An example of local analytic q-difference equation : Analytic classification.

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Abstract

Using the techniques developped by Jean Ecalle for the study of non-linear differential equations, we prove that the q-difference equation

$$x\sigma_q y = y + b(y,x)$$

with $(\sigma_q f)(x) = f(qx)$ $(q > 1)$ and $b(0,0) = \partial_y b(0,0) = 0$ is analytically conjugated to one of the following equations :

$$x\sigma_q y = y$$

or

$$x\sigma_q y = y + x$$

1 Introduction.

The aim of this paper is to study the formal or analytic conjugacy of q-difference equations, using formal or analytic power series in $\tilde{\mathcal{G}}$ or $\mathcal{G}$:

$$\begin{array}{rcl} \tilde{\mathcal{G}} &=& \{f(z,x) \in \mathbb{C}[[z,x]]\ ;\ \partial_z f(0,0) \neq 0,\ f(0,0) = 0\} \\ \mathcal{G} &=& \{f(z,x) \in \mathbb{C}\{z,x\}\ ;\ \partial_z f(0,0) \neq 0,\ f(0,0) = 0\} \end{array} \quad (1)$$

It is clear that, $\tilde{\mathcal{G}}$ and $\mathcal{G}$ are groups for the z-composition of series.

Let $\tilde{\mathcal{G}}_1$ (resp. $\mathcal{G}_1$) be the subgroup of elements f of $\tilde{\mathcal{G}}$ (resp. $\mathcal{G}$) such that $\partial_z f(0,0) = 1$. For to series $z + b(z,x)$ and $z + c(z,x)$ of $\mathcal{G}_1$, we will say that the equation

$$(E_b) \qquad x\sigma_q y = y + b(y,x)$$

is formally conjugate to the equation

$$(E_c) \qquad x\sigma_q z = z + c(z,x)$$

if there exists $f \in \tilde{\mathcal{G}}$ such that, if z satisfies (E_c), then $y = f(z,x)$ satisfies (E_b). Here $\sigma_q y(x) = y(qx)$, with $q > 1$ (Note that we chose $q > 1$ instead of $q \in \mathbb{C}$, $|q| > 1$ for the sake of simplicity but this wouldn't have change our results.).

The formal conjugacy is an equivalence relation and the equations are *analytically* conjugate if one can find a *conjugating series* $f \in \mathcal{G}$.

It was proven in [2] that, for any $z + b(z,x) \in \mathcal{G}_1$, the equation (E_b) is formally conjugate to

$$(E_0) \qquad x\sigma_q z = z$$

and that, if $b(0,x) = 0$, then this conjugacy is analytic. We prove in this paper that,

Theorem 1. *For any $z + b(z,x) \in \mathcal{G}_1$, there exists $\alpha(b) \in \mathbb{C}$ such that the equation (E_b) is analytically conjugate to the equation*

$$(E_{\alpha(b)}) \qquad x\sigma_q z = z + \alpha(b)x$$

Moreover, if $\alpha(b) \neq 0$, (E_b) is analytically conjugate to the equation

$$(E_1) \qquad x\sigma_q z = z + x$$

This means that there exists exactly two classes of analytic conjugacy. Using these results, as well as the theory of q-resummation (see [3]), one can then express some local solutions of (E_b) in terms of the solutions of (E_0) or (E_1).

In section 1, we remind the results of [2] on the formal conjugacy of (E_b) and (E_0). The main key to this result is that, to any formal (resp. analytic) series f of $\tilde{\mathcal{G}}_1$ (resp. $\mathcal{G}_1$), one can bijectively associate a formal (resp. analytic) substitution automorphism $\mathbb{F} \in \mathbb{C}[[z,x,\partial_z]]$ such that

$$\forall \varphi \in \mathbb{C}[[z,x]], \quad \mathbb{F}.\varphi(z,x) = \varphi(f(z,x),x) \tag{2}$$

Starting with $z + b(z,x) \in \mathcal{G}_1$, we prove that its associated analytic substitution automorphism $\mathbb{D}$ can be written

$$\mathbb{D} = \mathrm{Id} + \sum_{\eta \in H} x^{\sigma} \mathbb{D}_{\eta} \tag{3}$$

where H is a semigroup which elements are of type $\eta = \begin{pmatrix} n \\ \sigma \end{pmatrix}$ $(n \in \mathbb{Z},\ \sigma \in \mathbb{N})$ and

$$\forall \eta \in H, \quad \mathbb{D}_{\eta} \in \mathbb{C}[z, \partial_z] \tag{4}$$

It is then proven that the formal series $w \in \tilde{\mathcal{G}}_1$ that conjugates (E_b) to (E_0) is associated to a formal substitution automorphism $\mathbb{W}$ that can be written :

$$\mathbb{W} = \sum_{s \geq 0} \sum_{(\eta_1,\ldots,\eta_s) \in H^s} W^{\eta_1,\ldots,\eta_s} \mathbb{D}_{\eta_s} \ldots \mathbb{D}_{\eta_1} = \sum W^{\bullet} \mathbb{D}_{\bullet}$$

where $H^0 = \{\emptyset\}$, $W^{\emptyset} = 1$ and $\mathbb{D}_{\emptyset} = \mathrm{Id}$. The collection of weights $\{W^{\eta_1,\dots,\eta_s}\}$ is called a mould $W^\bullet$. We remind a formula on $W^\bullet$ that proves that its values are in $\mathbb{C}[[x]]$. We recall also that, as $W^\bullet$ satisfies symmetry relations (symmetrelity) : it automatically ensures that $\mathbb{W}$ is a formal substitution automorphism. When $b(0, x) = 0$, $\mathbb{W}$ turn to be an analytic substitution automorphism, or equivalently, $\mathbb{W}.z \in \mathcal{G}_1$. Otherwise, most of the coefficients of $W^\bullet$ are divergent and q-multisummable but, unfortunatly, the resummation of all the weights involves an infinite number of critical times in the resummation process (see section 2 below and [4] for details on q-multisummability).

To circumvent this difficulty, we prove in section 4 that the mould $W^\bullet$ can be twisted in a mould $U^\bullet$ whose coefficient are in $\mathbb{C}\{x\}$ ($U^\bullet$ is analytic). Because of the symmetries of $U^\bullet$ the operator

$$\mathbb{U} = \sum U^\bullet \mathbb{D}_\bullet \tag{5}$$

is an apriori formal substitution automorphism such that $y = u(z, x) = \mathbb{U}.z \in \tilde{\mathcal{G}}_1$ conjugates the equation

$$(E_b) \qquad x\sigma_q y = y + b(y, x)$$

to

$$(E_c) \qquad x\sigma_q z = z + c(x) = z + \mathbb{V}.z \in z + x\mathbb{C}[[x]]$$

where $\mathbb{V}$ is also defined by a mould $V^\bullet$ (see section 4).

In fact, $\mathbb{U}$ and $\mathbb{V}$ are analytic substitution automorphisms ($u(z, x) = \mathbb{U}.z \in \mathcal{G}_1$ and $c(x) \in x\mathbb{C}\{x\}$) but the analyticity of $U^\bullet$ does *not* implies the analyticity of $\mathbb{U}$. Despite nice estimates on $U^\bullet$ (section 4.4), which are derived from an explicit formula for the coefficients of $U^\bullet$ (section 4.3), there is no hope to prove that the series of analytic functions

$$u(z, x) = \mathbb{U}.z = \sum_{s \geq 0} \sum_{(\eta_1,\dots,\eta_s) \in H^s} (U^{\eta_1,\dots,\eta_s} \mathbb{D}_{\eta_s} \dots \mathbb{D}_{\eta_1}.z) = \sum (U^\bullet \mathbb{D}_\bullet.z)$$

is *normally* convergent in a (z, x)-neighborhood of $(0, 0)$.

To prove that $\mathbb{U}$ is analytic, we will need to reorganize the sum defining it, such that $\mathbb{U}.z$ appears as a normally convergent series of analytic functions in a neighborhood of (0, 0). This reorganization of terms is simply the arborification-coarborification defined by J. Ecalle (see [1]). The operator $\mathbb{U}$ can be written

$$\mathbb{U} = \sum (U^{\bullet^<} \mathbb{D}_{\bullet^<})$$

but here $\bullet^<$ means that the sum is over sequences of elements of H equipped with an arborescent order, instead of a total order. Each coefficient of the *arborescent* mould $U^{\bullet^<}$ depends on a combination of coefficients of $U^\bullet$: $U^{\bullet^<}$ is obtained by *arborification* of $U^\bullet$ and the operators $\mathbb{D}_{\bullet^<}$ are obtained from the operators $\mathbb{D}_\bullet$ by a dual (but not unique) operation : the *coarborification* (see section 5).

Using these operations, a formula is obtained for $U^{\bullet^<}$ in section 5.2. The estimates derived from this formula prove then that $\mathbb{U}$ is an *analytic* substitution automorphism (section 5.4), as well as $\mathbb{V}$ (section 6.1).

A last but simple remark (section 6.2) leads us to the attempted result, that is to say theorem 1.

2 Reminder about previous results and moulds.

We resume here the results obtained in [2].

2.1 The operators $\mathbb{D}_\eta$.

Let us first introduce some notations. We started with the equation

$$x\sigma_q y - y = b(y,x) \tag{6}$$

where $\sigma_q y(x) = y(qx)$, with $q>1$ and $b(y,x) \in \mathbb{C}\{y,x\}$ with the following conditions:

$$b(0,0)=0 \text{ and } (\partial_y b)(0,0)=0$$

The function b can be written :

$$b(z,x) = z\sum_{n\geq -1} b_n(x) z^n = z\sum_{n\geq -1}\sum_{\sigma\geq 0} b_{n,\sigma} x^\sigma z^n \qquad (b_{-1,0}=0,\ b_{0,0}=0) \tag{7}$$

If we consider the set

$$H_0 = \left\{ \eta = \begin{pmatrix} n \\ \sigma \end{pmatrix},\ n\geq -1,\ \sigma\geq 0 \right\} \Big/ \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = H_{-1} \cup H_{\geq 0} \tag{8}$$

where

$$H_{-1} = \{\eta \in H\,;\, n=-1\} \qquad H_{\geq 0} = \{\eta\in H\,;\, n\geq 0\}$$

then, one can define the following operators.

$$\forall \eta \in H_0\ ; \quad \mathbb{B}_\eta = b_\eta z^{n+1}\partial_z \tag{9}$$

where $b_{n,\sigma} = b_\eta$. This kind of operators are useful in the study of nonlinear differential equations but we will need here some slightly different operators.

Definition 2. *Let*

$$H = \{\eta_1 + \ldots + \eta_s\,;\, s\geq 1\,;\, \eta_i \in H_0\} \tag{10}$$

For $\eta\in H$, the operator $\mathbb{D}_\eta$ is defined by :

$$\mathbb{D}_\eta = \sum_{s\geq 1} \sum_{\substack{\eta_1+\ldots+\eta_s=\eta \\ \eta_i\in H_0}} \frac{1}{s!} b_{\eta_1}\ldots b_{\eta_s} z^{n_1+\ldots+n_s+s}\partial_z^s \tag{11}$$

It is important to notice that, for any $\eta = \begin{pmatrix} n \\ \sigma \end{pmatrix} \in H$, the operator $\mathbb{D}_\eta$ is a finite sum which degree in ∂_z is at most $n+2\sigma$. It is easy to see that :

$$\forall \eta\in H_0\ ; \quad \mathbb{D}_\eta . z = \mathbb{B}_\eta . z \tag{12}$$

and,

$$\begin{array}{rcccl} \mathbb{B}.z & = & \sum_{\eta\in H_0} x^\sigma \mathbb{B}_\eta .z & = & b(u,x) \\ \mathbb{D}.z & = & \left(\mathrm{Id} + \sum_{\eta\in H} x^\sigma \,\mathbb{D}_\eta\right).z & = & z + b(z,x) \end{array} \tag{13}$$

In fact, $\mathbb{D}$ is a "convergent" substitution automorphism :

$$\forall f, g \in u\mathbb{C}\{u\} \; ; \quad \mathbb{D}(f\,g) = (\mathbb{D}\,f)\,(\mathbb{D}\,g) \tag{14}$$

and the notion of convergence will be developed in section 4.4. We introduce also some notations :

Definition 3. *Let $\boldsymbol{H} = \cup_{s\geq 1} H^s$ be the set of sequences of elements of $\boldsymbol{H}$.*
If $\boldsymbol{\eta} = \eta_1, ..., \eta_s \in \boldsymbol{H}$, then

$$l(\boldsymbol{\eta}) = s \; ; \; \boldsymbol{\eta} = \left(\begin{array}{c} \boldsymbol{n} \\ \boldsymbol{\sigma} \end{array}\right) \; ; \; \|\boldsymbol{n}\| = n_1 + ... + n_s \tag{15}$$

Moreover $`\boldsymbol{\eta} = \eta_2, ..., \eta_s$, $\boldsymbol{\eta}_{\mathrm{in}} = \eta_1$ *and, finally,*

$$\mathbb{D}_{\boldsymbol{\eta}} = \mathbb{D}_{\eta_s} ... \mathbb{D}_{\eta_1} \tag{16}$$

If we refer to [1], the set $\{\mathbb{D}_\eta\}_{\eta\in H}$ defines a cosymmetrel comould. In fact the operator $\mathbb{D}$ is a first example of automorphism defined by a symmetrel mould contracted with $\{\mathbb{D}_\eta\}_{\eta\in H}$.

2.2 Moulds.

We add to $\boldsymbol{H}$ the empty sequence $\emptyset$ and $\mathbb{D}_\emptyset = \mathrm{Id}$. We consider moulds on $\boldsymbol{H}$, with values in $\mathbb{C}[[x]]$: a mould $M^\bullet$ is a collection $\{M^{\boldsymbol{\eta}} \in \mathbb{C}[[x]]\}_{\boldsymbol{\eta}\in\boldsymbol{H}}$.

2.2.1 Symmetrelity.

A mould $M^\bullet$ is *symmetrel* iff $M^\emptyset = 1$ and

$$\forall (\boldsymbol{\eta}, \boldsymbol{\mu}) \in \boldsymbol{H}^2, \qquad \sum_{\boldsymbol{\lambda}\in \mathrm{ctsh}(\boldsymbol{\eta},\boldsymbol{\mu})} M^{\boldsymbol{\lambda}} = M^{\boldsymbol{\eta}} M^{\boldsymbol{\mu}} \tag{17}$$

where the sum is over all the sequence $\boldsymbol{\lambda}$ obtained by contracting shuffling of the sequences $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$: one shuffles the two sequences $(\boldsymbol{\eta}, \boldsymbol{\mu}) \mapsto \boldsymbol{\lambda}^*$ and, eventually, one contracts pairs $(\lambda_i, \mu_j) \mapsto \lambda_i + \mu_j$ that are consecutive in the sequence $\boldsymbol{\lambda}^*$ (see [1]). We shall often deal with symmetrel moulds since, if $M^\bullet$ is symmetrel, then the operator

$$\mathbb{M} = \sum_{\boldsymbol{\eta}\in\boldsymbol{H}} M^{\boldsymbol{\eta}} \mathbb{D}_{\boldsymbol{\eta}} = \sum M^\bullet \mathbb{D}_\bullet$$

is a formal substitution automorphism :

$$\forall \varphi \in \mathbb{C}[[z,x]], \quad \mathbb{M}.\varphi(z,x) = \varphi(\mathbb{M}.z, x) = \varphi(m(z,x), x)$$

where $m(z,x)\in\mathbb{C}[[z,x]]$ and $\partial_z m(0,0)=1$.

Note that the mould $X^\bullet$ defined by $X^\emptyset=1$ and

$$\forall\boldsymbol{\eta}=(\eta_1,\ldots,\eta_s)\in\boldsymbol{H},\quad X^{\boldsymbol{\eta}}=\begin{cases} x^{\sigma_1} & \text{if } s=1\\ 0 & \text{if } s\geq 2\end{cases}\tag{18}$$

is symmetrel and $\mathbb{D}=\sum X^\bullet\mathbb{D}_\bullet$ (see equation 13).

2.2.2 Operations on moulds.

Product. Let $M^\bullet$ and $N^\bullet$ two symmetrel moulds. If $\mathbb{M}=\sum M^\bullet\mathbb{D}_\bullet$ and $\mathbb{N}=\sum N^\bullet\mathbb{D}_\bullet$ ($\mathbb{M}.z=m(z,x)$, $\mathbb{N}.z=n(z,x)$), then

$$\forall\varphi\in\mathbb{C}[[z,x]],\qquad \begin{aligned}\mathbb{N}.\mathbb{M}.\varphi(z,x) &= \varphi(m(n(z,x),x),x)\\ &= \sum_{\boldsymbol{\mu}\in\boldsymbol{H}} N^{\boldsymbol{\mu}}\mathbb{D}_{\boldsymbol{\mu}}.\sum_{\boldsymbol{\eta}\in\boldsymbol{H}} M^{\boldsymbol{\eta}}\mathbb{D}_{\boldsymbol{\eta}}.\varphi(z,x)\\ &= \sum_{\boldsymbol{\mu}\in\boldsymbol{H}}\sum_{\boldsymbol{\eta}\in\boldsymbol{H}} N^{\boldsymbol{\mu}}M^{\boldsymbol{\eta}}\mathbb{D}_{\boldsymbol{\eta}\boldsymbol{\mu}}.\varphi(z,x)\\ &= \sum_{\boldsymbol{\eta}\in\boldsymbol{H}}(M^\bullet\times N^\bullet)^{\boldsymbol{\eta}}\mathbb{D}_{\boldsymbol{\eta}}.\varphi(z,x)\end{aligned}$$

The associtative noncommutative product of $M^\bullet$ and $N^\bullet$ is defined by

$$(M^\bullet\times N^\bullet)^{\boldsymbol{\eta}}=\sum_{\boldsymbol{\eta}^1\boldsymbol{\eta}^2=\boldsymbol{\eta}} M^{\boldsymbol{\eta}^1}N^{\boldsymbol{\eta}^2}\tag{19}$$

where the sequences $\boldsymbol{\eta}^1$ or $\boldsymbol{\eta}^2$ can be empty. Moreover $M^\bullet\times N^\bullet$ is symmetrel. Note that the symmetrel mould $1^\bullet$ defined by $1^\emptyset=1$ and $1^{\boldsymbol{\eta}}=0$ if $\boldsymbol{\eta}$ is nonempty is the unity and every symmetrel mould $M^\bullet$ is invertible, of symmetrel inverse $\mathrm{inv}(M)^\bullet$. One can notice that the mould $I^\bullet$ defined by $I^\emptyset=1$, $I^{\eta_1}=1$ and $I^{\eta_1,\ldots,\eta_s}=0$ if $s\geq 2$ has a noticeable inverse $H^{\eta_1,\ldots,\eta_s}=(-1)^s$.

