APPROXIMATE SHOCK CURVES OF NON-CONSERVATIVE HYPERBOLIC EQUATIONS IN ONE SPACE DIMENSION

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Abstract

Abstract. In the case of non-conservative hyperbolic systems, several definitions of shock waves exist in the literature. In this paper, we propose a new, very simple one in the case of genuinely non-linear fields. On the other hand, taking a vanishing viscosity process, we prove the existence of shock curves when one considers viscosities given by a matrix which commutes with the matrix which defines the original system of equations. This setting generalizes a recent proof given by Bianchini and Bressan. At the end of the paper we prove that both definitions agree to third order near a given left state.

1 Introduction

We consider the strictly hyperbolic system in one space dimension

$$u_t + A(u)u_x = 0, \qquad u(x,t) \in \mathcal{U}, \qquad \forall t > 0,\, x \in \mathbf{R}, \tag{1}$$

where $\mathcal{U}$ is an open convex set of $\mathbf{R}^n$ and A a $n \times n$ strictly hyperbolic matrix with regular real coefficients.

Throughout this paper we will note $(\lambda_1(u), \ldots, \lambda_n(u))$ the n distinct eigenvalues of $A(u)$, and $(r_1(u), \ldots, r_n(u))$ a regular field of associated right eigenvectors such that $|r_j(u)| = 1$. Systems like (1) are called non-conservative and naturally arise in e.g. multiphasic flows modelizations (see [5] for instance). The mathematical theory of existence, uniqueness, regularity of their solutions remains quite open compared to systems of conservation laws

$$u_t + f(u)_x = 0, \qquad u(x,t) \in \mathcal{U}, \qquad \forall t > 0,\, x \in \mathbf{R}. \tag{2}$$

Indeed, in (2) solutions do exist (at least weakly) and discontinuities may appear in finite time. We refer to [10] for the theory concerning this system and in particular the notions of strict hyperbolicity, genuine non linearity, shock profiles and the Riemann problem. In systems such as (1), there is a lack of a natural definition for the non conservative product $A(u)u_x$ when u possesses discontinuities, whereas a term like $f(u)_x$ has a meaning in the sense of distributions.

Althought the theory is still quite open, systems written in the non-conservative form (1) have been recently studied in two different directions. The first one is given by Dal Maso, LeFloch and Murat in [4] and has been used by Le Floch and Tzavaras [7]. The idea is to define a family of paths

$$\phi : [0,1] \times \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}^n,$$

which satisfies $\phi(0, u^-, u^+) = u^-$ and $\phi(1, u^-, u^+) = u^+$. Then the term $A(u)u_x$ when u is a step function

$$u(x) = \begin{cases} u^- \text{ if } x < 0, \\ u^+ \text{ if } x > 0, \end{cases}$$

is understood as $\left(\int_0^1 A(\phi(\tau, u^-, u^+))\phi'(\tau, u^-, u^+)\, d\tau \right) \delta_0$, where δ_0 stands for the Dirac mass at 0. Furthermore, admissible paths must satisfy the Rankine-Hugoniot like condition

$$\int_0^1 A(\phi(\tau, u^-, u^+))\phi'(\tau, u^-, u^+)\, d\tau = s(u^+ - u^-). \tag{3}$$

It has been remarked [6] that shock waves in that sense are indeed vanishing viscosity limits of traveling-waves solutions to

$$u_t + A(u)u_x = \varepsilon \left(B(u)u_x\right)_x, \qquad \forall (x,t) \in \mathbf{R} \times \mathbf{R}_+, \tag{4}$$

when the viscosity ε tends to 0, provided that the paths ϕ is correctly chosen (defined by the viscous shock profiles, associated to B). The weakness of this theory is precisely that it does not give any "natural" or "predefined" path in the case where B is unknown.

On the other hand, using a regularizing technique, Bianchini and Bressan [2] have yet more recently defined solutions to (1) as unique limits of solutions to the viscous system

$$u_t + A(u)u_x = \varepsilon u_{xx}, \qquad \forall (x,t) \in \mathbf{R} \times \mathbf{R}_+, \tag{5}$$

when the viscosity ε tends to 0.

In particular, in this very technical work, they solve the Riemann problem associated to the original system for small values of $|u^+ - u^-|$

$$\begin{cases} u(x,0) = u^{\pm}, & \text{if } x \gtrless 0, \\ u = \lim\limits_{\varepsilon \to 0} u^{\varepsilon}, & \text{in } \mathbf{L}^1_{loc}, \qquad \text{where } u^{\varepsilon} \text{ solves (5).} \end{cases}$$

In this general setting, they recover the classical succession of auto-similar i-waves and give a characterization of them. As in the conservative case, with a genuine nonlinear hypothesis on the i-th eigenvalue, these elementar i-waves are shocks or rarefaction waves. Here, the weakness lies in the fact that their study is strictly limited to this kind of diffusion. In particular, the vanishing viscosity limit of (4) when the viscosity ε tends to 0 is not given by their theory.

Going further, traveling wave solutions of (4) have been shown to exist [8, 9] with very general viscosity matrices $B(u)$ in the neighborhood of a left state u^- and for speeds in a neighborhood of $\lambda_i(u^-)$ provided the i-characteristic field is genuinely nonlinear and with a few more non very restrictive assumptions.

Our aim is the numerical resolution of (1) and more precisely to construct a Riemann solver adapted to the numerical computation. In the non-conservative case, the classical notions of rarefaction waves or of i-waves when λ_i is linearly degenerate remain valid. But for λ_i genuinely nonlinear, the definition of i-shocks is not clear.

On the other hand, if $A = d_u f$, we know that a shock wave with left and right states (u^-, u^+) and speed σ solves the Rankine-Hugoniot condition

$$f(u^+) - f(u^-) = \sigma(u^+ - u^-). \tag{6}$$

Therefore the i-th shock curve starting from u^-: $u^+(\sigma) := \mathcal{S}_i(u^-, \sigma)$ may be parametrized by σ and satisfies (6). Differentiating (6) with respect to σ we get the ordinary differential system

$$\begin{cases} (A(u^+) - \sigma)u^+_{\sigma} &= u^+ - u^-, \\ u^+(\lambda_i(u^-)) &= u^-. \end{cases} \tag{7}$$

As remarked by [1], this last characterization of the shock curves does not rely on the fact that the system is in conservative form, therefore it is tempting to use it as a definition of shock curves in the non conservative case and set the following definition.

Definition 1 *We call approximate i-shock curve of (1) a nonconstant solution to (7) for* σ *in a neighborhood of* $\lambda_i(u^-)$.

We stress on the fact that this definition does not make any assumption on any kind of viscosity in the original system nor it gives the freedom to specify any path in the discontinuities. Nonetheless, from the computation above, we immediately recover the exact shock curves when the system is actually conservative.

In this paper, we give several arguments to justify the definition 1. First, we prove that solutions to the ODE (7) do actually exist (locally in a neighborhood of $\lambda_i(u^-)$). Then, we will compare our approximate shock curves with the shock curves found by a vanishing viscosity limit process. In that aim, let us introduce the following definition.

Definition 2 *Let B be a positive $n \times n$ matrix with constant coefficients, u is a vanishing viscosity solution to (1) associated to B if $u := \lim_{\varepsilon \to 0} u^\varepsilon$ in $\mathbf{L}^1_{loc}$ where*

$$u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x = \varepsilon B u^\varepsilon_{xx}, \qquad \forall (x,t) \in \mathbf{R} \times \mathbf{R}_+. \tag{8}$$

First, let us see if we could formally recover (7) in the setting of a general quasilinear hyperbolic system (1) with vanishing viscosity associated to B. In addition to a curve of final states $u^+(\sigma)$, we need a family of self-similar i-shock profiles.

