Global subanalytic solutions of Hamilton-Jacobi type equations

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Abstract

In the 80's Crandall and Lions introduced the concept of viscosity solution in order to get existence and/or uniqueness results for Hamilton-Jacobi equations. In this work we first investigate the Dirichlet and Cauchy-Dirichlet problems for such equations, where the Hamiltonian is associated to a problem of calculus of variations, and prove that if the data are analytic then the viscosity solution is moreover subanalytic. We then extend this result to Hamilton-Jacobi equations stemming from optimal control problems, in particular from sub-Riemannian geometry, which are generalized eikonal equations.

1 Introduction

1.1 Viscosity solutions

In the 80's Crandall and Lions [12] introduced the concept of viscosity solution in order to ensure uniqueness of solutions of Hamilton-Jacobi equations. Existence of viscosity solutions was also established under similar assumptions. A general definition of a viscosity solution of a first-order Hamilton-Jacobi equation is the following.

Let Ω be an open set in $\mathbb{R}^n$ and H be a continuous function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, called *Hamiltonian*, and g be a continuous function on $\partial\Omega$. Consider the first-order Hamilton-Jacobi equation on Ω

$$H(x, v(x), \nabla v(x)) = 0. \tag{1}$$

We first recall the notion of sub- and super-differential.

Definition 1.1. Let v be a scalar function on Ω. The *super-differential* at a point $x \in \Omega$ is defined as

$$D^+v(x) = \{p \in \mathbb{R}^n \mid \limsup_{y \to x} \frac{v(y) - v(x) - \langle p, y - x\rangle}{\|y - x\|} \leqslant 0\}.$$

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Similarly, the sub-differential at x is

$$D^- v(x) = \{p \in \mathbb{R}^n \mid \liminf_{y \to x} \frac{v(y) - v(x) - \langle p, y-x \rangle}{\|y-x\|} \geqslant 0\}.$$

We can now define the concept of viscosity solution introduced in [12].

Definition 1.2. Let v be a continuous function on Ω. The function v is a *viscosity super-solution* of equation (1) if

$$\forall x \in \Omega \quad \forall p \in D^+ v(x) \quad H(x, v(x), p) \leqslant 0.$$

Similarly, v is a *viscosity sub-solution* of equation (1) si

$$\forall x \in \Omega \quad \forall p \in D^- v(x) \quad H(x, v(x), p) \geqslant 0.$$

Finally, v is a *viscosity solution* of equation (1) if it is both a sub-solution and a super-solution.

This concept is adapted to get existence and uniqueness results, in particular for *Dirichlet problems* of the type

$$\begin{aligned} H(x, v(x), \nabla v(x)) &= 0 \ \text{ in } \Omega, \\ v_{|\partial\Omega} &= g, \end{aligned}$$

so as for many other problems (Cauchy problems, second-order equations, ...), see for instance [12, 20, 5, 6, 14]. Litterature on this subject is immense.

In the case of analytic Hamilton-Jacobi equations one could however expect these solutions to be more regular. Of course because of possible shocks one cannot expect to get global analytic solutions. For example in the case of the *eikonal equation*

$$\begin{aligned} \|\nabla v(x)\|^2 &= 1 \ \text{ in } \Omega, \\ v_{|\partial\Omega} &= 0, \end{aligned}$$

on a bounded analytic open set $\Omega \subset \mathbb{R}^n$, one can easily see that the unique viscosity solution is

$$v(x) = d(x, \partial\Omega).$$

Of course this function u is not analytic on Ω, due to intersection of characteristic curves (concerning the method of characteristics we refer the reader to the previously cited references). Anyway the function v is, in a sense, "analytic by parts". The right concept in order to describe such objects happens to be the concept of *subanalyticity*. In the next paragraph we recall a definition and several basic properties.

1.2 Subanalytic functions

Let us first define subanalytic sets, see [16], [17].

Definition 1.3. Let M be a real analytic finite dimensional manifold. A subset A of M is said to be *semi-analytic* if and only if, for any $x \in M$, there exists a neighborhood U of x in M and $2pq$ analytic functions g_{ij}, h_{ij} ($1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant q$), such that

$$A \cap U = \bigcup_{i=1}^{p} \{y \in U \mid g_{ij}(y) = 0 \text{ and } h_{ij}(y) > 0,\ j = 1 \ldots q\}.$$

Let SEM(M) denote the set of semi-analytic subsets of M.

