Integration of Brownian vector fields

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Summary. Using the Wiener chaos decomposition, we show that strong solutions of non Lipschitzian S.D.E.'s are given by random Markovian kernels. The example of Sobolev flows is studied in some detail, exhibiting interesting phase transitions.

Résumé. Intégration de champs de vecteurs browniens.

En utilisant la décomposition en chaos de Wiener, nous montrons que les solutions fortes d'E.D.S. non Lipschitziennes sont données par des noyaux markoviens aléatoires. L'exemple des flots de Sobolev est étudié, d'intéressantes transitions de phases sont observées.

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Introduction.

The purpose of this paper is to present an extended notion of strong solution to S.D.E.'s driven by Wiener processes. These solutions can be defined on rather general spaces.

They are not always given by flows of maps but by flows of Markovian kernels, which means splitting can occur. Coalescent flows also appear as solutions of these S.D.E.'s. Conditions are given under which coalescence and splitting occur or not.

A variety of examples are studied. The case of isotropic Sobolev flows shows in particular that splitting is related to hyperunstability and coalescence to hyperstability. We assume the diffusion induced by the flow is associated to a Dirichlet form. It is a convenient and well studied framework but this condition could clearly be relaxed (in particular to the framework of coercive forms).

The typical example we have in mind is the Brownian motion on a Riemannian manifold. We consider a covariance on vector fields which induces the Riemannian metric on each tangent space. When the covariance function has enough regularity, it is known that one can solve the linear S.D.E. driven by the canonical Wiener process associated to this covariance and get a multiplicative Brownian motion on the diffeomorphism group, which moves every point as a Brownian motion (see Le Jan and Watanabe [17] or Kunita [13]). But models related to turbulence theory produce natural examples where the regularity condition is not satisfied. Except for the work of Darling [5], where strong solutions are not considered, these S.D.E.'s have not been really studied. The idea is to define the solutions by their Wiener chaos expansion in terms of the heat semigroup. A similar expansion was given by Krylov and Veretennikov in [12], for S.D.E.'s with strong solutions.

In this form, they appear as a semigroup of operators, and the fact that these operators are Markovian is not clearly visible in the formula. To prove this, we consider an independent realization of the Brownian motion on the manifold and couple it with the Wiener process on vector fields using martingales. Then, the Markovian random operators which constitute the strong solution are obtained by filtering the Brownian motion with respect to the Wiener process. They determine the law of a canonical weak solution of the equation given the Wiener process on vector fields. This construction has been adequately generalized to be presented in the case of symmetric diffusions on a locally compact metric space. Relations with particle representations and filtering of S.P.D.E.'s can be observed (see Kurtz and Xiong [14]).

Criteria are provided to determine the nature of the solutions, namely if splitting or coalescence occur. The example of Sobolev flows is studied in details on Euclidean spaces and spheres and is of major interest especially in dimension 2 and 3 where an interesting phase diagram is given in terms of the two parameters determining the Sobolev norm on vector fields : The differentiability index and the relative weight of gradients and divergence free fields (compressibility).

Some of these results have been given in the note [16]. They are directly connected to a serie of works of Gawedzki and al on turbulent advection ([1], [9] and [10])

1 Covariance function on a manifold.

Let X be a manifold. A covariance function C on T^*X is a map from T^*X^2 in $\mathbb{R}$ such that, for any $(x,y) \in X^2$, C restricted to $T^*_xX \times T^*_yX$ is bilinear and such that for any n-uples $(\xi_1, ..., \xi_n)$ of T^*X,

$$\sum_{i,j} C(\xi_i, \xi_j) \geq 0. \tag{1.1}$$

For any $\xi = (x,u) \in T^*X$, let C_ξ be the vector field such that for any $\xi' = (y,v) \in T^*X$,

$$\langle C_\xi(y), v\rangle = C(\xi, \xi').$$

Let H_0 be the vector space generated by the vector fields C_ξ. Let us define the bilinear form on H_0, $\langle .,. \rangle_C$ such that

$$\langle C_\xi, C_{\xi'}\rangle_C = C(\xi, \xi'). \tag{1.2}$$

As equation (1.1) is satisfied, the bilinear form $\langle .,. \rangle_C$ is a scalar product on H_0. We denote $\|.\|_C$ the norm associated to $\langle .,. \rangle_C$.

Let H be the completion of H_0 with respect to $\|.\|_C$. $(H, \langle .,. \rangle_C)$ is a Hilbert space and we will design it as the self reproducing space associated to the covariance function C. H is constituted of vector fields on X and for any $h \in H$ and any $\xi = (x,u) \in T^*X$,

$$\langle C_\xi, h\rangle_C = \langle h(x), u\rangle. \tag{1.3}$$

Let $(e_k)_k$ be an orthonormal basis of H, then equation (1.3) implies that for any $\xi = (x,u) \in T^*X$,

$$C_\xi = \sum_k \langle e_k(x), u\rangle e_k. \tag{1.4}$$

This equation implies that for any $\xi' = (y,v) \in T^*X$,

$$C(\xi, \xi') = \sum_k \langle e_k(x), u\rangle \langle e_k(y), v\rangle. \tag{1.5}$$

Therefore

$$C = \sum_k e_k \otimes e_k. \tag{1.6}$$

Remark 1.1 *On the other hand, if we start with a countable family of vector fields $(V_k)_k$, such that for any $\xi = (x,u) \in T^*X$, $\sum_k \langle V_k(x), u\rangle^2 < \infty$, it is possible to define a covariance function on X by the formula*

$$C = \sum_k V_k \otimes V_k.$$

If a Riemannian metric $\langle .,. \rangle_x$ is given on X, we say that the covariance is bounded by the metric if $C(\xi, \xi) \leq \langle u, u\rangle_x$ for any $\xi = (x,u) \in T^*X$. Examples of isotropic covariances are given in section 9 and 10.

2 Covariance function bounded by a Dirichlet form.

Let X be a locally compact separable metric space and m be a positive Radon measure on X such that Supp$[m] = X$. On a metric space, as we do not have vector fields, we will have to define a Hilbert space that will play the role of the self-reproducing space associated to a covariance function.

Let $\mathcal{F}$ be a vector space of functions on X. Let $\mathcal{V} = L^2(m) \otimes \mathcal{F}$. A covariance function C on $\mathcal{F}$ is a bilinear mapping from $\mathcal{F} \times \mathcal{F}$ in $L^2(m \otimes m)$ such that for any $u_i \in L^2(m)$ and any $f_i \in \mathcal{F}$,

$$\int \sum_{i,j} u_i \otimes u_j \; C(f_i, f_j) \; dm^{\otimes 2} \geq 0. \tag{2.1}$$

This equation is the equivalent of equation (1.1). When X is a Riemannian manifold and m is the volume measure, then we can take $\mathcal{F} = H^1(X)$ and for $(f, g) \in \mathcal{F}^2$ and $(x, y) \in X^2$, $C(f, g)(x, y) = C((x, df(x)), (y, dg(y))$.

As equation (2.1) is satisfied, if $v = \sum_{i=1}^n u_i \otimes f_i \in \mathcal{V}$, we can define the C-norm of v,

$$\|v\|_C^2 = \int \sum_{i,j} u_i \otimes u_j \; C(f_i, f_j) \; dm^{\otimes 2}. \tag{2.2}$$

Let H be the separate completion of $\mathcal{V}$ with respect to $\|.\|_C$. If we denote $\langle ., . \rangle_C$ the scalar product associated to $\|.\|_C$, $(H, \langle ., . \rangle_C)$ is then a Hilbert space.

For any $h \in H$ and any $f \in \mathcal{F}$, let us denote $D_h f$, the unique function in $L^2(m)$ such that for any $u \in L^2(m)$,

$$\langle u, D_h f \rangle_{L^2(m)} = \langle h, u \otimes f \rangle_C. \tag{2.3}$$

It is clear that the map D from $H \times \mathcal{F}$ in $L^2(m)$ such that $D(h, f) = D_h f$ is bilinear. In the case where X is a differentiable manifold, H is a space of vector fields and $D_h f(x)$ is a derivative at x of f in the direction $h(x)$.

Let $(e_k)_k$ be an orthonormal basis of H, then as in section 1, for any $u \in L^2(m)$ and any $f \in \mathcal{F}$,

$$u \otimes f = \sum_k \langle e_k, u \otimes f \rangle_C \; e_k = \sum_k \left(\int u D_{e_k} f \; dm \right) \; e_k. \tag{2.4}$$

This equation implies that for any $(u, f) \in L^2(m) \times \mathcal{F}$ and any $(v, g) \in L^2(m) \times \mathcal{F}$,

$$\langle u \otimes f, v \otimes g \rangle_C = \int u \otimes v \; C(f, g) \; dm^{\otimes 2} \tag{2.5}$$

$$= \int u \otimes v \; (\sum_k D_{e_k} \otimes D_{e_k})(f, g) \; dm^{\otimes 2}. \tag{2.6}$$

Therefore

$$C = \sum_k D_{e_k} \otimes D_{e_k}. \tag{2.7}$$

Remark 2.1 *From equation (2.7), we see that to define a covariance C on $\mathcal{F}$, it is enough to be given a Hilbert space H and a bilinear mapping D from $H \times \mathcal{F}$ in $L^2(m)$ such that for any $f \in \mathcal{F}$, $\sum_k (D_{e_k} f)^2 \in L^1(m)$, where $(e_k)_k$ is an orthonormal basis of H.*

Proof. We just have to check that for any $(f,g) \in \mathcal{F}^2$, $C(f,g) \in L^2(m \otimes m)$, where C is defined by equation (2.7) and this is clearly true as $\sum_k (D_{e_k} f)^2 \in L^1(m)$ for any $f \in \mathcal{F}$. □

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet space, $\mathcal{F} \subset L^2(X,m)$. We will suppose that the Dirichlet form is local and conservative. We will note P_t, the associated Markovian semigroup, A its generator and $\mathcal{D}(A)$ the domain of A. We will also suppose that m is an invariant measure, making A and P_t self-adjoint in $L^2(m)$. As $\mathcal{E}$ is conservative, $P_t 1 = 1$. We will also assume that for any $f \in \mathcal{F}_b = L^\infty(m) \cap \mathcal{F}$, there exists $\Gamma(f,f) \in L^1(m)$ such that for any $g \in \mathcal{F}_b$,

$$2\mathcal{E}(fg, f) - \mathcal{E}(f^2, g) = \int g\Gamma(f,f)\, dm. \tag{2.8}$$

Γ can be extended to $\mathcal{F}$ and we note $\Gamma(f,g)$ the $L^1(m)$-valued bilinear form on $\mathcal{F}^2$, where for any $(f,g) \in \mathcal{F}^2$, $\Gamma(f,g) = \frac{1}{4}(\Gamma(f+g, f+g) - \Gamma(f-g, f-g))$. A sufficient condition of the existence of Γ (see corollary 4.2.3 in [2]) is that $\mathcal{D}(A)$ contains a subspace E of $\mathcal{D}(A)$, dense in $\mathcal{F}$, such that

$$\forall f \in E, \qquad f^2 \in \mathcal{D}(A).$$

Then, for $(f,g) \in E^2$,

$$\Gamma(f,g) = A(fg) - f\ Ag - g\ Af. \tag{2.9}$$

A necessary and sufficient condition for the existence of the energy density (or carré du champ operator) Γ is given in theorem 4.2.2 in [2].

Example 2.2 *M is a Riemannian manifold with the metric $\langle ., . \rangle$, m is the volume measure, $\mathcal{F} = H^1(M)$ and for any $(f,g) \in \mathcal{F} \otimes \mathcal{F}$,*

$$\mathcal{E}(f,g) = \frac{1}{2} \int_M \langle \nabla f, \nabla g \rangle\, dm.$$

In this case, $\Gamma(f,g) = \langle \nabla f, \nabla g \rangle$.

Assume we are given a covariance function on $\mathcal{F}$. For $f \in \mathcal{F}^2$, let us note $D(f,f)$, or $|Df|^2$, the series $\sum_k (D_{e_k} f)^2$. The definition of $D(f,f)$ is independent of the choice of the basis (as trace of a bilinear form).

Let us suppose that the covariance function is bounded by the Dirichlet form in the sense that

$$D(f,f) \leq \Gamma(f,f). \tag{2.10}$$

Note that the interesting case is when there is equality in equation (2.10). We can also define for any $(f,g) \in \mathcal{F}^2$, $D(f,g) = \sum_k D_{e_k} f D_{e_k} g \in L^1(m)$. When X is a manifold, (2.10) is

$$\sum_k (e_k(f)(x))^2 \leq \Gamma(f,f)(x).$$

Remark that condition (2.10) implies that for any finite family $(u_i, f_i) \in L^\infty(m) \times \mathcal{F}$,

$$\sum_{i,j} u_i u_j D(f_i, f_j) \leq \sum_{i,j} u_i u_j \Gamma(f_i, f_j). \tag{2.11}$$

When the u_i are step functions with discontinuities in a set of zero measure, (2.11) is satisfied as $\sum_{i,j} u_i u_j D(f_i, f_j) = |D(\sum_i u_i f_i)|^2$. Then, we can extend to any family (u_i) by density in $L^2(\Gamma(f,f)dm)$ for every f.

Remark 2.3 *The bilinear mapping D is a derivation : for any $h \in H$ and any $f \in \mathcal{F}$ such that $f^2 \in \mathcal{F}$,*

$$D_h f^2 = 2f D_h f. \tag{2.12}$$

Proof. We first make the remark that

$$\sum_k (D_{e_k} f^2 - 2f D_{e_k} f)^2 = D(f^2, f^2) - 4f D(f^2, f) + 4f^2 D(f, f).$$

Integrating this relation with respect to m and using (2.11), we get that

$$\int \sum_k (D_{e_k} f^2 - 2f D_{e_k} f)^2 \, dm \leq \int \left(\Gamma(f^2, f^2) - 4f\Gamma(f^2, f) + 4f^2 \Gamma(f, f) \right) \, dm = 0.$$

This implies that for every k, $D_{e_k} f^2 - 2f D_{e_k} f = 0$. □

3 Construction of the statistical solutions.

Let us given a covariance function C bounded by a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on a locally compact separable metric space as in the preceding section (equation (2.10) is satisfied). Let W_t be a cylindrical Brownian motion on H, i.e a Gaussian process indexed by $H \times \mathbb{R}^+$ with covariance matrix $\mathrm{cov}(W_t(h), W_s(h')) = s \wedge t \ \langle h, h' \rangle_C$. Set $W_t^k = W_t(e_k)$, $(W_t^k;\ k \in \mathbb{N})$ is a sequence of independent Wiener processes and we can represent W_t by $\sum_k W_t^k e_k$. Let $\mathcal{F}_t = \sigma(W_s^k;\ k \in \mathbb{N}; s \leq t) = \sigma(W_s;\ s \leq t)$.

Proposition 3.1 *Let $S_t^0 = P_t$. We can define a sequence S_t^n of random operators on $L^2(m)$ such that $E[(S_t^n f)^2] \leq P_t f^2$ and S_t^n is $\mathcal{F}_t$-measurable, by the recurrence formula*

$$S_t^{n+1} f = P_t f + \sum_k \int_0^t S_s^n (D_{e_k} P_{t-s} f)) \, dW_s^k. \tag{3.1}$$

Proof. Suppose we are given S_t^n, a $\mathcal{F}_t$-measurable random operator on $L^2(m)$ such that $E[(S_t^n f)^2] \leq P_t f^2$.

Let $f \in L^2(m)$, for any positive t, $P_t f \in \mathcal{F}$, so $D_{e_k} P_{t-s} f$ is well defined and belongs to $L^2(m)$.

$$\begin{aligned} E[(S_t^{n+1} f)^2] &= (P_t f)^2 + \sum_k \int_0^t E\left[(S_s^n (D_{e_k} P_{t-s} f))^2 \right] \, ds \\ &\leq (P_t f)^2 + \int_0^t P_s(|D P_{t-s} f|^2) \, ds \\ &\leq (P_t f)^2 + \int_0^t P_s(\Gamma(P_{t-s} f, P_{t-s} f)) \, ds. \end{aligned}$$

For $f \in L^\infty(m) \cap L^2(m)$, $\frac{\partial}{\partial s} P_s((P_{t-s} f)^2) = P_s(\Gamma(P_{t-s} f, P_{t-s} f))$ and

$$P_t f^2 = (P_t f)^2 + \int_0^t P_s(\Gamma(P_{t-s} f, P_{t-s} f)) \, ds. \tag{3.2}$$

An approximation by truncation shows that equation (3.2) remains true for $f \in L^2(m)$ and $E[(S_t^{n+1} f)^2] \leq P_t f^2$. □

Remark. *The definition of S_t^n is independent of the choice of the basis on H.*

In the following, we will use the canonical realization of the processes W_t^k. They will be defined as coordinate functions on $\Omega = C(\mathbb{R}^+, \mathbb{R})^{\mathbb{N}}$, with the product Wiener measure P. We note θ_t the natural shift on Ω, such that $W_{t+s}^k - W_t^k = W_s^k \circ \theta_t$.

Theorem 3.2 *The family of random operators S_t^n converge in $L^2(P)$ towards a one parameter family of $\mathcal{F}_t$-adapted Markovian operators S_t such that*

a) $S_{t+s} = S_t(S_s \circ \theta_t)$, for any $s, t \geq 0$;

b) $\forall f \in L^2(m)$, $S_t f$ is uniformly continuous with respect to t in $L^2(m \otimes P)$;

c) $E[(S_t f)^2] \leq P_t f^2$, for any $f \in L^2(m)$;

d) $S_t f = P_t f + \sum_k \int_0^t S_s(D_{e_k} P_{t-s} f)\, dW_s^k$, for any $f \in L^2(m)$;

e) $S_t f = f + \sum_k \int_0^t S_s(D_{e_k} f)\, dW_s^k + \int_0^t S_s(Af)\, ds$, for any $f \in \mathcal{D}(A)$.