Let $u^- \in \mathcal{U}$, and suppose we have a family $(U(\cdot,\sigma))_\sigma$, $\sigma \in (\lambda_i(u^-) - \alpha, \lambda_i(u^-) + \alpha)$ such that $U(\cdot,\sigma)$ solves

$$\left\{ \begin{array}{l} (A(U) - \sigma)U_x(x,\sigma) = (B(U)U_x)_x \qquad \forall x \in \mathbf{R}, \\ U(-\infty,\sigma) = u^- \qquad \forall \sigma \in (\lambda_i(u^-) - \alpha, \lambda_i(u^-) + \alpha), \\ U(\cdot, \lambda_i(u^-)) \equiv u^-. \end{array} \right. \tag{9}$$

For such a family of profiles, the map $U^\varepsilon(x,t,\sigma) := U\left(\frac{x-\sigma t}{\varepsilon}\right)$ solves (8) for all $\varepsilon > 0$.

Therefore $U(\cdot,\sigma)$ is a family of i-shock profiles starting from u^- with right states $u^+(\sigma) := \lim_{x\to+\infty} U(x,\sigma)$. Integrating (9) along the profile gives

$$\sigma(u^+ - u^-) = \int_{\mathbf{R}} A(U)U_x dx. \tag{10}$$

Differentiating this equation with respect to σ and integrating by parts yields

$$(A(u^+) - \sigma)u^+_\sigma(\sigma) \;=\; u^+ - u^- + \int_{\mathbf{R}} A(U)_x U_\sigma - A(U)_\sigma U_x dx, \tag{11}$$

and we recover (7) up to the default term

$$R(U,\sigma) := \int_{\mathbf{R}} A(U)_x U_\sigma - A(U)_\sigma U_x dx,$$

that we hope to be small at least for small jumps. Of course, if A is in fact the differential of a flux f, then $R(U,\sigma)$ clearly vanishes. We also remark that if the shock profiles are all on one fixed curve then R also vanishes. In that case, the shock curve is precisely this fixed curve. Therefore a natural way of finding the ordinary differential equation (7) is to assume that shock profiles and the shock curve are identical. We may also integrate (7) with respect to σ and obtain

$$\int_{\lambda_i(u^-)}^{\tau} A(u^+(\sigma))u^+_\sigma \, d\sigma = \int_{\lambda_i(u^-)}^{\tau} \frac{d}{d\sigma}\big(\sigma(u^+(\sigma)-u^-)\big)\, d\sigma = \tau(u^+(\tau)-u^-), \tag{12}$$

which means that we recover the correct Rankine-Hugoniot condition (3) along such profiles.

In addition to this quantitative difference (compare (7) ans (11)), there is a qualitative difference between the solutions of (7) and the i-shock curves. The space transformation $x \to -x$ changes the approximate i-shock curves. Namely, in the construction of an approximate shock curve, we have chosen to fix the left state but it could have been possible also to fix a right state u^+ and to define an approximate shock curve of left states $v(\sigma)$ by

$$\left\{\begin{array}{lcl} v(\lambda_i(u^+)) & = & u^+, \\ (A(v(\sigma))-\sigma)v_\sigma & = & u^+ - v. \end{array}\right. \tag{13}$$

Now, let $u^- \in \mathcal{U}$ and $u(\sigma)$ be a solution of (7), we pick up a speed σ^+ and note $u^+ := u(\sigma^+)$, then we construct v the approximate i-shock curve of left states satisfying (13). Unfortunately, contrarily to the conservative case, there is no reason that the equality $v(\sigma^+) = u^-$ holds. This point should be investigated in the near future.

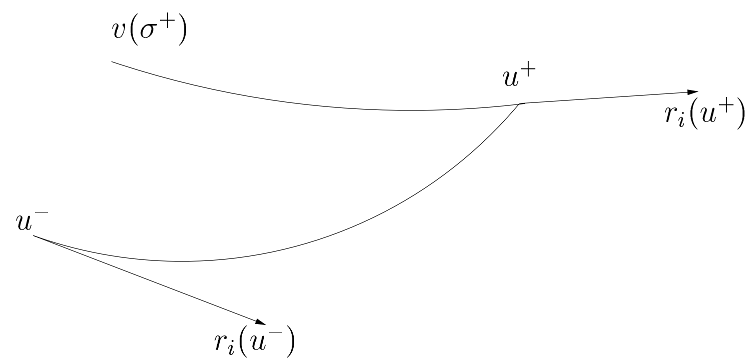

Figure 1: approximate i-shock curves: shocks starting from u^-, shocks ending at u^+.

The paper is organized as follows. In the next section, we prove the existence and uniqueness of approximate i-shock curves. Section 3 is devoted to the definition of shock curves according to the technique of Bianchini and Bressan. In particular, if no assumption is made about the genuine non-linearity of the i-th characteristic field, we give the setting to generalize their method to viscosities $B(u)$ which commute with $A(u)$. For the sake of simplicity, we give an alternate proof that supposes the i-th characteristic field to be genuinely non-linear. In section 4 we will estimate the term $R(\sigma)$ for σ close to the i-th characteristic speed $\lambda_i(u^-)$ and conclude by a comparison between approximate and true i-shock curves.

2 Existence and uniqueness of the approximate $i-$shock curve.

We first study the ordinary differential equation (7). The existence of solutions is not classical because of the degeneracy at $\sigma = \lambda_i(u^-)$. Moreover this equation does not apparently prescribe the first derivative of its solutions around this point. Let us first find what are the possible values of the first derivatives. Without loss of generality, we may suppose $i = 1$, $u^- = 0$ and $\lambda_1(0) = 0$. Then, we decompose the vector u in the basis $\mathcal{B} = (r_1(0), \cdots, r_n(0))$ and write $u = (v, w)$ with $v := u_1$ and $w := (u_2, \cdots, u_n)$. Finally, we write A in the basis $\mathcal{B}$:

$$A(v,w) := \begin{pmatrix} a(v,w) & b(v,w) \\ C(v,w) & D(v,w) \end{pmatrix},$$

where a is a real, b and C are respectively $(n-1)$-line and column vectors and D is a $(n-1)\times(n-1)$-square matrix. In particular, for $\sigma = 0$, we have $a(v,w) = 0$, $b(v,w) = 0$, $C(v,w) = 0$ and

$$D(v,w) \quad = \quad \begin{pmatrix} \lambda_2(0) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n(0) \end{pmatrix}.$$

We stress on the fact that D is invertible in the neighborhood of 0. Let us notice that $r_1 \cdot \nabla\lambda_1(0) = a_v(0)$, so that if we assume that A is $\mathbf{C}^1$ and that the i-th eigenvalue of A is genuinely nonlinear, we get $a_v(0) \neq 0$. In these coordinates $u(\sigma) = (v(\sigma), w(\sigma))$ is a solution of (7) if and only if

u is a differentiable function defined in a neighborhood I of 0 such that $v(0)=0$, $w(0)=0$ and $\forall\sigma\in I$,

$$\left\{\begin{array}{rcl} a(v,w)v_\sigma+\langle b(v,w);w_\sigma\rangle &=& (\sigma v)_\sigma \qquad (14)\\ C(v,w)v_\sigma+D(v,w)w_\sigma &=& (\sigma w)_\sigma. \qquad (15)\end{array}\right.$$

Only (14) is degenerate. From (15) and $w(0)=0$, we get $w_\sigma(0)=0$ and by Gronwall's lemma $w_\sigma=\mathcal{O}(\sigma v_\sigma)+\mathcal{O}(w)$, with $w=\mathcal{O}(\int_0^\sigma \tau v_\sigma(\tau))$. Keeping in mind these estimates, we write $v=v_\sigma(0)\sigma+o(\sigma)$ in (14), and we get

$$(a_v(0)v_\sigma(0)-1+o(1))\,\sigma v_\sigma=v+\mathcal{O}(\sigma w). \tag{16}$$

Dividing this equation by σ and letting σ go to 0, we are led to

$$\lim_{\sigma\to 0}v_\sigma(\sigma)=\frac{v_\sigma(0)}{a_v(0)v_\sigma(0)-1}. \tag{17}$$

Since v_σ has a limit at 0, we deduce that v must be of class $\mathbf{C}^1$ at 0, and (17) reads

$$v_\sigma(0)(a_v(0)v_\sigma(0)-2)=0. \tag{18}$$

We now consider the two cases.