The image of a semi-analytic subset by a proper analytic mapping is not in general semi-analytic, and thus this class has to be enlarged.

Definition 1.4. A subset A of M is said to be *subanalytic* if and only if, for any $x \in M$, there exists a neighborhood U of x in M and $2p$ couples $(\Phi_i^\delta, A_i^\delta)$ ($1 \leqslant i \leqslant p$ et $\delta = 1, 2$), where $A_i^\delta \in SEM(M_i^\delta)$, and for real analytic manifolds M_i^δ, the mappings $\Phi_i^\delta\ : M_i^\delta \to M$ are proper analytic, such that

$$A \cap U = \bigcup_{i=1}^{p} (\Phi_i^1(A_i^1) \backslash \Phi_i^2(A_i^2)).$$

Let SUB(M) denote the set of subanalytic subsets of M.

The subanalytic class is closed by union, intersection, complementary, inverse image by an analytic mapping, image by a proper analytic mapping. In brief, the subanalytic class is *o-minimal* (see [13]). Moreover subanalytic sets are *stratifiable* in the following sense.

Definition 1.5. Let M be a differentiable manifold. A *stratum* in M is a locally closed sub-manifold of M. A locally finite partition $\mathcal{S}$ of M is a *stratification* of M if any $S \in \mathcal{S}$ is a stratum such that

$$\forall T \in \mathcal{S} \quad T \cap \text{Fr } S \neq \emptyset \Rightarrow T \subset \text{Fr } S \text{ and } \dim T < \dim S.$$

Finally, a mapping $f : M \to N$ between two analytic manifolds is said to be *subanalytic* if its graph is a subanalytic subset of $M \times N$.

A basic property of subanalytic functions, which makes them very useful in calculus of variations, and more generally in optimal control theory, is the following, see [23].

Proposition 1.1. *Let M and N be real analytic finite dimensional manifolds, A be a subset of N, and $\Phi\ : N \to M$ and $f\ : N \to I\!R$ be subanalytic mappings. We define, for any $x \in M$*

$$\psi(x) = \inf\{f(y) \mid y \in \Phi^{-1}(x) \cap A\}.$$

If $\Phi_{|\bar{A}}$ is proper then ψ is subanalytic.

This crucial tool in order to establish subanalyticity, based on an infimum property of the solution, suggests to investigate Hamilton-Jacobi equations stemming from optimization problems.

First in the framework of the classical calculus of variations, where the Hamiltonian function is associated to an analytic Lagrangian function, we state existence, and in some cases, uniqueness of a subanalytic viscosity solution. These results precise those of Lions [20] in the analytic case.

We then extend these statements to cases where the Hamiltonian function is stemming from sub-Riemannian geometry, and more generally from an optimal control problem, and show how the subanalyticity status of solutions is related to the existence of *singular minimizing trajectories* of the underlying control problem.

2 Hamilton-Jacobi equations and calculus of variations

In this section we specialize results of [20] in an analytic framework.

2.1 A brief insight into calculus of variations

Let us first recall the classical framework of calculus of variations. Let $H(x,p)$ be a C^2 function on $\mathbb{R}^n \times \mathbb{R}^n$, called *Hamiltonian*, satisfying the following assumptions:

- H is uniformly superlinear, i.e.

$$\forall A>0 \quad \exists C(A)\in\mathbb{R} \mid \forall (x,p)\in\mathbb{R}^n\times\mathbb{R}^n \quad H(x,p)\geqslant A\|p\|-C(A), \tag{2}$$

- H is strictly convex in p, i.e. for all $(x,p)\in\mathbb{R}^n\times\mathbb{R}^n$ the second derivative

$$\frac{\partial^2 H}{\partial p^2}(x,p) \tag{3}$$

  is positive definite.

For any $(x,u)\in\mathbb{R}^n\times\mathbb{R}^n$ set

$$L(x,u)=\sup_{p\in\mathbb{R}^n}\left(\langle p,u\rangle - H(x,p)\right). \tag{4}$$

It is well-known that under the previous assumptions on H, this function, called the *Lagrangian*, is well-defined on $\mathbb{R}^n\times\mathbb{R}^n$, and moreover satisfies also assumptions (2), (3), see for instance [15]. Moreover the so-called *Legendre transformation*

$$\mathcal{T}(x,u)=\left(x,\frac{\partial L}{\partial u}(x,u)\right) \tag{5}$$

is a global C^1 diffeomorphism on $\mathbb{R}^n\times\mathbb{R}^n$.