S_t is uniquely characterized by c) and d) or by a), c) and e). We call it the statistical solution of the S.D.E. (see 3.22 below)

$$\forall f \in \mathcal{D}(A): \quad df(X_t) = \sum_k D_{e_k} f(X_t)\, dW_t^k + Af(X_t)\, dt. \tag{3.3}$$

Note that this S.D.E. does not always have a strong solution in the usual sense.

Proof. The convergence of S_t^n is immediate as for any $n \geq 1$, $J_t^n f = S_t^n f - S_t^{n-1} f$ is in the n-th Wiener chaos, $S_t f = P_t f + \sum_{n=1}^{\infty} J_t^n f$ and $(P_t f)^2 + \sum_{n \geq 1} E[(J_t^n f)^2] = \lim_{n \to \infty} E[(S_t^n f)^2] \leq P_t f^2$. It is clear that S_t is $\mathcal{F}_t$-adapted and satisfies c). d) is obtained taking the limit in the recurrence formula of the proposition. a) is deduced from the formula : $J_{t+s}^n = \sum_{k \leq n} J_t^k (J_s^{n-k} \circ \theta_t)$.

The uniqueness of a solution of d) verifying c) follows directly from the uniqueness of the Wiener chaos decomposition, obtained by iteration of d).

Proof of b). Let us remark that for any positive ε, $S_{t+\varepsilon} - S_t = S_t(S_\varepsilon \circ \theta_t - I)$. As S_t and $S_\varepsilon \circ \theta_t$ are independent and m is invariant under P_t , for any $f \in L^2(m)$

$$\begin{aligned} \int E[(S_{t+\varepsilon} f - S_t f)^2]\, dm &\leq \int E[P_t(S_\varepsilon \circ \theta_t f - f)^2]\, dm \\ &\leq \int E[(S_\varepsilon \circ \theta_t f - f)^2]\, dm \\ &\leq \int (P_\varepsilon f^2 - 2 f P_\varepsilon f + f^2)\, dm \\ &\leq 2 \|f\|_{L^2(m)} \|f - P_\varepsilon f\|_{L^2(m)}. \end{aligned} \tag{3.4}$$

Therefore, $\lim_{\varepsilon \to 0} \|S_{t+\varepsilon} f - S_t f\|_{L^2(m \otimes P)} = 0$, uniformly in t. □

Proof of e). Let us remark that for any ε and t positive,

$$\begin{aligned} S_{t+\varepsilon}f - S_t f &= S_t\left(P_\varepsilon f + \sum_k \int_0^\varepsilon S_u \circ \theta_t (D_{e_k} P_{\varepsilon-u} f)\, dW_u^k \circ \theta_t - f\right) \\ &= S_t(P_\varepsilon f - f) + \sum_k \int_t^{t+\varepsilon} S_s(D_{e_k} P_{t+\varepsilon-s} f)\, dW_s^k. \end{aligned} \tag{3.5}$$

Hence using (3.5) for $t = \frac{i}{n}t$ and $\varepsilon = \frac{1}{n}t$, for $f \in \mathcal{D}(A)$,

$$\begin{aligned} S_t f - f - \sum_k \int_0^t S_s(D_{e_k} f)\, dW_s^k - \int_0^t S_s(Af)\, ds \;=& \\ &= \sum_{i=0}^{n-1} \left[S_{\frac{i}{n}t}(P_{\frac{t}{n}} f - f) + \sum_k \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(D_{e_k} P_{\frac{i+1}{n}t - s} f)\, dW_s^k \right. \\ &\qquad \left. - \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(Af)\, ds - \sum_k \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(D_{e_k} f)\, dW_s^k \right] \\ &= A_1(n) + A_2(n) + A_3(n), \quad \text{with} \end{aligned}$$

$$A_1(n) = \sum_{i=0}^{n-1} S_{\frac{i}{n}t}(P_{\frac{t}{n}} f - f - \frac{t}{n} Af); \tag{3.6}$$

$$A_2(n) = \sum_{i=0}^{n-1} \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} (S_{\frac{i}{n}t}(Af) - S_s(Af))\, ds; \tag{3.7}$$

$$A_3(n) = \sum_{i=0}^{n-1} \sum_k \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(D_{e_k}(P_{\frac{i+1}{n}t - s} f - f))\, dW_s^k. \tag{3.8}$$

First, using the fact that m is P_t-invariant,

$$\|A_1(n)\|_{L^2(m\otimes P)} \le n \|P_{\frac{t}{n}} f - f - \frac{t}{n} Af\|_{L^2(m)} = \circ(1). \tag{3.9}$$

After, we remark that

$$\left\| \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} (S_{\frac{i}{n}t}(Af) - S_s(Af))\, ds \right\|^2_{L^2(m\otimes P)} \le \frac{t}{n} \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} \|S_{\frac{i}{n}t}(Af) - S_s(Af)\|^2_{L^2(m\otimes P)}\, ds.$$

As $S_t(Af)$ is uniformly continuous in $L^2(m\otimes P)$, there exists $\varepsilon(x)$ such that $\lim_{x\to 0} \varepsilon(x) = 0$ and $\|S_{\frac{i}{n}t}(Af) - S_s(Af)\|^2_{L^2(m\otimes P)} \le \varepsilon(\frac{t}{n})$ for any $s \in [\frac{i}{n}t, \frac{i+1}{n}t]$. Hence we get

$$\left\| \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} (S_{\frac{i}{n}t}(Af) - S_s(Af))\, ds \right\|^2_{L^2(m\otimes P)} \le \frac{t^2}{n^2} \varepsilon(\frac{t}{n})$$

and $\|A_2(n)\|_{L^2(m\otimes P)} = \circ(1)$.

At last, as the different terms in the sum in equation (3.8) are orthogonal,

$$\begin{aligned}
\|A_3(n)\|^2_{L^2(m\otimes P)} &= \sum_{i=0}^{n-1}\sum_k \int E\left[\left(\int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(D_{e_k}(P_{\frac{i+1}{n}t-s}f-f))dW_s^k\right)^2\right] dm \\
&\le \sum_{i=0}^{n-1}\int\int_{\frac{i}{n}t}^{\frac{i+1}{n}t} |D(P_{\frac{i+1}{n}t-s}f-f)|^2\ ds\ dm \\
&\le n\int_0^{\frac{t}{n}}\int |D(P_sf-f)|^2\ dm\ ds \\
&\le n\int_0^{\frac{t}{n}} \mathcal{E}(P_sf-f,P_sf-f)\ ds.
\end{aligned}$$

As $\lim_{s\to 0}\mathcal{E}(P_sf-f,P_sf-f)=0$, $\|A_3(n)\|_{L^2(m\otimes P)}=\circ(1)$.

Taking the limit as n goes to ∞, this shows that $\|S_tf-f-\sum_k\int_0^t S_s(D_{e_k}f)\ dW_s^k - \int_0^t S_s(Af)\ ds\|_{L^2(m\otimes P)}=0$. □

Proof that a), c) and e) imply d). Take $f\in L^2(m)$ and ε positive, assuming e),

$$\begin{aligned}
&S_tP_\varepsilon f - P_tP_\varepsilon f - \sum_k\int_0^t S_s(D_{e_k}P_{t-s}P_\varepsilon f)\ dW_s^k \ = \\
&\quad= \sum_{i=0}^{n-1}\left[S_{\frac{i+1}{n}t}(P_{t-\frac{i+1}{n}t}P_\varepsilon f) - S_{\frac{i}{n}t}(P_{t-\frac{i}{n}t}P_\varepsilon f) - \sum_k\int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(D_{e_k}(P_{t-s}P_\varepsilon f))\ dW_s^k\right] \\
&\quad= B_1(n)+B_2(n)+B_3(n), \quad \text{with}
\end{aligned}$$

$$B_1(n) = \sum_{i=0}^{n-1}\sum_k\int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(D_{e_k}(P_{t-\frac{i+1}{n}t}P_\varepsilon f - P_{t-s}P_\varepsilon f))dW_s^k; \tag{3.10}$$

$$B_2(n) = -\sum_{i=0}^{n-1} S_{\frac{i}{n}t}\left(P_{t-\frac{i}{n}t}P_\varepsilon f - P_{t-\frac{i+1}{n}t}P_\varepsilon f - \frac{t}{n}AP_{t-\frac{i+1}{n}t}P_\varepsilon f\right); \tag{3.11}$$

$$B_3(n) = \sum_{i=0}^{n-1}\int_{\frac{i}{n}t}^{\frac{i+1}{n}t}(S_s - S_{\frac{i}{n}t})(AP_{t-\frac{i+1}{n}t}P_\varepsilon f)\ ds, \tag{3.12}$$

as $S_{\frac{i+1}{n}t}(P_{t-\frac{i+1}{n}t}P_\varepsilon f) = S_{\frac{i}{n}t}(P_{t-\frac{i+1}{n}t}P_\varepsilon f) + \sum_{i=0}^{n-1}\int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(AP_{t-\frac{i+1}{n}t}P_\varepsilon f)\ ds$.

As the different terms in the sum in equation (3.10) are orthogonal,

$$\begin{aligned}
\|B_1(n)\|^2_{L^2(m\otimes P)} &\le \sum_{i=0}^{n-1}\int\int_{\frac{i}{n}t}^{\frac{i+1}{n}t} |D(P_{t-\frac{i+1}{n}t}P_\varepsilon f - P_{t-s}P_\varepsilon f)|^2\ ds\ dm \\
&\le \sum_{i=0}^{n-1}\int_{\frac{i}{n}t}^{\frac{i+1}{n}t} \mathcal{E}(P_{t-\frac{i+1}{n}t}P_\varepsilon f - P_{t-s}P_\varepsilon f, P_{t-\frac{i+1}{n}t}P_\varepsilon f - P_{t-s}P_\varepsilon f)\ ds \\
&\le n\int_0^{\frac{t}{n}} \mathcal{E}(P_sP_\varepsilon f - P_\varepsilon f, P_sP_\varepsilon f - P_\varepsilon f)\ ds,
\end{aligned} \tag{3.13}$$

as $\mathcal{E}(P_t f, P_t f) \leq \mathcal{E}(f,f)$, for any positive t and any $f \in L^2(m)$.

Equation (3.13) implies that $\|B_1(n)\|_{L^2(m\otimes P)} = \circ(1)$ (as $\lim_{s\to 0} \mathcal{E}(P_s P_\varepsilon f - P_\varepsilon f, P_s P_\varepsilon f - P_\varepsilon f) = 0$).

$$\begin{aligned}
\|B_2(n)\|_{L^2(m\otimes P)} &\leq \sum_{i=0}^{n-1} \|S_{\frac{i}{n}t}\left(P_{t-\frac{i}{n}t}P_\varepsilon f - P_{t-\frac{i+1}{n}t}P_\varepsilon f - \frac{t}{n}AP_{t-\frac{i+1}{n}t}P_\varepsilon f\right)\|_{L^2(m\otimes P)} \\
&\leq \sum_{i=0}^{n-1}\left(\int\left(P_{\frac{i+1}{n}t}P_\varepsilon f - P_{\frac{i}{n}t}P_\varepsilon f - \frac{t}{n}AP_{\frac{i}{n}t}P_\varepsilon f\right)^2 dm\right)^{\frac{1}{2}} \\
&\leq n\|P_{\frac{t}{n}}P_\varepsilon f - P_\varepsilon f - \frac{t}{n}AP_\varepsilon f\|_{L^2(m)}.
\end{aligned}$$

Hence, $\|B_2(n)\|_{L^2(m\otimes P)} = \circ(1)$.

Note that if $Q_t f = E[S_t f]$, e) implies that for any $f \in \mathcal{D}(A)$,

$$Q_t f = f + \int_0^t Q_s(Af)ds.$$

Then $\frac{\partial}{\partial_s} Q_s P_{t-s} f = 0$ for any $f \in L^2(m)$ and $0 < s < t$ (then $P_{t-s}f \in \mathcal{D}(A)$) and we have $Q_t f = P_t f$. With this remark and the fact that a) and c) are satisfied, we see that b) and equation (3.4) are satisfied (see the proof of b). Using (3.4), we have

$$\begin{aligned}
\|(S_s - S_{\frac{i}{n}t})(AP_{t-\frac{i+1}{n}t}P_\varepsilon f)\|^2_{L^2(m\otimes P)} &\leq \\
&\leq 2\|AP_{t-\frac{i+1}{n}t}P_\varepsilon f\|_{L^2(m)}\|AP_{t-\frac{i+1}{n}t}P_\varepsilon f - P_{s-\frac{i}{n}t}AP_{t-\frac{i+1}{n}t}P_\varepsilon f\|_{L^2(m)} \\
&\leq 2\|AP_\varepsilon f\|_{L^2(m)}\|AP_\varepsilon f - P_{s-\frac{i}{n}t}AP_\varepsilon f\|_{L^2(m)} \\
&\leq 4\|AP_\varepsilon f\|^2_{L^2(m)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|B_3(n)\|^2_{L^2(m\otimes P)} &\leq \sum_{i=0}^{n-1}\frac{t}{n}\int_{\frac{i}{n}t}^{\frac{i+1}{n}t}\|(S_s - S_{\frac{i}{n}t})(AP_{t-\frac{i+1}{n}t}P_\varepsilon f)\|^2_{L^2(m\otimes P)}\, ds \\
&\leq \frac{4t^2}{n}\|AP_\varepsilon f\|^2_{L^2(m)}.
\end{aligned}$$

Taking the limit as n goes to ∞, this shows that d) is satisfied for $P_\varepsilon f$, with $f \in L^2(m)$ and ε positive.

At last, as $\|S_t P_\varepsilon f - S_t f\|_{L^2(m\otimes P)} \leq \|P_\varepsilon f - f\|_{L^2(m)}$ (because c) is satisfied), $\|P_{t+\varepsilon}f - P_t f\|_{L^2(m\otimes P)} \leq \|P_\varepsilon f - f\|_{L^2(m)}$ and $\|\sum_k \int_0^t S_s(D_{e_k}P_{t-s}(P_\varepsilon f - f))dW_s^k\|^2_{L^2(m\otimes P)} \leq t\mathcal{E}(P_\varepsilon f - f, P_\varepsilon f - f)$, taking the limit when ε goes to 0, we prove that d) is satisfied for any $f \in L^2(m)$. □

Proof that S_t is Markovian.

A more concise proof of this fact has been given in [16], relying on Wiener exponentials and Girsanov formula. The advantage of the following proof is to be more explanatory, to

give a relation with weak solutions and to yield a construction of the process law associated with the statistical solution S_t.

Let $(\Omega', \mathcal{G}, \mathcal{G}_t, X_t, P_x)$ be a Hunt process associated to $(\mathcal{E}, \mathcal{F})$ (see [8]), we will take a canonical version with $\Omega' = C(\mathbb{R}^+, X)$. Let $W_t = \sum_k W_t^k e_k$ be a cylindrical Brownian motion on H, independent of the Markov process X_t.

Let $\mathcal{M}$ be the space of the martingales additive functionals, $\mathcal{G}_t$-adapted such that if $M \in \mathcal{M}$, $E_x[M_t^2] < \infty$, $E_x[M_t] = 0$ q.e. and if $e(M) = \sup_{t>0} \frac{1}{2t} E_m[M_t^2]$ (with $P_m = \int P_x \, dm(x)$), $e(M) < \infty$. $(\mathcal{M}, e)$ is a Hilbert space (see [8]).

For $f \in \mathcal{F}$, $M^f \in \mathcal{M}$ denotes the martingale part of the semi-martingale $f(X_t) - f(X_0)$. For $g \in C_K(X) \subset L^2(\Gamma(f,f)dm)$, we note $g.M^f \in \mathcal{M}$ the martingale $\int_0^t g(X_s) \, dM_s^f$, then $\mathcal{M}_0 = \{\sum_{i=1}^n g_i.M^{f_i};\ n \in \mathbb{N},\ g_i \in C_K(X),\ f_i \in \mathcal{F}\}$ is dense in $\mathcal{M}$ (see lemma 5.6.3 in [8]), and $e(\sum_i g_i.M^{f_i}) = \frac{1}{2} \sum_{i,j} \int g_i g_j \Gamma(f_i, f_j) \, dm$ (see theorem 5.2.3 and 5.6.1 in [8]).

Lemma 3.3 *For every $(M, N) \in \mathcal{M} \times \mathcal{M}$, there exists $\Gamma(M, N) \in L^1(m)$ such that*

$$\langle M, N \rangle_t = \int_0^t \Gamma(M, N)(X_s) \, ds, \tag{3.14}$$

where $\langle ., . \rangle_t$ is the usual martingale bracket. And for $(f, g) \in \mathcal{F}$, $\Gamma(M^f, M^g) = \Gamma(f, g)$.

Proof. When $f \in \mathcal{F}$, it follows from theorem 5.2.3 in [8] that

$$\langle M^f, M^f \rangle_t = \int_0^t \Gamma(f, f)(X_s) \, ds.$$

For $M = \sum_i h_i.M^{f_i}$, $N = \sum_j k_j.M^{g_j}$, two martingales of $\mathcal{M}_0$,

$$\langle M, N \rangle_t = \sum_{i,j} \int_0^t h_i k_j \Gamma(f_i, g_j)(X_s) \, ds = \int_0^t \Gamma(M, N)(X_s) \, ds, \tag{3.15}$$

with $\Gamma(M, N) = \sum_{i,j} h_i k_j \Gamma(f_i, g_j)$. Γ is a bilinear mapping from $\mathcal{M}_0 \times \mathcal{M}_0$ in $L^1(m)$. Γ is continuous since

$$\begin{aligned} \int |\Gamma(M, N)| \, dm &\leq \int \Gamma(M, M)^{\frac{1}{2}} \Gamma(N, N)^{\frac{1}{2}} \, dm \\ &\leq \left(\int \Gamma(M, M) \, dm \int \Gamma(N, N) \, dm \right)^{\frac{1}{2}} \\ &\leq e(M)^{\frac{1}{2}} e(N)^{\frac{1}{2}}. \end{aligned}$$

It follows that Γ can be extended to $\mathcal{M} \times \mathcal{M}$.

