}, \quad const.V_k^{\frac{1}{2}} \leq V_{k+1} - V_k \leq constV_k^{\frac{1}{2}}.$$

We also obtain that for $k \geq 0$, $const.k^2 + V_0 \leq V_k \leq const.k^2 + V_0$. It is easy then to get that for $1 \leq p \leq q$ and k large,

$$
\begin{aligned}
const.k^{2q+1} \leq \int_{T_k}^{T_{k+1}} |z|^q &\leq const.k^{2q+1}, \quad const.k^{q+1} \leq \int_{T_k}^{T_{k+1}} |y|^q \leq const.k^{q+1}, \\
const.k^p &\leq \int_{T_k}^{T_{k+1}} |u|^p \leq const.k^p.
\end{aligned}
$$

For $n \geq 0$, the input $u_n^{(1)}$ is defined as u on $[0, T_n]$ and 0 afterwards. If we use $(z_n^{(1)}, y_n^{(1)})$ to denote the resulting trajectory, then we get

$$
\begin{aligned}
const.n^{\frac{2(q+1)}{q}} \leq \|z_n^{(1)}\|_q &\leq const.n^{\frac{2(q+1)}{q}}, \quad const.n^{\frac{q+2}{q}} \leq \|y_n^{(1)}\|_q \leq const.n^{\frac{q+2}{q}}, \\
const.n^{\frac{p+1}{p}} &\leq \|u_n^{(1)}\|_p \leq const.n^{\frac{p+1}{p}}.
\end{aligned}
$$

5.2 Construction of $(u_n^{(2)})$

We already know that the shape found in Proposition 3 is the profile (3) in Theorem 1 thanks to Remark 1. But even though the L^p-norms of the inputs $\bar{u}_n$ involved to obtain the preceding result can be chosen increasing to ∞, their L^∞-norms are of course very small. For $n \geq 0$, the input $u_n^{(2)}$ is simply obtained as the concatenation of an input v_n and an input $\bar{u}_{n_k}$ given as follows: v_n is defined on some $[0, T_{v_n}]$ with large L^∞-norm and such that the resulting trajectory reaches the origin at time T_{v_n} and $\bar{u}_{n_k}$ is chosen such that its L^p-norm is bigger than the L^p-norm of v_n. Therefore, the sequence $(u_n^{(2)})$ completes the proof that the shape found in Proposition 3 is the profile (3) in Theorem 1 and the $u_n^{(2)}$ have arbitrarily large L^∞-norm.

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