Composition. The mould $M^\bullet\circ N^\bullet=P^\bullet$ is defined by $P^\emptyset=1$ and, if $\boldsymbol{\eta}\in\boldsymbol{H}$ is nonempty,

$$P^{\boldsymbol{\eta}}=\sum_{\boldsymbol{\eta}^1\ldots\boldsymbol{\eta}^t=\boldsymbol{\eta}} M^{\|\boldsymbol{\eta}^1\|,\ldots,\|\boldsymbol{\eta}^t\|}N^{\boldsymbol{\eta}^1}\ldots N^{\boldsymbol{\eta}^t}$$

where the sequences $\boldsymbol{\eta}^i$ are nonempty and $P^\bullet$ is symmetrel. The composition is associative and

$$(M^\bullet\times N^\bullet)\circ Q^\bullet=(M^\bullet\circ Q^\bullet)\times(N^\bullet\circ Q^\bullet)$$

In fact, for any symmetrel mould $M^\bullet$, $I^\bullet\circ M^\bullet=M^\bullet$ and

$$\mathrm{inv}(M)^\bullet=H^\bullet\circ M^\bullet\tag{20}$$

2.3 Previous results.

We remind here the results developped in [2]. We proved that the solution y of the equation

$$x\sigma_q y=y+b(y,x)$$

is formally conjugated to the the solution z of

$$x\sigma_q z = z$$

by a formal substitution automorphism $\mathbb{W} = \sum W^\bullet \mathbb{D}_\bullet$ where $W^\bullet$ is a symmetrel mould on $\boldsymbol{H}$ with values on $\mathbb{C}[[x]]$. To prove this, one notices that

$$\begin{aligned} y + b(y,x) &= \mathbb{W}z + b(\mathbb{W}.z, x) \\ &= \mathbb{W}.\mathbb{D}z \\ &= \sum (X^\bullet \times W^\bullet)\mathbb{D}_\bullet \end{aligned}$$

and

$$\begin{aligned} x\sigma_q y &= x\sigma_q\Big(\sum W^\bullet \mathbb{D}_\bullet.z\Big) \\ &= \sum s_q(W^\bullet)\mathbb{D}_\bullet.z \end{aligned}$$

where $s_q(W^{\boldsymbol{\eta}}) = x^{-\|\boldsymbol{n}\|}\sigma_q.W^{\boldsymbol{\eta}}$. We shall come back later on this last identity. The mould $W^\bullet$ is thus defined by the identities $W^\emptyset = 1$ and

$$s_q(W^\bullet) = X^\bullet \times W^\bullet \tag{21}$$

For a nonempty sequence $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) \in \boldsymbol{H}$, this reads

$$x^{-\|\boldsymbol{n}\|}\sigma_q.W^{\boldsymbol{\eta}} = W^{\boldsymbol{\eta}} + x^{\sigma_1} W^{`\boldsymbol{\eta}}$$

and it can be proved that

Theorem 4. *If* $\boldsymbol{\eta} = (\eta_1, \dots, \eta_s) = \begin{pmatrix} n_1, \dots, n_s \\ \sigma_1, \dots, \sigma_s \end{pmatrix} = \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{\sigma} \end{pmatrix} \in \boldsymbol{H}$, *then the monomial* $W^{\boldsymbol{\eta}}$ *is a formal series with the following expression :*

$$W^{\boldsymbol{\eta}}(x) = \varepsilon_{\boldsymbol{n}} \sum_{(k_1,\dots,k_s)\in\mathbb{Z}^{\boldsymbol{n}}} x^{\|\boldsymbol{\sigma}\| + \check{k}_1 n_1 + \dots + \check{k}_s n_s} q^{-\check{k}_1\sigma_1 - \dots - \check{k}_s\sigma_s} [\check{k}_1 + 1]_q^{n_1} \dots [\check{k}_s + 1]_q^{n_s} \tag{22}$$

with the notations

- $\varepsilon_{\boldsymbol{n}} = \prod_{1\le i\le s} \varepsilon_{\hat{n}_i}$ *with* $\hat{n}_i = n_i + \dots + n_s$ *with* $\varepsilon_n = 1$ *(resp.* $\varepsilon_n = -1$*) if* $n \ge 0$ *(resp.* $n < 0$*)*
- $\mathbb{Z}^{\boldsymbol{n}} = \mathbb{Z}^{\hat{n}_1} \times \dots \times \mathbb{Z}^{\hat{n}_s}$ *with* $\mathbb{Z}^n = \mathbb{Z}^{+*}$ *(resp.* $\mathbb{Z}^-$*) if* $n \ge 0$ *(resp.* $n < 0$*)*
- $\check{k}_i = k_1 + \dots + k_i$.
- $[k+1]_q = q^{-k(k+1)/2}$

Moreover, the mould $W^\bullet$ is symmetrel. As discussed in [2], there are severe obstacles to apply the techniques of q-resummation developped in [3] and [4] to this mould. It is important to notice that the divergence of $W^{\boldsymbol{\eta}}$ is due to possibly negative finishing sequences $\hat{n}_i = n_i + \dots + n_s$: if all the integers $\hat{n}_i$ are nonnegative, then $W^{\boldsymbol{\eta}}$ is convergent.

This last condition is always fulfilled when $b(0,x) = 0$ and then

Theorem 5. *The equation*

$$(x\sigma_q - 1)y = b(y,x) \tag{23}$$

with $b(0,x)=0$ *and* $\frac{\partial b}{\partial f}(0,0)=0$ *is analytically conjugate by* $y=\mathbb{W}.z\in\mathcal{G}_1$ *to the equation*

$$x\sigma_q z = z$$

The ideas that will now be developped are based on the two fundamental remarks

1. The divergence of $W^{\boldsymbol{\eta}}$ is due to possibly negative finishing sequences $\hat{n}_i = n_i + \ldots + n_s$.
2. Whenever, $b(x,\ 0)\neq 0$, divergence appears but we could try to skip it by analytic conjugation to an equation which solution contains all the divergence, that is to say of type :

$$x\sigma_q z = z + c(x)$$

3 Formal part : the moulds $\boldsymbol{W}^{\bullet}$, $\boldsymbol{J}^{\bullet}$ and $\boldsymbol{U}^{\bullet}$.

3.1 Compensating the divergence in $\boldsymbol{W}^{\bullet}$.

To skip the divergence, one can try to modify $W^{\bullet}$ into a symmetrel mould $U^{\bullet}$ which values are in $\mathbb{C}\{x\}$. Let

$$\boldsymbol{H}^+ = \{\boldsymbol{\eta}\in\boldsymbol{H}\,;\|\boldsymbol{n}\|\geq 0\} \quad \boldsymbol{H}^- = \{\boldsymbol{\eta}\in\boldsymbol{H}\,;\|\boldsymbol{n}\|<0\} \tag{24}$$

For a sequence of length 1, if $\eta_1\in\boldsymbol{H}^+$, W^{η_1} is convergent so $U^{\eta_1}=W^{\eta_1}$, otherwise, W^{η_1} is divergent and $U^{\eta_1}=0$.

Let $(\eta_1,\eta_2)\in\boldsymbol{H}$.

- If $(\eta_1,\eta_2)\in\boldsymbol{H}^+$ and $\eta_2\in\boldsymbol{H}^+$, W^{η_1,η_2} is convergent so $U^{\eta_1,\eta_2}=W^{\eta_1,\eta_2}$.
- If $(\eta_1,\eta_2)\in\boldsymbol{H}^+$ and $\eta_2\in\boldsymbol{H}^-$ (note that $\eta_1\in\boldsymbol{H}^+$), W^{η_1,η_2} is divergent but, because of the symmetrelity,

$$W^{\eta_1}W^{\eta_2}=W^{\eta_1,\eta_2}+W^{\eta_2,\eta_1}+W^{\eta_1+\eta_2}$$

  but $W^{\eta_2,\eta_1}=U^{\eta_2,\eta_1}$ and $W^{\eta_1+\eta_2}=U^{\eta_1+\eta_2}$ are convergent thus $U^{\eta_1,\eta_2}=W^{\eta_1,\eta_2}-W^{\eta_1}W^{\eta_2}$ is convergent.
- If $(\eta_1,\eta_2)\in\boldsymbol{H}^-$, then W^{η_1,η_2} is divergent and $U^{\eta_1,\eta_2}=0$.

We transformed $W^{\bullet}$ into a new mould that seems to remain symmetrel. In fact, this suggests the following definition.

Definition 6. *Let*

$$\alpha_+(n)=\begin{cases}1 & \text{if } n\geq 0\\ 0 & \text{if } n<0\end{cases} \qquad \alpha_-(n)=\begin{cases}0 & \text{if } n\geq 0\\ -1 & \text{if } n<0\end{cases} \tag{25}$$

Then the symmetrel mould $J^{\bullet}$ *on* $\boldsymbol{H}$ *is defined as follows :* $J^{\emptyset}=1$, $J^{\eta_1}=\alpha_+(n_1)$ *and, for* $s\geq 2$,

$$J^{\eta_1,\ldots,\eta_s}=\alpha_+(\hat{n}_1)\alpha_-(\hat{n}_2)\ldots\alpha_-(\hat{n}_s) \tag{26}$$

The above digression suggest to define

$$U^\bullet = J^\bullet \circ W^\bullet \tag{27}$$

3.2 Some results on the mould $J^\bullet$.

As $\alpha_+ = 1 + \alpha_-$, it is easy to check that

$$J^\bullet = I^\bullet \times K^\bullet \qquad K^\bullet = H^\bullet \times J^\bullet \tag{28}$$

where the symmetrel mould $K^\bullet$ is defined by

$$K^{\eta_1,\dots,\eta_s} = \alpha_-(\hat{n}_1)\alpha_-(\hat{n}_2)\dots\alpha_-(\hat{n}_s) \tag{29}$$

We give now some identities related to $J^\bullet$ and $K^\bullet$ (The very simple proofs are left to the reader).

$$J^\bullet \circ K^\bullet = K^\bullet \circ J^\bullet = 1^\bullet \tag{30}$$

One can deduce that

$$\begin{array}{rcl} J^\bullet \circ J^\bullet &=& (I^\bullet \times K^\bullet) \circ J^\bullet \\ &=& (I^\bullet \circ J^\bullet) \times (K^\bullet \circ J^\bullet) \\ &=& J^\bullet \times 1^\bullet \\ &=& J^\bullet \end{array} \tag{31}$$

and

$$\begin{array}{rcl} K^\bullet \circ K^\bullet &=& (H^\bullet \times J^\bullet) \circ K^\bullet \\ &=& (H^\bullet \circ K^\bullet) \times (J^\bullet \circ K^\bullet) \\ &=& \mathrm{inv}(K)^\bullet \times 1^\bullet \\ &=& \mathrm{inv}(K)^\bullet \end{array} \tag{32}$$

We end this section with a useful result :

Proposition 7. *For any moulds $A^\bullet$ and $B^\bullet$, we have*

$$J^\bullet \circ A^\bullet = J^\bullet \circ (A^\bullet \times (K^\bullet \circ B^\bullet)) \tag{33}$$

For the proof, see section 7.1.

3.3 Definition and properties of $U^\bullet$ and $V^\bullet$.

We define :

$$U^\bullet = J^\bullet \circ W^\bullet \tag{34}$$

From the definition of $J^\bullet$ one easily deduce that $U^\emptyset = 1$, and, if $\boldsymbol{\eta}$ is a nonempty sequence of $\boldsymbol{H}^-$, $U^{\boldsymbol{\eta}} = 0$.

We remind that, for any mould $A^\bullet$, $s_q(A^{\boldsymbol{\eta}}) = x^{-\|\boldsymbol{n}\|}\sigma_q.A^{\boldsymbol{\eta}}$ for a nonempty sequence and $s_q A^\emptyset = A^\emptyset$ if $A^\emptyset = 1$. The mould $W^\bullet$ is defined by the identities $W^\emptyset = 1$ and

$$s_q(W^\bullet) = X^\bullet \times W^\bullet \tag{35}$$

We define now the mould $V^\bullet$ by

$$s_q(U^\bullet) \times V^\bullet = X^\bullet \times U^\bullet \tag{36}$$

it is not difficult to check that $s_q(U^\bullet)$ is symmetrel, thus invertible, so that $V^\bullet$ is well defined and symmetrel.

Since $J^\bullet$ is a constant mould (with values in $\mathbb{C}$),

$$\begin{array}{rcc} s_q(U^\bullet) & = & s_q(J^\bullet \circ W^\bullet) \\ & = & J^\bullet \circ s_q(W^\bullet) \\ & = & J^\bullet \circ (X^\bullet \times W^\bullet) \\ & = & J^\bullet \circ (X^\bullet \times W^\bullet \times (K^\bullet \circ W^\bullet)) \\ & = & J^\bullet \circ (X^\bullet \times ((I^\bullet \times K^\bullet) \circ W^\bullet)) \\ & = & J^\bullet \circ (X^\bullet \times (J^\bullet \circ W^\bullet)) \\ & = & J^\bullet \circ (X^\bullet \times U^\bullet) \end{array} \tag{37}$$

thus $V^\bullet$ is defined by

$$(J^\bullet \circ (X^\bullet \times U^\bullet)) \times V^\bullet = X^\bullet \times U^\bullet$$

but

$$\begin{array}{rcc} (J^\bullet \circ (X^\bullet \times U^\bullet)) & = & ((I^\bullet \times K^\bullet) \circ (X^\bullet \times U^\bullet)) \\ & = & X^\bullet \times U^\bullet \times (K^\bullet \circ (X^\bullet \times U^\bullet)) \end{array}$$

thus

$$(K^\bullet \circ (X^\bullet \times U^\bullet)) \times V^\bullet = 1^\bullet$$

and, since the inverse of $K^\bullet \circ (X^\bullet \times U^\bullet)$ is $(\mathrm{inv}(K^\bullet)) \circ (X^\bullet \times U^\bullet)$,

$$V^\bullet = (\mathrm{inv}(K^\bullet)) \circ (X^\bullet \times U^\bullet) \tag{38}$$

but if $L^\bullet = \mathrm{inv}(K^\bullet)$ then $L^\emptyset = 1$, $L^{\eta_1} = -\alpha_-(n_1)$ and, for $(\eta_1, \ldots, \eta_s) \in \boldsymbol{H}$ $(s \geq 2)$:

$$L^{\eta_1,\ldots,\eta_s} = (-1)^s \alpha_+(\check{n}_1) \ldots \alpha_+(\check{n}_{s-1}) \alpha_-(\check{n}_s) \tag{39}$$

so it is clear that, if $\boldsymbol{\eta}$ is a nonempty sequence in $\boldsymbol{H}^+$, then $V^{\boldsymbol{\eta}} = 0$.

3.4 Back to the equations.

We have two moulds $U^\bullet$ and $V^\bullet$ on $\boldsymbol{H}$, with values in $\mathbb{C}[[x]]$, such that $U^\bullet$ (resp. $V^\bullet$) vanishes on $\boldsymbol{H}^-$ (resp. $\boldsymbol{H}^+$). Let us consider the analytic equation

$$x \sigma_q y = y + b(y, x)$$

and the a priori formal equation

$$x \sigma_q z = \mathbb{V}.z = \left(\sum V^\bullet \mathbb{D}_\bullet \right).z$$

where $\mathbb{V}$ is a formal automorphism. We will prove that these equation are conjugate by the a priori formal automorphism

$$\mathbb{U} = \sum U^\bullet \mathbb{D}_\bullet \tag{40}$$

If $y = \mathbb{U}.z$, then,

$$\begin{array}{rcl} x\sigma_q y & = & x\sigma_q(\mathbb{U}.z) \\ & = & \sum_{\boldsymbol{\eta}\in\boldsymbol{H}} x\sigma_q(U^{\boldsymbol{\eta}}\mathbb{D}_{\boldsymbol{\eta}}.z) \\ & = & \sum_{\boldsymbol{\eta}\in\boldsymbol{H}} x\sigma_q(U^{\boldsymbol{\eta}})\,\sigma_q(\mathbb{D}_{\boldsymbol{\eta}}.z) \end{array}$$

but $\mathbb{D}_\emptyset.z = z$, and, for a nonempty sequence $\boldsymbol{\eta}$, there exists a constant $\beta_{\boldsymbol{\eta}}$ such that $\mathbb{D}_{\boldsymbol{\eta}}.z = \beta_{\boldsymbol{\eta}} z^{1+\|\boldsymbol{n}\|}$:

$$\begin{array}{rcl} \sigma_q(\mathbb{D}_{\boldsymbol{\eta}}.z) & = & \sigma_q(\beta_{\boldsymbol{\eta}} z^{1+\|\boldsymbol{n}\|}) \\ & = & \beta_{\boldsymbol{\eta}}\sigma_q(z^{1+\|\boldsymbol{n}\|}) \\ & = & x^{-1-\|\boldsymbol{n}\|}\beta_{\boldsymbol{\eta}}(x\sigma_q z)^{1+\|\boldsymbol{n}\|} \\ & = & x^{-1-\|\boldsymbol{n}\|}\beta_{\boldsymbol{\eta}}(\mathbb{V}.z)^{1+\|\boldsymbol{n}\|} \\ & = & x^{-1-\|\boldsymbol{n}\|}\beta_{\boldsymbol{\eta}}\mathbb{V}.(z^{1+\|\boldsymbol{n}\|}) \\ & = & x^{-1-\|\boldsymbol{n}\|}\mathbb{V}.(\beta_{\boldsymbol{\eta}} z^{1+\|\boldsymbol{n}\|}) \\ & = & x^{-1-\|\boldsymbol{n}\|}\mathbb{V}.\mathbb{D}_{\boldsymbol{\eta}}.z \end{array}$$

this identity still holds for the empty sequence with $\|\boldsymbol{n}\| = 0$. One can deduce that,

$$\begin{array}{rcl} x\sigma_q y & = & \sum_{\boldsymbol{\eta}\in\boldsymbol{H}} x\sigma_q(U^{\boldsymbol{\eta}})\,\sigma_q(\mathbb{D}_{\boldsymbol{\eta}}.z) \\ & = & \sum_{\boldsymbol{\eta}\in\boldsymbol{H}} x\sigma_q(U^{\boldsymbol{\eta}})x^{-1-\|\boldsymbol{n}\|}\mathbb{V}.\mathbb{D}_{\boldsymbol{\eta}}.z \\ & = & \mathbb{V}.\sum_{\boldsymbol{\eta}\in\boldsymbol{H}} s_q(U^{\boldsymbol{\eta}})\mathbb{D}_{\boldsymbol{\eta}}.z \\ & = & \left(\sum V^\bullet\mathbb{D}_\bullet\right).\left(\sum s_q(U^\bullet)\mathbb{D}_\bullet\right).z \\ & = & \left(\sum (s_q(U^\bullet)\times V^\bullet)\mathbb{D}_\bullet\right).z \end{array} \tag{41}$$