Take $M \in \mathcal{M}$ and an approximating sequence $M_n \in \mathcal{M}_0$. $e(M_n - M)$ converges towards 0, M_n converges towards M in $L^2(P_x)$ and $\langle M_n, M_n \rangle_t$ converges in $L^1(P_x)$ towards $\langle M, M \rangle_t$ for a.e x (see section 5-2 in [8]). This proves that $\langle M, M \rangle_t = \int_0^t \Gamma(M, M)(X_s) \, ds$. □

Note that lemma 3.3 implies that $e(M, N) = \frac{1}{2} \int \Gamma(M, N) \, dm$.

Lemma 3.4 *If m is bounded, for any $h \in H$, there exists a unique continuous martingale in $\mathcal{M}$, N^h such that for any $f \in \mathcal{F}$, $e(N^h, M^f) = \frac{1}{2}\int D_h f \, dm$ and $\frac{d}{dt}\langle N^h, M^f\rangle_t = D_h f(X_t)$. In addition, $e(N^h) \leq \frac{1}{2} m(X)\|h\|^2$ and $\langle N^h\rangle_t \leq \|h\|^2 t$.*

Proof. For $h = \sum_k \lambda_k e_k \in H$, let us define a linear form, α_h on $\mathcal{M}_0$ such that for any $M = \sum_{i=1}^n g_i.M^{f_i} \in \mathcal{M}_0$, $\alpha_h(M) = \frac{1}{2}\sum_{i=1}^n \int g_i D_h f_i \, dm$.

$$\begin{aligned}(\alpha_h(M))^2 &= \left(\sum_k \lambda_k \frac{1}{2}\sum_{i=1}^n \int g_i D_{e_k} f_i \, dm\right)^2 \\ &\leq \frac{1}{4}\|h\|^2 m(X) \sum_{i,j} \int g_i g_j D(f_i, f_j) \, dm \leq \frac{1}{2}\|h\|^2 m(X) e(M).\end{aligned}$$

This proves that α_h is continuous on $\mathcal{M}_0$ and can be extended to a continuous linear form on $\mathcal{M}$ such that $\alpha_h(M) \leq \frac{1}{\sqrt{2}}\|h\|\sqrt{m(X)e(M)}$. To this form is associated a unique $N^h \in \mathcal{M}$ such that $\alpha_h(M) = e(N^h, M)$.

Note that for any $g \in C_K(X)$ and $f \in \mathcal{F}$, we have $\int g D_h f \, dm = \int \Gamma(N^h, g.M^f) \, dm = \int g\Gamma(N^h, M^f) \, dm$. This is satisfied for every $g \in C_K(X)$, therefore for any $f \in \mathcal{F}$, $\Gamma(N^h, M^f) = D_h f$.

Note that we also have, for $M \in \mathcal{M}_0$

$$\Gamma(N^h, M) \leq \|h\| \, \Gamma(M, M)^{\frac{1}{2}},$$

which implies that $\langle N^h\rangle_t \leq \|h\|^2 t$. □

Remark 3.5 *When m is not bounded, N^h can be defined as a local martingale such that for any compact K and any $f \in \mathcal{F}$, $1_K.N^h \in \mathcal{M}$, $e(1_K.N^h, M^f) = \frac{1}{2}\int_K D_h f \, dm$. In addition, $e(1_K.N^h) \leq \frac{1}{2} m(K)\|h\|^2$.*

Let γ_{kl} be a function on X such that $\frac{d}{dt}\langle N^{e_l}, N^{e_k}\rangle_t = \gamma_{kl}(X_t)$. Lemma 3.4 implies that the matrix $A = ((\delta_{kl} - \gamma_{kl}))$ is positive (as $\frac{d}{dt}\langle N^h\rangle_t \leq \|h\|^2$). Therefore, it is possible to find a matrix R such that $R^2 = A$.

Set $\widetilde{W}_t^k = N_t^{e_k} + \sum_l \int_0^t R_{kl}(X_s) dW_s^l$.

Lemma 3.6 $\widetilde{W}_t^k$ *is a sequence of independent Brownian motion.*

Proof. As $\widetilde{W}_t^k$ is a continuous martingale, we just have to compute $\frac{d}{dt}\langle \widetilde{W}_t^k, \widetilde{W}_t^l\rangle_t$:

$$\frac{d}{dt}\langle \widetilde{W}_t^k, \widetilde{W}_t^l\rangle_t = \gamma_{kl} + R_{kl}^2 = \delta_{kl}.$$

This implies the lemma. □

Let $\widetilde{S}_t$ be the family of Markovian operators on $L^2(m)$ such that for any $f \in L^2(m)$ and μ an initial distribution absolutely continuous with respect to m,

$$\int \widetilde{S}_t f \, d\mu = E_\mu[f(X_t)|\sigma(\widetilde{W}_s^k; \; k \in \mathbb{N}; \; s \leq t)]. \tag{3.16}$$

Remark that as X_t is Markovian and W_t has independent increments,

$$\int \widetilde{S}_t f \; d\mu = E_\mu[f(X_t)|\sigma(\widetilde{W}_s^k;\; k \in \mathbb{N};\; s \geq 0)]. \tag{3.17}$$

In the same way, we see that $\widetilde{S}_t$ satisfies the multiplicative cocycle property a).

Lemma 3.7 *For any $f \in \mathcal{D}(A)$ and μ an initial distribution absolutely continuous with respect to m,*

$$\widetilde{S}_t f = f + \sum_k \int_0^t \widetilde{S}_s(D_{e_k} f)\; d\widetilde{W}_s^k + \int_0^t \widetilde{S}_s(Af)\; ds, \quad P_\mu \; a.s.$$

Proof. For any $f \in \mathcal{D}(A)$, we have

$$f(X_t) = f(X_0) + M_t^f + \int_0^t Af(X_s)\; ds. \tag{3.18}$$

It is clear that $E_.[\int_0^t Af(X_s)\; ds|\sigma(\widetilde{W}_t^k;\; k \in \mathbb{N};\; s \leq t)] = \int_0^t \widetilde{S}_s Af(X_s)\; ds$, as (3.17) is satisfied. Let $Z_t = \sum_k \int_0^t H_s^k \; d\widetilde{W}_s^k \in L^2(\sigma(\widetilde{W}_s^k;\; k \in \mathbb{N};\; s \leq t))$,

$$\begin{aligned}
E_.[Z_t M_t^f] &= \sum_k E_. \left[\int_0^t H_s^k \; d\langle \widetilde{W}^k, M^f\rangle_s\right] \\
&= \sum_k E_. \left[\int_0^t H_s^k D_{e_k} f(X_s)\; ds\right] \\
&= \sum_k E_. \left[\int_0^t H_s^k \widetilde{S}_s(D_{e_k} f)\; ds\right] \\
&= E_. \left[Z_t \sum_k \int_0^t \widetilde{S}_s(D_{e_k} f)\; d\widetilde{W}_s^k\right].
\end{aligned}$$

This proves that $E_.[M_t^f|\sigma(\widetilde{W}_t^k;\; k \in \mathbb{N};\; s \leq t)] = \sum_k \int_0^t \widetilde{S}_s D_{e_k} f \; d\widetilde{W}_s^k$. □

Now, using uniqueness in theorem 3.2 and the isomorphism j between $L^2(\sigma(\widetilde{W}_t^k;\; t \geq 0;\; k \in \mathbb{N}))$ and $L^2(\sigma(W_t^k;\; t \geq 0;\; k \in \mathbb{N}))$, we see that $j\widetilde{S}_t = S_t$, which implies that S_t is Markovian. □

Proposition 3.8 *For any $f \in \mathcal{F}_b$, the martingale*

$$P_t^f = M_t^f - \sum_k \int_0^t D_{e_k} f(X_s) d\widetilde{W}_s^k \tag{3.19}$$

is orthogonal to the family of martingales $\{\widetilde{W}_t^k;\; k \in \mathbb{N}\}$, in the sense of the martingale bracket (i.e for any k, $\langle P^f, W^k\rangle_. = 0$). And for any $(f,g) \in \mathcal{F}_b^2$,

$$\langle P^f, P^g\rangle_t = \int_0^t (\Gamma(f,g)(X_s) - D(f,g)(X_s))\; ds. \tag{3.20}$$

Proof. We just have to show that $\langle P^f, \widetilde{W}^k\rangle_t = 0$ for every $f \in \mathcal{F}_b$ and every $k \in \mathbb{N}$ which is true as

$$\langle M^f, \widetilde{W}^k\rangle_t = \langle M^f, N^{e_k}\rangle_t = \int_0^t D_{e_k} f(X_s)\ ds.$$

Let $(f,g) \in \mathcal{F}_b^2$, then

$$\begin{aligned}
\langle P^f, P^g\rangle_t &= \langle P^f, M^g\rangle_t \\
&= \langle M^f, M^g\rangle_t - \sum_k \int_0^t D_{e_k} f(X_s) d\langle \widetilde{W}^k, M^g\rangle_s \\
&= \int_0^t \Gamma(f,g)(X_s)\ ds - \sum_k \int_0^t D_{e_k} f(X_s) D_{e_k} g(X_s)\ ds \\
&= \int_0^t \Gamma(f,g)(X_s)\ ds - \int_0^t D(f,g)(X_s)\ ds. \qquad \square
\end{aligned}$$

Remark 3.9 *If $\Gamma = D$, then for any $f \in \mathcal{F}_b$,*

$$M_t^f = \sum_k \int_0^t D_{e_k} f(X_s) dN_s^{e_k}, \tag{3.21}$$

$D_{e_k} f = \sum_l D_{e_l} f\ \gamma_{kl}(X_t)$ and the positive symmetric matrix $P = ((\gamma_{kl}))$ is a projector. In this case, $R = I - P$.

Proof. Let $P_t^f = \sum_k \int_0^t D_{e_k} f(X_s)\ dN_s^{e_k}$, $P^f \in \mathcal{M}$, then for any $M = \sum_{i=1}^n g_i . M^{f_i} \in \mathcal{M}_0$,

$$\begin{aligned}
\langle P^f, M\rangle_t &= \sum_k \int_0^t D_{e_k} f(X_s) d\langle N^{e_k}, M\rangle_s \\
&= \sum_k \sum_{i=1}^n \int_0^t D_{e_k} f(X_s) g_i(X_s) D_{e_k} f_i(X_s)\ ds \\
&= \sum_{i=1}^n \int_0^t g_i(X_s) D(f, f_i)(X_s)\ ds \\
&= \langle M^f, M\rangle_t.
\end{aligned}$$

This implies that for any $M \in \mathcal{M}$, $e(P^f, M) = e(M^f, M)$ and $P^f = M^f$.

As $\frac{d}{dt}\langle M^f, N^{e_k}\rangle_t = D_{e_k} f(X_t)$, we get that

$$D_{e_k} f(X_t) = \sum_l D_{e_l} f(X_t) \frac{d}{dt}\langle N^{e_l}, N^{e_k}\rangle_t = \left(\sum_l D_{e_l} f\ \gamma_{kl}\right)(X_t).$$

This relation implies that $N_t^{e_k} = \sum_l \int_0^t \gamma_{kl}(X_s)\ dN_s^{e_l}$ (this is easy to check, considering $\frac{d}{dt}\langle N^{e_k}, M\rangle_t$ with $M \in \mathcal{M}_0$). From this, we see that $\gamma_{kl} = \sum_i \gamma_{ki}\gamma_{il}$ (i.e $P^2 = P$). $\square$

Remark 3.10 *In the case $\Gamma = D$, proposition 3.8 implies that $P_t^f = 0$ and that*

$$M_t^f = \sum_k \int_0^t D_{e_k} f(X_s) d\widetilde{W}_s^k.$$

From this, we see that the diffusion X_t satisfies the S.D.E.

$$f(X_t) - f(X_0) = \sum_l \int_0^t D_{e_l} f(X_s)\ d\widetilde{W}_s^k + \int_0^t Af(X_s)\ ds \tag{3.22}$$

for every $f \in \mathcal{D}(A)$. Therefore $(X_t, \widetilde{W}_t)$ appears as a weak solution of this S.D.E. and $\widetilde{S}_t$ is defined by filtering X_t with respect to $\widetilde{W}_t$.

Let $P_{x,\widetilde{\omega}}(d\omega')$ be the conditional law of the diffusion X_t, given X_0 and $\{\widetilde{W}_t;\ t \in \mathbb{R}^+\}$ (it is independent of the choice of the initial distribution). Using the identity in law between W and $\widetilde{W}$, we get a family of conditional probabilities $P_{x,\omega}(d\omega')$ on $C(\mathbb{R}^+, X)$ defined $m \otimes P$ a.e.

Remark that (with $X_t(\omega') = \omega'(t)$)

$$S_t f(x,\omega) = \int f(X_t(\omega'))\ \widetilde{P}_x(\omega, d\omega') \qquad m \otimes P \quad a.s. \tag{3.23}$$

Under $P_{x,\omega}(d\omega')P(d\omega)$, $X_t(\omega')$ verifies the S.D.E. (3.3). It is a canonical weak solution of the S.D.E. (3.3) on a canonical extension of the probability space on which W is defined. S_t is obtained by filtering X_t with respect to W.

4 The n-point motion.

The n-point motion for the statistical solution S_t is described by the family of Markovian operators on $L^2(m^{\otimes n})$, $S_t^{\otimes n}$. Let $P_t^{(n)}$ be the family of operators on $L^2(m^{\otimes n})$ such that, for any $(f_i)_{1 \le i \le n} \in L^2(m)$,

$$P_t^{(n)} f_1 \otimes\ \ldots\ \otimes f_n = E[S_t f_1 \otimes\ \ldots\ \otimes S_t f_n]. \tag{4.1}$$

$P_t^{(n)}$ is a Markovian semigroup on $L^2(m^{\otimes n})$ as S_t is Markovian and satisfies a) in theorem 3.2.

Proposition 4.1 *$P_t^{(n)}$ is a semigroup associated to a Markov process on X^n.*

Proof. We use the conditional laws $P_{x,\omega}(d\omega')$ defined above. For every family of functions in $L^2(m)$, $(f_i;\ 1 \le i \le n)$, $m^{\otimes n} \otimes P$ a.e (with $X_t^i(\omega_i') = \omega_i'(t)$)

$$\begin{aligned} S_t^{\otimes n} f_1 \otimes \ldots \otimes f_n(x_1, \ldots, x_n, \omega) &= \int \prod_{i=1}^n f_i(X_t^i(\omega_i'))\ \otimes_{i=1}^n P_{x_i,\omega}(d\omega_i') \\ &= \prod_{i=1}^n S_t f_i(x_i, \omega). \end{aligned} \tag{4.2}$$

It is clear that for any family of probability laws on X absolutely continuous with respect to m, $(\mu_i;\ 1 \leq i \leq n)$,

$$P^{(n)}_{\mu_1,...,\mu_n}(d\omega'_1, ..., d\omega'_n) = \int_\Omega P(d\omega) \otimes_{i=1}^n \widetilde{P}_{\mu_i,\omega}(d\omega'_i) \tag{4.3}$$

defines a Markov process on X^n (here with initial distribution $\otimes_{i=1}^m \mu_i$). It is also clear that the associated semi-group of this Markov process is $P^{(n)}_t = E[S^{\otimes n}_t]$. □

Let $D^{(n)}$ be the linear map from $H \times \mathcal{F}^{\otimes n}$ in $L^2(m^{\otimes n})$ such that for any $(f_i)_{1\leq i\leq n} \in \mathcal{F}$ and $h \in H$,

$$D^{(n)}_h f_1 \otimes ... \otimes f_n = \sum_{i=1}^n f_1 \otimes ... \otimes D_h f_i \otimes ... \otimes f_n. \tag{4.4}$$

Lemma 4.2 *For any $(f_i)_{1\leq i\leq n} \in \mathcal{D}(A)$,*

$$\begin{aligned} S^{\otimes n}_t f_1 \otimes ... \otimes f_n &= f_1 \otimes ... \otimes f_n + \sum_k \int_0^t S^{\otimes n}_s (D^{(n)}_{e_k} f_1 \otimes ... \otimes f_n)\ dW^k_s \\ &\quad + \int_0^t S^{\otimes n}_s (A^{(n)} f_1 \otimes ... \otimes f_n), \end{aligned}$$

where $A^{(n)}$ is the generator of the semigroup $P^{(n)}_t$ of domain $\mathcal{D}(A^{(n)}) \supseteq \mathcal{D}(A)^{\otimes n}$. $A^{(n)}$ is such that

$$\begin{aligned} A^{(n)} f_1 \otimes ... \otimes f_n &= \sum_{i=1}^n f_1 \otimes ... \otimes Af_i \otimes ... \otimes f_n \\ &\quad + \sum_{1\leq i<j\leq n} \sum_k f_1 \otimes ... \otimes D_{e_k} f_i \otimes ... \otimes D_{e_k} f_j \otimes ... \otimes f_n. \end{aligned}$$

Remark. For $n = 2$, $A^{(2)} f \otimes g = Af \otimes g + f \otimes Ag + C(f, g)$.

Proof. This is just a straightforward application of Itô's formula applied to $S_t f_1 \otimes ... \otimes S_t f_n$, using the differential form of the equation satisfied by S_t, e) in theorem 3.2. And taking the expectation and differentiating with respect to t, we get

$$\begin{aligned} \frac{d}{dt}_{|t=0} P^{(n)}_t f_1 \otimes ... \otimes f_n &= \frac{d}{dt}_{|t=0} E[S^{\otimes n}_t f_1 \otimes ... \otimes f_n] \\ &= A^{(n)} f_1 \otimes ... \otimes f_n. \end{aligned}$$

The domain of $A^{(n)}$ contains $\mathcal{D}(A)^{\otimes n}$. □

$S^{\otimes n}_t$ satisfies a similar theorem as theorem 3.2 for S_t:

Proposition 4.3 *$S^{\otimes n}_t$ is the unique one parameter family of $\mathcal{F}_t$-adapted Markovian operators on $L^2(m^{\otimes n})$ such that*

a) $S^{\otimes n}_{t+s} = S^{\otimes n}_t (S^{\otimes n}_s \circ \theta_t)$, for any $s, t \geq 0$;

b) $\forall u \in L^2(m^{\otimes n})$, $S^{\otimes n}_t u$ is uniformly continuous with respect to t in $L^2(m^{\otimes n} \otimes P)$;

c) $E[(S_t^{\otimes n}u)^2] \leq P_t^{(n)}u^2$, *for any* $u \in L^2(m^{\otimes n})$;

d) $S_t^{\otimes n}u = P_t^{(n)}u + \sum_k \int_0^t S_s^{\otimes n}(D_{e_k}^{(n)}P_{t-s}^{(n)}u)\ dW_s^k$, *for any* $u \in L^2(m^{\otimes n})$;

e) $S_t^{\otimes n}u = u + \sum_k \int_0^t S_s^{\otimes n}(D_{e_k}^{(n)}u)\ dW_s^k + \int_0^t S_s^{\otimes n}(A^{(n)}u)\ ds$, *for any* $u \in \mathcal{D}(A)^{\otimes n}$.

