On the Wulff crystal in the Ising model

Raphaël Cerf[1] and Ágoston Pisztora[2]

Abstract

We study the phase separation phenomenon in the Ising model in dimensions $d \geq 3$. To this end we work in a large box with plus boundary conditions and we condition the system to have an excess amount of negative spins so that the empirical magnetization is smaller than the spontaneous magnetization m^*. We confirm the prediction of the phenomenological theory by proving that with high probability a single droplet of the minus phase emerges surrounded by the plus phase. Moreover, the rescaled droplet is asymptotically close to a definite deterministic shape – the Wulff crystal – which minimizes the surface free energy. In the course of the proof we establish a surface order large deviation principle for the magnetization. Our results are valid for temperatures T below a limit of slab-thresholds $\widehat{T}_c$ conjectured to agree with the critical point T_c. Moreover, T should be such that there exist only two extremal translation invariant Gibbs states at that temperature; a property which can fail for at most countably many values and which is conjectured to be true for every T. The proofs are based on the Fortuin-Kasteleyn representation of the Ising model along with coarse-graining techniques. To handle the emerging macroscopic objects we employ tools from geometric measure theory which provide an adequate framework for the large deviation analysis. Finally, we give a heuristic argument that for subcritical temperatures close enough to T_c, the dominant minus spin cluster of the Wulff droplet *permeates* the entire box and and has a strictly positive local density everywhere.

Key Words: phase separation, Wulff crystal, Ising model, large deviations, FK percolation
AMS (1991) subject classifications: 60K35, 82Bxx, 60F10
Running Head: Wulff crystal in the Ising model

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