But $s_q(U^\bullet)\times V^\bullet = X^\bullet \times U^\bullet$ thus

$$\begin{array}{rcl} x\sigma_q y & = & \left(\sum (X^\bullet\times U^\bullet)\mathbb{D}_\bullet\right).z \\ & = & \mathbb{U}.\left(\sum X^\bullet\mathbb{D}_\bullet\right).z \\ & = & \mathbb{U}.(z + b(z,x)) \\ & = & \mathbb{U}.z + b(\mathbb{U}.z, x) \\ & = & y + b(y,x) \end{array} \tag{42}$$

Finally, $\mathbb{D}_\emptyset.z = z$, and, for a nonempty sequence $\boldsymbol{\eta}$, there exists a constant $\beta_{\boldsymbol{\eta}}$ such that $\mathbb{D}_{\boldsymbol{\eta}}.z = \beta_{\boldsymbol{\eta}} z^{1+\|\boldsymbol{n}\|}$ but since $\mathbb{D}_{\boldsymbol{\eta}}$ is a differential operator in z, if $\|\boldsymbol{n}\| < -1$, then $\beta_{\boldsymbol{\eta}} = 0$. As $V^\bullet$ vanishes on $\boldsymbol{H}^+$, it is clear that

$$\mathbb{V}.z = z + c(x) \in z + x\mathbb{C}[[x]] \tag{43}$$

In conclusion

Theorem 8. *Let us consider the equation*

$$(x\sigma_q - 1)y = b(y, x) \tag{44}$$

with $b(0,0)=0$ *and* $\frac{\partial b}{\partial f}(0,0)=0$*. This equation is conjugate by the a priori formal automorphism* $\mathbb{U}$ *:*

$$y = \mathbb{U}.z = \Big(\sum U^\bullet \mathbb{D}_\bullet\Big).z \in \mathbb{C}[[z,x]] \tag{45}$$

to the a priori formal equation

$$x\sigma_q z = \mathbb{V}.z = \Big(\sum V^\bullet \mathbb{D}_\bullet\Big).z \in z + x\mathbb{C}[[x]] \tag{46}$$

It remains to prove that $\mathbb{U}.z \in \mathbb{C}\{z, x\}$ and $\mathbb{V}.z \in z + x\mathbb{C}\{x\}$. We will first prove that the values of $U^\bullet$ and $V^\bullet$ are in $\mathbb{C}\{x\}$ (i.e. $U^\bullet$ and $V^\bullet$ are analytic) then we will use the arborification process to prove that $\mathbb{U}$ and $\mathbb{V}$ are *convergent* automorphisms, namely, $\mathbb{U}.z \in \mathbb{C}\{z,x\}$ and $\mathbb{V}.z \in z + x\mathbb{C}\{x\}$.

4 Analyticity of $U^\bullet$ and $V^\bullet$.

4.1 The mould $U^\bullet$.

We prove that

$$\forall \boldsymbol{\eta} \in \boldsymbol{H}, \qquad U^{\boldsymbol{\eta}} \in x^{\|\boldsymbol{\sigma}\|}\mathbb{C}\{x\} \tag{47}$$

We already know that $\forall \boldsymbol{\eta} \in \boldsymbol{H},\ U^{\boldsymbol{\eta}} \in \mathbb{C}[[x]]$. Let us remind (see [2]) that if $n > 0$ and $g \in \mathbb{C}\{x\}$, then the equation

$$(x^{-n}\sigma_q - 1)f = g$$

has a unique solution in $\mathbb{C}\{x\}$ of greater valuation than g and if $g \in x\mathbb{C}\{x\}$, then the equation

$$(\sigma_q - 1)f = g$$

has a unique solution in $x\mathbb{C}\{x\}$ of same valuation.

Let us prove 47 by induction on the length of the sequences in $\boldsymbol{H}$. The result is obvious for the empty sequence ($U^\emptyset = 1$). We remind that :

$$s_q(U^\bullet) = J^\bullet \circ (X^\bullet \times U^\bullet)$$

For a sequence $\boldsymbol{\eta} \in \boldsymbol{H}$ of length $l(\boldsymbol{\eta}) = 1$ ($\boldsymbol{\eta} = \eta_1$), if $\eta_1 \in \boldsymbol{H}^-$ then $U^{\eta_1} = 0$ and, otherwise,

$$\begin{aligned} s_q(U^{\eta_1}) &= J^{\eta_1}(x^{\sigma_1} + U^{\eta_1}) \\ x^{-n_1}\sigma_q U^{\eta_1} &= x^{\sigma_1} + U^{\eta_1} \end{aligned}$$

note that $n_1 \geq 0$, $x^{\sigma_1} \in \mathbb{C}\{x\}$ and if $n_1 = 0$ then, by definition of $\boldsymbol{H}$, $\sigma_1 \geq 1$ and $x^{\sigma_1} \in x\mathbb{C}\{x\}$. So, in any case,

$$U^{\eta_1} \in x^{\sigma_1}\mathbb{C}\{x\}.$$

Let $l_0 \geq 1$ and suppose that

$$\forall \boldsymbol{\eta} \in \boldsymbol{H}\,;\, l(\boldsymbol{\eta}) < l_0 \qquad U^{\boldsymbol{\eta}} \in x^{\|\boldsymbol{\sigma}\|}\mathbb{C}\{x\}$$

and $\boldsymbol{\eta} \in \boldsymbol{H}$ such that $l(\boldsymbol{\eta}) = l_0$ ($\boldsymbol{\eta} = \eta_1, \ldots, \eta_{l_0}$). If $\boldsymbol{\eta} \in \boldsymbol{H}^-$ then $U^{\boldsymbol{\eta}} = 0$, otherwise

$$\begin{aligned}
s_q(U^{\boldsymbol{\eta}}) &= (J^\bullet \circ (X^\bullet \times U^\bullet))^{\boldsymbol{\eta}} \\
x^{-\|\boldsymbol{n}\|}\sigma_q(U^{\boldsymbol{\eta}}) &= J^{\|\boldsymbol{\eta}\|}(U^{\boldsymbol{\eta}} + x^{\sigma_1} U^{`\boldsymbol{\eta}}) \\
&\quad + \sum_{\substack{\boldsymbol{\eta}^1 \ldots \boldsymbol{\eta}^t = \boldsymbol{\eta} \\ t \geq 2}} J^{\|\boldsymbol{\eta}^1\|, \ldots, \|\boldsymbol{\eta}^t\|}(X^\bullet \times U^\bullet)^{\boldsymbol{\eta}^1} \ldots (X^\bullet \times U^\bullet)^{\boldsymbol{\eta}^t} \\
&= U^{\boldsymbol{\eta}} + x^{\sigma_1} U^{`\boldsymbol{\eta}} \\
&\quad + \sum_{\substack{\boldsymbol{\eta}^1 \ldots \boldsymbol{\eta}^t = \boldsymbol{\eta} \\ t \geq 2}} J^{\|\boldsymbol{\eta}^1\|, \ldots, \|\boldsymbol{\eta}^t\|}(X^\bullet \times U^\bullet)^{\boldsymbol{\eta}^1} \ldots (X^\bullet \times U^\bullet)^{\boldsymbol{\eta}^t} \\
&= U^{\boldsymbol{\eta}} + P^{\boldsymbol{\eta}}
\end{aligned}$$

By induction, it is clear the $P^{\boldsymbol{\eta}}$ depends on values of $U^\bullet$ for sequences of length $< l_0$, thus, $P^{\boldsymbol{\eta}} \in x^{\|\boldsymbol{\sigma}\|}\mathbb{C}\{x\}$. If $\|\boldsymbol{n}\| > 0$, then $U^{\boldsymbol{\eta}} \in x^{\|\boldsymbol{\sigma}\|}\mathbb{C}\{x\}$, if $\|\boldsymbol{n}\| = 0$, by construction of $\boldsymbol{H}$, $\|\boldsymbol{\sigma}\| > 0$ and, once again $U^{\boldsymbol{\eta}} \in x^{\|\boldsymbol{\sigma}\|}\mathbb{C}\{x\}$.

This ends the proof by induction.

4.2 The mould $V^\bullet$.

There is not much to say to prove that,

$$\forall \boldsymbol{\eta} \in \boldsymbol{H}, \qquad V^{\boldsymbol{\eta}} \in x^{\|\boldsymbol{\sigma}\|}\mathbb{C}\{x\} \tag{48}$$

since this is already true for $U^\bullet$ and

$$V^\bullet = (\mathrm{inv}(K^\bullet)) \circ (X^\bullet \times U^\bullet)$$

We end this section by giving an interesting formula for $U^\bullet$ which is similar to the one given in theorem 4.

4.3 A formula for $U^\bullet$.

Theorem 9. *If* $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_s) = \begin{pmatrix} n_1, \ldots, n_s \\ \sigma_1, \ldots, \sigma_s \end{pmatrix} = \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{\sigma} \end{pmatrix} \in \boldsymbol{H}$, *then the monomial* $U^{\boldsymbol{\eta}}$ *is a convergent series with the following expression :*

$$U^{\boldsymbol{\eta}}(x) = \varepsilon_{\boldsymbol{n}} \sum_{(k_1, \ldots, k_s) \in \mathbb{Z}^{\boldsymbol{n}} \cap P^{\boldsymbol{n}}} x^{\|\boldsymbol{\sigma}\| + \check{k}_1 n_1 + \ldots + \check{k}_s n_s} q^{-\check{k}_1 \sigma_1 - \ldots - \check{k}_s \sigma_s} [\check{k}_1 + 1]_q^{n_1} \ldots [\check{k}_s + 1]_q^{n_s} \tag{49}$$

with the notations

- $\varepsilon_{\boldsymbol{n}} = \prod_{1 \leq i \leq s} \varepsilon_{\hat{n}_i}$ *with* $\hat{n}_i = n_i + \ldots + n_s$ *with* $\varepsilon_n = 1$ *(resp.* $\varepsilon_n = -1$*) if* $n \geq 0$ *(resp.* $n < 0$*)*
- $\mathbb{Z}^{\boldsymbol{n}} = \mathbb{Z}^{\hat{n}_1} \times \ldots \times \mathbb{Z}^{\hat{n}_s}$ *with* $\mathbb{Z}^n = \mathbb{Z}^{+*}$ *(resp.* $\mathbb{Z}^-$*) if* $n \geq 0$ *(resp.* $n < 0$*)*
- $P^{\boldsymbol{n}} = \{(k_1, \ldots, k_s) \in \mathbb{Z}^s\,;\, \check{k_i} > 0 \text{ if } \varepsilon_{\hat{n}_i} = -1\}$

- $\check{k_i} = k_1 + \ldots + k_i$.
- $[k+1]_q = q^{-k(k+1)/2}$

Note that if $\|\boldsymbol{n}\| < 0$, then $\varepsilon_{\hat{n}_i} = \varepsilon_{\|\boldsymbol{n}\|} = -1$ thus $k_1 \in \mathbb{Z}^-$ but, in $P^{\boldsymbol{n}}$, $k_1 > 0$, thus $\mathbb{Z}^{\boldsymbol{n}} \cap P^{\boldsymbol{n}} = \emptyset$ and we recover that $U^{\boldsymbol{\eta}} = 0$. We shall focus on sequences in $\boldsymbol{H}^+$. The complete proof can be found in section 7.2. We just give an example here to catch the idea of the proof.

For a sequence $(k_1, \ldots, k_s) \in \mathbb{Z}^s$, let $l_i = \check{k_i}$, thus $k_1 = l_1$ and, for i greater than 1, $k_i = l_i - l_{i-1}$. For a sequence $\boldsymbol{\eta} = (\eta_1, \ldots., \eta_s) \in \boldsymbol{H}^+$ and $\boldsymbol{l} = (l_1, \ldots, l_s) \in \mathbb{Z}^s$ $(s \geq 1)$, let

$$T_{\boldsymbol{l}}^{\boldsymbol{\eta}} = x^{\|\boldsymbol{\sigma}\| + l_1 n_1 + \ldots + l_s n_s} q^{-l_1\sigma_1 - \ldots - l_s\sigma_s} [l_1+1]_q^{n_1} \ldots [l_s+1]_q^{n_s} = \prod_{i=1}^{s} T_{l_i}^{\eta_i} \tag{50}$$

With this notation, the formula in theorem 4 becomes

$$W^{\boldsymbol{\eta}}(x) = \varepsilon_{\boldsymbol{n}} \sum_{(l_1, l_2 - l_1 \ldots, l_s - l_{s-1}) \in \mathbb{Z}^{\boldsymbol{n}}} T_{\boldsymbol{l}}^{\boldsymbol{\eta}}$$

using the function $\rho_1(n) = \begin{cases} 1 & \text{if } n > 0 \\ 0 & \text{if } n \leq 0 \end{cases}$ and $\rho_{-1} = 1 - \rho_1$, we get

$$W^{\boldsymbol{\eta}}(x) = \varepsilon_{\boldsymbol{n}} \sum_{\boldsymbol{l} \in \mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1) \rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1) \ldots \rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1}) T_{\boldsymbol{l}}^{\boldsymbol{\eta}} = \sum_{\boldsymbol{l} \in \mathbb{Z}^s} w_{\boldsymbol{l}}^{\boldsymbol{\eta}} T_{\boldsymbol{l}}^{\boldsymbol{\eta}} \tag{51}$$

and we have to prove that

$$U^{\boldsymbol{\eta}}(x) = \varepsilon_{\boldsymbol{n}} \sum_{\boldsymbol{l} \in \mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1) \rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1) \ldots \rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1}) \left(\prod_{\substack{1 \leq i \leq s \\ \varepsilon_{\hat{n}_i} = -1}} \rho_1(l_i) \right) T_{\boldsymbol{l}}^{\boldsymbol{\eta}} = \sum_{\boldsymbol{l} \in \mathbb{Z}^s} \tilde{u}_{\boldsymbol{l}}^{\boldsymbol{\eta}} T_{\boldsymbol{l}}^{\boldsymbol{\eta}} \tag{52}$$

Let $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3) \in \boldsymbol{H}^+$ such that $\hat{n}_1 \geq 0$, $\hat{n}_2 < 0$, $\hat{n}_3 < 0$, then,

$$\begin{aligned}
\tilde{u}_{\boldsymbol{l}}^{\boldsymbol{\eta}} &= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)\rho_1(l_2)\rho_1(l_3) \\
&= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)\rho_1(l_2)(1 - \rho_{-1}(l_3)) \\
&= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)\rho_1(l_2) \\
&\qquad - \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)\rho_1(l_2)\rho_{-1}(l_3) \\
&= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)\rho_1(l_2) - \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_1(l_2)\rho_{-1}(l_3) \\
&= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)(1 - \rho_{-1}(l_2)) \\
&\qquad - \rho_1(l_1)\rho_{-1}(l_2 - l_1)(1 - \rho_{-1}(l_2))\rho_{-1}(l_3) \\
&= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2) - \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2)\rho_{-1}(l_2) \\
&\qquad - \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3) + \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_2)\rho_{-1}(l_3) \\
&= \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3 - l_2) - \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3 - l_2) \\
&\qquad - \rho_1(l_1)\rho_{-1}(l_2 - l_1)\rho_{-1}(l_3) + \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3)
\end{aligned}$$

On the other hand,

$$U^{\boldsymbol{\eta}} = (J^\bullet \circ W^\bullet)^{\boldsymbol{\eta}} = W^{\eta_1,\eta_2,\eta_3} - W^{\eta_1,\eta_2} W^{\eta_3} - W^{\eta_1} W^{\eta_2,\eta_3} + W^{\eta_1} W^{\eta_2} W^{\eta_3} = \sum_{\boldsymbol{l}\in\mathbb{Z}^s} u_{\boldsymbol{l}}^{\boldsymbol{\eta}} T_{\boldsymbol{l}}^{\boldsymbol{\eta}}$$

It is clear that $n_1 \geq 0$, $n_1 + n_2 \geq 0$.

- If $n_2 < 0$, then

$$\begin{aligned} u_{\boldsymbol{l}}^{\boldsymbol{\eta}} &= \rho_1(l_1)\rho_{-1}(l_2-l_1)\rho_{-1}(l_3-l_2) - \rho_1(l_1)\rho_{-1}(l_2-l_1)\rho_{-1}(l_3) \\ &\quad - \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3-l_2) + \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3) \\ &= \tilde{u}_{\boldsymbol{l}}^{\boldsymbol{\eta}} \end{aligned}$$

- If $n_2 \geq 0$, then

$$\begin{aligned} u_{\boldsymbol{l}}^{\boldsymbol{\eta}} &= \rho_1(l_1)\rho_{-1}(l_2-l_1)\rho_{-1}(l_3-l_2) + \rho_1(l_1)\rho_1(l_2-l_1)\rho_{-1}(l_3) \\ &\quad - \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3-l_2) - \rho_1(l_1)\rho_1(l_2)\rho_{-1}(l_3) \\ &= \rho_1(l_1)\rho_{-1}(l_2-l_1)\rho_{-1}(l_3-l_2) + \rho_1(l_1)(1-\rho_{-1}(l_2-l_1))\rho_{-1}(l_3) \\ &\quad - \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3-l_2) - \rho_1(l_1)(1-\rho_{-1}(l_2))\rho_{-1}(l_3) \\ &= \rho_1(l_1)\rho_{-1}(l_2-l_1)\rho_{-1}(l_3-l_2) - \rho_1(l_1)\rho_{-1}(l_2-l_1)\rho_{-1}(l_3) \\ &\quad - \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3-l_2) + \rho_1(l_1)\rho_{-1}(l_2)\rho_{-1}(l_3) \\ &= \tilde{u}_{\boldsymbol{l}}^{\boldsymbol{\eta}} \end{aligned}$$

and that proves that the formula holds in this case.

4.4 Estimates for $U^\bullet$ and "divergence" of $\sum U^\bullet \mathbb{D}_\bullet$.

Using the above formula for $U^\bullet$, one gets

Proposition 10. *Let $0 \leq |x| \leq \varepsilon \leq q^{-1/2}$. For $\boldsymbol{\eta} = (\eta_1, ..., \eta_s) \in \boldsymbol{H}^+$,*

$$|U^{\boldsymbol{\eta}}(x)| \leq C_q^s |x|^{\|\boldsymbol{n}\| + \|\boldsymbol{\sigma}\|} q^{-(\|\boldsymbol{n}\| + \|\boldsymbol{\sigma}\|)} \tag{53}$$

with $C_q = \dfrac{1}{1-q^{-1/2}}$.