$S_t^{\otimes n}$ *is uniquely characterized by c) and d) or by a), c) and e). We call it the statistical solution of the S.D.E.*

$$\forall f \in \mathcal{D}(A^{(n)}): \quad df(X_t^{(n)}) = \sum_k D_{e_k}^{(n)}f(X_t)\ dW_t^k + A^{(n)}f(X_t)\ dt. \tag{4.5}$$

Note that this S.D.E. does not always have a strong solution in the usual sense.

Proof. a), b) and c) are clearly satisfied. e) follows from the proposition.

It can be proved that a), c) and e) imply d) in a similar way as in theorem 3.2. As the solutions of c) and d) are unique (because they give the chaos expansion of $S_t^{\otimes n}u$), the proof is complete. □

Remark 4.4 *In general,* $m^{\otimes n}$ *is not invariant under* $P_t^{(n)}$. *In the measure preserving case,* m *is invariant under* S_t *and* $m^{\otimes n}$ *is invariant under* $P_t^{(n)}$.

5 Measure preserving flows.

For every measure μ, we will note μS_t the measure such that $\mu S_t(f) = \int S_t f\ d\mu$.

Proposition 5.1 *Assume that* $\int C(f,g)\ dm \otimes dm$ *vanishes for all* f, g *in* $\mathcal{F}$. *Then* $mS_t = m$ *a.s (i.e* m *is an invariant measure for* S_t*). Moreover, define* r_t *on* $L^2(\mathcal{F}_t)$ *by* $W_s^k \circ r_t = W_{t-s}^k - W_t^k$. *Then the adjoint of* S_t *in* $L^2(m)$ *is* $S_t^* = S_t \circ r_t$.

Remark 5.2 *In the Riemannian manifold case, the condition that* $\int C(f,g)\ dm \otimes dm$ *vanishes for all* f, g *in* $\mathcal{F}$ *is equivalent to assume that* W *is divergent free in the weak sense.*

Proof. W is divergent free in the weak sense if and only if for any $f \in \mathcal{F}$, $\int \langle W, \nabla f\rangle\ dm = 0$. This is also equivalent to the condition that for any $f \in \mathcal{F}$, $E[(\int \langle W, \nabla f\rangle\ dm)^2] = 0$. From this, it is easy to conclude as

$$E\left[\left(\int \langle W, \nabla f\rangle\ dm\right)^2\right] = \int C(f,f)\ dm^{\otimes 2}. \qquad \square$$

Lemma 5.3 *Assume that* $\int C(f,g)\ dm \otimes dm$ *vanishes for all* f, g *in* $\mathcal{F}$, *then for every* $h \in H$, f, g *in* $\mathcal{F}$,

$$\int gD_h f\ dm = -\int fD_h g\ dm. \tag{5.1}$$

Proof. Take f, g in $\mathcal{F}$ such that $fg \in \mathcal{F}$, then, as D_{e_k} is a derivation, $D_{e_k}(fg) = gD_{e_k}f + fD_{e_k}g$. Using this property, we get

$$\begin{aligned}\sum_k \left(\int (gD_{e_k}f + fD_{e_k}g)\, dm\right)^2 &= \sum_k \left(\int D_{e_k}(fg)\, dm\right)^2 \\ &= \int C(fg, fg)\, dm^{\otimes 2} = 0.\end{aligned}$$

This implies that for every k, $\int gD_{e_k}f\, dm = -\int fD_{e_k}g\, dm$. □

Proof of proposition 5.1. Let us remark that the expression of the chaos of $S_t f$ are given by the expression

$$J_t^n f = \int_{0\le s_1\le s_2\le \ldots\le s_n\le t} \sum_{k_1,\ldots,k_n} P_{s_1}D_{e_{k_1}}P_{s_2-s_1}D_{e_{k_2}}\ldots P_{t-s_n}D_{e_{k_n}}f\; dW_{s_1}^{k_1}\ldots dW_{s_n}^{k_n}. \tag{5.2}$$

From this expression, using lemma 5.3 and the fact that P_t is self-adjoint in $L^2(m)$, we get that for f and g in $L^2(m)$,

$$\begin{aligned}\int gJ_t^n f\, dm &= \\ &= \int\int_{0\le s_1\ldots\le s_n\le t} f \sum_{k_1,\ldots,k_n} (-1)^n P_{t-s_n}D_{e_{k_n}}P_{s_{n-1}-s_n}\ldots P_{s_2-s_1}D_{e_{k_1}}P_{s_1}g\; dW_{s_1}^{k_1}\ldots dW_{s_n}^{k_n}\; dm.\end{aligned} \tag{5.3}$$

Making the change of variable $u_{n-i+1} = t - s_i$, we get that the adjoint of J_t^n is given by

$$(J^n)_t^* g = \int_{0\le u_1\le u_2\le\ldots\le u_n\le t} \sum_{k_1,\ldots,k_n} P_{u_1}D_{e_{k_1}}P_{u_2-u_1}D_{e_{k_2}}\ldots P_{t-u_n}D_{e_{k_n}}g\, dW_{u_1}^{k_1}\circ r_t\ldots dW_{u_n}^{k_n}\circ r_t. \tag{5.4}$$

From this it is easy to see that $S_t^* g = (S_t \circ r_t)g$ (as they have the same chaos expansion).

Notice that $S_t^* 1 = 1$. A priori the constant functions are not in $L^2(m)$, but there exists an increasing sequence in $L^2(m)$, g_n such that g_n converges towards 1. For any nonnegative function $f \in L^2(m)$,

$$\int S_t f g_n\, dm = \int f S_t^* g_n\, dm. \tag{5.5}$$

This equation implies, taking the limit as n goes to ∞, that

$$mS_t(f) = \int f S_t^* 1\, dm = m(f). \tag{5.6}$$

And we get that $mS_t = m$ a.s. □

Remark 5.4 *The adjoint in $L^2(m^{\otimes n})$ of $S_t^{\otimes n}$ is $S_t^{*\otimes n}$. And as in the measure preserving case, $E[S_t^{*\ \otimes n}] = P_t^{(n)}$, we have that $P_t^{(n)}$ is self-adjoint in $L^2(m^{\otimes n})$.*

6 Existence of a flow of maps.

We say that S_t is a flow of maps if and only if there exists a family of measurable mappings $(\varphi_t)_{t\geq 0}$ from $X \times \Omega$ in X such that for any $f \in L^2(m)$, $S_t f = f \circ \varphi_t$.

Lemma 6.1 *S_t is a flow of maps if and only if for any $f \in L^2(m)$, $E[(S_t f)^2] = P_t f^2$.*

Proof. It is clear that there exists Markovian kernels on X, $s_t(x, \omega, dy)$ such that $S_t f(x) = \int f(y) s_t(x, \omega, dy)$. As $m \otimes P$ a.e,

$$(S_t f)^2(x) - S_t f^2(x) = \int \left(f(y) - \int f(z) s_t(x, \omega, dz) \right)^2 s_t(x, \omega, dy), \tag{6.1}$$

if $E[(S_t f)^2] = P_t f^2$, $\int (f(y) - \int f(z) s_t(x, \omega, dz))^2 s_t(x, \omega, dy) = 0$ and $s_t(x, \omega, dz)$ is a Dirac measure $\delta_{\varphi_t(x,\omega)}$, where $\varphi_t(x, \omega)$ is defined $m \otimes P$ a.e. □

Let $h \in L^1(m)$ be a positive function such that $\int h \, dm = 1$. For any positive t, let μ_t be a probability on the Borel sets of $X \times X$ such that for any $(f, g) \in L^2(m) \times L^2(m)$, $\mu_t(f \otimes g) = \int E[S_t f(x) S_t g(x)] \, h \, dm$.

Remark 6.2 *S_t is a flow of maps if and only if $\mu_t(\Delta) = 1$, where $\Delta = \{(x, x);\ x \in X\}$.*

Proof. If S_t is a flow of maps, there exists φ_t such that $S_t f = f \circ \varphi_t$. If A and B are disjoints Borel sets of finite measure,

$$\mu_t(A \times B) = \int E[1_A(\varphi_t(x)) 1_B(\varphi_t(x))] \, h \, dm(x) = 0.$$

This implies that $\mu_t(X \times X - \Delta) = 0$ and as μ_t is a probability that $\mu_t(\Delta) = 1$.

If $\mu_t(\Delta) = 1$, for $f \in L^2(m)$, let $h = f^2 \otimes 1 - 2f \otimes f + 1 \otimes f^2$, then $\mu_t(h) = 0$. This implies that

$$\int_X P_t f^2 \, h \, dm = \int_X E[(S_t f)^2(x)] \, h \, dm \tag{6.2}$$

and that $E[(S_t f)^2] = P_t f^2$. Hence S_t is a flow of maps. □

Let (X_t, Y_t) be the Markov process associated to the semigroup $P_t^{(2)}$. Recall that we denoted $P_{(.,.)}^{(2)}$ the law of this Markov process.

Proposition 6.3 *S_t is a flow of maps if for any positive r and any t,*

$$\lim_{d(x,y)\to 0} P_{(x,y)}^{(2)}[d(X_t, Y_t) \geq r] = 0.$$

Proof. For $\varepsilon > 0$, let ν_ε be the measure on $X \times X$ such that for any $(f, g) \in L^2(m) \times L^2(m)$, $\nu_\varepsilon(f \otimes g) = \int f \, P_\varepsilon g \, h \, dm$. For any $(f, g) \in L^2(m) \times L^2(m)$,

$$\nu_\varepsilon P_t^{(2)}(f \otimes g) = \int E[S_t f(x) P_\varepsilon S_t g(x)] \, h \, dm. \tag{6.3}$$

As $P_\varepsilon S_t g$ converges in $L^2(m \otimes P)$ towards $S_t g$,

$$\lim_{\varepsilon \to 0} \nu_\varepsilon P_t^{(2)}(f \otimes g) = \int E[S_t f(x) S_t g(x)]\ h\ dm. \tag{6.4}$$

Therefore, the family of measure $(\nu_\varepsilon P_t^{(2)})_{\varepsilon > 0}$ converges weakly as ε goes to 0 towards μ_t.

Assume that for any positive r and any t, $\lim_{d(x,y)\to 0} P_{(x,y)}^{(2)}[d(X_t, Y_t) \geq r] = 0$. Let A and B be two disjoint Borel sets of finite measure such that $d(A, B) \geq r$, then

$$\nu_\varepsilon P_t^{(2)}(A \times B) = \int_X f_\varepsilon(x)\ dm(x),$$

with

$$f_\varepsilon(x) = \int P_{(x,y)}^{(2)}[X_t \in A \text{ and } Y_t \in B]\ p_\varepsilon(x, dy),$$

where $p_\varepsilon(x, dy)$ is the kernel given by P_ε. Remark that $f_\varepsilon(x) \leq P_x[X_t \in A] \in L^1(m)$ (as $\int P_x[X_t \in A]\ dm(x) = \int P_t 1_A\ dm \leq m(A) < \infty$).

As $d(A, B) \geq r$,

$$\begin{aligned} f_\varepsilon(x) &\leq \int P_{(x,y)}^{(2)}[d(X_t, Y_t) \geq r]\ p_\varepsilon(x, dy) \\ &\leq \int_{\{d(x,y)>\alpha\}} p_\varepsilon(x, dy) + \int_{\{d(x,y)\leq\alpha\}} P_{(x,y)}^{(2)}[d(X_t, Y_t) \geq r]\ p_\varepsilon(x, dy). \end{aligned}$$

For any positive β, there exists α such that $d(x, y) \leq \alpha$ implies that $P_{(x,y)}^{(2)}[d(X_t, Y_t) \geq r] \leq \beta$. For this α, it is clear that $\lim_{\varepsilon\to 0} \int_{\{d(x,y)>\alpha\}} p_\varepsilon(x, dy) = 0$ and that $\int_{\{d(x,y)\leq\alpha\}} P_{(x,y)}^{(2)}[d(X_t, Y_t) \geq r]\ p_\varepsilon(x, dy) \leq \beta$. Therefore, we see that

$$\lim_{\varepsilon \to 0} \nu_\varepsilon P_t^{(2)}(A \times B) = 0.$$

This implies that $\mu_t(X \times X - \Delta) = 0$ and that S_t is a flow of maps. □

Proposition 6.4 *If there exists a positive t, a positive r and $p \in]0, 1]$ such that for $m^{\otimes 2}$-every (x, y), $P_{(x,y)}^{(2)}[d(X_t, Y_t) > r] \geq p$, then S_t is not a flow of maps.*

Proof. Suppose there exists a positive t, a positive r and $p \in]0, 1]$ such that for $m^{\otimes 2}$-every (x, y), $P_{(x,y)}^{(2)}[d(X_t, Y_t) > r] \geq p$.

Let $(B_i)_{i\in\mathbb{N}}$ be a partition of X such that the diameter of B_i is lower than r.

Let us suppose that $\mu_t(\Delta) = 1$ (or that S_t is a flow of maps). Then we have $\sum_i \mu_t(B_i \times B_i) = 1$ and for any positive α, there exists N such that

$$\sum_{i=1}^{N} \mu_t(B_i \times B_i) \geq 1 - \alpha.$$

As $\nu_\varepsilon P_t^{(2)}$ converges weakly towards μ_t,

$$\begin{aligned}
\sum_{i=1}^{N} \mu_t(B_i \times B_i) &= \lim_{\varepsilon\to 0} \nu_\varepsilon P_t^{(2)}(B_i \times B_i) \\
&= \lim_{\varepsilon\to 0} \sum_{i=1}^{N} \int_{X\times X} P_{(x,y)}^{(2)}[(X_t, Y_t) \in B_i \times B_i]\; p_\varepsilon(x, dy)\; h(x)dm(x) \\
&\leq \lim_{\varepsilon\to 0} \sum_{i=1}^{N} \int_{X\times X} P_{(x,y)}^{(2)}[X_t \in B_i;\; d(X_t, Y_t) \leq r]\; p_\varepsilon(x, dy)\; h(x)dm(x) \\
&\leq \lim_{\varepsilon\to 0} \int_{X\times X} P_{(x,y)}^{(2)}[d(X_t, Y_t) \leq r]\; p_\varepsilon(x, dy)\; h(x)dm(x) \leq 1 - p
\end{aligned}$$

Choosing $\alpha < p$, we get a contradiction. Hence $\mu(\Delta) < 1$ and S_t is not a flow of maps. □

7 A one dimensional example.

Let $X = \mathbb{R}$, P_t be the semigroup of the Brownian motion on $\mathbb{R}$ and the covariance function $C(x, y) = \mathrm{sgn}(x)\mathrm{sgn}(y)$ (where $\mathrm{sgn}(x)$ denotes the sign of x with the convention $\mathrm{sgn}(0) = 1$). Here, we have $W_t(x) = \mathrm{sgn}(x)B_t$, where B_t is a Brownian motion starting from 0. Let $L_t^x = \sup_{s\leq t}\{-\mathrm{sgn}(x)(x + B_s)\} \vee 0$ and $R_t^x = x + B_t + \mathrm{sgn}(x)L_t^x$, the Brownian motion starting from x, reflected at 0.

Proposition 7.1 *The statistical solution S_t can be written the following way*

$$S_t f(x) = f(R_t^x)1_{L_t^x=0} + \frac{1}{2}\left[f(R_t^x) + f(-R_t^x)\right] 1_{L_t^x>0}. \tag{7.1}$$

Proof. Changing of probability space, it is possible to build a Brownian motion starting from x, β_t such that $B_t = \int_0^t \mathrm{sgn}(\beta_s)d\beta_s$ (then β_t is a weak solution of the S.D.E. $dX_t = \mathrm{sgn}(X_t)dB_t$). Let $S_t f(x) = E[f(\beta_t)|\mathcal{F}^B]$, with $\mathcal{F}^B = \sigma(B_u; u \geq 0)$. Let us remark that L_t^x is the local time of β at 0 and $R_t^x = \mathrm{sgn}(x)|\beta_t|$. As

$$\begin{aligned}
f(\beta_t) &= P_t f(x) + \int_0^t (P_{t-s}f)'(\beta_s)d\beta_s \\
&= P_t f(x) + \int_0^t \mathrm{sgn}(\beta_s)(P_{t-s}f)'(\beta_s)dB_s, \\
S_t f(x) &= P_t f(x) + \int_0^t E[\mathrm{sgn}(\beta_s)(P_{t-s}f)'(\beta_s)|\mathcal{F}^B]dB_s \\
&= P_t f(x) + \int_0^t S_s\left(\mathrm{sgn}()(P_{t-s}f)'\right)dB_s. \qquad \square
\end{aligned}$$

8 The Lipschitz case in $\mathbb{R}^d$.

Let P_t be the semigroup of a symmetric diffusion in $\mathbb{R}^d$ of generator A. Let $C(x,y)$ be a covariance matrix such that for any (x,y),

$$\sum_{i=1}^{d}(C^{ii}(x,x)+C^{ii}(y,y)-2C^{ii}(x,y)) \leq \frac{C}{2}\, d(x,y)^2, \tag{8.1}$$

where C is a positive constant.