Definition 2.1. Let $\mathcal{AC}$ denote the set of absolutely continuous curves in $\mathbb{R}^n$.

- Let $x(.) \in \mathcal{AC}$ be defined on $[0, T]$. The *action* of $x(.)$ on $[0, T]$ is defined as
$$\mathcal{A}_T(x(.)) = \int_0^T L(x(t), \dot{x}(t))dt. \tag{6}$$

- For all $x, y \in \mathbb{R}^n$ define the *value function* at (x, y) as
$$S(x, y) = \inf \{\mathcal{A}_T(x(.)) \mid T > 0, x(.) \in \mathcal{AC}, x(0) = y, x(T) = x\}. \tag{7}$$

Assumption (3) actually implies that for all $x, y \in \mathbb{R}^n$ there exists an absolutely continuous curve joining y to x and minimizing the action. This result is known as *Tonnelli Theorem*, see [15]. In particular the infimum (7) is a minimum.

On the other part minimizing curves are solutions of *Euler-Lagrange* equations
$$\frac{d}{dt}\frac{\partial L}{\partial u}(x(t), \dot{x}(t)) = \frac{\partial L}{\partial x}(x(t), \dot{x}(t)),$$
which are equivalent, using the Legendre mapping, to the Hamilton equations
$$\dot{x}(t) = \frac{\partial H}{\partial p}(x(t), p(t)),\ \dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), p(t)), \tag{8}$$
where $p(t)$, called *adjoint vector*, is defined as
$$(x(t), p(t)) = \mathcal{T}(x(t), \dot{x}(t)).$$
Moreover since the final time is not fixed the identity
$$H(x(t), p(t)) = 0 \tag{9}$$
holds along the trajectory.

A curve $x(.) \in \mathcal{AC}$ for which there exists $p(.) \in \mathcal{AC}$ such that equations (8) and (9) hold is called an *extremal*.

Remark 2.1. If in the definition (11) of $S(x, y)$ the final time T is fixed then Eq. (9) does not hold.

2.2 The Dirichlet problem

The following theorem is an analytic version of [20, Th. 5.3 p. 132].

Theorem 2.1. *Let Ω denote a bounded subanalytic open subset of $\mathbb{R}^n$, c be a real number, and let $H(x, p)$ be an analytic Hamiltonian function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying assumptions (2), (3), and such that moreover*
$$\exists \alpha < c \mid \forall x \in \mathbb{R}^n \quad H(x, 0) \leqslant \alpha. \tag{10}$$

Let $\mathcal{AC}$ denote the set of absolutely continuous curves in $I\!R^n$. For all $x, y \in \overline{\Omega}$ set

$$S(x,y) = \inf \left\{ \int_0^T \big(L(x(t),\dot{x}(t)) + c\big) dt \ / \ T > 0, x(.) \in \mathcal{AC}, x(0) = y, x(T) = x \right\}, \tag{11}$$

where L is the Lagrangian associated to the Hamiltonian H. Then for all $y_0 \in \overline{\Omega}$ the function $x \mapsto S(x, y_0)$ (resp. for all $x_0 \in \overline{\Omega}$ the function $y \mapsto S(x_0, y)$) is a viscosity solution of

$$H(x, \nabla v(x)) - c = 0 \ \ in \ \Omega \backslash \{y_0\}, \ \ v(y_0) = 0, \tag{12}$$

(resp. $H(x, -\nabla v(x)) - c = 0$ in $\Omega \backslash \{x_0\}$, $v(x_0) = 0$), which is subanalytic on $\overline{\Omega}$.

Let g denote a subanalytic function on $\Sigma = \partial\Omega$, satisfying the so-called compatibility condition

$$\forall x, y \in \Sigma \quad g(x) - g(y) \leqslant S(x, y). \tag{13}$$

Let us finally define, for all $x \in \overline{\Omega}$

$$S(x) = \inf_{y \in \Sigma} \left(g(y) + S(x, y)\right). \tag{14}$$

Then S is the unique viscosity solution of the Dirichlet problem

$$\begin{aligned} H(x, \nabla v(x)) - c = 0 \ \ &in \ \Omega, \\ v_{|\Sigma} = g&, \end{aligned} \tag{15}$$

and moreover is continuous and subanalytic on $\overline{\Omega}$.