Case 1: If $v_\sigma(0)=0$. Since v_σ is continuous at 0, integrating (14), gives $v=o(\sigma\sup_{|\tau|\leq|\sigma|}|v_\sigma(\tau)|)$ and replacing this estimate in (16), yields $\sigma v_\sigma=o(\sigma\sup_{|\tau|\leq|\sigma|}|v_\sigma(\tau)|)$. Therefore $v\equiv 0$ and from the non degeneracy of D, $w\equiv 0$.

Case 2: If $v_\sigma(0)\neq 0$, we deduce $v_\sigma(0)=\dfrac{2}{a_v(0)}$.

The first case is the null solution and the second case corresponds to the solution we are looking for. Let us show that this solution exists and that there is no other one.

Proposition 1 *Let $A:\mathcal{U}\to\mathbf{R}^{n^2}$ be a $\mathbf{C}^2$ function such that for all $u\in\mathcal{U}$, $A(u)$ is a strictly hyperbolic matrix with eigenvalues $(\lambda_1(u),\cdots,\lambda_n(u))$ and corresponding eigenvectors $(r_1(u),\cdots,r_n(u))$, $|r_i(u)|=1$. We suppose that for a given $i\in(1,\cdots,n)$ and $u^-\in\mathcal{U}$, $r_i\cdot\nabla\lambda_i(u^-)\neq 0$ (i.e: the i-th eigenvalue is genuinely non linear) then (7) has a local unique non trivial solution u in the neighborhood of $\lambda_i(u^-)$. Moreover*

$$u-u^-=\frac{2}{r_i\cdot\nabla\lambda_i(u^-)}(\sigma-\lambda_i(u^-))+o(\sigma-\lambda_i(u^-)).$$

*Proof :*We suppose again $i=1$, $u^-=0$ and $\lambda_1(0)=0$ and we will use the same notations as before.

If the system was conservative (in the form (2)), then u would solve the following system

$$\left\{\begin{array}{rcl} f^1(v(\sigma),w(\sigma)) & = & \sigma v(\sigma) \qquad (19)\\ \tilde{f}(v(\sigma),w(\sigma)) & = & \sigma w(\sigma) \qquad (20)\end{array}\right.$$

where $v:=u^1$, $w:=(u^2,\cdots,u^n)$ and $\tilde{f}:=(f^2,\cdots,f^n)$.

We fix (σ,v) close to $(0,0)$, since $\mathrm{d}_w\tilde{f}$ is invertible, using the implicit function theorem, we may solve w with (20) as $w=W(v,\sigma)$. Now a non trivial solution of (19,20) $(\sigma,v,W(v,\sigma))$ would solve

$$F(v,\sigma):=\frac{f^1(v,W(v,\sigma))}{v}-\sigma=0 \qquad (21)$$

and thanks to the the genuine non linearity hypothesis, it is easily seen that d_vF is invertible. Using again the implicit function theorem, the solution v to (21) may be defined as a function of σ. Hence, in a neighbourhood of $(0,0)$, (u,σ) solves (6) with $u\not\equiv 0$ if and only if $u=(v(\sigma),W(v(\sigma),\sigma))$.

We will follow the same strategy in the general case. Let δ be a positive real such that $\delta<\dfrac{1}{a_v(0)}$, we define

$$\begin{array}{l} E_w:=\{w\in\mathbf{C}^1((-1,1),\,\mathbf{R}^{n-1})\,;\ w(0)=0\}\,,\\ E_v:=\{v\in\mathbf{C}^1((-1,1),\,\mathbf{R})\,;\ v(0)=0\}\,,\\ \Omega_\delta:=\{v\in E_v\,;\ ||v_\sigma-v^0_\sigma||_\infty<\delta\}\ \text{ where } v^0(\sigma):=\dfrac{2}{a_v(0)}\sigma.\end{array}$$

Let u be a solution to (7) and $u_\varepsilon=\frac{1}{\varepsilon}u(\varepsilon\sigma)$. Writing the system under the form (14), (15), $u_\varepsilon=(v,w)$ verifies

$$\left\{\begin{array}{rcl} a(\varepsilon v,\varepsilon w)v_\sigma+\langle b(\varepsilon v,\varepsilon w);w_\sigma\rangle & = & \varepsilon(\sigma v)_\sigma \qquad (22)\\ (\varepsilon v,\varepsilon w)v_\sigma+D(\varepsilon v,\varepsilon w)w_\sigma & = & \varepsilon(\sigma w)_\sigma. \qquad (23)\end{array}\right.$$

Our aim is to solve (22,23) for small ε with (v,w) in $\Omega_\delta\times E_w$. We first define

$$\begin{array}{rcl} W\ :\ \Omega_\delta\times(-\alpha,\alpha) & \longrightarrow & E_w\\ (v,\varepsilon) & \longmapsto & W(v,\varepsilon)\end{array}$$

by $w:=W(v,\varepsilon)$ is such that (23) holds, i.e.

$$\left\{\begin{array}{rcl} w(0) & = & 0\\ w_\sigma & = & (D(\varepsilon v,\varepsilon w)-\varepsilon\sigma)^{-1}(\varepsilon w-C(\varepsilon v,\varepsilon w)v_\sigma).\end{array}\right.$$

Since D is invertible in the neighborhood of $(0,0)$, we see that W is well defined. Moreover this is a $\mathbf{C}^1$ function with respect to the $\mathbf{C}^1$ norms. Let us also remark that $W(v,0)=0$.

Then, we define

$$F \,:\, \Omega_\delta \times (-\alpha, \alpha) \quad \longrightarrow \quad E_v$$

by

$$F(v,\varepsilon)(\sigma) := \begin{cases} \dfrac{1}{\varepsilon v(\sigma)} \displaystyle\int_0^\sigma (a(\varepsilon v,\, \varepsilon W) v_\sigma + \langle b(\varepsilon v,\, \varepsilon W); W_\sigma \rangle)\; d\tau - \sigma & \text{if } \varepsilon \neq 0, \\ \dfrac{a_v(0,0) v(\sigma)}{2} - \sigma & \text{if } \varepsilon = 0, \end{cases}$$

where we have set $W = W(v,\varepsilon)$ to lighten the notations.

We notice that for $v \in \Omega_\delta$ and a fixed small real ε, $(v,\, w := W(v,\varepsilon))$ solves (22) if and only if $F(v,\varepsilon) = 0$.

We compute

$$F(v_0, 0) = 0,\;\; D_v F(v_0, 0)(\delta v) = \frac{a_v(0,0)}{2} \delta v$$

and $D_v F(v_0, 0)$ is invertible. Assuming that F is a $\mathbf{C}^1$ function, we can apply the implicit function theorem in Banach spaces and there is a $\mathbf{C}^1$ function V and a sufficiently small $\alpha > 0$ so that: $\forall (v,w) \in \Omega_\delta \times E_w$, $\forall \varepsilon \in (-\alpha,\, \alpha)$,

$$(v, w, \varepsilon) \text{ solves } (22, 23) \quad \Leftrightarrow \quad (v = V(\varepsilon) \text{ and } w = W(V(\varepsilon), \varepsilon))\,.$$

We conclude that (7) has a unique non locally constant $\mathbf{C}^1$ solution in the neighborhood of 0.

Proving that $F : \Omega_\delta \times (-\alpha,\, \alpha) \to E_w$ is $\mathbf{C}^1$ with respect to $\mathbf{C}^1$ norm is tedious but direct and not difficult. This follows from the fact that

$$\begin{array}{rcl} \Omega_\delta & \longrightarrow & C^0((-1,1)) \\ v & \longmapsto & \left(\sigma \mapsto \dfrac{v(\sigma)}{\sigma}\right) \end{array}$$

is a C^1 function with respect to C^1 norm.