Suppose that $A=\sum_{i,j}C^{ij}(x,x)\partial_i\partial_j+\sum_i b^i(x)\partial_i$ and that b^i is a Lipschitz function (i.e. $\|b(x)-b(y)\| \leq \frac{C}{4}d(x,y)$).

Let S_t be the statistical solution associated to P_t and C, $P_t^{(2)}$ be the semigroup of the 2-point process and (X_t,Y_t) be the associated Markov process. Let $h(x,y)=d(x,y)^2$,

Lemma 8.1 $P_t^{(2)}h(x,y) \leq h(x,y)e^{Ct}$.

Proof. Remark first that

$$A^{(2)}h(x,y)=\sum_i(C^{ii}(x,x)+C^{ii}(y,y)-2C^{ii}(x,y))+2\langle b(x)-b(y),x-y\rangle. \tag{8.2}$$

The Lipschitz conditions imply that

$$A^{(2)}h(x,y) \leq C\ h(x,y).$$

Using Itô's formula, there exists a martingale M_t such that

$$h(X_t,Y_t)-h(x,y)=M_t+\int_0^t A^{(2)}h(X_s,Y_s)\,ds \leq M_t+C\int_0^t h(X_s,Y_s)\,ds.$$

This implies that $P_t^{(2)}h(x,y)-h(x,y) \leq C\int_0^t P_s^{(2)}h(x,y)ds$. Hence the lemma. □

Theorem 8.2 *The statistical solution is a flow of maps.*

Proof. This is clear since $P^{(2)}_{(x,y)}[d(X_t,Y_t)\geq r] \leq \frac{1}{r^2}P_t^{(2)}h(x,y) \leq \frac{1}{r^2}d(x,y)^2e^{Ct}$, which goes to 0 as $d(x,y)$ goes to 0. This implies the theorem. □

9 Isotropic statistical solution on S^d.

9.1 Isotropic covariance function on S^d.

On S^d, the isotropic covariance function C are given by the formula (see Raimond [19])

$$C((x,u),(y,v))=\alpha(t)\langle u,v\rangle+\beta(t)\langle u,y\rangle\langle v,x\rangle, \tag{9.1}$$

with $(x,y) \in S^d \times S^d$, $t = \langle x,y \rangle = \cos\varphi$ and $(u,v) \in T_x S^d \times T_y S^d$. α and β are given by

$$\alpha(t) = \sum_{l=1}^{\infty} a_l \gamma_l(t) + \sum_{l=1}^{\infty} b_l \left(t\gamma_l(t) - \frac{1-t^2}{d-1}\gamma_l'(t) \right), \tag{9.2}$$

$$\beta(t) = \sum_{l=1}^{\infty} a_l \gamma_l'(t) + \sum_{l=1}^{\infty} b_l \left(-\gamma_l(t) - \frac{t}{d-1}\gamma_l'(t) \right), \tag{9.3}$$

where $\gamma_l(t) = C_{l-1}^{\frac{d+1}{2}}(t)/C_{l-1}^{\frac{d+1}{2}}(1)$, C_l^p is a Gegenbauer polynomial, a_l and b_l are nonnegative such that $\sum_l a_l < \infty$ and $\sum_l b_l < \infty$. Using the integral form of the Gegenbauer polynomials (see [21] p. 496):

$$\gamma_l(\cos\varphi) = \int_0^{\pi} [z(\varphi,\theta)]^{l-1} \sin^d\theta \, \frac{d\theta}{c_d}, \tag{9.4}$$

with $c_d = \int_0^{\pi} \sin^d\theta \, d\theta$ and $z(\varphi,\theta) = \cos\varphi - i\sin\varphi\cos\theta$.

For $\alpha > 0$, set $G(\varphi) = \sum_{l\geq 2} \frac{1}{(l-1)^{\alpha+1}} \gamma_l(\cos\varphi)$. G is well defined on $[0,\pi]$ as $|\gamma_l| \leq 1$.

Lemma 9.1 *For $\alpha \in]0,2[$, we have*

- *G is differentiable on $]0,\pi[$.*
- $\lim_{\varphi\to 0+} \frac{G(0)-G(\varphi)}{\varphi^\alpha} = \int_0^{\pi}\int_0^{\infty} \frac{\cos^2\theta}{t^2+\cos^2\theta} t^{\alpha-1}\sin^d\theta \, \frac{dt \; d\theta}{\Gamma(\alpha+1)c_d} = KG(0)$.
- $\lim_{\varphi\to 0+} \frac{G'(\varphi)}{\varphi^{\alpha-1}} = -\alpha \int_0^{\pi}\int_0^{\infty} \frac{\cos^2\theta}{t^2+\cos^2\theta} t^{\alpha-1}\sin^d\theta \, \frac{dt \; d\theta}{\Gamma(\alpha+1)c_d} = -\alpha K G(0)$.

Proof. Take $\varphi \in]0,\pi[$. At first, we are going to prove that $I(\varphi) = \sum_{l\geq 2} \frac{1}{(l-1)^{\alpha+1}} \left| \frac{d}{d\varphi}\gamma_l(\cos\varphi) \right|$ is finite. As $\frac{1}{l^\alpha} = \int_0^{\infty} e^{-ls} s^{\alpha-1} \, \frac{ds}{\Gamma(\alpha)}$,

$$I(\varphi) \leq \int_0^{\pi}\int_0^{\infty} \sum_{l\geq 1} [e^{-s}|z(\varphi,\theta)|]^l \frac{\left|\frac{d}{d\varphi} z(\varphi,\theta)\right|}{|z(\varphi,\theta)|} s^{\alpha-1}\sin^d\theta \, \frac{ds \; d\theta}{\Gamma(\alpha)c_d} \tag{9.5}$$

$$\leq \int_0^{\pi}\int_0^{\infty} f_{\varphi,\theta}(s) \; ds \; d\theta = 2\int_0^{\frac{\pi}{2}}\int_0^{\infty} f_{\varphi,\theta}(s) \; ds \; d\theta, \tag{9.6}$$

with $f_{\varphi,\theta}(s) = \frac{e^{-s}\left|\frac{d}{d\varphi} z(\varphi,\theta)\right|}{1-e^{-s}|z(\varphi,\theta)|} \frac{s^{\alpha-1}}{\Gamma(\alpha)c_d}$.

It is easy to see that

$$\int_1^{\infty} f_{\varphi,\theta}(s) \; ds \leq \frac{1}{\Gamma(\alpha)c_d} \int_1^{\infty} \frac{e^{-s}s^{\alpha-1}}{(1-e^{-s})} ds < \infty. \tag{9.7}$$

On the other hand,

$$\int_0^1 f_{\varphi,\theta}(s) \; ds \leq \frac{1}{\Gamma(\alpha)c_d} \int_0^1 \frac{ds}{1-e^{-s}|z(\varphi,\theta)|} = \frac{1}{\Gamma(\alpha)c_d} F_\varphi(\theta). \tag{9.8}$$

Let $x_\varphi(\theta) = -\log|z(\varphi,\theta)|$, then $F_\varphi(\theta) = \int_{x_\varphi(\theta)}^{x_\varphi(\theta)+1} \frac{dt}{1-e^{-t}}$. As $\lim_{\theta\to 0+} x_\varphi(\theta) = 0$, we have $F_\varphi(\theta) \sim -\log x_\varphi(\theta)$ as θ goes to 0. From this, we see that $F_\varphi(\theta) = O(\log\theta)$ as θ goes to 0. This implies that $I(\varphi)$ is finite.

Now, applying the derivation under the integral theorem, we prove that G is differentiable on $]0,\pi[$ and that for $\varphi \in]0,\pi[$,

$$\begin{aligned} G'(\varphi) &= \sum_{l\geq 1} \int_0^\pi \frac{[z(\varphi,\theta)]^{l-1} \frac{d}{d\varphi} z(\varphi,\theta)}{l^\alpha} \sin^d\theta \, \frac{d\theta}{c_d} && (9.9)\\ &= \int_0^\pi \int_0^\infty \sum_{l\geq 1} [e^{-s} z(\varphi,\theta)]^l \frac{\frac{d}{d\varphi} z(\varphi,\theta)}{z(\varphi,\theta)} s^{\alpha-1} \sin^d\theta \, \frac{ds\, d\theta}{\Gamma(\alpha) c_d} && (9.10)\\ &= \int_0^\pi \int_0^\infty \frac{e^{-s} \frac{d}{d\varphi} z(\varphi,\theta)}{1-e^{-s} z(\varphi,\theta)} s^{\alpha-1} \sin^d\theta \, \frac{ds\, d\theta}{\Gamma(\alpha) c_d}. && (9.11) \end{aligned}$$

As $z(\varphi,\pi-\theta) = \overline{z(\varphi,\theta)}$,

$$G'(\varphi) = -\int_0^{\frac{\pi}{2}} \int_0^\infty \frac{a(s,\varphi) - \sin^2\theta}{b(s,\varphi) + \cos^2\theta} \times \frac{\cos\varphi}{\sin\varphi} s^{\alpha-1} \sin^d\theta \, \frac{2ds\, d\theta}{\Gamma(\alpha) c_d},$$

with $a(s,\varphi) = \frac{1}{e^{-s}\cos\varphi}$ and $b(s,\varphi) = \frac{(1-e^{-s}\cos\varphi)^2}{e^{-2s}\sin^2\varphi}$. Changing of variables ($s = t\varphi$),

$$\begin{aligned} -\frac{G'(\varphi)}{\varphi^{\alpha-1}} &= \int_0^{\frac{\pi}{2}} \int_0^\infty \frac{a(t\varphi,\varphi) - \sin^2\theta}{b(t\varphi,\varphi) + \cos^2\theta} \times \frac{\varphi\cos\varphi}{\sin\varphi} t^{\alpha-1} \sin^d\theta \, \frac{2dt\, d\theta}{\Gamma(\alpha) c_d} && (9.12)\\ &= \int_0^{\frac{\pi}{2}} \int_0^\infty I(t,\varphi,\theta)\, dt\, d\theta. && (9.13) \end{aligned}$$

Let $\varepsilon > 0$, there exists a positive constant C_ε such that for any $t \in [0,\varepsilon]$,

$$0 \leq I(t,\varphi,\theta) \leq C_\varepsilon t^{\alpha-1}. \tag{9.14}$$

Remark also that

$$I(t,\varphi,\theta) \leq C_{d,\alpha} \frac{t^2\varphi^2 e^{-t\varphi}}{(1-e^{-t\varphi})^2} \times t^{\alpha-3}, \tag{9.15}$$

where $C_{d,\alpha}$ is a positive constant. Let $C = C_{d,\alpha} \sup_{x>0} \frac{x^2 e^{-x}}{(1-e^{-x})^2} < \infty$, then, for any positive t

$$0 \leq I(t,\varphi,\theta) \leq C\ t^{\alpha-3}. \tag{9.16}$$

As $F(t) = C_\varepsilon t^{\alpha-1} 1_{0<t\leq\varepsilon} + C\ t^{\alpha-3} 1_{t>\varepsilon}$ belongs to $L^1(d\theta\otimes dt)$ for $\alpha \in]0,2[$, $\lim_{\varphi\to 0} a(t\varphi,\varphi) = 1$ and $\lim_{\varphi\to 0} b(t\varphi,\varphi) = t^2$, by the Lebesgue dominated convergence theorem,

$$\lim_{\varphi\to 0} \frac{G'(\varphi)}{\varphi^{\alpha-1}} = -\int_0^{\frac{\pi}{2}} \int_0^\infty \frac{\cos^2\theta}{t^2+\cos^2\theta} t^{\alpha-1} \sin^d\theta \, \frac{2d\theta\, dt}{c_d\Gamma(\alpha)} = -\alpha K. \tag{9.17}$$

We have proved the second limit. The first limit is easy to obtain as

$$\begin{aligned} G(0) - G(\varphi) &= -\int_0^\varphi G'(x)\, dx \\ &= -K\ \varphi^\alpha + \circ(\varphi^\alpha). \end{aligned}$$

This finishes the proof of the lemma. □

Let us introduce a covariance function analogous to the Sobolev covariance of chapter 10. Let a and b be nonnegative reals. Take $a_l = \frac{a}{(l-1)^{\alpha+1}}$ and $b_l = \frac{b}{(l-1)^{\alpha+1}}$ for $l \geq 1$ and $a_1 = b_1 = 0$. Then,

$$\alpha(\cos\varphi) = aG(\varphi) + b\left(\cos\varphi\, G(\varphi) + \frac{\sin\varphi}{d-1} \times G'(\varphi)\right) \tag{9.18}$$

$$\beta(\cos\varphi) = -\frac{a}{\sin\varphi}G'(\varphi) + b\left(-G(\varphi) + \frac{\cos\varphi}{(d-1)\sin\varphi} \times G'(\varphi)\right). \tag{9.19}$$

In the case $d = 2$, let $(\Xi_m^l;\ l \in \mathbb{N};\ -l \leq m \leq l)$ be the spherical harmonics (with the spherical coordinates, $\Xi_m^l(x) = C_{l-|m|}^{\frac{1}{2}+|m|}(\cos\theta)(\sin\theta)^{|m|}e^{im\varphi}$). They constitute an orthonormal basis of $L^2(S^2)$. Let Δ be the De Rham Laplacian, $\Delta = \delta d + d\delta$, where d and δ denote respectively the derivative and the coderivative on S^2. This basis is an eigenbasis for Δ and

$$\Delta\Xi_m^l = l(l+1)\Xi_m^l.$$

Let $*\omega$ be the adjoint of the form ω (note that the adjoint of a 0-form is a 2-form and the adjoint of a 1-form is a 1-form), then $\delta = -*d*$. $(U_m^l = \frac{d\Xi_m^l}{\sqrt{l(l+1)}};\ l \in \mathbb{N},\ -l \leq m \leq l)$ and $(V_m^l = \frac{*d\Xi_m^l}{\sqrt{l(l+1)}};\ \in \mathbb{N},\ -l \leq m \leq l)$ form respectively an orthonormal basis of the L^2-space of gradient vector fields on S^2 and of the L^2-space of divergent free vector fields on S^2 (note that the coderivative of a 1-form is the divergence of the corresponding vector field). The union of this two basis form an orthonormal basis of the L^2-space of vector fields on S^2.

Let us remark that

$$\begin{aligned}\Delta d\ \Xi_m^l &= d\delta d\ \Xi_m^l = d\Delta\ \Xi_m^l = l(l+1)d\Xi_m^l \\ \Delta * d\ \Xi_m^l &= \delta d * d\ \Xi_m^l = -*d*d*d\ \Xi_m^l = *\Delta d\ \Xi_m^l = l(l+1) * d\Xi_m^l.\end{aligned}$$

Therefore this basis is an eigenbasis for the operator Δ.

Let $(z_m^{1,l})_{l,m}$ and $(z_m^{2,l})_{l,m}$ be two independent sequences of independent of Gaussian variables with mean 0 and variance 1. Then, for $i \in \{1,2\}$,

$$\sum_m z_m^{i,l}\Xi_m^l(x)$$

is an isotropic Gaussian scalar field on S^2 with covariance matrix $\sum_m \Xi_m^l(x)\Xi_m^l(y)$.

For $g \in SO(3)$, we denote $T_{mn}^l(g)$ the matrix element in the basis $(\Xi_m^l)_{-l\leq m\leq m}$ of the irreducible representation T^l (with $T^l(g)h(x) = h(g^{-1}x)$), then (see [21] p.483)

$$\Xi_m^l(x) = \sqrt{2l+1}T_{m0}^l(g_x),$$

where g_x is an element of $SO(3)$ such that $g_x p = x$, with $p = (0,0,1)$ in the canonical basis of $\mathbb{R}^3$. Then

$$\begin{aligned}\sum_m \Xi_m^l(x)\Xi_m^l(y) &= (2l+1)\sum_m T_{m0}^l(g_x)T_{m0}^l(g_y) \\ &= (2l+1)T_{00}^l(g_y^{-1}g_x) \\ &= (2l+1)C_l^{\frac{1}{2}}(\langle x,y\rangle),\end{aligned}$$

where $\langle .,. \rangle$ denotes the scalar product in $\mathbb{R}^3$. For (x,u) and (x,v) in T^*S^2, (we have $\frac{d}{dt}C_l^{\frac{1}{2}} = C_{l-1}^{\frac{3}{2}}$ and $C_{l-1}^{\frac{3}{2}}(1) = \frac{l(l+1)}{2}$)

$$\begin{aligned}\sum_m d\Xi_m^l(x) \otimes d\Xi_m^l(y)(u,v) &= (2l+1)(d\otimes d)C_l^{\frac{1}{2}}(\langle x,y\rangle)(u,v) \\ &= (2l+1)\frac{l(l+1)}{2}\left[\gamma_l(t)\langle u,v\rangle + \gamma_l'(t)\langle u,y\rangle\langle v,x\rangle\right].\end{aligned}$$

From this we also get that

$$\begin{aligned}\sum_m *d\ \Xi_m^l(x) \otimes *d\ \Xi_m^l(y)(u,v) &= (2l+1)\frac{l(l+1)}{2}\left[(t\gamma_l(t) - (1-t^2)\gamma_l'(t))\langle u,v\rangle + \right. \\ &\qquad \left. (-\gamma_l(t) - t\gamma_l'(t))\langle u,y\rangle\langle v,x\rangle\right].\end{aligned}$$

Therefore, we get that

$$\xi = \sum_l \sum_m \sqrt{\frac{2a_l}{2l+1}} z_m^{1,l} U_m^l + \sum_l \sum_m \sqrt{\frac{2b_l}{2l+1}} z_m^{2,l} V_m^l$$

is an isotropic Gaussian vector field associated to the covariance function C given by (9.1) and

$$C = \sum_l \frac{2a_l}{2l+1} \sum_m U_m^l \otimes U_m^l + \sum_l \frac{2b_l}{2l+1} \sum_m V_m^l \otimes V_m^l.$$

Here, we have $a_l = \frac{a}{(l-1)^{\alpha+1}}$ and $b_l = \frac{b}{(l-1)^{\alpha+1}}$. Note that $\frac{2a_l}{2l+1} = aF(l(l+1))$ and $\frac{2b_l}{2l+1} = bF(l(l+1))$, where $F(s) = \left(\frac{\sqrt{1+4s}-1}{2} - 1\right)^{-(\alpha+1)} \times \frac{2}{\sqrt{1+4s}}$. Therefore, the norm on the self-reproducing space associated to C is

$$\|V\|_C^2 = \frac{1}{a}\|\pi V\|_s^2 + \frac{1}{b}\|(I-\pi)V\|_s^2,$$

where π is the orthogonal projection on the L^2-space of gradient vector fields and

$$\|V\|_s^2 = \langle F(\Delta)^{-1}V, V\rangle_2.$$

Therefore, the self-reproducing space appears to be an L^2-Sobolev space of order $s = \frac{\alpha+2}{2}$.