Remark 2.2. If the Hamiltonian H is defined on $\Omega \times I\!R^n$ and moreover assumptions (2), (3) and (10) hold only for all $x \in \Omega$, then the conclusion on $S(x)$ still holds. However the definition of $S(x, y)$ has to be modified: the path joining y to x should be included in $\overline{\Omega}$, as done in [20] (see in particular [20, Rem. 5.4 p. 118]. Moreover in this case we cannot prove the subanalyticity of the mappings $x \mapsto S(x, y_0)$ (resp. $y \mapsto S(x_0, y)$).

Remark 2.3. As noticed in [20, Rem. 5.8 p. 128], this result also holds on an unbounded open set Ω, except for uniqueness.

Example 2.1. Let Ω be a subanalytic bounded open set in $I\!R^n$, $\Sigma = \partial\Omega$, let $m(.) : I\!R^n \to I\!R^n$ be an analytic mapping, and $f(.)$ be an analytic function on $I\!R^n$. We assume that there exists a positive real number ε such that

$$\|m(x)\|^2 \leqslant f(x)^2 - \varepsilon$$

for all $x \in I\!R^n$. Then there exists an unique viscosity solution S of the Dirichlet problem

$$\begin{aligned} \|\nabla v(x) - m(x)\| - |f(x)| = 0 \ \ &\text{in } \Omega, \\ v_{|\Sigma} = 0&, \end{aligned} \tag{16}$$

which is continuous and subanalytic on $\overline{\Omega}$.

This is indeed an easy consequence of Th. 2.1, with the Lagrangian function

$$L(x,u)=\frac{1}{2}\|u\|^2+\langle m(x),u\rangle+\frac{1}{2}f(x)^2,$$

for all $x,u\in\mathbb{R}^n$.

Proof of Theorem 2.1. For all $x\in\mathbb{R}^n$ and $u\in\mathbb{R}^n$, the assumption (10) implies the inequality $L(x,u)+c>0$, and thus $S(x,y)$ for all $x,y\in\overline{\Omega}$.

Let $y_0\in\overline{\Omega}$ be fixed. It is proved in [20] that the function $x\mapsto S(x,y_0)$ is a viscosity solution of (12). Let us prove that it is moreover subanalytic.

In order to do this we shall express $S(x,y_0)$ using *extremal curves*. Absolutely continuous curves on $[0,T]$ such that $x(0)=y_0$ and minimizing the action are extremal, namely solutions of (8), and thus are parametrized by T and $p(0)$, where $p(.)$ is an adjoint vector associated to $x(.)$. This suggests to introduce the mapping

$$\begin{array}{rcl}\Phi:\mathbb{R}^+\times\overline{\Omega}\times\mathbb{R}^n & \longrightarrow & \mathbb{R}^n\\ (T,y,\psi) & \longmapsto & x(T)\end{array}$$

where $(x(.),p(.))$ is the solution of equations (8) such that $x(0)=y$ and $p(0)=\psi$. The system (8) is analytic and hence Φ is an analytic mapping. We also set, with the same notations, for all $T\in\mathbb{R}^+,y\in\overline{\Omega},\psi\in\mathbb{R}^n$,

$$\sigma(T,y,\psi)=\int_0^T\big(L(x(t),\dot{x}(t))+c\big)dt=\int_0^T\big(L(\mathcal{T}^{-1}(x(t),p(t)))+c\big)\,dt.$$

The Legendre mapping $\mathcal{T}$ is analytic and thus σ is also analytic. With these notations we have

$$S(x,y_0)=\inf\{\sigma(T,y_0,\psi)\mid\exists T\geqslant 0,\ \exists\psi\in\mathbb{R}^n,\ \Phi(T,y_0,\psi)=x\}.$$

In order to apply Prop. 1.1 we have to prove that the couples (T,ψ) in the last formula can be chosen in a compact set of $\mathbb{R}^n\times\mathbb{R}^n$ as $x\in\overline{\Omega}$. By definition of the minimal action (11), condition (9) leads to

$$H(y_0,\psi)=c.$$

Using assumption (H_1), we infer that ψ belongs to a compact set of $\mathbb{R}^n$ as $x\in\overline{\Omega}$. Morever assumption (10) yields

$$\sigma(T,y_0,\psi)\geqslant(c-\alpha)T,$$

and since $\alpha<c$ we also get that T is bounded. Finally, there exists a compact subset $K\subset\mathbb{R}^+\times\mathbb{R}^n$ such that

$$S(x,y_0)=\inf\left\{\sigma(T,y_0,\psi)\mid\exists\ (T,\psi)\in K,\ \Phi(T,y_0,\psi)=x\right\}.$$

Prop. 1.1 now asserts that the function $x\mapsto S(x,y_0)$ is subanalytic on $\overline{\Omega}$.