For the proof see section 7.3.

Unfortunatly, this is not sufficient to assume that $\sum U^\bullet \mathbb{D}_\bullet$ is "convergent" : in order to study the convergence of such operators as $\sum U^\bullet \mathbb{D}_\bullet$, we need to define some seminorms on differential operators in ∂_z, acting on $\mathbb{C}\{x, z\}$. Let $\varphi \in \mathbb{C}\{x, z\}$, and P an operator from $\mathbb{C}\{x, z\}$ into itself. We have the seminorms

$$\|\varphi\|_U = \sup_{(x,z)\in U} |\varphi(x, z)| \tag{54}$$

$$\|P\|_{U,V} = \sup_{\varphi; \|\varphi\|_U \leq 1} \|P.\varphi\|_V \tag{55}$$

Where U and V are to (x, z)-neighborhood of $(0, 0)$. A series $\sum P_n$ is normally convergent if $\sum \|P_n\|_{U,V}$ converges for at least a pair U, V. We should now only consider the following neighborhoods :

$$\forall \alpha, \beta > 0, \quad U_{\alpha,\beta} = \{(x, z) \in \mathbb{C}^2 \ ; \ |x| < \alpha, \ |z| < \beta\} \tag{56}$$

We can't establish directly the convergence of the a priori formal substitution automorphism $\mathbb{U}=\sum U^\bullet \mathbb{D}_\bullet$ because, for $\boldsymbol{\eta}=(\eta_1,\ldots,\eta_s)\in\boldsymbol{H}^+$ the only estimates we got are

$$\|U^{\boldsymbol{\eta}}\|_U \leq C_U^s D_U^{\|\boldsymbol{n}\|+\|\boldsymbol{\sigma}\|}$$

whereas (see [1]) we can't hope to improve lower estimate such as

$$\|\mathbb{D}_{\boldsymbol{\eta}}\|_{U,V} \geq C_{U,V}^s\, s! \tag{57}$$

Note that this does not prove that $\mathbb{U}$ remains formal but that its expansion $\sum U^\bullet \mathbb{D}_\bullet$ is not normally convergent. In fact, we prove in the next section that $\mathbb{U}$ is analytic, using a different expansion, obtained by arborification and that appears to be normally convergent.

5 Analytic part : arborification.

5.1 Reminder on arborification.

5.1.1 Contracting arborification.

We follow the definitions of J. Ecalle [1]. Let us consider an additive semigroup H. The set $\boldsymbol{H}$ is the set of sequences on H, where a sequence is a totally ordered sequence of elements of H, with possible repetitions.

An *arborescent sequence* on H is a sequence $\boldsymbol{\eta}^<=(\eta_1,\ldots,\eta_s)^<\in\boldsymbol{H}^<$ of elements of H with an arborescent order on the indices $\{1,\ldots,s\}$: each $i\in\{1,\ldots,s\}$ possess at most one predecessor i_-. We note $\boldsymbol{\eta}^<=\boldsymbol{\eta}'^<\oplus\boldsymbol{\eta}''^<$ the disjoint union of $\boldsymbol{\eta}'^<$ and $\boldsymbol{\eta}''^<$, the partial orders of $\boldsymbol{\eta}'^<$ and $\boldsymbol{\eta}''^<$ being preserved and the elements $\boldsymbol{\eta}'^<$ are not comparable with those of $\boldsymbol{\eta}''^<$. $\emptyset$ is the empty sequence. A sequence $\boldsymbol{\eta}^<$ is irreducible if it is not a disjoint union of smaller nontrivial sequences ; that is to say that it has exactly one least element.

We remind here that a mould $A^\bullet=\{A^{\boldsymbol{\eta}}\}$ on H with values in a commutative algebra is a family of elements $A^{\boldsymbol{\eta}}$ indexed by the sequences $\boldsymbol{\eta}\in\boldsymbol{H}$ of H. For example, $U^\bullet$ is a mould on H with values in $\mathbb{C}\{x\}$. Moreover, this mould is symmetrel : $U^\emptyset=1$ and, for any pair $(\boldsymbol{\eta}',\boldsymbol{\eta}'')$, we get

$$U^{\boldsymbol{\eta}'}U^{\boldsymbol{\eta}''}=\sum \operatorname{ctsh}\begin{pmatrix}\boldsymbol{\eta}',\boldsymbol{\eta}''\\ \boldsymbol{\eta}\end{pmatrix}U^{\boldsymbol{\eta}} \tag{58}$$

Where $\operatorname{ctsh}\begin{pmatrix}\boldsymbol{\eta}',\boldsymbol{\eta}''\\ \boldsymbol{\eta}\end{pmatrix}$ is the number of ways to get $\boldsymbol{\eta}$ by contracting shuffling of $\boldsymbol{\eta}'$ and $\boldsymbol{\eta}''$.

We also remind that an arborescent mould $A^{\bullet^<}=\{A^{\boldsymbol{\eta}^<}\}$ on H with values in a commutative algebra is a family of elements $A^{\boldsymbol{\eta}^<}$ indexed by the arborescent sequences $\boldsymbol{\eta}^<\in\boldsymbol{H}^<$ of H. Such an arborescent mould $A^{\bullet^<}$is separative if :

$$A^\emptyset=1 \text{ and } \quad \forall \boldsymbol{\eta}'^<,\boldsymbol{\eta}''^<, \quad A^{\boldsymbol{\eta}'^<\oplus\boldsymbol{\eta}''^<}=A^{\boldsymbol{\eta}'^<}A^{\boldsymbol{\eta}''^<} \tag{59}$$

We get such arborescent separative moulds by *contracting arborification* of symmetrel moulds. This operation is defined as follows.

Let $\boldsymbol{\eta}^{<} = (\eta_1, \ldots, \eta_s)^{<}$ be an arborescent sequence and $\boldsymbol{\eta}' = (\eta'_1, \ldots, \eta'_{s'})$ a totally ordered sequence. Let $\operatorname{cont}\begin{pmatrix} \boldsymbol{\eta}^{<} \\ \boldsymbol{\eta}' \end{pmatrix}$ be the number of monotonic contractions of $\boldsymbol{\eta}^{<}$ on $\boldsymbol{\eta}'$, that is to say the number of surjections σ from $\{1, \ldots, s\}$ into $\{1, \ldots, s'\}$ such that :

$$(i_1 < i_2 \text{ in } \boldsymbol{\eta}^{<}) \Longrightarrow (\sigma(i_1) < \sigma(i_2) \text{ in } \boldsymbol{\eta}') \tag{60}$$

$$\forall j \in \{1, \ldots, s'\} \ ; \quad \eta'_j = \sum_{\sigma(i) = j} \eta_i \tag{61}$$

The relation

$$A^{\boldsymbol{\eta}^{<}} = \sum \operatorname{cont}\begin{pmatrix} \boldsymbol{\eta}^{<} \\ \boldsymbol{\eta}' \end{pmatrix} A^{\boldsymbol{\eta}'} \tag{62}$$

defines a homomorphism from the algebra of moulds into the algebra of arborescent moulds. Moreover, the contracting arborification of a symmetrel mould is separative. One can also notice that, if $\boldsymbol{\eta}$ is a totally ordered sequence and $\boldsymbol{\eta}^{<}$ is that arborescent sequence with the same order (total), then $A^{\boldsymbol{\eta}^{<}} = A^{\boldsymbol{\eta}}$.

5.1.2 Product and composition.

We give here some formulas for the arborification of a product and of a composition of moulds.

Product. Let $M^\bullet$ and $N^\bullet$ two moulds on H, then

$$P^{\bullet^{<}} = (M^\bullet \times N^\bullet)^{<} = M^{\bullet^{<}} \times N^{\bullet^{<}} \Longleftrightarrow P^{\boldsymbol{\eta}^{<}} = \sum M^{\boldsymbol{\eta}'^{<}} N^{\boldsymbol{\eta}''^{<}} \tag{63}$$

with a sum extended to all the monotonic partitions of $\boldsymbol{\eta}^{<}$ in $\boldsymbol{\eta}'^{<}$ and $\boldsymbol{\eta}''^{<}$, that is to say all the partitions such that no element in $\boldsymbol{\eta}'^{<}$ is greater than any element of $\boldsymbol{\eta}''^{<}$. The order of $\boldsymbol{\eta}'^{<}$ and $\boldsymbol{\eta}''^{<}$ is of course inherited from the one of $\boldsymbol{\eta}^{<}$. For example, if $\boldsymbol{\eta}^{<} = \eta_1 \bullet (\eta_2 \oplus \eta_3)$ (η_2 and η_3 are not comparable but have η_1 as a common predecessor), then

$$P^{\boldsymbol{\eta}^{<}} = M^{\boldsymbol{\eta}^{<}} N^{\emptyset} + M^{\emptyset} N^{\boldsymbol{\eta}^{<}} + M^{\eta_1} N^{\eta_2 \oplus \eta_3} + M^{\eta_1 \bullet \eta_2} N^{\eta_3} + M^{\eta_1 \bullet \eta_3} N^{\eta_2}$$

The arborescent separative moulds are stable by multiplication and the multiplicative inverse $N^{\bullet^{<}}$ of a separative arborescent mould $M^{\bullet^{<}}$ can be computed with the formula :

$$N^{\boldsymbol{\eta}^{<}} = \sum (-1)^s M^{\boldsymbol{\eta}^{1<}} \ldots M^{\boldsymbol{\eta}^{s<}} \tag{64}$$

where the sum is extended to the monotonic partitions $\boldsymbol{\eta}^{1<}, \ldots, \boldsymbol{\eta}^{s<}$ of $\boldsymbol{\eta}^{<}$, with no empty sequence $\boldsymbol{\eta}^{i<}$, counting separately partitions which differs only by the order of the $\boldsymbol{\eta}^{i<}$.

Composition. Let $M^\bullet$ and $N^\bullet$ two moulds on H, then

$$P^{\bullet^{<}} = (M^\bullet \circ N^\bullet)^{<} \Longleftrightarrow P^{\boldsymbol{\eta}^{<}} = \sum M^{\|\boldsymbol{\eta}^{1<}\|, \ldots, \|\boldsymbol{\eta}^{s<}\|} N^{\boldsymbol{\eta}^{1<}} \ldots N^{\boldsymbol{\eta}^{s<}} \tag{65}$$

where the sum is over the same set as in the formula for the mulitplicative inverse and $\|\boldsymbol{\eta}^<\| = \eta_1 + \ldots + \eta_s$ if $\boldsymbol{\eta}^< = (\eta_1, \ldots, \eta_s)$.

5.2 The arborified mould $\boldsymbol{U}^{\bullet^<}$.

We will first change some notations. Let $\boldsymbol{\eta}^< = (\eta_1, \ldots, \eta_s)^<$ be an arborescent sequence of length s and of sum $\|\boldsymbol{\eta}^<\| = \eta_1 + \ldots + \eta_s$. We redefine the partial sums :

$$\begin{aligned} \check{n}_i &= \sum_{j \leq i} n_j \\ \hat{n}_i &= \sum_{j \geq i} n_j \end{aligned} \tag{66}$$

where the orders $\leq$ and $\geq$ are now relative to the partial order on $\{1, \ldots, s\}$. We have the following theorem :

Theorem 11. *If* $\boldsymbol{\eta}^< = (\eta_1, \ldots, \eta_s)^< = \begin{pmatrix} n_1, \ldots, n_s \\ \sigma_1, \ldots, \sigma_s \end{pmatrix}^< = \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{\sigma} \end{pmatrix}^< \in \boldsymbol{H}^<$, *then the monomial* $W^{\boldsymbol{\eta}^<}$ *is a formal series and :*

$$W^{\boldsymbol{\eta}^<}(x) = \varepsilon_{\boldsymbol{n}^<} \sum_{(k_1, \ldots, k_s)^< \in \mathbb{Z}^{\boldsymbol{n}^<}} x^{\|\boldsymbol{\sigma}\| + \check{k}_1 n_1 + \ldots + \check{k}_s n_s} q^{-\check{k}_1 \sigma_1 - \ldots - \check{k}_s \sigma_s} [\check{k}_1 + 1]_q^{n_1} \ldots [\check{k}_s + 1]_q^{n_s} \tag{67}$$

with the following rules

- *The sequences* $\boldsymbol{n}^<$ *and* $(k_1, \ldots, k_s)^<$ *inherit the partial order of* $\boldsymbol{\eta}^<$.
- $\varepsilon_{\boldsymbol{n}} = \prod_{1 \leq i \leq s} \varepsilon_{\hat{n}_i}$ *with* $\hat{n}_i = \sum_{j \geq i} n_j$ *and* $\varepsilon_n = 1$ *(resp.* $\varepsilon_n = -1$*) if* $n \geq 0$ *(resp.* $n < 0$*).*
- $\mathbb{Z}^{\boldsymbol{n}^<} = (\mathbb{Z}^{\hat{n}_1} \times \ldots \times \mathbb{Z}^{\hat{n}_s})^<$ *and* $\mathbb{Z}^n = \mathbb{Z}^{+*}$ *(resp.* $\mathbb{Z}^-$*) if* $n \geq 0$ *(resp.* $n < 0$*).*
- $\check{k}_i = \sum_{j \leq i} k_j$.

This result was given in [2]. The formula for $W^\bullet$ was derived from the fact that

$$s_q(W^\bullet) = X^\bullet \times W^\bullet$$

As $X^\bullet$ and $W^\bullet$ are symmetrel the arborescent moulds $X^{\bullet^<}$ and $W^{\bullet^<}$ are separative. Moreover, for an irreducible sequence $\boldsymbol{\eta}^< = (\eta_1, \ldots, \eta_s)^< = \eta_1 \bullet \boldsymbol{\eta}'^< \in \boldsymbol{H}^<$, it is easy to check that

$$X^{\boldsymbol{\eta}^<} = \begin{cases} x^{\sigma_1} & \text{if } s = 1 \\ 0 & \text{otherwise} \end{cases}$$

and then

$$s_q(W^{\boldsymbol{\eta}^<}) = W^{\boldsymbol{\eta}^<} + x^{\sigma_1} W^{\boldsymbol{\eta}'^<}$$

It is then easy to check that the formula works. From this one can deduce a formula for $U^{\bullet^<}$:

Theorem 12. *If* $\boldsymbol{\eta}^< = (\eta_1, \ldots, \eta_s)^< = \begin{pmatrix} n_1, \ldots, n_s \\ \sigma_1, \ldots, \sigma_s \end{pmatrix}^< = \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{\sigma} \end{pmatrix}^< \in \boldsymbol{H}^<$, *then the monomial* $U^{\boldsymbol{\eta}^<}$ *is a convergent series with the following expression :*

$$U^{\boldsymbol{\eta}^<}(x) = \varepsilon_{\boldsymbol{n}^<} \sum_{(k_1, \ldots, k_s)^< \in \mathbb{Z}^{\boldsymbol{n}^<} \cap P^{\boldsymbol{n}^<}} x^{\|\boldsymbol{\sigma}\| + \check{k}_1 n_1 + \ldots + \check{k}_s n_s} q^{-\check{k}_1 \sigma_1 - \ldots - \check{k}_s \sigma_s} [\check{k}_1 + 1]_q^{n_1} \ldots [\check{k}_s + 1]_q^{n_s} \quad (68)$$

with the notations

- *The sequences* $\boldsymbol{n}^<$ *and* $(k_1, \ldots, k_s)^<$ *inherit the partial order of* $\boldsymbol{\eta}^<$.
- $\varepsilon_{\boldsymbol{n}} = \prod_{1 \leq i \leq s} \varepsilon_{\hat{n}_i}$ *with* $\hat{n}_i = \sum_{j \geq i} n_j$ *with* $\varepsilon_n = 1$ *(resp.* $\varepsilon_n = -1$*) if* $n \geq 0$ *(resp.* $n < 0$*)*
- $\mathbb{Z}^{\boldsymbol{n}^<} = \mathbb{Z}^{\hat{n}_1} \times \ldots \times \mathbb{Z}^{\hat{n}_s}$ *with* $\mathbb{Z}^n = \mathbb{Z}^{+*}$ *(resp.* $\mathbb{Z}^-$*) if* $n \geq 0$ *(resp.* $n < 0$*)*
- $P^{\boldsymbol{n}^<} = \{(k_1, \ldots, k_s)^< \in \mathbb{Z}^s \,;\, \check{k}_i > 0 \text{ if } \varepsilon_{\hat{n}_i} = -1\}$
- $\check{k}_i = \sum_{j \leq i} k_j$.
- $[k+1]_q = q^{-k(k+1)/2}$

The proof is essentially the same as in theorem 9, using the formula linking arborification to composition.