Note that if a or b vanishes, the self-reproducing space is restricted to divergent free or gradient vector fields.

9.2 An example of phase transitions for the statistical solution.

Let P_t be the semigroup of the Brownian motion of variance $(a+b)G(0)$ and S_t be the statistical solution associated to P_t and C. If $\alpha \geq 2$, (8.1) is satisfied and the statistical solution is a flow of maps. Let $\eta = \frac{b}{a+b}$.

Theorem 9.2 *For any $\alpha \in]0,2[$,*

- *For $d = 2$ or 3 and $\eta < 1 - \frac{d}{\alpha^2}$, the statistical solution is a coalescing flow of maps.*
- *For $d = 2$ or 3 and $1 - \frac{d}{\alpha^2} < \eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$, the statistical solution is not a flow of maps and $P_t^{(2)}$ is a Markovian semigroup of a 2-point process hitting the diagonal.*
- *For $d = 2$ or 3 and $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$ or for $d \geq 4$, S_t is not a flow of maps and the 2-point process does not hit the diagonal.*

Proof. Let us study the Markovian semigroup $P_t^{(2)}$, and the associated Markov process (X_t, Y_t) on $S^d \times S^d$ starting from (x, y) (with $x \neq y$). Let $\psi_t = d(X_t, Y_t)$. As $h(x, y) = d(x, y)^2$ is a C^2-function and belongs to $\mathcal{D}(A^{(2)})$ and as X_t and Y_t are solution of an S.D.E. like (3.3), ψ_t^2 is a diffusion on $[0, \pi^2]$ and is solution of an S.D.E. ψ_t is also a diffusion on $[0, \pi]$ (note that $d(x, y)$ a priori does not belong to $\mathcal{D}(A^{(2)})$). This diffusion is eventually reflected (or absorbed) in 0 and π. Its generator is $L = \sigma^2(\varphi)\frac{d^2}{d\varphi^2} + b(\varphi)\frac{d}{d\varphi}$ (see Raimond [19]), with

$$\sigma^2(\varphi) = \alpha(1) - \alpha(\cos\varphi)\cos\varphi + \beta(\cos\varphi)\sin^2\varphi, \tag{9.20}$$

$$b(\varphi) = \frac{(d-1)}{\sin\varphi}(\alpha(1)\cos\varphi - \alpha(\cos\varphi)). \tag{9.21}$$

The generator of ψ_t^2 is $L' = \widetilde{\sigma}^2(x)\frac{d^2}{dx^2} + \widetilde{b}(x)\frac{d}{dx}$, with

$$\widetilde{\sigma}^2(x) = 4x\sigma^2(\sqrt{x}), \tag{9.22}$$

$$\widetilde{b}(x) = 2\sigma^2(\sqrt{x}) + 2\sqrt{x}b(\sqrt{x}). \tag{9.23}$$

Let us study s, the scale function of the diffusion ψ_t,

$$s(x) = \int_{x_0}^{x} \exp\left[-\int_{x_0}^{y} \frac{b(\varphi)}{\sigma^2(\varphi)}\, d\varphi\right]\, dy, \text{ with } (x_0, x) \in]0, \pi[^2.$$

From the lemma, we see that as φ goes to 0,

$$\alpha(\cos\varphi) = (a+b)G(0) - \left(a + \left(1 + \frac{\alpha}{d-1}\right)b\right)KG(0)\varphi^\alpha + \circ(\varphi^\alpha) \tag{9.24}$$

$$\beta(\cos\varphi) = \alpha\left(a - \frac{b}{d-1}\right)KG(0)\varphi^{\alpha-2} + \circ(\varphi^{\alpha-2}). \tag{9.25}$$

Hence,

$$\sigma^2(\varphi) = (a+b)KG(0)(\alpha + 1 - \alpha\eta)\varphi^\alpha(1 + \circ(1)) \tag{9.26}$$

$$b(\varphi) = (a+b)KG(0)(d - 1 + \alpha\eta)\varphi^{\alpha-1}(1 + \circ(1)). \tag{9.27}$$

Let us note $\mu = \frac{d-1+\alpha\eta}{\alpha+1-\alpha\eta}$. Then, we have that $\frac{b(\varphi)}{\sigma^2(\varphi)} = \frac{\mu}{\varphi}(1 + \circ(1))$ and for any positive ε, there exist positive constants C_1 and C_2 such that for $y \leq x_0$,

$$C_1 y^{-\mu+\varepsilon} \leq \exp\left[-\int_{x_0}^{y} \frac{b(\varphi)}{\sigma^2(\varphi)}\, d\varphi\right] \leq C_2 y^{-\mu-\varepsilon}. \tag{9.28}$$

From this, we see that $s(0+) = -\infty$ if $\mu > 1$ (or if $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$) and $s(0+)$ is finite if $\mu < 1$ (or if $\eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$). It is easy to check that $s(\pi-) = +\infty$. Hence (see theorem 3.1 in [11]), if $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$, $\lim_{t\to\infty} \psi_t = 0$ a.s. and if $\eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$, $\liminf \psi_t = 0$ and $\limsup \psi_t = \pi$ a.s.. Note that for $d \geq 4$ and $\alpha \in]0, 2[$, $\frac{1}{2} - \frac{(d-2)}{2\alpha} < 0$, therefore, for $d \geq 4$, $\liminf \psi_t = 0$ and $\limsup \psi_t = \pi$ a.s.

As $s(\pi-) = +\infty$, for every positive t, $\psi_t \in [0, \pi[$, and ψ_t^2 is a solution of the S.D.E.

$$d\psi_t^2 = \sqrt{2}\widetilde{\sigma}(\psi_t^2)\ dB_t + \widetilde{b}(\psi_t^2)\ dt. \tag{9.29}$$

Note that, as $\widetilde{\sigma}(0) = \widetilde{b}(0) = 0$, 0 is a solution, and the solutions of this S.D.E. might be not unique.

Let dm be the speed measure of the diffusion, with

$$m(x) = \int_{x_0}^{x} \exp\left[\int_{x_0}^{y} \frac{b(\varphi)}{\sigma^2(\varphi)}\ d\varphi\right]\ \frac{dy}{\sigma^2(y)}, \text{ with } (x_0, x) \in]0, \pi[^2.$$

- **Case $s(0+) > -\infty$ (i.e $\mu < 1$ or $\eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$).**

  In this case, $\liminf \psi_t = 0$. Let $T = \inf\{t > 0;\ \psi_t = 0\}$, from Proposition 16.43 p.366 in [3], T is finite or the boundary point 0 is accessible if and only if for any positive b, $\int_0^b |s(x) - s(0)| dm(x)$ is finite.

  For any positive ε, there exists a constant C such that for any $y \in]0, x_0[$

  $$\frac{1}{\sigma^2(y)} \exp\left[\int_{x_0}^{y} \frac{b(\varphi)}{\sigma^2(\varphi)}\ d\varphi\right] \leq C\ y^{\mu-\alpha-\varepsilon}.$$

  Hence, for $x \in]0, x_0[$, using (9.28), $0 \leq |s(x) - s(0)| m'(x) \leq CC_2\ x^{(1-\alpha-2\varepsilon)\wedge 0}$, which implies that $\int_0^b |s(x) - s(0)| dm(x)$ is finite (choose ε such that $2\varepsilon < 2 - \alpha$). This proves that T is finite a.s. Using Breiman's terminology (see [3] p.368-369), 0 is a closed boundary point.

  0 is an exit boundary point if and only if $m(0+) = -\infty$. This is the case if $\mu - \alpha < -1$ (or if $\eta < 1 - \frac{d}{\alpha^2}$). Note that for $d = 2$ or 3 and $\alpha \in]0, 2[$, $1 - \frac{d}{\alpha^2} < \frac{1}{2} - \frac{(d-2)}{2\alpha}$. Hence, if $m(0+) = -\infty$, the diffusion ψ_t is absorbed in 0.

- If $s(0+) > -\infty$ and $m(0+) = -\infty$ (i.e if $\mu < \alpha - 1 < 1$ or if $\eta < 1 - \frac{d}{\alpha^2} < \frac{1}{2} - \frac{(d-2)}{2\alpha}$), 0 is an exit boundary point and the diffusion ψ_t is absorbed in 0, and for any positive r,

  $$\lim_{d(x,y)\to 0} P^{(2)}_{(x,y)}[d(X_t, Y_t) > r] = \lim_{\varphi\to 0} P_\varphi[\psi_t > r] = 0.$$

  Now, applying proposition 6.3, we prove that the statistical solution is a flow of maps.

- $s(0+) > -\infty$ and $m(0+) > -\infty$ (i.e $\alpha - 1 < \mu < 1$ or $\eta \in \left]1 - \frac{d}{\alpha^2}, \frac{1}{2} - \frac{(d-2)}{2\alpha}\right[$).

  In this case, there is no uniqueness of the solution of the S.D.E. satisfied by ψ_t as there is a solution starting from 0, which is not identically 0.

We now have to discuss the absorption or the reflection in 0 of the diffusion ψ_t. In order to do this, for $\varepsilon \in]0,1[$, let us introduce the covariance $C_\varepsilon = (1-\varepsilon)^2 C$ (then, if W_t is the cylindrical Brownian motion associated to C, $(1-\varepsilon)W_t$ is the cylindrical Brownian motion associated to C_ε) and S_t^ε be the statistical solution associated to P_t and C_ε.

For $f \in L^2(m)$, $S_t^\varepsilon f = \sum_{n\geq 0} J_t^{n,\varepsilon} f$, where $J_t^{n,\varepsilon} f$ is the nth chaos in the chaos expansion of $S_t^\varepsilon f$. Note that $S_t^\varepsilon = Q_{\log(1-\varepsilon)} S_t$, where Q_α is the Ornstein-Uhlenbeck operator on the Wiener space (used in Malliavin calculus [18]). It is easy to see that $J_t^{n,\varepsilon} f = (1-\varepsilon)^n J_t^n f$, where $J_t^n f$ is the nth chaos in the chaos expansion of $S_t f$, hence

$$E[(S_t^\varepsilon f - S_t f)^2] = \sum_{n\geq 1} (1-(1-\varepsilon)^{2n}) E[(J_t^n f)^2]. \tag{9.30}$$

Hence it is clear that the $L^2(P)$-limit as ε goes to 0 of $S_t^\varepsilon f$ is $S_t f$.

Let $(X_t^\varepsilon, Y_t^\varepsilon)$ be the Markov process associated to $P_t^{(2),\varepsilon} = E[S_t^{\varepsilon\ \otimes 2}]$ and $\psi_t^\varepsilon = d(X_t^\varepsilon, Y_t^\varepsilon)$. ψ_t^ε is a diffusion with generator L_ε. It is easy to see that $L_\varepsilon = (1-(1-\varepsilon)^2)L_1 + (1-\varepsilon)^2 L$ (note that $A_\varepsilon^{(2)} = A\otimes I + I \otimes A + (1-\varepsilon)^2 C = A_1^{(2)} + (1-\varepsilon)^2 (A^{(2)} - A_1^{(2)})$), and $L_\varepsilon = \sigma_\varepsilon^2(\varphi)\frac{d^2}{d\varphi^2} + b_\varepsilon(\varphi)\frac{d}{d\varphi}$, with

$$\sigma_\varepsilon^2(\varphi) = (1-(1-\varepsilon)^2)\sigma_1^2(\varphi) + (1-\varepsilon)^2\sigma^2(\varphi), \tag{9.31}$$

$$b_\varepsilon(\varphi) = (1-(1-\varepsilon)^2) b_1(\varphi) + (1-\varepsilon)^2 b(\varphi). \tag{9.32}$$

Let us remark that L_1 is the generator of the diffusion distance between two independent Brownian motions on S^d. Note that as φ goes to 0,

$$\sigma_1^2(\varphi) \sim \sigma_1^2(0) = 2 \qquad \text{and} \qquad b_1(\varphi) \sim \frac{2(d-1)}{\varphi} \tag{9.33}$$

and $\sigma^2(\varphi) = (1-(1-\varepsilon)^2)\sigma_1^2(\varphi)(1+O(\varphi^\alpha))$ and $b_\varepsilon(\varphi) = (1-(1-\varepsilon)^2) b_1(\varphi)(1+O(\varphi^\alpha))$. Studying the scale function s_ε of ψ_t^ε, we get that $s_\varepsilon(0+) = s_1(0+) = -\infty$ (as two independent Brownian motions cannot meet each other on S^d). We still have $s_\varepsilon(\pi-) = \infty$.

Let dm_ε be the speed measure of ψ_t^ε. As $m_\varepsilon(0+) > -\infty$ and $m_\varepsilon(\pi-) < \infty$, dm_ε is an invariant finite measure for the diffusion ψ_t^ε. As $\lim_{\varepsilon\to 0} \sigma_\varepsilon^2 = \sigma^2$ and $\lim_{\varepsilon\to 0} b_\varepsilon = b$, we get that $\lim_{\varepsilon\to 0} m_\varepsilon'(x) = m'(x)$. Let us note $\varepsilon' = 1-(1-\varepsilon)^2$ and let

$$f(\varepsilon', \varphi) = \frac{\varepsilon' b_1(\varphi) + (1-\varepsilon') b(\varphi)}{\varepsilon'\sigma_1^2(\varphi) + (1-\varepsilon')\sigma^2(\varphi)}. \tag{9.34}$$

This function is increases with ε' if $\frac{b_1(\varphi)}{\sigma_1^2(\varphi)} \leq \frac{b(\varphi)}{\sigma^2(\varphi)}$. As $\frac{b_1(\varphi)}{\sigma_1^2(\varphi)} - \frac{b(\varphi)}{\sigma^2(\varphi)} \sim (d-1-\mu)\frac{1}{\varphi}$ as φ goes to 0 and as $(d-1-\mu)$ is positive, there exists φ_0 such that for any $\varphi < \varphi_0$, $f(\varepsilon', \varphi) \geq \frac{b(\varphi)}{\sigma^2(\varphi)} = f(0,\varphi)$ and for $\varepsilon' < 1/2$,

$$m_\varepsilon'(x) \leq \frac{2}{\sigma^2(x)} \exp\left(-\int_x^{\varphi_0} \frac{b(\varphi)}{\sigma^2(\varphi)}\, d\varphi \right) \times C_{\varphi_0},$$

where $C_{\varphi_0} = \sup_{\varepsilon\in[0,1]} \exp\left(\int_{\varphi_0}^{x_0} f(\varepsilon',\varphi)\, d\varphi\right) < \infty$. The Lebesgue's dominated convergence theorem implies that m'_ε converges in $L^1([0,\pi])$ towards m'.

Let us remark that the $L^2(P)$ convergence of $S_t^\varepsilon f$ towards $S_t f$ for every $f \in L^2(m)$ implies that ψ_t^ε converges in distribution towards ψ_t as ε goes to 0. As dm_ε is an invariant measure, for any continuous function f on $[0,\pi]$, we have

$$\int E[f(\psi_t^\varepsilon)|\psi_0^\varepsilon = x]\; dm_\varepsilon(x) = \int f\; dm_\varepsilon. \tag{9.35}$$

As

$$\left|\int E[f(\psi_t^\varepsilon)|\psi_0^\varepsilon = x]\; dm_\varepsilon(x) - \int E[f(\psi_t)|\psi_0 = x]\; dm(x)\right| \;\leq$$
$$\leq\; \|f\|_\infty \int_0^\pi |m'_\varepsilon(x) - m'(x)|\; dx + \left|\int E[f(\psi_t^\varepsilon)|\psi_0^\varepsilon = x] - E[f(\psi_t)|\psi_0 = x]\; dm(x)\right|,$$

we get that (because m'_ε converges in $L^1([0,\pi])$ towards m' and ψ_t^ε converges in distribution toward ψ_t.)

$$\int E[f(\psi_t)|\psi_0 = x]\; dm(x) = \lim_{\varepsilon\to 0} \int E[f(\psi_t^\varepsilon)|\psi_0^\varepsilon = x]\; dm_\varepsilon(x) = \lim_{\varepsilon\to 0} \int f\; dm_\varepsilon = \int f\; dm.$$

This implies that dm is an invariant measure for ψ_t. As $m(0+) > -\infty$, the diffusion ψ_t is not absorbed in 0 and is reflected in 0.

In this case, 0 is a closed regular boundary point. This point is instantaneously reflecting as $m(\{0\}) = 0$. This implies the existence of a positive t, a positive r and $p \in]0,1]$ such that for any $x \in]0,\pi[$, $P_x[\psi_t \geq r] \geq p$. Then, applying proposition 6.4, S_t is not a flow of maps.