3 Construction of shock waves with the center manifold technique of Bianchini and Bressan.

In [2] S. Bianchini and A. Bressan solve the Riemann problem with left and right states (u^-, u^+)

$$\left\{\begin{array}{l} \omega(x,t) = \omega(x/t,1),\ \omega(x,0) = u^{\pm} \text{ if } x \gtrless 0, \\ \omega \text{ is a vanishing viscosity solution of (9) with the viscosity matrix } B \equiv \mathrm{Id}. \end{array}\right. \tag{24}$$

As usual, for each $u^- \in \mathcal{U}$ they construct n independant curves $\Psi_i(s)(u^-)$ such that the Riemann problem with left and right states $(u^-, \Psi_i(s)(u^-))$ is solved with a i-wave. Then using an implicit function theorem, they find for each right state u^+ sufficiently close to u^-, a sequence $(s_1, \cdots, s_n)$ such that

$$u^+ = \Psi_n(s_n) \circ \cdots \circ \Psi_1(s_1)(u^-).$$

(Here we suppose that the eigenvalues are numbered such that $\lambda_1(u) < \cdots < \lambda_n(u)$.) Our aim in the next section is to estimate the difference between $\Psi_i(s)(u^-)$ and the approximate shock curve that we have defined in the case of genuinely nonlinear fields. In a first step, we will generalize the construction made by Bianchini and Bressan to more general viscosity matrices $B(u)$. Actually, we will first need $B(u)$ to be uniformly invertible and positive, but also to commute with $A(u)$ for all u. These hypotheses are more general than the case studied in [2] but still rather strong. In particular, the fact that $B(u)$ and $A(u)$ commute does not seem to be just a technical assumption. It is used many times at several crucial points. At the end of this section, we evaluate the difference between the shock curves and the approximate shock curves for shocks of small amplitude. Again, we will often use this assumption and we expect this result about the distance between approximate and exact shock curves to be wrong for general viscosity matrices (or at least less precise).

3.1 The center manifold

From now on, we assume $B(u)$ is a $n \times n$ viscosity matrix, smooth in u that satisfies

$$\left\{\begin{array}{ll} \exists c > 0,\ \forall u \in \mathcal{U}, \forall r \in \mathbf{R}^n,\ < B(u)r; r > \geq C|r|^2, & (25) \\ \forall u \in \mathcal{U},\ A(u)B(u) = B(u)A(u). & (26) \end{array}\right.$$

Remark 1 *Since the system has been assumed to be strictly hyperbolic, a natural consequence of the fact that $A(u)$ and $B(u)$ commute is that $B(u)$ is also diagonalisable in the same eigenvectors basis than $A(u)$. Indeed from*

$$A(u)B(u)r_i(u) = B(u)A(u)r_i(u),$$

we obtain

$$A(u)B(u)r_i(u) = \lambda_i(u)B(u)r_i(u),$$

which means that $B(u)r_i(u) \in \mathbf{R}r_i(u)$.

We recall that the shock profiles satisfy the equation

$$(A(U) - \sigma)U_x = (B(U)U_x)_x, \qquad x \in \mathbf{R}. \tag{27}$$

We want to use the center manifold theorem (see [11]) to select among the solutions of (27) the ones which may remain bounded. As in [2], we rewrite (27) as the first order system

$$\left\{ \begin{array}{rcl} u' & = & B(u)^{-1}w, \\ w' & = & (A(u) - \sigma)B(u)^{-1}w, \\ \sigma' & = & 0. \end{array} \right. \tag{28}$$

Now, let $u^- \in \mathcal{U}$. We are looking for i-shock profiles passing in the neighbourhood of u^-. Of course, the constant shock wave $(u^-, 0, \lambda_i(u^-))$ is an equilibrium point of the sytem (28). We rewrite (28) as

$$\left\{ \begin{array}{rcl} u' & = & B(u^-)^{-1}w + (B(u)^{-1} - B(u^-)^{-1})\, w, \\ w' & = & (A(u^-) - \lambda_i(u^-))B(u^-)^{-1}w \\ & & \qquad + ((A(u) - \sigma)B(u)^{-1} - (A(u^-) - \lambda_i(u^-))B(u^-)^{-1})\, w, \\ \sigma' & = & 0, \end{array} \right.$$

which has the form $X' = MX + f(X)$, with $f(X) = \mathcal{O}(|X - X_0|^2)$ and $X_0 := (u^-, 0, \lambda_i(u^-))$. We then apply the center manifold theorem. Let $\mathcal{N} := \bigcup_k \mathrm{Ker}(M^k)$, there exists a smooth manifold $\mathcal{M}$ and $\Omega \subset \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ a neighborhood of $(u^-, 0, \lambda_i(u^-))$ such that

1. $\mathcal{M} \subset \Omega$ is tangent to $\mathcal{N}$ at the stationary point X_0,
2. $\mathcal{M}$ is locally invariant for the system (28),
3. $\mathcal{M}$ contains all the small trajectories of (28) around $(u^-, 0, \lambda_i(u^-))$.

Here, we have

$$M = \begin{pmatrix} 0 & B(u^-)^{-1} & 0 \\ 0 & (A(u^-) - \lambda_i(u^-))B(u^-)^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

from which we may compute the tangent manifold

$$\mathcal{N} := \left\{(u, w, \sigma);\ w \in \mathbf{R}B(u^-)r_i(u^-)\right\}.$$

Here, we already stress on the fact that without (26) the tangent manifold is more complicated and may not have the same dimension for instance.

From the tangent space $\mathcal{N}$, we may parametrize $\mathcal{M}$ as

$$\mathcal{M} = \{(u, w, \sigma) \in \Omega,\ w = w_i q_i(u, w_i, \sigma)\},$$

where $w_i \in \mathbf{R}$ and q_i is a smooth vector field, normalized ($|q_i(u, w_i, \sigma)| = 1$) which satisfies $q_i(u^-, 0, \lambda_i(u^-)) \in \mathbf{R}B(u^-)r_i(u^-)$.

Denoting $\tilde{r}_i(u, w_i, \sigma) := B(u)^{-1}q_i(u, w_i, \sigma)$, we have $\tilde{r}_i(u^-, 0, \lambda_i(u^-)) \in \mathbf{R}r_i(u^-)$.

Actually, for u close to u^-, the above construction may be used to define the center manifold around the point $(u, 0, \lambda_i(u))$ instead of $(u^-, 0, \lambda_i(u^-))$. This leads to $\tilde{r}_i(u, 0, \lambda_i(u)) \in \mathbf{R}r_i(u)$ for all u in a neighborhood of u^-.

Now, since we have assume in (26) that $A(u)$ and $B(u)$ commute for all u in Ω, $A(u)$ and $B(u)$ share common eigenvectors and

$$q_i(u, 0, \lambda_i(u)) \in \mathbf{R}r_i(u). \tag{29}$$

We will show now that this relation is not only true for $\sigma = \lambda_i(u)$, but also for any σ. This will be absolutely crucial for the convergence of the method below.