Finally, let S, the value function, be defined as in (14). It is continuous on $\overline{\Omega}$ because $S(x,y)$ is continuous. From [20] S is a viscosity solution of (15). Uniqueness comes from assumption (10) and [6, Th. 2.7 p. 37]. Subanalyticity of S on $\overline{\Omega}$ is a consequence of Prop. 1.1. $\square$

2.3 The Cauchy problem

The following theorem is an analytic version of [20, Th. 11.1 p. 217].

Theorem 2.2. *Let $T > 0$ be fixed, let Ω denote a bounded subanalytic open subset of $\mathbb{R}^n$, and $H(x,p)$ be an analytic Hamiltonian function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying assumptions (2), (3). For all $s,t \in [0,T]$ such that $s < t$ and $x,y \in \overline{\Omega}$ set*

$$S(t,x,s,y) = \inf\left\{\int_s^t L(x(t),\dot{x}(t))\,dt \mid x(.) \in \mathcal{AC}, x(s) = y, x(t) = x\right\}. \tag{17}$$

Then for all $s_0 \in [0,T[$ and $y_0 \in \overline{\Omega}$ the function $S(.,.,s_0,y_0)$ is a viscosity solution of

$$\begin{aligned} \frac{\partial v}{\partial t} + H(x, \frac{\partial v}{\partial x}) = 0 \quad &in\ ]s_0,T] \times \Omega\backslash\{y_0\}, \\ \lim_{t\to s_0} v(t,y_0) = 0, \end{aligned} \tag{18}$$

which is subanalytic on $]s_0,T]\times\overline{\Omega}$. Let g be a subanalytic function on $\left(\{0\}\times\overline{\Omega}\right)\cup([0,T]\times\partial\Omega)$ satisfying the compatibility condition

$$\begin{aligned} \forall (s,y) \in \left(\{0\}\times\overline{\Omega}\right)\cup([0,T[\times\partial\Omega) \quad \forall (t,x) \in]s,T]\times\partial\Omega \\ g(t,x) - g(s,y) \leqslant S(t,x,s,y). \end{aligned} \tag{19}$$

For all $t \in]0,T]$ and $x \in \overline{\Omega}$, set

$$S(t,x) = \inf\left\{S(t,x,s,y) + g(s,y) \mid (s,y) \in \left(\{0\}\times\overline{\Omega}\right)\cup([0,t[\times\partial\Omega)\right\}. \tag{20}$$

Then S is continuous on $]0,T]\times\overline{\Omega}$, is the unique viscosity solution of the Cauchy-Dirichlet problem

$$\begin{aligned} \frac{\partial v}{\partial t} + H(x, \frac{\partial v}{\partial x}) = 0 \quad &in\ ]0,T[\times\Omega, \\ v = g \quad &on\ ]0,T]\times\partial\Omega, \\ \lim_{t\to 0} v(t,x) = g(0,x), \end{aligned} \tag{21}$$

and moreover S is subanalytic on $]0,T]\times\overline{\Omega}$.

Proof. The proof is similar to the proof of Th. 2.1, except for the compactness argument needed to apply Prop. 1.1. We have with similar notations

$$S(t,x,s_0,y_0) = \inf\{\sigma(t,s_0,y_0,\psi) \mid \Phi(t,s_0,y_0,\psi) = x\},$$

where σ and Φ are analytic mappings. Let us prove that $S(.,.,s_0,y_0)$ is subanalytic on any compact subset $I\times K$ of $]s_0,T]\times\overline{\Omega}$. First of all notice that extremals are C^1 functions of t for the Legendre mapping is C^1. In particular the function $t \mapsto L(x(t),\dot{x}(t))$ is C^1 along an extremal curve.