One can notice that, for an irreducible sequence $\boldsymbol{\eta}^<$, if $\|\boldsymbol{\eta}^<\| < 0$ then $U^{\boldsymbol{\eta}^<} = 0$ and as $U^{\bullet^<}$ is separative, for a sequence $\boldsymbol{\eta}^< = \boldsymbol{\eta}^{\boldsymbol{1}<} \oplus \ldots \oplus \boldsymbol{\eta}^{\boldsymbol{s}<}$ ($\boldsymbol{\eta}^{\boldsymbol{i}<}$ irreducible), if there exist i_0 such that $\|\boldsymbol{\eta}^{\boldsymbol{i_0}<}\| < 0$ then $U^{\boldsymbol{\eta}^<} = 0$. It means that we can restrict ourselves to arborescent sequences

$$\boldsymbol{\eta}^< \in \boldsymbol{H}^{+^<} = \{\boldsymbol{\eta}^{\boldsymbol{1}<} \oplus \ldots \oplus \boldsymbol{\eta}^{\boldsymbol{s}<} \,;\, \boldsymbol{\eta}^{\boldsymbol{i}<} \text{ irreducible and } \|\boldsymbol{\eta}^{\boldsymbol{i}<}\| \geq 0\} \quad (69)$$

Using this theorem, it is easy to obtain, as in proposition 10, that

Proposition 13. *Let* $0 \leq |x| \leq q^{-1/2}$. *For* $\boldsymbol{\eta}^< = (\eta_1, \ldots, \eta_s)^< \in \boldsymbol{H}^{+^<}$,

$$|U^{\boldsymbol{\eta}^<}(x)| \leq C_q^s |x|^{\|\boldsymbol{n}^<\| + \|\boldsymbol{\sigma}^<\|} q^{-\|\boldsymbol{n}^<\| - \|\boldsymbol{\sigma}^<\|} \quad (70)$$

5.3 Coarborification.

Theorem 14. *There exists a unique arborescent comould* $\mathbb{D}_{\bullet^<}$ *with the three following properties :*

i. $\mathbb{D}_{\bullet^<}$ *is coseparative :* $\mathbb{D}_\emptyset = 1$ *and*

$$\mathrm{col}(\mathbb{D}_{\boldsymbol{\eta}^<}) = \sum \mathbb{D}_{\boldsymbol{\eta}'^<} \otimes \mathbb{D}_{\boldsymbol{\eta}''^<} \quad (\boldsymbol{\eta}'^< \oplus \boldsymbol{\eta}''^< = \boldsymbol{\eta}^<) \quad (71)$$

with a sum extended to the arborescent sequences $\boldsymbol{\eta}'^<$, $\boldsymbol{\eta}''^<$ *(even the empty sequences) which disjoint union is* $\boldsymbol{\eta}^<$.

ii. If $\deg(\boldsymbol{\eta}^<) = d$, $\mathbb{D}_{\boldsymbol{\eta}^<}$ *is a differential operator of degree* d *in* ∂_z *: if the sequence* $\boldsymbol{\eta}^<$ *has exactly* d *minimal elements and thus :*

$$\boldsymbol{\eta}^< = \boldsymbol{\eta}^{\boldsymbol{1}<} \oplus \ldots \oplus \boldsymbol{\eta}^{\boldsymbol{d}<} \quad \textit{(with } \boldsymbol{\eta}^{\boldsymbol{i}<} \textit{ irreducible and } \neq \emptyset\textit{)} \quad (72)$$

the operator $\mathbb{D}_{\boldsymbol{\eta}^<}$ *can be written :*

$$\mathbb{D}_{\boldsymbol{\eta}^<} = b(z)\partial_z^d \tag{73}$$

iii. *If* $\boldsymbol{\eta}^< = \eta_1.\boldsymbol{\eta}^{*<}$ *(*$\boldsymbol{\eta}^<$ *has a least element* η_1 *followed by an arborescent sequence* $\boldsymbol{\eta}^{*<}$*) we get :*

$$\mathbb{D}_{\boldsymbol{\eta}^<}.u = \mathbb{D}_{\boldsymbol{\eta}^{*<}}.\mathbb{D}_{\eta_1}.z \tag{74}$$

Moreover, as $\mathbb{D}_\bullet$ *is cosymmetrel*

$$\mathbb{D}_{\boldsymbol{\eta}'} = \sum \operatorname{cont}\begin{pmatrix} \boldsymbol{\eta}^< \\ \boldsymbol{\eta}' \end{pmatrix} \mathbb{D}_{\boldsymbol{\eta}^<} \tag{75}$$

These results were proven by Jean Ecalle (see [1]). Note that $\mathbb{D}_\emptyset = 1$ and

$$\mathbb{D}_{\eta_1^<} = (\mathbb{D}_{\eta_1}.z)\partial_z = \begin{cases} \mathbb{B}_{\eta_1} & \text{if } \eta_1 \in H_0 \\ 0 & \text{otherwise} \end{cases} \tag{76}$$

and if the length of $\boldsymbol{\eta}^<$ is greater than two :

- Either $\boldsymbol{\eta}^<$ is irreducible : $\boldsymbol{\eta}^< = \eta_1.\boldsymbol{\eta}^{*<}$ and of degree $d = 1$. Thus :

$$\mathbb{D}_{\boldsymbol{\eta}^<} = (\mathbb{D}_{\boldsymbol{\eta}^{*<}}.\mathbb{D}_{\eta_1}.z)\partial_z \tag{77}$$

- Either $\boldsymbol{\eta}^<$ is reducible of degree $d \geq 2$ and $\boldsymbol{\eta}^< = \boldsymbol{\eta}^{\mathbf{1}<} \oplus \ldots \oplus \boldsymbol{\eta}^{\boldsymbol{d}<}$ (with $\boldsymbol{\eta}^{\boldsymbol{i}<}$ irreducible and $\neq \emptyset$), in this case :

$$\mathbb{D}_{\boldsymbol{\eta}^<} = \frac{1}{d_1!\ldots d_s!}(\mathbb{D}_{\boldsymbol{\eta}^{\mathbf{1}<}}.z)\ldots(\mathbb{D}_{\boldsymbol{\eta}^{\boldsymbol{d}<}}.z)\partial_z^d \tag{78}$$

  where $d_1, \ldots, d_s$ are the numbers of identical arborescent sequences $\boldsymbol{\eta}^{\boldsymbol{j}<}$ in the decomposition into irreducible sequences, of course $\sum d_i = d$.

One can also notice that if a sequence $\boldsymbol{\eta}^<$ est irreducible of sum $\|\boldsymbol{\eta}^<\| \notin H_0$, then $\mathbb{D}_{\boldsymbol{\eta}^<} = 0$. This property remains valid if $\boldsymbol{\eta}^<$ has at least a monotonic partition $\boldsymbol{\eta}^{\mathbf{1}<}, \ldots, \boldsymbol{\eta}^{\boldsymbol{s}<}$ with an irreducible part $\boldsymbol{\eta}^{\boldsymbol{i}<}$ such that $\|\boldsymbol{\eta}^{\boldsymbol{i}<}\| \notin H_0$. It means that we can restrict ourselves to the arborescent sequences $(\eta_1, \ldots, \eta_s)^<$ such that, for $1 \leq i \leq s$, $\hat{\eta}_i = \sum_{j \geq i} \eta_j \in H_0$. We note $\boldsymbol{H}_{\mathbf{0}}^<$ this set of sequences. For details, see [1].

Because of equation 75, it is easy to check that, formally,

$$\mathbb{U} = \sum U^\bullet \mathbb{D}_\bullet = \sum U^{\bullet^<} \mathbb{D}_{\bullet^<} \tag{79}$$

5.4 Analyticity of $\mathbb{U}$.

As $U^\bullet$ is symmetrel, $\mathbb{U}$ is a *formal* substitution automorphism :

$$\forall \varphi \in \mathbb{C}[[z, x]], \quad \mathbb{U}.\varphi(z, x) = \varphi(u(z, x), x)$$

with $u \in \mathbb{C}[[z, x]]$, $u(0, 0) = 0$ and $\partial_z u(0, 0) = 1$. It is then easy to see that $\mathbb{U}$ is analytic iff $u \in \mathbb{C}\{z, x\}$. We will prove now that the series of analytic functions

$$u(z,x) = \mathbb{U}.z = \sum_{\boldsymbol{\eta}^< \in \boldsymbol{H}^<} U^{\boldsymbol{\eta}^<} \mathbb{D}_{\boldsymbol{\eta}^<}.z$$

is normally convergent in a neighborhood of $(0, 0)$. First of all, we remind that, because of proposition 13, if $0 \leq |x| \leq q^{-1/2}$ and $\boldsymbol{\eta}^< = (\eta_1, \ldots, \eta_s)^< \in \boldsymbol{H}^<$,

$$|U^{\boldsymbol{\eta}^<}(x)| \leq C_q^s |x|^{\|\boldsymbol{\sigma}^<\|} \tag{80}$$

In section 2.1, starting with

$$b(z,x) = z \sum_{\eta \in H_0} b_\eta x^\sigma z^n \in \mathbb{C}\{z, x\}$$

we built on $H = \{\eta_1 + \ldots + \eta_s\,;\, s \geq 1\,;\, \eta_i \in H_0\}$ the operators $\mathbb{D}_\eta$ is defined by :

$$\mathbb{D}_\eta = \sum_{s \geq 1} \sum_{\substack{\eta_1 + \ldots + \eta_s = \eta \\ \eta_i \in H_0}} \frac{1}{s!} b_{\eta_1} \ldots b_{\eta_s} z^{n_1 + \ldots + n_s + s} \partial_z^s \tag{81}$$

Similarly, if

$$\bar{b}(t,x) = t \sum_{\eta \in H_0} |b_\eta| x^\sigma t^n \in \mathbb{C}\{t, x\}$$

one can build the operators on H :

$$\bar{\mathbb{D}}_\eta = \sum_{s \geq 1} \sum_{\substack{\eta_1 + \ldots + \eta_s = \eta \\ \eta_i \in H_0}} \frac{1}{s!} |b_{\eta_1}| \ldots |b_{\eta_s}| t^{n_1 + \ldots + n_s + s} \partial_t^s$$

It is clear that, if $|z| \leq t$, then,

$$\forall \boldsymbol{\eta} \in \boldsymbol{H}, \quad |\mathbb{D}_{\boldsymbol{\eta}}.z| \leq \bar{\mathbb{D}}_{\boldsymbol{\eta}}.t$$

and, because of the chosen rule of coarborification, if, once again, $|z| \leq t$, then,

$$\forall \boldsymbol{\eta}^< \in \boldsymbol{H}^<, \quad |\mathbb{D}_{\boldsymbol{\eta}^<}.z| \leq \bar{\mathbb{D}}_{\boldsymbol{\eta}^<}.t \tag{82}$$

In conclusion, $u(z, x) = \mathbb{U}.z = \sum_{\boldsymbol{\eta}^< \in \boldsymbol{H}^<} U^{\boldsymbol{\eta}^<} \mathbb{D}_{\boldsymbol{\eta}^<}.z$ is a series of analytic functions such that, for $|x| \leq q^{-1/2}$ and $|z| \leq t$,

$$\forall \boldsymbol{\eta}^< \in \boldsymbol{H}^<, \quad |U^{\boldsymbol{\eta}^<} \mathbb{D}_{\boldsymbol{\eta}^<}.z| \leq C_q^{s(\boldsymbol{\eta}^<)} |x|^{\|\boldsymbol{\sigma}^<\|} \bar{\mathbb{D}}_{\boldsymbol{\eta}^<}.t \tag{83}$$

Let

$$\bar{u}(t, |x|) = \sum_{\boldsymbol{\eta}^< \in \boldsymbol{H}^<} C_q^{s(\boldsymbol{\eta}^<)} |x|^{\|\boldsymbol{\sigma}^<\|} \bar{\mathbb{D}}_{\boldsymbol{\eta}^<}.t = \bar{\mathbb{U}}.z = \sum C^{\bullet^<} \bar{\mathbb{D}}_{\bullet^<} \tag{84}$$

where $s(\boldsymbol{\eta}^<) = s$ if $\boldsymbol{\eta}^< = (\eta_1, \ldots, \eta_s)^<$ and $C^\emptyset = 1$ ($\bar{\mathbb{D}}_\emptyset = \mathrm{Id}$). It is clear that if $\bar{u}$ is convergent for $|x|$ and t small enough, then $u(z, x) = \mathbb{U}.z = \sum_{\boldsymbol{\eta}^< \in \boldsymbol{H}^<} U^{\boldsymbol{\eta}^<} \mathbb{D}_{\boldsymbol{\eta}^<}.z$ is normally convergent in a neighborhood of $(0, 0)$, thus $u \in \mathbb{C}\{z, x\}$ and $\mathbb{U}$ is an analytic substitution automorphism.

We have

$$\bar{\mathbb{U}} = \bar{\mathbb{U}}_0 + \bar{\mathbb{U}}_1 + \bar{\mathbb{U}}_2 + \bar{\mathbb{U}}_3$$

where $\bar{\mathbb{U}}_0 = \mathrm{Id}$, and

$$\begin{aligned}
\bar{\mathbb{U}}_1 &= \sum_{\substack{\boldsymbol{\eta}^< \in \boldsymbol{H}^< \\ \boldsymbol{\eta}^< \text{ irreducible} \\ s(\boldsymbol{\eta}^<)=1}} C^{\boldsymbol{\eta}^<} \bar{\mathbb{D}}_{\boldsymbol{\eta}^<} = \sum_{\boldsymbol{\eta}^< \in \boldsymbol{H}_1^<} C^{\boldsymbol{\eta}^<} \bar{\mathbb{D}}_{\boldsymbol{\eta}^<} \\
\bar{\mathbb{U}}_2 &= \sum_{\substack{\boldsymbol{\eta}^< \in \boldsymbol{H}^< \\ \boldsymbol{\eta}^< \text{ irreducible} \\ s(\boldsymbol{\eta}^<)\geq 2}} C^{\boldsymbol{\eta}^<} \bar{\mathbb{D}}_{\boldsymbol{\eta}^<} = \sum_{\boldsymbol{\eta}^< \in \boldsymbol{H}_2^<} C^{\boldsymbol{\eta}^<} \bar{\mathbb{D}}_{\boldsymbol{\eta}^<} \\
\bar{\mathbb{U}}_3 &= \sum_{\substack{\boldsymbol{\eta}^< \in \boldsymbol{H}^< \\ \boldsymbol{\eta}^< \text{ not irreducible}}} C^{\boldsymbol{\eta}^<} \bar{\mathbb{D}}_{\boldsymbol{\eta}^<} = \sum_{\boldsymbol{\eta}^< \in \boldsymbol{H}_3^<} C^{\boldsymbol{\eta}^<} \bar{\mathbb{D}}_{\boldsymbol{\eta}^<}
\end{aligned} \tag{85}$$

Because of the coarborification rule (applied to $\bar{\mathbb{D}}_\bullet$), we get the following identities

$$\bar{\mathbb{U}}_1 = C_q \sum_{\eta \in H_0} |b_\eta| \, |x|^\sigma t^{n+1} \partial_t \tag{86}$$

and $\bar{\mathbb{U}}_1.t = C_q \bar{b}(t, |x|) = c(t, |x|)$. Using the rule 77, we get,

$$\begin{aligned}
\bar{\mathbb{U}}_2 &= \sum_{\substack{\boldsymbol{\eta}^< \in \boldsymbol{H}^< \\ \boldsymbol{\eta}^< \text{ irreducible} \\ s(\boldsymbol{\eta}^<)\geq 2}} C^{\boldsymbol{\eta}^<} \bar{\mathbb{D}}_{\boldsymbol{\eta}^<} \\
&= \sum_{\boldsymbol{\eta}^< \in \boldsymbol{H}_3^<} C^{\boldsymbol{\eta}^<} \bar{\mathbb{D}}_{\boldsymbol{\eta}^<} \\
&= \sum_{\substack{\boldsymbol{\eta}^< = \eta_1 \bullet \boldsymbol{\eta}^{*<} \\ \eta_1 \in \boldsymbol{H}_1^< \\ \boldsymbol{\eta}^{*<} \in \boldsymbol{H}_1^< \cup \boldsymbol{H}_2^< \cup \boldsymbol{H}_3^<}} C^{\boldsymbol{\eta}^{*<}} C^{\eta_1} (\bar{\mathbb{D}}_{\boldsymbol{\eta}^{*<}}.\bar{\mathbb{D}}_{\eta_1}.t) \partial_t \\
&= \big((\bar{\mathbb{U}}_1 + \bar{\mathbb{U}}_2 + \bar{\mathbb{U}}_3).\bar{\mathbb{U}}_1.t\big) \partial_t
\end{aligned} \tag{87}$$

and finally, using the rule 78,

$$\begin{aligned}
\bar{\mathbb{U}}_3 &= \sum_{\substack{\boldsymbol{\eta}^< \in \boldsymbol{H}^< \\ \boldsymbol{\eta}^< \text{ not irreducible}}} C^{\boldsymbol{\eta}^<} \bar{\mathbb{D}}_{\boldsymbol{\eta}^<} = \sum_{\boldsymbol{\eta}^< \in \boldsymbol{H}_3^<} C^{\boldsymbol{\eta}^<} \bar{\mathbb{D}}_{\boldsymbol{\eta}^<} \\
&= \sum_{\substack{\boldsymbol{\eta}^< = \boldsymbol{\eta}^{1<} \oplus \ldots \oplus \boldsymbol{\eta}^{d<} \\ d \geq 2 \,;\, \boldsymbol{\eta}^{i<} \in \boldsymbol{H}_1^< \cup \boldsymbol{H}_2^<}} \frac{1}{d!} \frac{d!}{d_1! \ldots d_s!} (\bar{\mathbb{D}}_{\boldsymbol{\eta}^{1<}}.t) \ldots (\bar{\mathbb{D}}_{\boldsymbol{\eta}^{d<}}.t) \partial_t^d \\
&= \sum_{d \geq 2} \frac{1}{d!} \big((\bar{\mathbb{U}}_1 + \bar{\mathbb{U}}_2).t\big)^d \partial_t^d
\end{aligned} \tag{88}$$

But

$$\begin{aligned}
\bar{u}(t,|x|) &= \bar{\mathbb{U}}.t\\
&= (\bar{\mathbb{U}}_0+\bar{\mathbb{U}}_1+\bar{\mathbb{U}}_2+\bar{\mathbb{U}}_3).t\\
&= t+\bar{\mathbb{U}}_1.t+\bar{\mathbb{U}}_2.t\\
&= t+\bar{\mathbb{U}}_1.t+(\bar{\mathbb{U}}_1+\bar{\mathbb{U}}_2+\bar{\mathbb{U}}_3).\bar{\mathbb{U}}_1.t\big)\\
&= t+\bar{\mathbb{U}}_1.t+\sum_{d\geq 1}\frac{1}{d!}\big((\bar{\mathbb{U}}_1+\bar{\mathbb{U}}_2).t\big)^d\partial_t^d(\bar{\mathbb{U}}_1.t)\\
&= t+\sum_{d\geq 0}\frac{1}{d!}\big((\bar{\mathbb{U}}_1+\bar{\mathbb{U}}_2).t\big)^d\partial_t^d(c(t,|x|))\\
&= t+\sum_{d\geq 0}\frac{1}{d!}(\bar{u}(t,|x|)-t)^d\partial_t^d(c(t,|x|))\\
&= t+c(\bar{u}(t,|x|),|x|)
\end{aligned}$$

But, as $c\in\mathbb{R}^+\{t,|x|\}$ and $c(0,0)=\partial_t c(0,0)=0$, using the implicit function theorem and majorant series, it can easily be proved that $\bar{u}\in\mathbb{R}^+\{t,|x|\}$. Thus,

Theorem 15. *There exists $\alpha>0$ such that the series of analytic functions*

$$u(z,x)=\mathbb{U}.z=\sum_{\boldsymbol{\eta}^<\in\boldsymbol{H}^<}U^{\boldsymbol{\eta}^<}\mathbb{D}_{\boldsymbol{\eta}^<}.z$$

is normally convergent in $V_\alpha=\{|x|\leq\alpha\,,\,|z|\leq\alpha\}$ which means that $\mathbb{U}$ is an analytic substitution automorphism : $u\in\mathbb{C}\{z,x\}$.

6 Analytic classification.

We first prove that $\mathbb{V}$ is also an analytic substitution automorphism.