Lemma 1 *For all u in a neighborhood of u^- and all σ in a neighborhood of $\lambda_i(u)$, one has*

$$q_i(u, 0, \sigma) \in \mathbf{R}r_i(u). \tag{30}$$

Proof :Let $(u^*, w^*, \sigma) \in \mathcal{M}$, from the center manifold theorem, there exists a curve $\{(u(x), w_i(x), \sigma), x \in (-\alpha, \alpha)\}$ such that $u(0) = u^*$, $w(0) = w^*$ and

$$(B(u)u_x)_x = (A(u) - \sigma)u_x, \qquad x \in (-\alpha, \alpha).$$

Writing $w = w_i q_i(u, w_i, \sigma)$ and $u_x = w_i\tilde{r}_i(u, w_i, \sigma)$, we obtain

$$w_{i,x}q_i(u, w_i, \sigma) + w_i q_{i,x}(u, w_i, \sigma) = (A(u) - \sigma)w_i\tilde{r}_i(u, w_i, \sigma). \tag{31}$$

But, since q_i is normalized, taking the dot product with q_i, we get

$$w_{i,x} = w_i(\tilde{\lambda}_i(u,w_i,\sigma) - \sigma) < \tilde{r}_i(u,w_i,\sigma); q_i(u,w_i,\sigma) >, \tag{32}$$

where we have defined

$$\tilde{\lambda}_i(u,v_i,\sigma) := \frac{< A(u)\tilde{r}_i(u,w_i,\sigma) \ ; \ q_i(u,w_i,\sigma) >}{< \tilde{r}_i(u,w_i,\sigma) \ ; \ q_i(u,w_i,\sigma) >}. \tag{33}$$

Reintroducing (32) in (31) and dividing by w_i yields

$$q_{i,x} = \Big(A(u) - \tilde{\lambda}_i < \tilde{r}_i; q_i > B(u)\Big)\, \tilde{r}_i - \sigma\,(Id - < \tilde{r}_i; q_i > B(u))\, \tilde{r}_i. \tag{34}$$

But $q_{i,x} = w_i \tilde{r}_i \cdot \nabla_u q_i + w_{i,x} \cdot \nabla_v q_i = \mathcal{O}(w_i)$ and letting w_i go to 0, we are led to

$$\Big(A(u) - \sigma Id + (\sigma - \tilde{\lambda}_i(u,0,\sigma)) < \tilde{r}_i \ ; \ q_i > B(u)\Big)\, \tilde{r}_i(u,0,\sigma) = 0. \tag{35}$$

From the assumption (26), the linear operator in the left-hand side is diagonal in the basis $(r_1(u), \ldots, r_n(u))$. The corresponding eigenvalues are $(\nu_k(u,\sigma))_{1 \le k \le n} := (\lambda_k(u) - \sigma + (\sigma - \tilde{\lambda}_i(u,0,\sigma)) < \tilde{r}_i \ ; \ q_i >)\mu_k(u))_{1 \le k \le n}$ where $(\mu_k(u))_k$ are the eigenvalues of $B(u)$.

For $\sigma = \lambda_i(u)$ we know that $\tilde{r}_i(u,0,\lambda_i(u)) \in \mathbf{R} r_i(u)$ and then $\nu_k(u,\lambda_i(u)) = 0 \Leftrightarrow k = i$. We get from a continuity argument

$$\tilde{r}_i(u,0,\sigma) \in \mathbf{R} r_i(u) \tag{36}$$

as claimed.

In particular, we also have

$$q_i(u,0,\sigma) = B(u)^{-1} \tilde{r}_i(u,0,\sigma) \in \mathbf{R} r_i(u) \tag{37}$$

and since we have normalized it, $q_i(u,0,\sigma)$ and therefore $\tilde{r}_i(u,0,\sigma)$ don't depend on σ. Hence, we may rewrite Eq. (35) as

$$\Big(A(u) - \tilde{\lambda}_i(u,0,\sigma) < \tilde{r}_i \ ; \ q_i > B(u)\Big)\, \tilde{r}_i(u,0,\sigma) = 0, \tag{38}$$

where $\tilde{\lambda}_i(u,0,\sigma)$ does not depend on σ.

Remark 2 *In the preceding proof, we have written a characterization of the solutions to (27). Let I be an open subset of $\mathbf{R}$, let $\{(u(x), w(x), \sigma),\ x \in I\} \subset \mathcal{M}$, then u satisfies*

$$(B(u)u_x)_x = (A(u) - \sigma)u_x, \qquad x \in I,$$

if and only if (32) holds on I.

3.2 The fixed point method

In this section, we generalize the approach of Bianchini and Bressan [2] to our situation $B(u) \neq Id$. Althought we don't give here the proof explicitely, the theorem of existence of a fixed point to the map defined below is true and we leave the reader follow exactly the strategy used in [2] in this context. However, in the following section using the stronger hypothesis of genuine non-linearity, we give a more elementary proof to existence of shock profiles that is sufficient for our purpose.

Let us introduce the following notations:

$$\begin{aligned}
&\Gamma := \left\{\gamma = (u, v, \sigma) \in \mathcal{C}^0([0,1], \Omega)\,;\, u(0) = u^-\right\},\\
&\rho(\gamma(t)) := < \tilde{r}_i(\gamma(t))\ ;\ q_i(\gamma(t)) >,\\
&f_i(t,\gamma) := \int_0^t \tilde{\lambda}_i(\gamma(\tau))\rho(\gamma(\tau))d\tau,
\end{aligned}$$

$$\begin{aligned}
g_i(t,\gamma) \ := \ \sup\Big\{ &\int_0^t \lambda(\tau)\rho(\gamma(\tau))d\tau\ ;\ \lambda \text{ increasing and } \forall 0 \le T \le 1 :\\
&\int_0^T \lambda(\tau)\rho(\gamma(\tau))d\tau \ \le\ f_i(T,\gamma).\Big\}.
\end{aligned}$$

Let us remark that from (25) ρ is positive and that in the case $B \equiv Id$, the function $g_i(\cdot,\gamma)$ is the convex envelop of $f_i(\cdot,\gamma)$. We are now ready to describe the fixed point method of Bianchini and Bressan: let us fix a small positive parameter $s > 0$ and let γ be an element of Γ, then we define $\hat{\gamma}(\gamma, s) = (\hat{u}, \hat{w}_i, \hat{\sigma})$ by

$$\left\{\begin{array}{rcl}
\hat{u}(t) & := & u^- + s\displaystyle\int_0^t \tilde{r}_i(\gamma(\tau))d\tau,\\
\hat{w}_i(t) & := & s\left(f_i(t,\gamma(\cdot)) - g_i(t,\gamma(\cdot))\right),\\
\hat{\sigma}(t) & := & \frac{1}{\rho(\gamma(\tau))}\dfrac{\mathrm{d}}{\mathrm{d}t} g_i(t,\gamma(\cdot)),
\end{array}\right. \tag{39}$$

In the case $B \equiv Id$, Bianchini and Bressan [2] show that there exists a unique $\gamma(\cdot, s)$ in Γ such that $\hat{\gamma}(\gamma(\cdot,s),s) = \gamma(\cdot,s)$. The result is still true in our case, but for shortness and since we are only interested with shock curves we don't give the proof here, the only additional difficulty is the definition of $g_i(\cdot,\gamma)$ which is no more the convex envelop of $f_i(\cdot,\gamma)$.

Assuming the existence and uniqueness of solution to (39) for small s, we define a family $\Psi_i(s)(u^-) := u(1,s)$ of right states joined from the left state u^- by a i-wave.

$$\omega^s(x,t) \ = \ \left\{\begin{array}{rl}
u^- & \text{if } x/t \le \sigma(0,s),\\
u(\tau,s) & \text{if } x/t = \sigma(\tau,s),\\
\Psi_i(s)(u^-) & \text{if } x/t \ge \sigma(1,s).
\end{array}\right.$$

For simplicity let us suppose that $\lambda_i(u^-)$ is genuinely non linear and let us formally describe the i-wave. Let $\gamma(t,s) = {}^t(u,v,\sigma)$ be a solution to the fixed point (39) in a neighborhood of ${}^t(u^-,0,\lambda_i(u^-))$, a simple calculation, (30) and (39) yields

$$\frac{d}{dt}\tilde{\lambda}_i(\gamma(t,s))) = sr_i(u^-)\cdot\nabla_u\lambda_i(u^-) + o(s).$$

Now, if the i-field is genuinely nonlinear

$$r_i(u^-)\cdot\nabla_u\lambda_i(u^-) \neq 0,$$

and we have the following alternative :

(a) if $r_i(u^-)\cdot\nabla\lambda_i(u^-) > 0$, then the $t \mapsto \tilde{\lambda}_i(\gamma(t,s))$ is increasing and we have from (39) and (30), $\sigma(t,s) = \lambda_i(u(t,s))$, the function $\omega^s(x,t)$ is therefore a classical rarefaction wave.