- **Case $s(0+) = -\infty$ (i.e $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$).**

  In this case, 0 is an open boundary point or inaccessible. From Proposition 16.45 in [3], 0 is an entrance boundary point if and only if $\int_{0+} |s(x)|dm(x) < \infty$. For any positive ε, there exists a positive constant D such that, for any $x \in]0, x_0[$,

  $$|s(x)m'(x)| \leq D\; x^{(\frac{\varepsilon}{c}-\alpha-\varepsilon)\wedge 0} x^{-\frac{\varepsilon}{c}-\varepsilon+1} \leq D\; x^{1-\alpha-2\varepsilon}.$$

  This shows that $\int_{0+} s(x)dm(x) < \infty$ (choose ε such that $2\varepsilon \leq 2-\alpha$). Therefore, we have proved that 0 is an entrance point for the diffusion, and in particular, there exists a positive t, a positive α and $p \in]0,1[$ such that for any $x \in]0,\pi[$, $P_x[\psi_t > \alpha] > p$. Proposition 6.4 implies that S_t is not a flow of maps. □

For $\alpha > 2$, the statistical solution is an isotropic Brownian flow of diffeomorphisms. In Raimond [19], the Lyapounov exponents of this flow are computed. The sign of the first

Lyapounov exponent $\lambda_1(\alpha, d)$ describes the stability of the flow. It is unstable if $\lambda_1 \geq 0$ and stable if $\lambda_1 < 0$. The computation of $\lambda_1(\alpha, d)$ gives

$$\lambda_1 = \frac{(d-4)a+db}{d+2}\zeta(\alpha-1) + \left(\frac{d-1}{d+2}\right)[(d-4)a+db]\zeta(\alpha) - d\left(\frac{2(d-1)a+db}{d+2}\right)\zeta(\alpha+1), \tag{9.36}$$

where $\zeta(\alpha) = \sum_{l\geq 1}\frac{1}{l^\alpha}$ is the zeta function. Therefore, we have $\lambda_1(\alpha, d) = 0$ if and only if

$$\eta = \eta(\alpha, d) = \frac{-(d-4)\zeta(\alpha-1) - (d-1)(d-4)\zeta(\alpha) + 2d(d-1)\zeta(\alpha+1)}{4\zeta(\alpha-1) + 4(d-1)\zeta(\alpha) + d(d-2)\zeta(\alpha+1)}. \tag{9.37}$$

It is easy to see that for fixed η, $\lim_{\alpha\to 2+}\lambda_1(\alpha, d) = +\infty$ if $d \geq 4$ or if $\eta < \frac{1}{2} - \frac{d-2}{4} = \frac{4-d}{4}$ and that $\lim_{\alpha\to 2+}\lambda_1(\alpha, d) = -\infty$ if $\eta > \frac{4-d}{4}$. Remark that $\lim_{\alpha\to 2-} 1 - \frac{d}{\alpha^2} = \lim_{\alpha\to 2-}\frac{1}{2} - \frac{(d-2)}{2\alpha} = \frac{4-d}{4}$. This shows that coalescence appear when λ_1 goes to $+\infty$ and splitting appear when λ_1 goes to $-\infty$.

The results of this section is given by phase diagrams in the appendix.

10 Isotropic statistical solution on $\mathbb{R}^d$.

10.1 Stationary and isotropic covariance functions on $\mathbb{R}^d$.

On $\mathbb{R}^d$, the stationary isotropic covariance function C are (see Le Jan [15]) such that $C^{ij}(x, y) = C^{ij}(x-y)$, for $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, with

$$C^{ij}(z) = \delta^{ij}B_N(\|z\|) + \frac{z^i z^j}{\|z\|^2}(B_L(\|z\|) - B_N(\|z\|)), \tag{10.1}$$

with

$$B_L(r) = \iint \cos(\rho u_1 r)u_1^2\omega(du)(F_L(d\rho) - F_N(d\rho)) + \iint \cos(\rho u_1 r)\omega(du)F_N(d\rho), \tag{10.2}$$

$$B_N(r) = \iint \cos(\rho u_1 r)u_2^2\omega(du)(F_L(d\rho) - F_N(d\rho)) + \iint \cos(\rho u_1 r)\omega(du)F_N(d\rho), \tag{10.3}$$

F_L and F_N being finite positive measures on $\mathbb{R}^+$. $\omega(du)$ is the normalized Lebesgue measure on S^{d-1}. F_L and F_N represent respectively the gradient part and the zero divergence part of the associated Gaussian vector field.

For α and m positive reals, let $F(d\rho) = \frac{\rho^{d-1}}{(\rho^2+m^2)^{\frac{d+\alpha}{2}}}d\rho$, $F_L(d\rho) = aF(d\rho)$ and $F_N(d\rho) = \frac{b}{d-1}F(d\rho)$, where a and b are nonnegative. In the Fourier representation, (C is a positive constant)

$$\hat{C}^{ij}(k) = C(\|k\|^2 + m^2)^{-\frac{d+\alpha}{2}}\left(a\frac{k^i k^j}{\|k\|^2} + \frac{b}{d-1}\left(\delta_{ij} - \frac{k^i k^j}{\|k\|^2}\right)\right). \tag{10.4}$$

Notice that in the Fourier representation, the Laplace operator on vector fields is given by the multiplication by $-\|k\|^2$ and the projection π on gradient vector fields (in the L^2 space) by $\frac{k^i k^j}{\|k\|^2}$. (i.e if V is a vector field and $\hat{V}^i(k)$ its Fourier transform, $(\pi\hat{V})^i(k) = \sum_j \frac{k^i k^j}{\|k\|^2}\hat{V}^j(k)$.)

Therefore, given a L^2 vector field, $\int \sum_i C^{ij}(x-y)V^i(x)\,dx$ is also $C(-\Delta+m^2)^{-\frac{d+\alpha}{2}}(a\pi V+\frac{b}{d-1}(I-\pi)V)$. The self-reproducing space appears to be the L^2-Sobolev space of order $s=\frac{d+\alpha}{2}$ equipped with the norm

$$\|V\|^2=\frac{1}{a}\|\pi V\|_s^2+\frac{d-1}{b}\|(I-\pi)V\|_s^2,$$

where

$$\|V\|_s^2=\langle(-\Delta+m^2)^sV,V\rangle_2.$$

Note that if a or b vanishes, the self-reproducing space is restricted to divergence free or gradient vector fields.

Lemma 10.1 *For $\alpha\in]0,2[$, as r goes to 0,*
i) $\iint\cos(\rho u_1 r)\omega(du)F(d\rho)=F(\mathbb{R}^+)-\alpha_1 r^\alpha+\circ(r^\alpha)$.
ii) $\iint\cos(\rho u_1 r)u_1^2\omega(du)F(d\rho)=\frac{F(\mathbb{R}^+)}{d}-\alpha_2 r^\alpha+\circ(r^\alpha)$.
iii) $\iint\cos(\rho u_1 r)u_2^2\omega(du)F(d\rho)=\frac{F(\mathbb{R}^+)}{d}-\alpha_3 r^\alpha+\circ(r^\alpha)$.
with $\alpha_2=\frac{\alpha+1}{d+\alpha}\alpha_1$, $\alpha_3=\frac{1}{d+\alpha}\alpha_1$ and

$$\alpha_1=c_d\left(\int_0^\infty(1-\cos x)\frac{dx}{x^{\alpha+1}}\right)\left(\int_0^{\frac{\pi}{2}}(\cos\theta)^\alpha(\sin\theta)^{d-2}d\theta\right).$$

Proof. For $r>0$, making the change of variable $x=\rho u_1 r$,

$$\begin{aligned}\iint(1-\cos(\rho u_1 r))\omega(du)F(d\rho) &= c_d\int_0^1\int_0^\infty(1-\cos(\rho u_1 r))(1-u_1^2)^{\frac{d-2}{2}}du_1\frac{\rho^{d-1}d\rho}{(\rho^2+m^2)^{\frac{d+\alpha}{2}}}\\ &= r^\alpha c_d\int_0^1\left(\int_0^\infty(1-\cos x)\frac{x^{d-1}dx}{(x^2+r^2u_1^2m^2)^{\frac{d+\alpha}{2}}}\right)u_1^\alpha(1-u_1^2)^{\frac{d-2}{2}}du_1.\end{aligned}$$

As $\lim_{r\to0}\int_0^\infty(1-\cos x)\frac{x^{d-1}}{(x^2+r^2u_1^2m^2)^{\frac{d+\alpha}{2}}}dx=\int_0^\infty(1-\cos x)\frac{dx}{x^{\alpha+1}}<\infty$, we get that

$$\lim_{r\to0}\frac{1}{r^\alpha}\iint(1-\cos(\rho u_1 r))\omega(du)F(d\rho)=c_d\left(\int_0^\infty(1-\cos x)\frac{dx}{x^{\alpha+1}}\right)I(d-2,\alpha)=\alpha_1,$$

with $I(n,t)=\int_0^{\frac{\pi}{2}}(\cos\theta)^t(\sin\theta)^n\,d\theta=\frac{1}{2}B(\frac{n+1}{2},\frac{t+1}{2})$ for $t\geq0$ and $n\in\mathbb{N}$, and $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. This shows i). ii) and iii) can be obtained the same way with

$$\alpha_2=c_d\int_0^\infty(1-\cos x)\frac{dx}{x^{\alpha+1}}I(d-2,\alpha+2)$$

and $\alpha_1=\alpha_2+(d-1)\alpha_3$ (note that $\int u_1^2\omega(du)=\frac{1}{d}$). It is easy to see that for $\alpha>0$ and $d\geq1$,

$$I(d-2,\alpha+2)=\frac{\alpha+1}{d+\alpha}I(d-2,\alpha).$$

Therefore, $\alpha_2=\frac{\alpha+1}{d+\alpha}\alpha_1$. With the relation $\alpha_1=\alpha_2+(d-1)\alpha_3$, we get that $\alpha_3=\frac{1}{d+\alpha}\alpha_1$. □

Remark 10.2 *As z goes to 0,*

$$C^{ij}(z) = B\delta^{ij} - \frac{\alpha_1}{d-1}\left[((d-1)a + (d+\alpha-1)b)\delta^{ij} - \alpha((d-1)a-b)\frac{z^i z^j}{\|z\|^2}\right]\|z\|^\alpha(1+\circ(1)),$$

with $B = B_L(0) = B_N(0) = \frac{a+b}{d}F(\mathbb{R}^+)$. Let us note that the dependence on m only appear in B.

10.2 An example of phase transitions for the statistical solution.

Let P_t be the semigroup of a Brownian motion on $\mathbb{R}^d$ with variance $(a+b)F(\mathbb{R}^+)$. Let S_t be the statistical solution associated to P_t and C. If $\alpha \geq 2$, equation (8.1) is satisfied and S_t is a flow of maps.

Let $\eta = \frac{b}{a+b}$, then we have the theorem.

Theorem 10.3 *For any $\alpha \in]0,2[$,*

- *For $d = 2$ or 3 and $\eta < 1 - \frac{d}{\alpha^2}$, the statistical solution is a coalescing flow of maps.*
- *For $d = 2$ or 3 and $1 - \frac{d}{\alpha^2} < \eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$, the statistical solution is not a flow of maps and $P_t^{(2)}$ is a Markovian semigroup of a 2-point process hitting the diagonal.*
- *For $d = 2$ or 3 and $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$ or for $d \geq 4$, S_t is not a flow of maps and the 2-point process does not hit the diagonal.*

Remark. Related results have been independently given by Gawedzki and Vergassola in [10].

Remark. The results of this theorem are exactly the same as for the sphere.

Proof. Let us study the Markovian semigroup $P_t^{(2)}$, and the associated Markov process (X_t, Y_t) on $\mathbb{R}^d \times \mathbb{R}^d$ starting from (x,y) (with $x \neq y$). Then $r_t = d(X_t, Y_t)$ is a diffusion in $\mathbb{R}^+$ (eventually reflected in 0), with generator $L = \sigma^2(r)\frac{d^2}{dr^2} + b(r)\frac{d}{dr}$ (see Le Jan [15]), with

$$\sigma^2(r) = B - B_L(r), \tag{10.5}$$

$$b(r) = (d-1)\frac{B - B_N(r)}{r}. \tag{10.6}$$

From the lemma, it is easy to see that as r goes to 0,

$$\sigma^2(r) = \frac{(a+b)\alpha_1}{d+\alpha}(\alpha + 1 - \alpha\eta)r^\alpha(1+\circ(1)), \tag{10.7}$$

$$b(r) = \frac{(a+b)\alpha_1}{d+\alpha}(d - 1 + \alpha\eta)r^{\alpha-1}(1+\circ(1)). \tag{10.8}$$

Note that we get the same behaviour of σ and b around 0 as in section 9.2.

As in section 9.2, let us study s, the scale function of the diffusion. As r goes to ∞, $s'(r) \sim r^{1-d}$, therefore $s(+\infty)$ is finite if and only if $d \geq 3$.

We also see that $s(0+) = -\infty$ if $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$ and $s(0+)$ is finite if $\eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$.

Let dm be the speed measure of the diffusion. Let us study the boundary point 0.

As $m(0+) > -\infty$ if $\eta > 1 - \frac{d}{\alpha^2}$, as in section 9.2, with a similar proof, we can prove that the diffusion r_t is instantaneously reflecting at 0 if $\eta \in]1 - \frac{d}{\alpha^2}, \frac{1}{2} - \frac{(d-2)}{2\alpha}[$.

0 is an exit boundary point if $\eta < 1 - \frac{d}{\alpha^2}$ (note that $1 - \frac{d}{\alpha^2} \leq \frac{1}{2} - \frac{d-2}{\alpha}$) and the diffusion is absorbed by 0.

Therefore, we get that

- If $d \geq 3$ and $\eta \in]1 - \frac{d}{\alpha^2}, \frac{1}{2} - \frac{(d-2)}{2\alpha}[$, r_t is instantaneously reflecting at 0 and is transient. In this case, as in section 9.2, S_t is not a flow of maps.

- If $d = 2$ and $\eta \in]1 - \frac{d}{\alpha^2}, \frac{1}{2} - \frac{(d-2)}{2\alpha}[$, r_t is instantaneously reflecting at 0 and is recurrent. In this case, as in section 9.2, S_t is not a flow of maps.

- If $d \geq 3$ and $\eta < 1 - \frac{d}{\alpha^2}$, r_t is absorbed at 0 with probability $\frac{s(\infty)-s(r_0)}{s(\infty)-s(0)}$ and converges towards $+\infty$ with probability $\frac{s(r_0)-s(0)}{s(\infty)-s(0)}$. In this case, as in the section 9.2, S_t is a coalescent flow of maps.

- If $d \leq 2$ and $\eta < 1 - \frac{d}{\alpha^2}$, r_t is absorbed at 0. In this case, as in section 9.2, S_t is a coalescent flow of maps.

If $\eta > \frac{1}{2} - \frac{d-2}{2\alpha}$, then we have that $s(0) = -\infty$. In this case, 0 is an entrance boundary point as $\int_{0+} |s(x)| dm(x) < \infty$. r_t is recurrent if $d \leq 2$ and transient if $d \geq 3$. As in the section 9.2, we prove that S_t is not a flow of maps. □

For $\alpha > 2$, the statistical solution is a stationary isotropic Brownian flow of diffeomorphisms. In Le Jan [15], the Lyapounov exponents of this flow are computed. The sign of the first Lyapounov exponent $\lambda_1(\alpha, d)$ describes the stability of the flow. It is unstable if $\lambda_1 \geq 0$ and stable if $\lambda_1 < 0$. The computation of $\lambda_1(\alpha, d)$ gives (see [15])

$$\lambda_1 = \frac{1}{2(d+2)}((d-4)a + db) \int \rho^2 F(d\rho), \tag{10.9}$$

Therefore, we have $\lambda_1(\alpha, d) = 0$ if and only if $d \leq 4$ and

$$\eta = \eta(d) = \frac{4-d}{4} \tag{10.10}$$

As in the section 9.2, we see that for fixed η, $\lim_{\alpha \to 2+} \lambda_1(\alpha, d) = +\infty$ if $d \geq 4$ or if $\eta < \frac{1}{2} - \frac{d-2}{4} = \frac{4-d}{4}$ and that $\lim_{\alpha \to 2+} \lambda_1(\alpha, d) = -\infty$ if $\eta > \frac{4-d}{4}$. This shows that coalescence appear when λ_1 goes to $+\infty$ and splitting appear when λ_1 goes to $-\infty$.

Remark that $\lim_{\alpha \to 2-} 1 - \frac{d}{\alpha^2} = \lim_{\alpha \to 2-} \frac{1}{2} - \frac{(d-2)}{2\alpha} = \frac{4-d}{4}$.

The results of this section is given by phase diagrams in the appendix.

11 Reflecting flows.

Let D be an open convex domain in $\mathbb{R}^d$ with C^1 boundary ∂D. Let d be the Euclidean metric in $\mathbb{R}^d$. For any $x \in \partial D$, we note $n(x)$ the directed inward unit normal vector to ∂D.

Let $C(x,y)$ be a covariance function satisfying (8.1) for any $(x,y) \in D \times D$. Let P_t be the semigroup of a symmetric diffusion of generator $A = \sum_{i,j} C^{ij}(x,x)\partial_i\partial_j + \sum_i b^i(x)\partial_i$ in D reflected on ∂D. Assume that b^i is a Lipschitz function (i.e. $\|b(x) - b(y)\| \leq \frac{C}{4} d(x,y)$). P_t is associated to the Dirichlet form $(\mathcal{E}, \mathcal{F})$, where $\mathcal{F} = H^1(D)$ and $\mathcal{E}(f,g) = \frac{1}{2}\sum_{i,j} \int_D C^{ij}(x,x)\frac{\partial}{\partial x^i} f(x) \frac{\partial}{\partial x^j} g(x)\, dx$.

We can construct a statistical solution associated to P_t and C. Let $P_t^{(2)}$ be the semigroup of the 2-point process and (X_t, Y_t) be the associated Markov process, of law $P^{(2)}_{(.,.)}$.

We know that X_t and Y_t are two diffusions in D reflected on ∂D. Let φ_t and ψ_t denote the local times of X_t and Y_t on ∂D.

Lemma 11.1 $P_t^{(2)} h(x,y) \leq h(x,y) e^{Ct}$.