6.1 Analyticity of $\mathbb{V}$.

Let $\mathbb{U}_q=\sum s_q(U^\bullet)\mathbb{D}_\bullet=\sum$

$$\mathbb{U}_q=\sum s_q(U^\bullet)\mathbb{D}_\bullet=\sum_{\boldsymbol{\eta}\in\boldsymbol{H}}x^{-\|\boldsymbol{n}\|}U^{\boldsymbol{\eta}}(qx)\mathbb{D}_{\boldsymbol{\eta}}$$

The operator $\mathbb{U}_q$ is a formal substitution automorphism, but, after arborification,

$$\forall\boldsymbol{\eta}^<\in\boldsymbol{H}^<,\quad s_q(U^\bullet)^{\boldsymbol{\eta}^<}(x)=\begin{cases}0 & \text{if }\ \boldsymbol{\eta}^<\in\boldsymbol{H}^</\boldsymbol{H}^{+<}\\ x^{-\|\boldsymbol{n}^<\|}U^{\boldsymbol{\eta}^<}(qx) & \text{if }\ \boldsymbol{\eta}^<\in\boldsymbol{H}^{+<}\end{cases}$$

thus, for $|x|\leq q^{-3/2}$, using proposition 13,

$$\forall\boldsymbol{\eta}^<\in\boldsymbol{H}^<,\quad |s_q(U^\bullet)^{\boldsymbol{\eta}^<}(x)|\leq C_q^s|x|^{\|\boldsymbol{\sigma}^<\|}\tag{89}$$

this proves once again that $\mathbb{U}_q$ is an analytic substitution automorphism, such that

$$u_q(z,x)=\mathbb{U}_q.z\in\mathbb{C}\{z,x\}\tag{90}$$

and,

$$\forall \varphi \in \mathbb{C}\{z,x\}, \quad \mathbb{U}_q.\varphi(z,x) = \varphi(u_q(z,x),x)$$

Finally (see section 3.4) $\mathbb{V} = \sum V^\bullet \mathbb{D}_\bullet$ is a formal substitution such that

$$v(z,x) = \mathbb{V}.z \in z + x\mathbb{C}[[x]]$$

and

$$\mathbb{V}.\mathbb{U}_q = \mathbb{U}.\mathbb{D} \tag{91}$$

where $\mathbb{D} = \sum X^\bullet \mathbb{D}_\bullet$ is an analytic substitution automorphism such that

$$\forall \varphi \in \mathbb{C}\{z,x\}, \quad \mathbb{D}.\varphi(z,x) = \varphi(z + b(z,x),x)$$

but it is clear that, as $\mathbb{U}_q$ is analytic, its inverse substitution automorphism $\mathbb{U}_q^{-1}$ is also analytic and $\mathbb{V} = \mathbb{U}.\mathbb{D}.\mathbb{U}_q^{-1}$ is analytic :

$$v(z,x) = \mathbb{V}.z \in z + x\mathbb{C}\{x\}$$

We can resume this in the following theorem

Theorem 16. *Let us consider the equation*

$$(x\sigma_q - 1)y = b(y,x) \tag{92}$$

with $b(0,0) = 0$ *and* $\frac{\partial b}{\partial f}(0,0) = 0$. *This equation is conjugate by the analytic automorphism* $\mathbb{U}$ *:*

$$y = \mathbb{U}.z = \Big(\sum U^\bullet \mathbb{D}_\bullet\Big).z = u(z,x) \in \mathbb{C}\{z,x\} \tag{93}$$

to the equation

$$x\sigma_q z = \mathbb{V}.z = v(z,x) \in z + x\mathbb{C}\{x\} \tag{94}$$

To prove theorem 1, it remains to prove that the equation above is analytically conjugate to

$$x\sigma_q z = z + \alpha x \tag{95}$$

where $\alpha \in \mathbb{C}$ depends on v.

6.2 The equation $x\sigma_q z = \mathbb{V}.z = v(z,x) \in z + x\mathbb{C}\{x\}$.

We have

$$v(z,x) = z + \sum_{n \geq 1} v_n x^n \in z + x\mathbb{C}\{x\}$$

If

$$\alpha = \sum_{n \geq 1} v_n q^{-n(n-1)/2} \in \mathbb{C} \tag{96}$$

the equation

$$x\sigma_q(f) = v(f,x) - \alpha x = f + (v_1 - \alpha)x + \sum_{n \geq 2} v_n x^n \in z + x\mathbb{C}\{x\} \tag{97}$$

possess an analytic solution : If $f(x)=\sum_{n\geq 1} f_n x^n$,

$$\begin{aligned} x\sigma_q f - f &= \sum_{n\geq 2} f_{n-1}q^{n-1}x^n - \sum_{n\geq 1} f_n x^n \\ &= (v_1-\alpha)x + \sum_{n\geq 2} v_n x^n \\ &= -\left(\sum_{k\geq 2} v_k q^{-k(k-1)/2}\right)x + \sum_{n\geq 2} v_n x^n \end{aligned}$$

thus $f_1=\sum_{k\geq 2} v_k q^{-k(k-1)/2}$ and

$$\forall n\geq 2, \quad f_n = f_{n-1}q^{n-1} - v_n$$

but it is easy to check that

$$\forall n\geq 1, \quad f_n = \sum_{k\geq n+1} v_k q^{-(k+n-1)(k-n)/2} = \sum_{k\geq n+1} v_k [k]_q [n]_q^{-1}$$

and $f\in\mathbb{C}\{x\}$: there exists $C>0$ such that

$$\forall n\geq 1, \quad |v_n|\leq C^n$$

thus, for $n\geq 1$,

$$\begin{aligned} |f_n| &\leq \sum_{k\geq n+1} C^k q^{-(k+n-1)(k-n)/2} \\ &\leq C^n \sum_{k\geq n+1} C^{k-n} q^{-(k+n-1)(k-n)/2} \\ &\leq C^n \sum_{k\geq 1} C^k q^{-(k+2n-1)k/2} \\ &\leq C^n \sum_{k\geq 1} C^k q^{-(k-1)k/2} \\ &\leq M_C C^n \end{aligned} \tag{98}$$

But if $\tilde{z}=z-f$, then

$$\begin{aligned} x\sigma_q(\tilde{z}) &= x\sigma_q(z) - x\sigma_q(f) \\ &= z + \sum_{n\geq 1} v_n x^n - f - (v_1-\alpha)x - \sum_{n\geq 2} v_n x^n \\ &= \tilde{z} + \alpha x \end{aligned}$$

This proves finally theorem 1 :

Let us consider the equation

$$(x\sigma_q - 1)y = b(y,x) \tag{99}$$

with $b(0,0)=0$ *and* $\frac{\partial b}{\partial f}(0,0)=0$. *There exists* $\alpha(b)\in\mathbb{C}$, *such that this equation is analytically conjugate to the equation*

$$x\sigma_q z = z + \alpha(b)x \tag{100}$$

and, if $\alpha(b)\neq 0$, *this last equation is analytically conjugate to*

$$x\sigma_q z_0 = z_0 + x \tag{101}$$

(Take $z=\alpha(b)z_0$*).*

7 Proofs.

7.1 Proof of proposition 7.

Proposition 7. For any moulds $A^\bullet$ and $B^\bullet$, we have

$$J^\bullet \circ A^\bullet = J^\bullet \circ (A^\bullet \times (K^\bullet \circ B^\bullet)) \tag{102}$$

Let $C^\bullet = K^\bullet \circ B^\bullet$ and $D^\bullet = A^\bullet \times (K^\bullet \circ B^\bullet)$ and let $\boldsymbol{\eta} \in \boldsymbol{H}$ nonempty. We can restrict ourselves to the case $\boldsymbol{\eta} \in \boldsymbol{H}^+$ because, otherwise, the equation becomes $0=0$. We remind that :

$$(J^\bullet \circ D^\bullet)^{\boldsymbol{\eta}} = \sum_{\boldsymbol{\eta}^1 \dots \boldsymbol{\eta}^t = \boldsymbol{\eta}} J^{\|\boldsymbol{\eta}^1\|, \dots, \|\boldsymbol{\eta}^t\|} D^{\boldsymbol{\eta}^1} \dots D^{\boldsymbol{\eta}^t}$$

One can expand the products $D^{\boldsymbol{\eta}^1} \dots D^{\boldsymbol{\eta}^t}$ such that we only get 5 type of monomials :

1. The monomials $A^{\boldsymbol{\alpha}^1} \dots A^{\boldsymbol{\alpha}^r}$ where $\boldsymbol{\alpha}^1 \dots \boldsymbol{\alpha}^r = \boldsymbol{\eta}$,
2. The monomials $A^{\boldsymbol{\alpha}^1} \dots A^{\boldsymbol{\alpha}^r} C^{\boldsymbol{\beta}}$ where $\boldsymbol{\alpha}\boldsymbol{\beta} = \boldsymbol{\eta}$ and $\boldsymbol{\alpha}^1 \dots \boldsymbol{\alpha}^r = \boldsymbol{\alpha}$,
3. The monomials $A^{\boldsymbol{\alpha}^1} \dots A^{\boldsymbol{\alpha}^r} C^{\boldsymbol{\beta}}\, D^{\boldsymbol{\gamma}^1} \dots D^{\boldsymbol{\gamma}^s}$ where $\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma} = \boldsymbol{\eta}$ and $\boldsymbol{\alpha}^1 \dots \boldsymbol{\alpha}^r = \boldsymbol{\alpha}$ and $\boldsymbol{\gamma}^1 \dots \boldsymbol{\gamma}^s = \boldsymbol{\gamma}$,
4. The monomials $C^{\boldsymbol{\beta}^1} \dots C^{\boldsymbol{\beta}^r}\, D^{\boldsymbol{\gamma}^1} \dots D^{\boldsymbol{\gamma}^s}$ where $\boldsymbol{\beta}\boldsymbol{\gamma} = \boldsymbol{\eta}$ and $\boldsymbol{\beta}^1 \dots \boldsymbol{\beta}^r = \boldsymbol{\beta}$ and $\boldsymbol{\gamma}^1 \dots \boldsymbol{\gamma}^s = \boldsymbol{\gamma}$,
5. The monomials $C^{\boldsymbol{\beta}^1} \dots C^{\boldsymbol{\beta}^r}$ where $\boldsymbol{\beta}^1 \dots \boldsymbol{\beta}^r = \boldsymbol{\eta}$.

Let us now take look at the contribution to each type of monomial of the expression $(J^\bullet \circ D^\bullet)^{\boldsymbol{\eta}}$.

1. The first type gives :
$$\sum_{\boldsymbol{\alpha}^1 \dots \boldsymbol{\alpha}^r = \boldsymbol{\eta}} J^{\|\boldsymbol{\alpha}^1\|, \dots, \|\boldsymbol{\alpha}^t\|} A^{\boldsymbol{\alpha}^1} \dots A^{\boldsymbol{\alpha}^r} = (J^\bullet \circ A^\bullet)^{\boldsymbol{\eta}}$$
2. When $\boldsymbol{\alpha}\boldsymbol{\beta} = \boldsymbol{\eta}$ and $\boldsymbol{\alpha}^1 \dots \boldsymbol{\alpha}^r = \boldsymbol{\alpha}$, the monomial $A^{\boldsymbol{\alpha}^r} \dots A^{\boldsymbol{\alpha}^r} C^{\boldsymbol{\beta}}$ has, as a coefficient :
$$J^{\|\boldsymbol{\alpha}^1\|, \dots, \|\boldsymbol{\alpha}^r\|, \|\boldsymbol{\beta}\|} + J^{\|\boldsymbol{\alpha}^1\|, \dots, \|\boldsymbol{\alpha}^r\| + \|\boldsymbol{\beta}\|} = J^{\|\boldsymbol{\alpha}^1\|, \dots, \|\boldsymbol{\alpha}^r\| + \|\boldsymbol{\beta}\|}(1 + \alpha_-(\|\boldsymbol{\beta}\|))$$
but, either $\|\boldsymbol{\beta}\| < 0$ and it vanishes, either $\|\boldsymbol{\beta}\| \geq 0$ and then $C^{\boldsymbol{\beta}} = (K^\bullet \circ B^\bullet)^{\boldsymbol{\beta}} = 0$ (see the definition of $K^\bullet$).
3. When $\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma} = \boldsymbol{\eta}$ and $\boldsymbol{\alpha}^1 \dots \boldsymbol{\alpha}^r = \boldsymbol{\alpha}$ and $\boldsymbol{\gamma}^1 \dots \boldsymbol{\gamma}^s = \boldsymbol{\gamma}$, the monomial $A^{\boldsymbol{\alpha}^1} \dots A^{\boldsymbol{\alpha}^r} C^{\boldsymbol{\beta}}\, D^{\boldsymbol{\gamma}^1} \dots D^{\boldsymbol{\gamma}^s}$ has, as coefficient :
$$J^{\|\boldsymbol{\alpha}^1\|, \dots, \|\boldsymbol{\alpha}^r\|, \|\boldsymbol{\beta}\|, \|\boldsymbol{\gamma}^1\|, \dots, \|\boldsymbol{\gamma}^s\|} + J^{\|\boldsymbol{\alpha}^1\|, \dots, \|\boldsymbol{\alpha}^r\| + \|\boldsymbol{\beta}\|, \|\boldsymbol{\gamma}^1\|, \dots, \|\boldsymbol{\gamma}^s\|}$$
for the same reason, if $\|\boldsymbol{\beta}\| + \|\boldsymbol{\gamma}^1\| + \dots + \|\boldsymbol{\gamma}^s\| < 0$, this term vanishes and if $\|\boldsymbol{\beta}\| + \|\boldsymbol{\gamma}^1\| + \dots + \|\boldsymbol{\gamma}^s\| \geq 0$, this term still vanishes if $\|\boldsymbol{\gamma}^1\| + \dots + \|\boldsymbol{\gamma}^s\| \geq 0$ (definition of $J^\bullet$) and otherwise $\|\boldsymbol{\gamma}^1\| + \dots + \|\boldsymbol{\gamma}^s\| < 0$, which means that $\|\boldsymbol{\beta}\| \geq 0$ and then $C^{\boldsymbol{\beta}} = (K^\bullet \circ B^\bullet)^{\boldsymbol{\beta}} = 0$.

4. When $\boldsymbol{\beta}\boldsymbol{\gamma}=\boldsymbol{\eta}$ and $\boldsymbol{\beta^1...\beta^r}=\boldsymbol{\beta}$ and $\boldsymbol{\gamma^1...\gamma^s}=\boldsymbol{\gamma}$, the monomial $C^{\boldsymbol{\beta^1}}...C^{\boldsymbol{\beta^r}}$ $D^{\boldsymbol{\gamma^1}}...D^{\boldsymbol{\gamma^s}}$ has, as a factor :
$$J^{\|\boldsymbol{\beta^1}\|,...,\|\boldsymbol{\beta^r}\|,\|\boldsymbol{\gamma^1}\|,...,\|\boldsymbol{\gamma^s}\|}$$
this vanishes if $\|\boldsymbol{\beta^1}\|+...+\|\boldsymbol{\beta^r}\|+\|\boldsymbol{\gamma^1}\|+...+\|\boldsymbol{\gamma^s}\|<0$ or $\|\boldsymbol{\gamma^1}\|+...+\|\boldsymbol{\gamma^s}\|\geq 0$, otherwise $\|\boldsymbol{\gamma^1}\|+...+\|\boldsymbol{\gamma^s}\|<0$ and $\|\boldsymbol{\beta^1}\|+...+\|\boldsymbol{\beta^r}\|+\|\boldsymbol{\gamma^1}\|+...+\|\boldsymbol{\gamma^s}\|\geq 0$ thus $\|\boldsymbol{\beta^1}\|+...+\|\boldsymbol{\beta^r}\|\geq 0$ which means that at least one of the $\|\boldsymbol{\beta^i}\|$ is nonnegatrive and so $C^{\boldsymbol{\beta^1}}...C^{\boldsymbol{\beta^r}}$ vanishes.
5. The fifth type gives 0 for the same reason.