(b) if $r_i(u^-)\cdot\nabla\lambda_i(u^-) < 0$, then $t \mapsto \tilde{\lambda}_i(\gamma(t,s))$ is decreasing. Coming back to (39), we find that $\sigma(t,s) = \sigma(s)$ is actually constant in t, and $w_i(t,s) > 0$ for $0 < t < 1$. We then make the change of variable $dt = w_i(t,s)dx$, which is increasing and maps $(0,1)$ onto $\mathbf{R}$. Setting $U(x) := u(t(x),s)$, we have

$$U_x(x) = u_t(t,s)\frac{dt}{dx} = w_i s\tilde{r}_i(U(x), w_i(t(x),s), \sigma(s))$$

from (39). We also have from (39) that

$$w_{i,x} = w_{i,t}\frac{dt}{dx} = sw_i\rho(\tilde{\lambda}_i - \sigma(s))$$

which are the conditions for U to solve

$$(B(U)U_x)_x = (A(U) - \sigma(s))U_x, \qquad x \in \mathbf{R}. \tag{40}$$

In this case, $\omega^s(x,t)$ is a vanishing viscosity limit of the viscous traveling waves of speed $\sigma(s)$, $u_\varepsilon(x,t) := U\left(\dfrac{x}{\varepsilon t}\right)$.

For $s < 0$ small enough in magnitude, the same method may be used replacing sup by inf and $\geq$ by $\leq$ in the definition of g_i.

Remark 3 *In the case $B \equiv Id$ studied in [2], the parameter s is the length of the path $\{\omega^s(z,1),\ z \in \mathbf{R}\}$ and $\Psi_i(\cdot)(u^-)$ form a curve of right states reached from u^- by means of i-waves. Without any genuine nonlinearity hypothesis, these i-waves are not necessarily i-shocks or i-rarefaction curves but can be made of a possibly infinite succession of these two kinds of waves and the function $\Psi_i(\cdot)(u^-)$ is merely Lipschitz. In the sequel, we avoid this situation.*

3.3 Shock curves

Here, we give an alternate proof of existence in the case where the i-th characteristic field is moreover genuinely nonlinear. Clearly this result is weaker than the profile construction of Sainsaulieu [8] which has been rephrased using the center manifold technique by Schecter [9]. Nonetheless, we give it here for the sake of completeness and because our assumption that $A(u)$ and $B(u)$ commute will be absolutely crucial in the comparison of exact and approximate shock curves in the next section. In particular, if it is not the case then there is no reason that (30) holds. We will come back to that point later.

Therefore, we now choose $u^- \in \Omega$, $i \in \{1, \cdots, n\}$ and make the genuine nonlinearity hypothesis, replacing r_i by $-r_i$ if necessary, we may assume

$$r_i(u^-) \cdot \nabla\lambda_i(u^-) < 0. \tag{41}$$

For data close to $(u^-, 0, \lambda_i(u^-))$, and $s > 0$ small enough, the fixed point $\gamma(t, s)$ satisfies

$$\left\{\begin{array}{rcl} u(t,s) &=& u^- + s\displaystyle\int_0^t \tilde{r}_i(\gamma(\tau,s))\mathrm{d}\tau, \\ w_i(t,s) &=& s\displaystyle\int_0^t \rho(\gamma(\tau,s))(\tilde{\lambda}_i(\gamma(\tau,s)) - \sigma(s))d\tau, \\ \sigma(s) &=& \dfrac{f_i(1,\gamma(1,s))}{\displaystyle\int_0^1 \rho(\gamma(\tau,s))d\tau}, \end{array}\right. \tag{42}$$

and describes a i-shock profile. Indeed, if $dt := w_i(t,s)dx$ and $U(x) := u(t(x), s)$, then (27) holds. Moreover we will show in the next section that $\sigma_s(0) = \dfrac{r_i \cdot \nabla\lambda_i(u^-)}{2} \neq 0$, hence, for the parametrization of the profiles $u(\cdot, s)$ one can take σ instead of s. From these observations we introduce the following

Definition 3 *We will call i-shock curve:*

$$\begin{array}{rcl} \mathcal{S}_i(\cdot, u^-) : (\lambda_i(u^-) - \alpha_0\,,\, \lambda_i(u^-) + \alpha_0) &\longrightarrow& \mathcal{U} \\ \sigma(s) &\longmapsto& u(1,s), \end{array}$$

where u, w_i, σ solves (42).

In our case, the situation is easier than the problem without (41) and we prove the existence of such shock curves.

Proposition 2 *Suppose A satisfies (41) for u^- in $\mathcal{U}$, B is regular, $B(u)$ and $A(u)$ commute for all u in $\mathcal{U}$ and (25) holds for some $c > 0$. Then $\mathcal{S}_i(\cdot, u^-)$ is uniquely defined and the corresponding functions $u(t,s)$, $w_i(t,s)$ and $\sigma(s)$ are regular.*

Proof :Let k be a positive integer. Suppose that A and thus $\tilde{r}_i$ are $\mathbf{C}^k$. We want to apply an implicit function theorem, let us introduce the space

$$E := \left\{ {}^t(u, w_i, \sigma) \in \mathbf{C}^k\left([0,\, 1], \Omega\right) \; ; \; \sigma \text{ constant} \right\},$$

and the function

$$\begin{array}{rccl} F: & (-\alpha,\, \alpha) \times E & \longrightarrow & E \\ & (s, \gamma) & \longmapsto & \overline{\gamma} = {}^t(\overline{u}, \overline{w_i}, \overline{\sigma}) \end{array}$$

where

$$\left\{ \begin{array}{rclcl} \overline{u}(t) & := & u(t,s) & - & u^- - s \displaystyle\int_0^t \tilde{r}_i(\gamma(\tau))\mathrm{d}\tau, \\ \overline{\sigma} & := & \sigma & - & \dfrac{\displaystyle\int_0^1 \rho(\gamma(\tau))\tilde{\lambda}_i(\gamma(\tau))d\tau}{\displaystyle\int_0^1 \rho(\gamma(\tau))d\tau}, \\ \overline{w_i}(t) & := & w_i(t) & - & s \displaystyle\int_0^t \rho(\gamma(\tau))(\tilde{\lambda}_i(\gamma(\tau)) - \sigma)d\tau. \end{array} \right. \tag{43}$$

We remark that $F \in \mathbf{C}^k((-\alpha,\, \alpha) \times E, E)$ and that $F(0, \gamma_0) = 0$ where γ_0 is the constant function $\gamma_0 \equiv {}^t(u^-, 0, \lambda_i(u^-))$.

Then we compute

$$\mathrm{D}F_\gamma(\gamma_0) \cdot \begin{pmatrix} \delta u \\ \delta v \\ \delta\sigma \end{pmatrix} = \begin{pmatrix} \delta u \\ \delta v \\ \delta\sigma \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ D_u\overline{\sigma}(\delta u) + D_{w_i}\overline{\sigma}(\delta w_i) + D_\sigma\overline{\sigma}(\delta\sigma) \end{pmatrix},$$

but, since (30) holds for all u, σ, we get that $\rho(u, 0, \sigma)$ and $\tilde{\lambda}_i(u, 0, \sigma)$ are independant from σ and then $D_\sigma\overline{\sigma}(\gamma_0) = 0$. Therefore $\mathrm{D}F_\gamma(\gamma_0)$ is invertible and by the implicit function theorem, we get a unique $s \to u(\cdot, s)$ defined on a neighborhood of 0 such that u is $\mathbf{C}^k$ and u solves (42). $\square$