Proof. Let us note

$$L^{(2)} = A_x + A_y + \sum_{i,j} C^{ij}(x,y)\partial_{x_i}\partial_{x_j}.$$

From (8.1) and the Lipschitz conditions, we get that for $h(x,y) = d(x,y)^2$,

$$L^{(2)} h(x,y) \leq C\ h(x,y).$$

Using Tanaka's formula, there exists a martingale M_t such that

$$\begin{aligned} h(X_t, Y_t) - h(x,y) &= M_t + \int_0^t L^{(2)} h(X_s, Y_s)\, ds & (11.1) \\ &+ \int_0^t \langle \nabla_x h(X_s, Y_s), n(X_s)\rangle d\varphi_s + \int_0^t \langle \nabla_y h(X_s, Y_s), n(Y_s)\rangle d\psi_s & (11.2) \end{aligned}$$

As $\nabla_x h(x,y) = 2(x-y)$, using the fact that D is convex, we get that for $x \in \partial D$

$$\langle \nabla_x h(x,y), n(x)\rangle < 0.$$

This implies that

$$h(X_t, Y_t) - h(x,y) \leq M_t + C \int_0^t h(X_s, Y_s)\, ds.$$

Taking the expectation, we get that $P_t^{(2)} h(x,y) - h(x,y) \leq C \int_0^t P_s^{(2)} h(x,y)\, ds$. Hence the lemma. □

Theorem 11.2 *The statistical solution is a flow of maps.*

Proof. This is the same proof as the proof of theorem 8.2. □

Appendix: Phase diagrams for the statistical solutions studied in section 9 and 10.

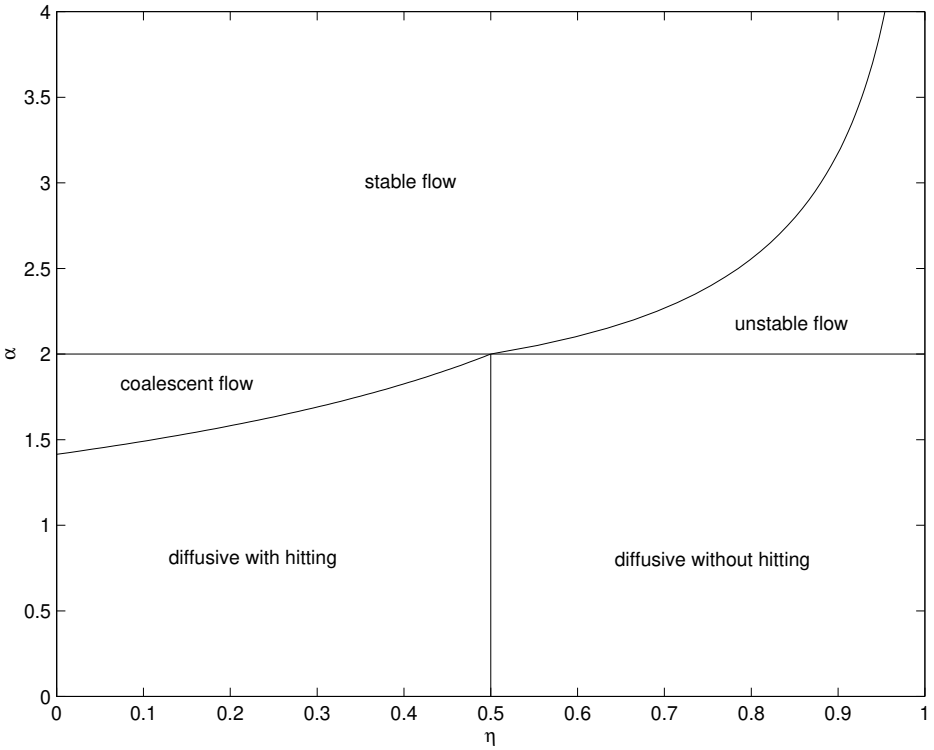

Figure 1 : Phase diagram on S^2.

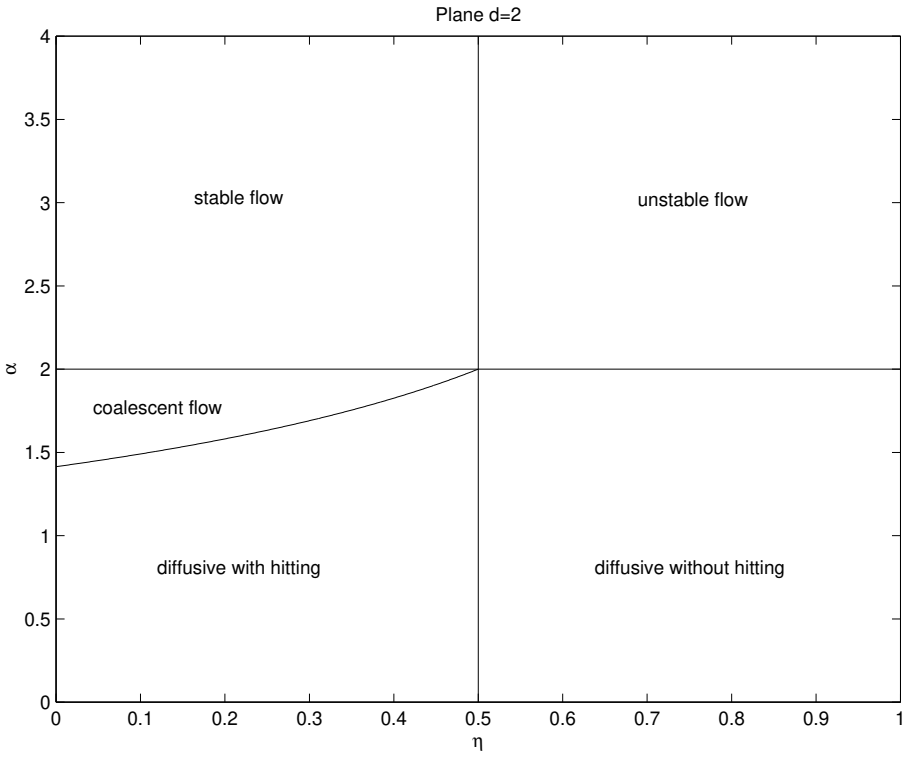

Figure 2 : Phase diagram on $\mathbb{R}^2$.

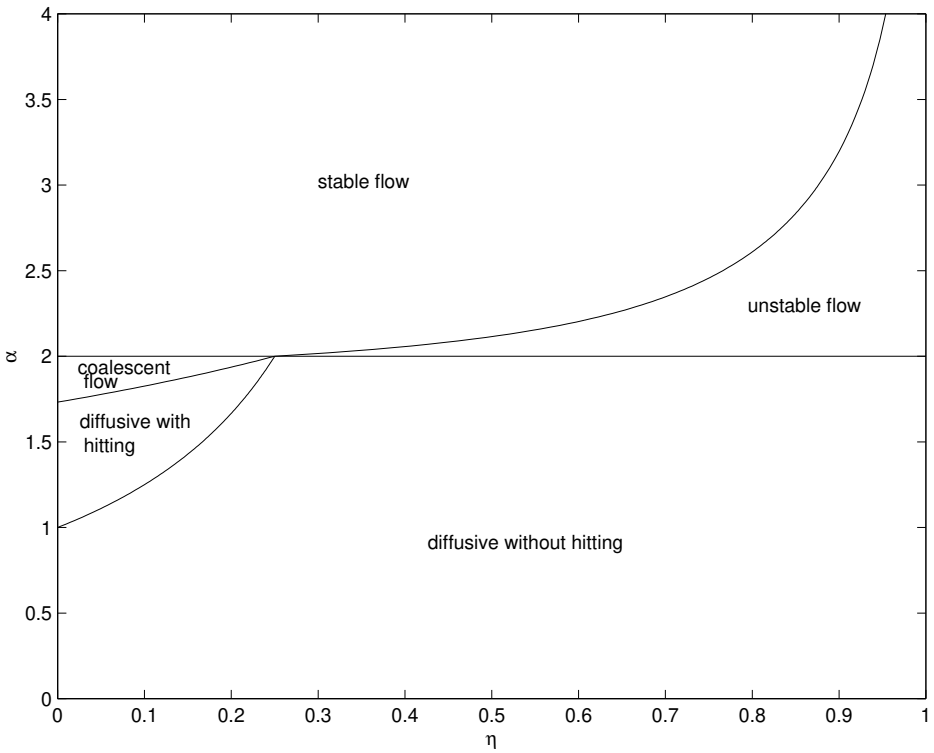

Figure 3 : Phase diagram on S^3.

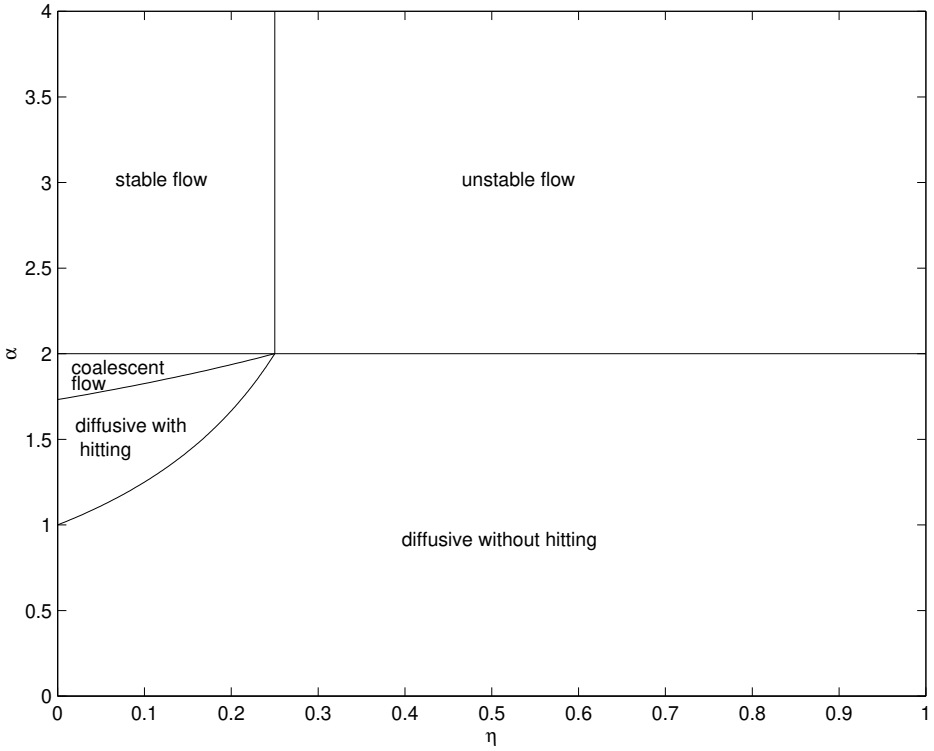

Figure 4 : Phase diagram on $\mathbb{R}^3$.

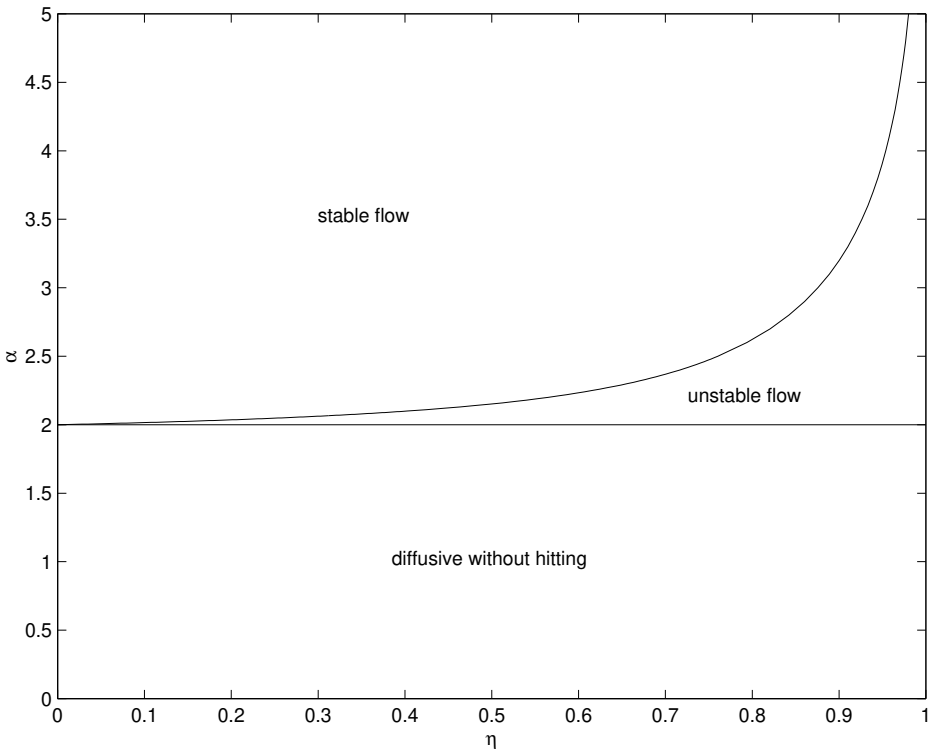

Figure 5 : Phase diagram on S^4.

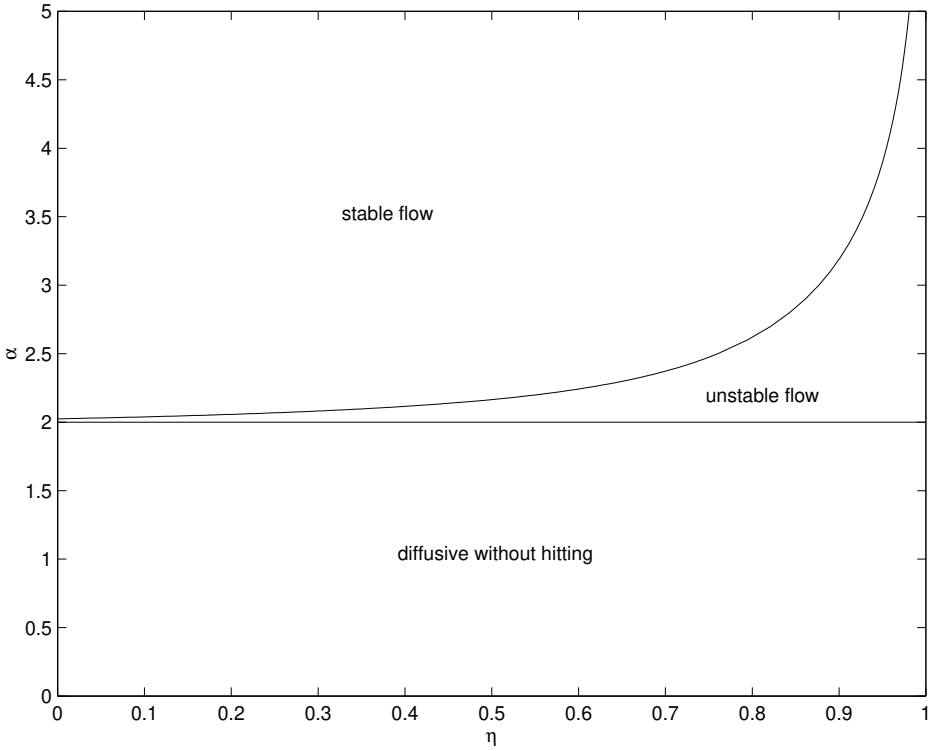

Figure 6 : Phase diagram on S^5.

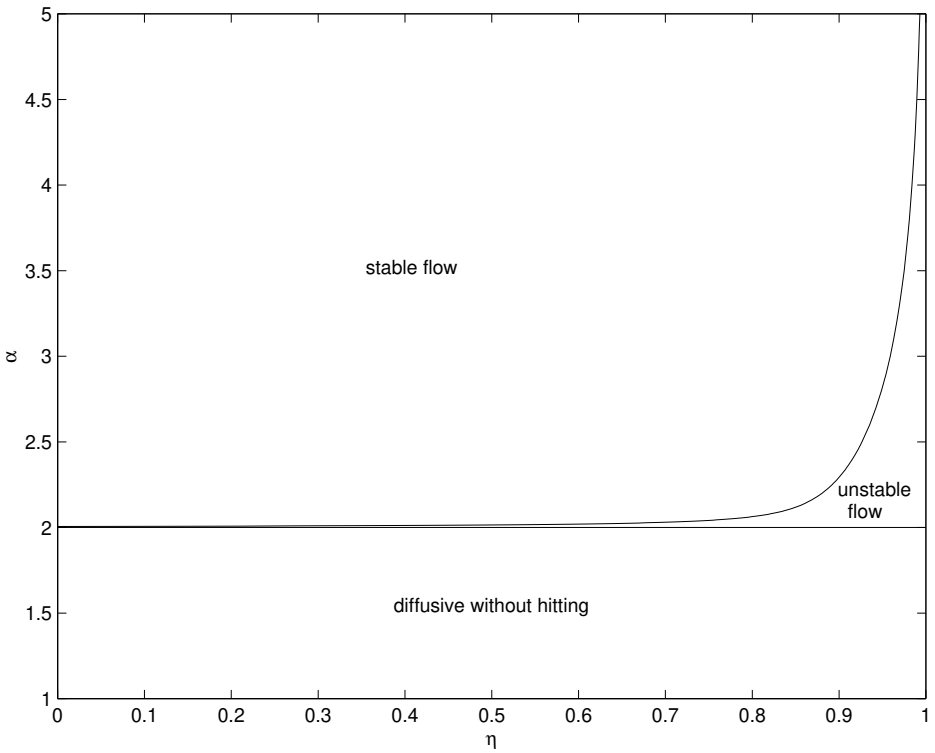

Figure 7 : Phase diagram on S^{50}.

Let us remark that when $\alpha < 2$, the diagrams are exactely the same for the sphere and for the plane. For the sphere, we see that, for $\alpha > 2$ and $\eta \leq 2 - \frac{\zeta(\alpha)}{\zeta(\alpha+1)}$, the flow gets stable when d goes to ∞ : (9.37) implies that $\lim_{d\to\infty} \eta(\alpha, d) = 2 - \frac{\zeta(\alpha)}{\zeta(\alpha+1)}$ for $\alpha > 2$. We see that, for any d and $\eta \in [0, 1[$, the flow gets stable when α goes to ∞ : (9.37) implies that $\lim_{\alpha\to\infty} \eta(\alpha, d) = 1$.

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