In conclusion, the only nonzero contribution is $(J^\bullet\circ A^\bullet)^{\boldsymbol{\eta}}$ and we proved that
$$J^\bullet\circ A^\bullet = J^\bullet\circ(A^\bullet\times(K^\bullet\circ B^\bullet))$$

7.2 Proof of theorem 9.

Theorem 9. If $\boldsymbol{\eta}=(\eta_1,...,\eta_s)=\begin{pmatrix} n_1,...,n_s\\ \sigma_1,...,\sigma_s\end{pmatrix}=\begin{pmatrix}\boldsymbol{n}\\ \boldsymbol{\sigma}\end{pmatrix}\in\boldsymbol{H}$, then the monomial $U^{\boldsymbol{\eta}}$ is a convergent series with the following expression :
$$U^{\boldsymbol{\eta}}(x)=\varepsilon_{\boldsymbol{n}}\sum_{(k_1,...,k_s)\in\mathbb{Z}^{\boldsymbol{n}}\cap P^{\boldsymbol{n}}} x^{\|\boldsymbol{\sigma}\|+\check{k}_1n_1+...+\check{k}_sn_s}q^{-\check{k}_1\sigma_1-...-\check{k}_s\sigma_s}[\check{k}_1+1]_q^{n_1}...[\check{k}_s+1]_q^{n_s}\quad(103)$$
with the notations

- $\varepsilon_{\boldsymbol{n}}=\prod_{1\leq i\leq s}\varepsilon_{\hat{n}_i}$ with $\hat{n}_i=n_i+...+n_s$ with $\varepsilon_n=1$ (resp. $\varepsilon_n=-1$) if $n\geq 0$ (resp. $n<0$)
- $\mathbb{Z}^{\boldsymbol{n}}=\mathbb{Z}^{\hat{n}_1}\times...\times\mathbb{Z}^{\hat{n}_s}$ with $\mathbb{Z}^n=\mathbb{Z}^{+*}$ (resp. $\mathbb{Z}^-$) if $n\geq 0$ (resp. $n<0$)
- $P^{\boldsymbol{n}}=\{(k_1,...,k_s)\in\mathbb{Z}^s\,;\check{k}_i>0\ \text{ if }\varepsilon_{\hat{n}_i}=-1\}$
- $\check{k}_i=k_1+...+k_i$.
- $[k+1]_q=q^{-k(k+1)/2}$

Proof. To prove this theorem we will introduce some notations. For a sequence $(k_1,...,k_s)\in\mathbb{Z}^s$, let $l_i=\check{k}_i$, thus $k_1=l_1$ and, for $i\geq 2$, $k_i=l_i-l_{i-1}$. For a sequence $\boldsymbol{\eta}=(\eta_1,....,\eta_s)\in\boldsymbol{H}^+$ and $\boldsymbol{l}=(l_1,...,l_s)\in\mathbb{Z}^s$ $(s\geq 1)$, let
$$T_{\boldsymbol{l}}^{\boldsymbol{\eta}}=x^{\|\boldsymbol{\sigma}\|+l_1n_1+...+l_sn_s}q^{-l_1\sigma_1-...-l_s\sigma_s}[l_1+1]_q^{n_1}...[l_s+1]_q^{n_s}=\prod_{i=1}^{s}T_{l_i}^{\eta_i}\qquad(104)$$
With this notation, the formula in theorem 4 becomes
$$W^{\boldsymbol{\eta}}(x)\ =\ \varepsilon_{\boldsymbol{n}}\sum_{(l_1,l_2-l_1...,l_s-l_{s-1})\in\mathbb{Z}^{\boldsymbol{n}}}T_{\boldsymbol{l}}^{\boldsymbol{\eta}}$$
using the function $\rho_1(n)=\begin{cases}1 & \text{if } n>0\\ 0 & \text{if } n\leq 0\end{cases}$ and $\rho_{-1}=1-\rho_1$, we get
$$W^{\boldsymbol{\eta}}(x)\ =\ \varepsilon_{\boldsymbol{n}}\sum_{\boldsymbol{l}\in\mathbb{Z}^s}\rho_{\varepsilon_{\hat{n}_1}}(l_1)\rho_{\varepsilon_{\hat{n}_2}}(l_2-l_1)...\rho_{\varepsilon_{\hat{n}_s}}(l_s-l_{s-1})T_{\boldsymbol{l}}^{\boldsymbol{\eta}}=\sum_{\boldsymbol{l}\in\mathbb{Z}^s}w_{\boldsymbol{l}}^{\boldsymbol{\eta}}T_{\boldsymbol{l}}^{\boldsymbol{\eta}}\qquad(105)$$

and we have to prove that

$$\begin{aligned} U^{\boldsymbol{\eta}}(x) &= \varepsilon_{\boldsymbol{n}} \sum_{\boldsymbol{l}\in\mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1)\rho_{\varepsilon_{\hat{n}_2}}(l_2-l_1)\ldots\rho_{\varepsilon_{\hat{n}_s}}(l_s-l_{s-1})\left(\prod_{\substack{1\le i\le s\\ \varepsilon_{\hat{n}_i}=-1}} \rho_1(l_i)\right)T_{\boldsymbol{l}}^{\boldsymbol{\eta}} \\ &= \sum_{\boldsymbol{l}\in\mathbb{Z}^s} \tilde{u}_{\boldsymbol{l}}^{\boldsymbol{\eta}} T_{\boldsymbol{l}}^{\boldsymbol{\eta}} \end{aligned} \tag{106}$$

Let $\tilde{U}^{\boldsymbol{\eta}}$ be the second term of this equation. It is clear that the identities hold if $\hat{n}_1<0$ because $(J\circ W)^{\boldsymbol{\eta}}=0$ and in $U^{\boldsymbol{\eta}}$ we have the factor $\rho_{-1}(l_1)\rho_1(l_1)=0$. Let us suppose now that $\hat{n}_1\ge 0$. If $I=\{1\le i\le s\ ;\ \varepsilon_{\hat{n}_i}=-1\}=\emptyset$ then, once again, we obtain the identity because

$$\prod_{\substack{1\le i\le s\\ \varepsilon_{\hat{n}_i}=-1}} \rho_1(l_i)=1 \quad \text{and} \quad (J\circ W)^{\boldsymbol{\eta}}=W^{\boldsymbol{\eta}}$$

We need some notations to prove the identity. For $\boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_t)\in\boldsymbol{H}$, $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_t)\in\{-1,+1\}^t$ and $\boldsymbol{m}=(m_1,\ldots,m_t)\in\mathbb{Z}^t$, then

$$w_{\boldsymbol{m}}^{\boldsymbol{\gamma},\boldsymbol{\alpha}}=\left(\prod \alpha_i\right)\rho_{\alpha_1}(m_1)\rho_{\alpha_2}(m_2-m_1)\ldots\rho_{\alpha_t}(m_t-m_{t-1})$$

and if $\boldsymbol{\varepsilon}(\boldsymbol{\gamma})=(\varepsilon_{\hat{p}_1},\ldots,\varepsilon_{\hat{p}_t})$ with $\boldsymbol{\gamma}=\begin{pmatrix} p_1,\ldots,p_t\\ \sigma_1,\ldots,\sigma_t\end{pmatrix}$, then

$$w_{\boldsymbol{m}}^{\boldsymbol{\gamma}}=w_{\boldsymbol{m}}^{\boldsymbol{\gamma},\boldsymbol{\varepsilon}(\boldsymbol{\gamma})}$$

If $\boldsymbol{\eta}=(\eta_1,\ldots,\eta_s)\in\boldsymbol{H}^+$ $(s\ge 1)$ then

$$U^{\boldsymbol{\eta}}(x) = \sum_{\boldsymbol{\eta}^1\ldots\boldsymbol{\eta}^t=\boldsymbol{\eta}} J^{\|\boldsymbol{\eta}^1\|,\ldots,\|\boldsymbol{\eta}^s\|}W^{\boldsymbol{\eta}^1}(x)\ldots W^{\boldsymbol{\eta}^t}(x)$$

If $I=\{1\le i\le s\ ;\ \varepsilon_{\hat{n}_i}=-1\}=\{i_1<\ldots<i_q\}$ (note that $1\notin I$), because of the definition of $\boldsymbol{J}^{\bullet}$, we get

$$U^{\boldsymbol{\eta}}(x)=\sum_{K=\{k_1<\ldots<k_j\}\subset I} (-1)^{|K|}W^{\boldsymbol{\eta}_{(k_0,k_1)}}\ldots W^{\boldsymbol{\eta}_{(k_j,k_{j+1})}}$$

where $|K|$ is the cardinality of K, $k_0=1$, $k_{j+1}=s+1$ and $\boldsymbol{\eta}_{(k_i,k_{i+1})}=(\eta_{k_i},\ldots,\eta_{k_{i+1}-1})$. Using formula 105, we get

$$\begin{aligned} U^{\boldsymbol{\eta}}(x) &= \sum_{\boldsymbol{l}\in\mathbb{Z}^s}\ \sum_{K=\{k_1<\ldots<k_j\}\subset I} (-1)^{|K|} w_{\boldsymbol{l}_{(k_0,k_1)}}^{\boldsymbol{\eta}_{(k_0,k_1)}}\ldots w_{\boldsymbol{l}_{(k_j,k_{j+1})}}^{\boldsymbol{\eta}_{(k_j,k_{j+1})}} T_{\boldsymbol{l}}^{\boldsymbol{\eta}} \\ &= \sum_{\boldsymbol{l}\in\mathbb{Z}^s}\ \sum_{K=\{k_1<\ldots<k_j\}\subset I} (-1)^{|K|} w_{\boldsymbol{l}_{(k_0,k_1)}}^{\boldsymbol{\eta}_{(k_0,k_1)},\boldsymbol{\varepsilon}(\boldsymbol{\eta}_{(k_0,k_1)})}\ldots w_{\boldsymbol{l}_{(k_j,k_{j+1})}}^{\boldsymbol{\eta}_{(k_j,k_{j+1})},\boldsymbol{\varepsilon}(\boldsymbol{\eta}_{(k_j,k_{j+1})})} T_{\boldsymbol{l}}^{\boldsymbol{\eta}} \end{aligned}$$

For $0\le r\le q$, let $I_r=\{i_1<\ldots<i_r\}$ ($I_q=I$ and $I_0=\emptyset$). If

$$\tilde{U}_r^{\boldsymbol{\eta}}=\varepsilon_{\boldsymbol{n}}\sum_{\boldsymbol{l}\in\mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1)\rho_{\varepsilon_{\hat{n}_2}}(l_2-l_1)\ldots\rho_{\varepsilon_{\hat{n}_s}}(l_s-l_{s-1})\left(\prod_{i\in I_r}\rho_1(l_i)\right)T_{\boldsymbol{l}}^{\boldsymbol{\eta}}$$

then $\tilde{U}_0^{\boldsymbol{\eta}} = W^{\boldsymbol{\eta}}$ and $\tilde{U}_q^{\boldsymbol{\eta}} = \tilde{U}^{\boldsymbol{\eta}}$. For $1 \le r \le q$, we get

$$\begin{aligned}
\tilde{U}_r^{\boldsymbol{\eta}} &= \varepsilon_{\boldsymbol{n}} \sum_{\boldsymbol{l}\in\mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1)\rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1)\dots\rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1})\left(\prod_{i\in I_r} \rho_1(l_i)\right)T_{\boldsymbol{l}}^{\boldsymbol{\eta}} \\
&= \varepsilon_{\boldsymbol{n}} \sum_{\boldsymbol{l}\in\mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1)\rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1)\dots\rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1})\left(\prod_{i\in I_{r-1}} \rho_1(l_i)\right)(1 - \rho_{-1}(l_{i_r}))T_{\boldsymbol{l}}^{\boldsymbol{\eta}} \\
&= \tilde{U}_{r-1}^{\boldsymbol{\eta}} - \varepsilon_{\boldsymbol{n}} \sum_{\boldsymbol{l}\in\mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1)\rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1)\dots\rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1})\left(\prod_{i\in I_{r-1}} \rho_1(l_i)\right)\rho_{-1}(l_{i_r})T_{\boldsymbol{l}}^{\boldsymbol{\eta}}
\end{aligned}$$

One can notice that, in the sum, $l_{i_{r-1}} > 0$ (take $i_{r-1} = i_0 = 1$ if $r = 1$) and $(l_{i_{r-1}+1} - l_{i_{r-1}})$, ..., $(l_{i_r-1} - l_{i_r-2})$ must be positive, thus $l_{i_r-1} > 0$. As l_{i_r} is non positive, this automatically ensures that $(l_{i_r} - l_{i_r-1})$ is non positive, thus

$$\begin{aligned}
\tilde{U}_r^{\boldsymbol{\eta}} &= \sum_{\boldsymbol{l}\in\mathbb{Z}^s} w_{\boldsymbol{l}}^{\boldsymbol{\eta},\boldsymbol{\varepsilon}(\boldsymbol{\eta})}\left(\prod_{i\in I_r} \rho_1(l_i)\right)T_{\boldsymbol{l}}^{\boldsymbol{\eta}} \\
&= \sum_{\boldsymbol{l}\in\mathbb{Z}^s} w_{\boldsymbol{l}}^{\boldsymbol{\eta},\boldsymbol{\varepsilon}(\boldsymbol{\eta})}\left(\prod_{i\in I_{r-1}} \rho_1(l_i)\right)T_{\boldsymbol{l}}^{\boldsymbol{\eta}} - \sum_{\boldsymbol{l}\in\mathbb{Z}^s} w_{\boldsymbol{l}}^{\boldsymbol{\eta},\boldsymbol{\varepsilon}(\boldsymbol{\eta})}\left(\prod_{i\in I_{r-1}} \rho_1(l_i)\right)\rho_{-1}(l_{i_r})T_{\boldsymbol{l}}^{\boldsymbol{\eta}} \\
&= \tilde{U}_{r-1}^{\boldsymbol{\eta}} \\
&\quad - \varepsilon_{\boldsymbol{n}} \sum_{\boldsymbol{l}\in\mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1)\rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1)\dots\rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1})\left(\prod_{i\in I_{r-1}} \rho_1(l_i)\right)\rho_{-1}(l_{i_r})T_{\boldsymbol{l}}^{\boldsymbol{\eta}} \\
&= \tilde{U}_{r-1}^{\boldsymbol{\eta}} \\
&\quad - \varepsilon_{\boldsymbol{n}} \sum_{\boldsymbol{l}\in\mathbb{Z}^s} \left(\prod_{i<i_r} \rho_{\varepsilon_{\hat{n}_i}}(l_i - l_{i-1})\right)\rho_{-1}(l_{i_r})\left(\prod_{i>i_r} \rho_{\varepsilon_{\hat{n}_i}}(l_i - l_{i-1})\right)\left(\prod_{i\in I_{r-1}} \rho_1(l_i)\right)T_{\boldsymbol{l}}^{\boldsymbol{\eta}} \\
&= \sum_{\boldsymbol{l}\in\mathbb{Z}^s} w_{\boldsymbol{l}}^{\boldsymbol{\eta},\boldsymbol{\varepsilon}(\boldsymbol{\eta})}\left(\prod_{i\in I_{r-1}} \rho_1(l_i)\right)T_{\boldsymbol{l}}^{\boldsymbol{\eta}} \\
&\quad - \sum_{\boldsymbol{l}\in\mathbb{Z}^s} w_{\boldsymbol{l}_{(i_0,i_r)}}^{\boldsymbol{\eta}_{(i_0,i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(i_0,i_r)}}\left(\prod_{i\in I_{r-1}} \rho_1(l_i)\right)w_{\boldsymbol{l}_{(i_r,i_{q+1})}}^{\boldsymbol{\eta}_{(i_r,i_{q+1})},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(i_r,i_{q+1})}}T_{\boldsymbol{l}}^{\boldsymbol{\eta}}
\end{aligned}$$

Where $i_0 = 1$ and $i_{q+1} = s$. Using this formula recursively proves that

$$\tilde{U}^{\boldsymbol{\eta}}(x) = \sum_{\boldsymbol{l}\in\mathbb{Z}^s} \sum_{K=\{k_1<\dots<k_j\}\subset I} (-1)^{|K|} w_{\boldsymbol{l}_{(k_0,k_1)}}^{\boldsymbol{\eta}_{(k_0,k_1)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_0,k_1)}} \dots w_{\boldsymbol{l}_{(k_j,k_{j+1})}}^{\boldsymbol{\eta}_{(k_j,k_{j+1})},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_j,k_{j+1})}} T_{\boldsymbol{l}}^{\boldsymbol{\eta}}$$

which must be compared to

$$U^{\boldsymbol{\eta}}(x) = \sum_{\boldsymbol{l}\in\mathbb{Z}^s} \sum_{K=\{k_1<\dots<k_j\}\subset I} (-1)^{|K|} w_{\boldsymbol{l}_{(k_0,k_1)}}^{\boldsymbol{\eta}_{(k_0,k_1)},\boldsymbol{\varepsilon}(\boldsymbol{\eta}_{(k_0,k_1)})} \dots w_{\boldsymbol{l}_{(k_j,k_{j+1})}}^{\boldsymbol{\eta}_{(k_j,k_{j+1})},\boldsymbol{\varepsilon}(\boldsymbol{\eta}_{(k_j,k_{j+1})})} T_{\boldsymbol{l}}^{\boldsymbol{\eta}}$$

For a given $K=\{k_1<\ldots<k_j\}\subset I$, it is clear that

$$\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_j,k_{j+1})}=\boldsymbol{\varepsilon}(\boldsymbol{\eta}_{(k_j,k_{j+1})})=\varepsilon_{\hat{n}_{k_j}},\ldots,\varepsilon_{\hat{n}_s}\quad(k_{j+1}=s)$$

but generally speaking, we can have

$$\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_{j_0},k_{j_0+1})}\neq\boldsymbol{\varepsilon}(\boldsymbol{\eta}_{(k_{j_0},k_{j_0+1})})$$

for $0\leq j_0<j$. We shall prove that $\tilde{U}^{\boldsymbol{\eta}}=U^{\boldsymbol{\eta}}$. Let $1\leq i\leq s$ and $i_r\in I$ (or $r=q+1$ and $i_{q+1}=s+1$) such that $i<i_r$. Let also

$$\mathcal{K}_{i,i_r}=\{K=\{k_1<\ldots<k_j\}\subset I\ ;\exists 1\leq j_0\leq j;\ k_{j_0}\leq i<k_{j_0+1}=i_r\}$$

For the partitions $K\in\mathcal{K}_{i,i_r}$, the i^{th} sign of $\tilde{\boldsymbol{\alpha}}_K=\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_0,k_1)}\ldots\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_j,k_{j+1})}=\boldsymbol{\varepsilon}(\boldsymbol{\eta})$ is $\varepsilon_{n_i+\ldots+n_s}$ whereas, in $\boldsymbol{\alpha}_K=\boldsymbol{\varepsilon}(\boldsymbol{\eta}_{(k_0,k_1)})\ldots\boldsymbol{\varepsilon}(\boldsymbol{\eta}_{(k_j,k_{j+1})})$ is $\varepsilon_{n_i+\ldots+n_{i_r-1}}$. If $r=q+1$ these signs are equal. Otherwise, if $\varepsilon_{n_i+\ldots+n_s}=+1$, since $\varepsilon_{n_{i_r}+\ldots n_s}=-1$, is implies that $\varepsilon_{n_i+\ldots+n_{i_r-1}}=+1$ so that the sign at i^{th} position in $\tilde{\boldsymbol{\alpha}}_K$ and $\boldsymbol{\alpha}_K$ is the same $(+1)$. But, if ever $\varepsilon_{n_i+\ldots+n_s}=-1$, we can't conclude : $\varepsilon_{n_i+\ldots+n_{i_r-1}}$ could be $+1$ or -1. The main fact to prove is that in $\tilde{U}^{\boldsymbol{\eta}}$, we can, if necessary, change $\varepsilon_{n_i+\ldots+n_s}=-1$ into $\varepsilon_{n_i+\ldots+n_{i_r-1}}=+1$, for the terms corresponding to $K\in\mathcal{K}_{i,i_r}$, without changing the value of $\tilde{U}^{\boldsymbol{\eta}}$. It is clear that, iterating this result proves finally that $\tilde{U}^{\boldsymbol{\eta}}=U^{\boldsymbol{\eta}}$. So, consider the case when $\varepsilon_{n_i+\ldots+n_s}=-1$ and $\varepsilon_{n_i+\ldots+n_{i_r-1}}=+1$ that can occur iff $r\leq q$ and $i=i_t\in I$ $(1\leq t<r)$. The terms where we want to change this sign are

$$\tilde{U}^{\boldsymbol{\eta}}_{i,i_r}=\sum_{\boldsymbol{l}\in\mathbb{Z}^s}\sum_{K=\{k_1<\ldots<k_j\}\subset\mathcal{K}_{i,i_r}}(-1)^{|K|}w_{\boldsymbol{l}_{(k_0,k_1)}}^{\boldsymbol{\eta}_{(k_0,k_1)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_0,k_1)}}\ldots w_{\boldsymbol{l}_{(k_j,k_{j+1})}}^{\boldsymbol{\eta}_{(k_j,k_{j+1})},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_j,k_{j+1})}}T_{\boldsymbol{l}}^{\boldsymbol{\eta}}$$