4 Order of the remaining term

We now study $R(\sigma)$ defined in the introduction and show that it is a term of third order in σ. The definition of i-shock profiles of the previous section allows us to derive rigorous estimates of R and then of the difference between the approximate i-shock curve and the i-shock curve defined in the definition 3. More precisely, let $u^- \in \mathcal{U}$ and $i \in \{1, \cdots, n\}$ such that the inequality (41) holds, and consider the functions $u(t,s)$, $w_i(t,s)$ and $\sigma(s)$ constructed in the proposition 2. We can define $\tilde{u}(t, \sigma(s)) := u(t,s)$ and

$$U(x,\sigma) := \tilde{u}(\psi(x,\sigma),\sigma), \qquad \forall x \in \mathbf{R},$$

with ψ defined by

$$\begin{aligned} \psi_x(x,\sigma) &= w_i(\psi(x,\sigma(s)),s), \qquad \forall x \in \mathbf{R}, \\ \psi(0,\sigma) &= \frac{1}{2}. \end{aligned}$$

Then $(U(\cdot,\sigma))$ is a regular family of i-shock profiles for which the formal calculus of the introduction which yields (11) is valid. We have

$$\big(A(\tilde{u}^+) - \sigma\big)\, \tilde{u}^+_\sigma(\sigma) \;\; = \;\; \tilde{u}^+ - u^- + \tilde{R}(\sigma), \tag{44}$$

with

$$\tilde{R}(\sigma) := \int_0^1 A(\tilde{u})_\tau \tilde{u}_\sigma - A(\tilde{u})_\sigma \tilde{u}_\tau \, d\tau, \tag{45}$$

and $\tilde{u}^+(\sigma) := \tilde{u}(1,\sigma)$.

Proposition 3 *Under the above regularity and genuine nonlinearity hypothesis:*

$$\tilde{R}(\sigma) = \mathcal{O}(\left|\sigma - \lambda_i(u^-)\right|^3).$$

Proof :Since $\sigma_s \neq 0$, we have to prove that

$$R(s) := \int_0^1 A(u)_\tau u_s - A(u)_s u_\tau \, d\tau = \mathcal{O}(s^3).$$

Since $u(\tau,0) = u^-$, we have $R(0) = 0$. Let us note $(\cdot\,,\,\cdot) : \nabla_{u,u} A(u)$ the second order derivative of A at u. We compute

$$\begin{aligned} R_s(s) \;\; = \;\; & \int_0^1 u_{\tau s} \cdot \nabla_u A(u) u_s - u_s \cdot \nabla_u A(u) u_{\tau s} \, d\tau \\ & + \int_0^1 u_\tau \cdot \nabla_u A(u) u_{ss} - u_{ss} \cdot \nabla_u A(u) u_\tau \, d\tau \\ & + \int_0^1 (u_\tau, u_s) : \nabla_{u,u} A(u) u_s - (u_s, u_s) : \nabla_{u,u} A(u) u_\tau \, d\tau. \end{aligned}$$

Since $u_\tau(\tau,0)=0$, we have

$$\begin{aligned}
R_{ss}(0) &= \int_0^1 u_{\tau ss}\cdot\nabla_u A(u^-)u_s - u_s\cdot\nabla_u A(u^-)u_{\tau ss}\,d\tau \\
&\quad +2\int_0^1 u_{\tau s}\cdot\nabla_u A(u^-)u_{ss} - u_{ss}\cdot\nabla_u A(u^-)u_{\tau s}\,d\tau \qquad (46)\\
&\quad +2\int_0^1 (u_{\tau s},\,u_s):\nabla_{u,u}A(u^-)u_s - (u_s,\,u_s):\nabla_{u,u}A(u^-)u_{\tau s}\,d\tau.
\end{aligned}$$

From the differentiation with respect to s of (42), we get (we have denoted $\gamma=\gamma(\tau,s)$)

$$\begin{aligned}
u_s(t,s) &= s\int_0^t u_s\cdot\nabla_u\tilde r_i(\gamma)+w_{i,s}\cdot\nabla_{w_i}\tilde r_i(\gamma)+\sigma_s\cdot\nabla_\sigma\tilde r_i(\gamma)d\tau+\int_0^t\tilde r_i(\gamma)d\tau,\\
\sigma_s(s) &= \frac{\displaystyle\int_0^1 u_s\cdot\nabla_u(\rho\tilde\lambda_i)(\gamma)+w_{i,s}\cdot\nabla_{w_i}(\rho\tilde\lambda_i)(\gamma)+\sigma_s\cdot\nabla_\sigma(\rho\tilde\lambda_i)(\gamma)d\tau}{\displaystyle\int_0^1\rho(\gamma)d\tau}\\
&\quad -\frac{\displaystyle\int_0^1\rho\tilde\lambda_i(\gamma)d\tau}{\left(\displaystyle\int_0^1\rho(\gamma)d\tau\right)^2}\int_0^1 u_s\cdot\nabla_u\rho(\gamma)+w_{i,s}\cdot\nabla_{w_i}\rho(\gamma)+\sigma_s\cdot\nabla_\sigma\rho(\gamma)d\tau,
\end{aligned}$$

$$\begin{aligned}
w_{i,s}(t,s) &= s\left(\int_0^t u_s\cdot\nabla_u(\rho\tilde\lambda_i)(\gamma)+w_{i,s}\cdot\nabla_{w_i}(\rho\tilde\lambda_i)(\gamma)+\sigma_s\cdot\nabla_\sigma(\rho\tilde\lambda_i)(\gamma)d\tau-t\sigma_s(s)\right)\\
&\quad+\left(f_i(t,\gamma)-\sigma(s)\int_0^t\rho(\gamma)d\tau\right).
\end{aligned}$$

For $s=0$, we compute $w_{i,s}(t,0)=0$, $u_s(t,0)=tr_i(u^-)$ and we easily deduce that $R_s(0)=0$. Moreover since we have proved that $q_i(u,0,\sigma)\in\mathbf{R}r_i(u)$, we have $\nabla_\sigma\tilde\lambda_i(\gamma_0)=0$ and $\nabla_\sigma\rho(\gamma_0)=0$. Therefore, $\sigma_s(0)=\frac{1}{2}r_i\cdot\nabla_u\lambda_i(u^-)$ as claimed earlier. We insist on the fact that the hypothesis that $A(u)$ and $B(u)$ commute is crucial here. In particular, if it was not the case (for example if one considers traveling wave solutions constructed by Sainsaulieu [8] or Schecter [9]) the computation above would be wrong.

Another differentiation of (42) leads to

$$\begin{aligned}
u_{ss}(t,0) &= 2\int_0^t u_s\cdot\nabla_u\tilde r_i(\gamma(\tau,0))+v_s\cdot\nabla_v\tilde r_i(\gamma(\tau,0))+\sigma_s\cdot\nabla_\sigma\tilde r_i(\gamma(\tau,0))d\tau,\\
&= t^2 r_i(u^-)\cdot\nabla_u r_i(u^-).
\end{aligned}$$

Substituting in (46) $u_{ss}(\tau,0)=\tau^2 r_i(u^-)\cdot\nabla_u r_i(u^-)$,$u_{\tau ss}(\tau,0)=2\tau r_i(u^-)\cdot\nabla_u r_i(u^-)$, $u_s(\tau,0)=\tau r_i(u^-)$ and $u_{\tau s}(\tau,0)=r_i(u^-)$, we get that the first and the second term in the right hand side cancelled each other and that the last term vanishes. Finally $R_{ss}(0)=0$. □

We are now able to compare the approximate i-shock curve and the i-shock curve of the defintion 3, namely, we prove the following proposition.

Proposition 4 *Let us call $\tilde{u}(\sigma)$ the reparametrization by σ of the family of i-shock curves found above, and $u(\sigma)$ the approximate i-shock curve of the section 2. We define the vector $\overline{u}:=u-\tilde{u}$ whose components in the basis $(r_1(u^-),\cdots,r_n(u^-))$ are denoted $\overline{u}=(\overline{u}_1,,\cdots,\overline{u}_n)$. Then we have*

$$\left\{\begin{array}{lcl}\overline{u}_i &=& \mathcal{O}(|\sigma-\lambda_i(u^-)|^3),\\ \overline{u}_j &=& \mathcal{O}(|\sigma-\lambda_i(u^-)|^4),\ \forall j\neq i.\end{array}\right. \tag{47}$$

In other words $u(\sigma)=\tilde{u}(\sigma)+\mathcal{O}(|\sigma-\lambda_i(u^-)|^3)$ and there exists a regular reparametrization of u: $u^*(\sigma):=u(\psi(\sigma))$ with $\psi_\sigma(\lambda_i(u^-))=1$ such that $u^*(\sigma)=\tilde{u}(\sigma)+\mathcal{O}(|\sigma-\lambda_i(u^-)|^4)$.

*Proof :*Without loss of generality, we may suppose again that $i=1$, $u^-=0$ and $\lambda_1(0)=0$. We substract the equation $(A(\tilde{u})-\sigma)\,\tilde{u}_\sigma=\tilde{u}+\mathcal{O}(\sigma^3)$ from $(A(u)-\sigma)\,u_\sigma=u$ and get

$$(A(u)-\sigma)\,\overline{u}_\sigma=\overline{u}+(A(\tilde{u})-A(u))\,\tilde{u}_\sigma+\mathcal{O}(\sigma^3). \tag{48}$$

We use the same decomposition of the vectors in the basis $(r_1(0),\cdots,r_n(0))$ as in the proof of proposition 1. We write $\overline{u}=(v,w)$ with $v:=\overline{u}_1$, $w:=(\overline{u}_2,\cdots,\overline{u}_n)$ and

$$A(u):=\left(\begin{array}{cc} a(u) & b(u)\\ C(u) & D(u)\end{array}\right).$$

Now Eq. (48) reads

$$\left(\begin{array}{c}(a(u)-\sigma)v_\sigma+<b(u);w_\sigma>\\ C(u)v_\sigma+(D(u)-\sigma)w_\sigma\end{array}\right)=\left(\begin{array}{c}v\\ w\end{array}\right)+(A(\tilde{u})-A(u))\tilde{u}_\sigma+\mathcal{O}(\sigma^3). \tag{49}$$

On one hand, we have the expansion

$$(A(\tilde{u})-A(u))\tilde{u}_\sigma=\ -\overline{u}\cdot\nabla_u A(u)\tilde{u}_\sigma+\ \underbrace{\mathcal{O}(|\overline{u}|^2)}_{\mathcal{O}(\sigma\overline{u})}.$$

Since $\tilde{u}_\sigma = \dfrac{2r_1(0)}{r_1(0)\cdot\nabla_u\lambda_1(0)} + \mathcal{O}(\sigma)$ and $r_1(0)\cdot\nabla_u a(0) = r_1(0)\cdot\nabla_u\lambda_1(0) r_1(0)$, we get

$$(A(\tilde{u}) - A(u))\tilde{u}_\sigma = \begin{pmatrix} -2v + \mathcal{O}(\sigma v) + \mathcal{O}(w) \\ \mathcal{O}(v) + \mathcal{O}(w) \end{pmatrix}. \tag{50}$$

On the other hand, let us recall that $D(u)$ is invertible and that $a(u) - \sigma = \sigma(1 + \mathcal{O}(\sigma))$, $b(u) = \mathcal{O}(\sigma)$, $C(u) = \mathcal{O}(\sigma)$. Substituting the preceding equalities and (50) in (49), we deduce the estimates

$$v + \sigma v_\sigma = \mathcal{O}(\sigma v) + \mathcal{O}(\sigma^2 v_\sigma) + \mathcal{O}(w) + \mathcal{O}(\sigma w_\sigma) + \mathcal{O}(\sigma^3), \tag{51}$$

$$w_\sigma = \mathcal{O}(v) + \mathcal{O}(\sigma v_\sigma) + \mathcal{O}(w) + \mathcal{O}(\sigma^3). \tag{52}$$

From (52) and the regularity of $\overline{u}$, we deduce $w_\sigma = \mathcal{O}(\sigma v_\sigma) + \mathcal{O}(w) + \mathcal{O}(\sigma^3)$, then by Gronwall's lemma we get $w_\sigma = \mathcal{O}(\sigma v_\sigma) + \mathcal{O}(\sigma^3)$. Now from (51), we get

$$(\sigma v)_\sigma = \mathcal{O}(\sigma^2 v_\sigma) + \mathcal{O}(\sigma^3),$$

which yields $v = \mathcal{O}(\sigma^3)$ and consequently $w = \mathcal{O}(\sigma^4)$. $\square$

This last proposition shows that the approximate i-shock curve is strictly closer to the i-shock curve than any reparametrization of the i-rarefaction curve $\mathcal{R}_i(u^-)(\sigma)$ defined by

$$\begin{cases} \big(A(\mathcal{R}_i(u^-)(\sigma)) - \sigma\big) \dfrac{d}{d\sigma}\mathcal{R}_i(u^-)(\sigma) = 0, \\ \mathcal{R}_i(u^-)(\lambda_i(u^-)) = u^-. \end{cases}$$

Indeed, in the conservative case, it is well known that $\mathcal{R}_i(u^-)(\lambda_i(u^-) + 2(\sigma - \lambda_i(u^-))) = \mathcal{S}_i(u^-)(\sigma) + \mathcal{O}(|\sigma - \lambda_i(u^-)|^2)$ with in general no better estimates.

5 conclusion

Differentiating the Rankine-Hugoniot condition with respect to the shock speed leads to a natural definition of shock curves for non conservative hyperbolic systems along genuinely non-linear fields. On the other hand, defining the shock curves through a vanishing viscosity method in the case where the viscosity matrix commute with the matrix of the system gives a solution which agrees with the preceding one up to third order. We stress on the fact that this property is crucially linked to the fact that the viscosity matrix and the matrix of the system commute, and we might lose one order in the general case. Nonetheless, we remark that such viscosities arise in the numerical computation of solutions to Eq. (1) when one uses a Roe approximate Riemann solver or VFFC method [5]. Indeed, it is well known that in these methods the numerical dissipation is $\Delta x|A(u)|$ where Δx is the space step. This obviously commutes with $A(u)$, and therefore, the shock curves generated by such schemes might be wrong and at least as far from the "good curves" than our approximate shock curves. It is indeed expected, from the above study, that for general dissipations, the shock curves agree to the approximate shock curves only to second order.

References

[1] F. Alouges, F. De Vuyst, G. Le Coq, and E. Lorin. Personnal communication.

[2] S. Bianchini and A. Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Preprint.*, pages 1–99.

[3] S. Bianchini and A. Bressan. A center manifold technique for tracing viscous waves. *Commun. Pure Appl. Anal.*, 1(2):161–190, 2002.

[4] G. Dal Maso, P. G. LeFloch, and F. Murat. Definition and weak stability of nonconservative products. *J. Math. Pures Appl.*, 74:483–548, 1995.

[5] J.-M. Ghidaglia, A. Kumbaro, and G. Le Coq. On the numerical solution to two fluid models via a cell centered finite volume method. *Eur. J. Mech. B Fluids*, 20(6):841–867, 2001.

[6] P. G. LeFloch. Shock waves for nonlinear hyperbolic systems in nonconservative form. *preprint 593, Institute for Mathematics and its Applications, University of Minnesota, Minneapolis*, 1989.

[7] P. G. LeFloch and A E. Tzavaras. Representation of weak limits and definition of nonconservative products. *SIAM J. Math. Anal.*, 30(6):1309–1342 (electronic), 1999.

[8] L. Sainsaulieu. Traveling-wave solutions of convection-diffusion systems in nonconservation form. *SIAM J. Math. Anal.*, 27(5):1286–1310, 1996.

[9] S. Schecter. Traveling-wave solutions of convection-diffusion systems by center manifold reduction. *Nonlinear Anal. Ser. A: Theory Methods*, 49(1):35–59, 2002.

[10] D. Serre. *Systems of conservation laws. 1.* Cambridge University Press, Cambridge, 1999. Hyperbolicity, entropies, shock waves, Translated from the 1996 French original by I. N. Sneddon.

[11] A. Vanderbauwhede and S. A. van Gils. Center manifolds and contractions on a scale of Banach spaces. *J. Funct. Anal.*, 72(2):209–224, 1987.