Once again, let $I_{t-1}=\{i_1<\ldots<i_{t-1}\}$ and ${}_rI=\{i_r<\ldots<i_q\}$. It is clear that $\mathcal{K}_{i_t,i_r}=\mathcal{K}^1_{i_t,i_r}\cup\mathcal{K}^2_{i_t,i_r}$ $(\mathcal{K}^1_{i_t,i_r}\cap\mathcal{K}^2_{i_t,i_r}=\emptyset)$ where

$$\mathcal{K}^1_{i_t,i_r}=\{K=K^1\cup K^2\ ;\ K^1=\{k^1_1<\ldots<k^1_{j_1}\}\subset I_{t-1}\ K^2=\{i_r=k^2_1<\ldots<k^2_{j_2}\}\subset{}_rI\ \}$$

$$\mathcal{K}^2_{i_t,i_r}=\{K=K'\cup\{i_t\}\ ;\ K'\in\mathcal{K}^1_{i_t,i_r}\}$$

thus, if

$$w_{K^{1\prime}}=w_{\boldsymbol{l}_{(k^1_0,k^1_1)}}^{\boldsymbol{\eta}_{(k^1_0,k^1_1)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k^1_0,k^1_1)}}\ldots w_{\boldsymbol{l}_{(k^1_{j_1-1},k^1_{j_1})}}^{\boldsymbol{\eta}_{(k^1_{j_1-1},k^1_{j_1})},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k^1_{j_1-1},k^1_{j_1})}}$$

and

$$w_{K^2}=w_{\boldsymbol{l}_{(k^2_1,k^2_2)}}^{\boldsymbol{\eta}_{(k^2_1,k^2_2)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k^2_1,k^2_2)}}\ldots w_{\boldsymbol{l}_{(k^2_{j_2},s)}}^{\boldsymbol{\eta}_{(k^2_{j_2},s)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k^2_{j_2},s)}}$$

then

$$\begin{aligned}\tilde{U}^{\boldsymbol{\eta}}_{i,i_r}\ =\ &\sum_{\boldsymbol{l}\in\mathbb{Z}^s}\sum_{\substack{K^1=\{k^1_1<\ldots<k^1_{j_1}\}\subset I_{t-1}\\K^2=\{i_r=k^2_1<\ldots<k^2_{j_2}\}\subset{}_rI}}(-1)^{|K^1|+|K^2|}w_{K^{1\prime}}w_{\boldsymbol{l}_{(k^1_{j_1},i_r)}}^{\boldsymbol{\eta}_{(k^1_{j_1},i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k^1_{j_1},i_r)}}w_{K^2}T_{\boldsymbol{l}}^{\boldsymbol{\eta}}\\&+\sum_{\substack{\boldsymbol{l}\in\mathbb{Z}^s\\K^1=\{k^1_1<\ldots<k^1_{j_1}\}\subset I_{t-1}\\K^2=\{i_r=k^2_1<\ldots<k^2_{j_2}\}\subset{}_rI}}(-1)^{|K^1|+|K^2|+1}w_{K^{1\prime}}w_{\boldsymbol{l}_{(k^1_{j_1},i_t)}}^{\boldsymbol{\eta}_{(k^1_{j_1},i_t)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k^1_{j_1},i_t)}}w_{\boldsymbol{l}_{(i_t,i_r)}}^{\boldsymbol{\eta}_{(i_t,i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k^1_{j_1},i_r)}}w_{K^2}T_{\boldsymbol{l}}^{\boldsymbol{\eta}}\end{aligned}$$

If $y_{\boldsymbol{l}_{(k_{j_1}^1,i_r)}}^{\boldsymbol{\eta}_{(k_{j_1}^1,i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_{j_1}^1,i_r)}} = \prod_{k_{j_1}^1 \le i \le i_r-1, i \ne i_t} \varepsilon_{\hat{n}_i}\rho_{\varepsilon_{\hat{n}_i}}(l_i - l_{i-1})$ (with $l_{k_{j_1}^1-1}=0$), then

$$\begin{aligned}
w_{\boldsymbol{l}_{(k_{j_1}^1,i_r)}}^{\boldsymbol{\eta}_{(k_{j_1}^1,i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_{j_1}^1,i_r)}} &= y_{\boldsymbol{l}_{(k_{j_1}^1,i_r)}}^{\boldsymbol{\eta}_{(k_{j_1}^1,i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_{j_1}^1,i_r)}} \varepsilon_{\hat{n}_{i_t}}\rho_{\varepsilon_{\hat{n}_{i_t}}}(l_{i_t} - l_{i_t-1}) \\
&= -\rho_{-1}(l_{i_t} - l_{i_t-1})\, y_{\boldsymbol{l}_{(k_{j_1}^1,i_r)}}^{\boldsymbol{\eta}_{(k_{j_1}^1,i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_{j_1}^1,i_r)}} \\
&= \rho_1(l_{i_t} - l_{i_t-1})\, y_{\boldsymbol{l}_{(k_{j_1}^1,i_r)}}^{\boldsymbol{\eta}_{(k_{j_1}^1,i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_{j_1}^1,i_r)}} - y_{\boldsymbol{l}_{(k_{j_1}^1,i_r)}}^{\boldsymbol{\eta}_{(k_{j_1}^1,i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_{j_1}^1,i_r)}}
\end{aligned}$$

and

$$\begin{aligned}
w_{\boldsymbol{l}_{(k_{j_1}^1,i_t)}}^{\boldsymbol{\eta}_{(k_{j_1}^1,i_t)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_{j_1}^1,i_t)}} w_{\boldsymbol{l}_{(i_t,i_r)}}^{\boldsymbol{\eta}_{(i_t,i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_{j_1}^1,i_r)}} &= y_{\boldsymbol{l}_{(k_{j_1}^1,i_r)}}^{\boldsymbol{\eta}_{(k_{j_1}^1,i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_{j_1}^1,i_r)}} \varepsilon_{\hat{n}_{i_t}}\rho_{\varepsilon_{\hat{n}_{i_t}}}(l_{i_t}) \\
&= -\rho_{-1}(l_{i_t})\, y_{\boldsymbol{l}_{(k_{j_1}^1,i_r)}}^{\boldsymbol{\eta}_{(k_{j_1}^1,i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_{j_1}^1,i_r)}} \\
&= \rho_1(l_{i_t})\, y_{\boldsymbol{l}_{(k_{j_1}^1,i_r)}}^{\boldsymbol{\eta}_{(k_{j_1}^1,i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_{j_1}^1,i_r)}} - y_{\boldsymbol{l}_{(k_{j_1}^1,i_r)}}^{\boldsymbol{\eta}_{(k_{j_1}^1,i_r)},\boldsymbol{\varepsilon}(\boldsymbol{\eta})_{(k_{j_1}^1,i_r)}}
\end{aligned}$$

thus, using this in the expression $\tilde{U}_{i,i_r}^{\boldsymbol{\eta}}$ we changed $\varepsilon_{\hat{n}_{i_t}} = -1$ into $\varepsilon_{n_{i_t}+\ldots+n_{i_r}} = +1$ without changing the value of $\tilde{U}_{i,i_r}^{\boldsymbol{\eta}}$ and thus of $\tilde{U}^{\boldsymbol{\eta}}$. This ends the proof of the theorem.

□

7.3 Proof of Proposition 10.

Proposition 10. Let $0 \le |x| \le \varepsilon \le q^{-1/2}$. For $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_s) \in \boldsymbol{H}^+$,

$$|U^{\boldsymbol{\eta}}(x)| \le C_q^s |x|^{\|\boldsymbol{n}\|+\|\boldsymbol{\sigma}\|} q^{-(\|\boldsymbol{n}\|+\|\boldsymbol{\sigma}\|)} \tag{107}$$

with $C_q = \dfrac{1}{1-q^{-1/2}}$.

Proof. We remind that, for $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_s) \in \boldsymbol{H}^+$

$$T_{\boldsymbol{l}}^{\boldsymbol{\eta}}(x) = x^{\|\boldsymbol{\sigma}\|+l_1 n_1+\ldots+l_s n_s} q^{-l_1\sigma_1-\ldots-l_s\sigma_s} [l_1+1]_q^{n_1} \ldots [l_s+1]_q^{n_s} = \prod_{i=1}^{s} T_{l_i}^{\eta_i}(x)$$

and

$$U^{\boldsymbol{\eta}}(x) = \varepsilon_{\boldsymbol{n}} \sum_{\boldsymbol{l} \in \mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1)\rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1)\ldots\rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1}) \left(\prod_{\substack{1 \le i \le s \\ \varepsilon_{\hat{n}_i}=-1}} \rho_1(l_i) \right) T_{\boldsymbol{l}}^{\boldsymbol{\eta}}(x)$$

thus

$$|U^{\boldsymbol{\eta}}(x)| \le |U^{\boldsymbol{\eta}}(|x|)| = \sum_{\boldsymbol{l} \in \mathbb{Z}^s} \rho_{\varepsilon_{\hat{n}_1}}(l_1)\rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1)\ldots\rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1}) \left(\prod_{\substack{1 \le i \le s \\ \varepsilon_{\hat{n}_i}=-1}} \rho_1(l_i) \right) T_{\boldsymbol{l}}^{\boldsymbol{\eta}}(|x|)$$

Let

$$R_l^{\boldsymbol{\eta}} = \rho_{\varepsilon_{\hat{n}_1}}(l_1)\,\rho_{\varepsilon_{\hat{n}_2}}(l_2 - l_1)\ldots\rho_{\varepsilon_{\hat{n}_s}}(l_s - l_{s-1})\left(\prod_{\substack{1\le i\le s\\ \varepsilon_{\hat{n}_i}=-1}} \rho_1(l_i)\right)$$

and $L^{\boldsymbol{\eta}} = \{\boldsymbol{l} \in \mathbb{Z}^s\ ;\ R_{\boldsymbol{l}}^{\boldsymbol{\eta}} = 1\} \subset \mathbb{Z}^s$. First of all, one can notice that

$$L^{\boldsymbol{\eta}} \subset (\mathbb{N}^*)^s \tag{108}$$

because, for $1 \le i \le s$, if $\varepsilon_{\hat{n}_i} = -1$, then the factor $\rho_1(l_i)$ in $R_l^{\boldsymbol{\eta}}$ ensures that $l_i > 0$ and if $\varepsilon_{\hat{n}_i} = +1$, then

- if $i = 1$ then $\rho_1(l_1) = 1$ and $l_1 > 0$,
- if $i > 1$ and all the signs $\varepsilon_{\hat{n}_j}$ $(j \le i)$ are $+1$, then $l_1 > 0$, $l_2 > l_1$, ..., $l_i > l_{i-1}$ thus $l_i > 0$,
- otherwise $i > 1$ and there exist $i_0 = \max\{i; \varepsilon_{\hat{n}_i} = -1\}$. Thanks to the factor $\rho_1(l_{i_0})$, once again, $l_{i_0} > 0$, $l_{i_0+1} > l_{i_0}$, ..., $l_i > l_{i-1}$ and $l_i > 0$.

Based on a similar discussion, we get that

$$\forall \boldsymbol{\eta} = (\eta_1, \ldots, \eta_s) \in \boldsymbol{H}^+, \quad n(\boldsymbol{\eta}) = \min_{\boldsymbol{l} \in L^{\boldsymbol{\eta}}} (n_1 l_1 + \ldots + n_s l_s) \ge (n_1 + \ldots + n_s) \ge 0 \tag{109}$$

and

$$\forall \boldsymbol{\eta} = (\eta_1, \ldots, \eta_s) \in \boldsymbol{H}^+, \quad \begin{array}{rcl} p_q(\boldsymbol{\eta}) & = & \max\limits_{\boldsymbol{l} \in L^{\boldsymbol{\eta}}} ([l_1+1]_q^{n_1} \ldots [l_s+1]_q^{n_s}) \\ & \le & [1]_q^{n_1+\ldots+n_s} = q^{-\|\boldsymbol{n}\|} \end{array} \tag{110}$$

These inequalities can be proved by induction on the length s of the sequence $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_s) \in \boldsymbol{H}^+$. These results are obvious for $\boldsymbol{\eta} = (\eta_1) \in \boldsymbol{H}^+$ since $l_1 > 0$. Let $s \ge 2$, we remind that $L^{\boldsymbol{\eta}} \subset (\mathbb{N}^*)^s$ thus,

- If $\varepsilon_{\hat{n}_s} = +1$, then $n_s \ge 0$ and $l_s > l_{s-1} > 0$. It implies that $n_s l_s \ge n_s l_{s-1}$ and $[l_s+1]_q^{n_s} \le [l_{s-1}+1]_q^{n_s}$ $(l_s(l_s+1) \ge l_{s-1}(l_{s-1}+1))$ so

$$\begin{array}{rcl} n((\eta_1, \ldots, \eta_s)) & \ge & n((\eta_1, \ldots, \eta_{s-1} + \eta_s)) \\ p_q((\eta_1, \ldots, \eta_s)) & \le & p_q((\eta_1, \ldots, \eta_{s-1} + \eta_s)) \end{array} \tag{111}$$

- If $\varepsilon_{\hat{n}_s} = -1$, then $n_s < 0$ and $0 < l_s \le l_{s-1}$. It implies that $n_s l_s \ge n_s l_{s-1}$ and $[l_s+1]_q^{n_s} \le [l_{s-1}+1]_q^{n_s}$ $(l_s(l_s+1) \le l_{s-1}(l_{s-1}+1))$ so, once again

$$\begin{array}{rcl} n((\eta_1, \ldots, \eta_s)) & \ge & n((\eta_1, \ldots, \eta_{s-1} + \eta_s)) \\ p_q((\eta_1, \ldots, \eta_s)) & \le & p_q((\eta_1, \ldots, \eta_{s-1} + \eta_s)) \end{array}$$

Since $(\eta_1, \ldots, \eta_{s-1} + \eta_s) \in \boldsymbol{H}^+$, the induction is then trivial. This result implies that, for $\boldsymbol{l} \in L^{\boldsymbol{\eta}}$,

$$\begin{array}{rcl} T_{\boldsymbol{l}}^{\boldsymbol{\eta}}(|x|) & \le & q^{-\|\boldsymbol{n}\|} |x|^{\|\boldsymbol{\sigma}\|} |x|^{l_1 n_1 + \ldots + l_s n_s} q^{-l_1 \sigma_1 - \ldots - l_s \sigma_s} \\ & \le & |x|^{\|\boldsymbol{\sigma}\| + \|\boldsymbol{n}\|} q^{-\|\boldsymbol{\sigma}\| - \|\boldsymbol{n}\|} |x|^{l_1 n_1 + \ldots + l_s n_s - \|\boldsymbol{n}\|} \prod\limits_{i=1}^{s} q^{-\sigma_i (l_i - 1)} \\ & \le & |x|^{\|\boldsymbol{\sigma}\| + \|\boldsymbol{n}\|} q^{-\|\boldsymbol{\sigma}\| - \|\boldsymbol{n}\|} \prod\limits_{i=1}^{s} |x|^{n_i (l_i - 1)} q^{-\sigma_i (l_i - 1)} \end{array}$$

but, if $|x| \leq q^{-1/2}$, since $l_1 n_1 + \ldots + l_s n_s - \|\boldsymbol{n}\| \geq 0$,

$$\begin{aligned} T_{\boldsymbol{l}}^{\boldsymbol{\eta}}(|x|) &\leq |x|^{\|\boldsymbol{\sigma}\|+\|\boldsymbol{n}\|} q^{-\|\boldsymbol{\sigma}\|-\|\boldsymbol{n}\|} (q^{-1/2})^{l_1 n_1+\ldots+l_s n_s-\|\boldsymbol{n}\|} \prod_{i=1}^{s} q^{-\sigma_i(l_i-1)} \\ &\leq |x|^{\|\boldsymbol{\sigma}\|+\|\boldsymbol{n}\|} q^{-\|\boldsymbol{\sigma}\|-\|\boldsymbol{n}\|} \prod_{i=1}^{s} q^{-1/2(n_i+2\sigma_i)(l_i-1)} \end{aligned}$$

and, as for $\boldsymbol{\eta} \in \boldsymbol{H}^{+}$ and $\boldsymbol{l} \in L^{\boldsymbol{\eta}} \in (\mathbb{N}^*)^s$, $n_i + 2\sigma_i \geq 1$,

$$T_{\boldsymbol{l}}^{\boldsymbol{\eta}}(|x|) \leq |x|^{\|\boldsymbol{\sigma}\|+\|\boldsymbol{n}\|} q^{-\|\boldsymbol{\sigma}\|-\|\boldsymbol{n}\|} \prod_{i=1}^{s} q^{-1/2(l_i-1)}$$

and, finally

$$\begin{aligned} |U^{\boldsymbol{\eta}}(|x|)| &= \sum_{\boldsymbol{l} \in L^{\boldsymbol{\eta}}} T_{\boldsymbol{l}}^{\boldsymbol{\eta}}(|x|) \\ &\leq |x|^{\|\boldsymbol{\sigma}\|+\|\boldsymbol{n}\|} q^{-\|\boldsymbol{\sigma}\|-\|\boldsymbol{n}\|} \sum_{\boldsymbol{l} \in L^{\boldsymbol{\eta}}} \prod_{i=1}^{s} q^{-1/2(l_i-1)} \\ &\leq |x|^{\|\boldsymbol{\sigma}\|+\|\boldsymbol{n}\|} q^{-\|\boldsymbol{\sigma}\|-\|\boldsymbol{n}\|} \sum_{\boldsymbol{l} \in (\mathbb{N}^*)^s} \prod_{i=1}^{s} q^{-1/2(l_i-1)} \\ &\leq |x|^{\|\boldsymbol{\sigma}\|+\|\boldsymbol{n}\|} q^{-\|\boldsymbol{\sigma}\|-\|\boldsymbol{n}\|} C_q^s \end{aligned}$$

□

8 Conclusion.

We shall prove in a forthcoming paper that this result still holds for systems of nonlinear q-difference equations. This means that q-difference nonlinear equations are almost always conjugated to linear q-difference equations : This situation is totally different from the one encountered in the case of singular irregular equations.

We did not say much about the solutions of the equation

$$(x\sigma_q - 1)y = b(y, x)$$

but, since it is analytically conjugated to an equation

$$x\sigma_q z = z + \alpha(b)x$$

it would be sufficient to study the "small" solutions of such an equation, which can be obtained by q-resummation of the formal solution :

$$\tilde{z}(x) = -\alpha(b) \sum_{n \geq 1} q^{-n(n-1)/2} x^n$$

See [3] for details. One should also deduce some results on the Stokes phenomenon.

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