The following argument can for instance be found in [15]. By a continuity argument we can assert that the set

$$\left\{ \int_{s_0}^{t} L(x(t), \dot{x}(t))\, dt \mid (t, x(t)) \in I \times K,\ x(.) \text{ extremal},\ x(s_0) = y_0 \right\}$$

is bounded. On the other part, for all extremal curve $x(.)$, there exists from the Mean Value Theorem a real number $t_0 \in [0, T]$ such that

$$\int_{s_0}^{t} L(x(t), \dot{x}(t))\, dt = (t - s_0) L(x(t_0), \dot{x}(t_0)).$$

By continuity of the extremal flow we infer easily that the set

$$\{\dot{x}(0) \mid x(.) \text{ extremal},\ x(s_0) = y_0,\ x(t) = x,\ (t, x) \in I \times K\}$$

is bounded in $\mathbb{R}^n$ (see [15, Chap. 4] for all details). Therefore the same holds for the corresponding initial adjoint vectors $p(0)$. The conclusion is then similar to the proof of Th. 2.1. □

3 Hamilton-Jacobi equations and sub-Riemannian geometry

In this section we investigate Dirichlet problems stemming from sub-Riemannian geometry.

3.1 A brief insight into sub-Riemannian geometry

We first recall a general definition of a sub-Riemannian distance, due to [7]. Let M be a smooth n-dimensional manifold, m an integer such that $1 \leqslant m \leqslant n$, and $f_1, \ldots, f_m$ be smooth vector fields on a manifold M. For all $x \in M$ and $v \in T_x M$, set

$$g(x, v) = \inf \left\{ u_1^2 + \cdots + u_m^2 \mid u_1, \ldots, u_m \in \mathbb{R},\ u_1 f_1(x) + \cdots + u_m f_m(x) = v \right\}.$$

Then $g(x, .)$ is a positive definite quadratic form on the subspace of $T_x M$ spanned by $f_1(x), \ldots, f_m(x)$. Outside this subspace we set $g(x, v) = +\infty$. The form g is called the *sub-Riemannian metric associated to the m-uple of vector fields* $(f_1, \ldots, f_m)$.

Let $\mathcal{AC}([0, 1], M)$ denote the set of absolutely continuous paths in M, defined on $[0, 1]$. We define the length of $\gamma \in \mathcal{AC}([0, 1], M)$ as

$$l(\gamma) = \int_0^1 \sqrt{g(\gamma(t), \dot{\gamma}(t))}\, dt.$$

Finally, the *sub-Riemannian distance associated to the m-uple of vector fields* $(f_1, \ldots, f_m)$, between two points x_0, x_1 in M, is defined as

$$d_{SR}(x_0, x_1) = \inf\{l(\gamma) \mid \gamma \in \mathcal{AC}([0, 1], M),\ \gamma(0) = x_0, \gamma(1) = x_1\}.$$

Consider on the other part the differential system on TM

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i(x(t)) \quad \text{a.e. on } [0,1], \tag{22}$$

where the function $u(.) = (u_1(.), \ldots, u_m(.))$, called *control function*, belongs to $L^2([0,1], \mathbb{R}^m)$. Let $x_0 \in \mathbb{R}^n$, and let $\mathcal{U}$ denote the (open) subset of $L^2([0,1], \mathbb{R}^m)$ such that the solution of (22) starting from x_0 and associated to a control $u \in \mathcal{U}$ is well-defined on $[0,1]$. The mapping $E_{x_0} : u \in \mathcal{U} \longmapsto x(1) \in \mathbb{R}^n$, which to a control u associates the extremity $x(1)$ of the corresponding solution $x(.)$ of (22) such that $x(0) = x_0$, is called *end-point mapping* at the point x_0. It is a smooth mapping. The trajectory $x(.)$ is said to be *singular* if the associated control u is a singular point of the end-point mapping. It is said *minimizing* if it realizes the sub-Riemannian distance between its extremities.

Remark 3.1. A sub-Riemannian manifold is often defined as a triple (M, Δ, g), where M is a n-dimensional manifold, Δ is a distribution of rank m (with $m \leqslant n$), and g is a Riemannian metric on Δ. Here, as in [7], and for the needs of applications, the point of view is more general and the assumption of constant rank is relaxed.

Notice that if the vector fields $(f_1, \ldots, f_m)$ are everywhere linearly independent, then controlled paths solutions of (22) coincide with absolutely continuous paths tangent to the distribution Δ, where

$$\Delta(x) = \text{Span } \{f_1(x), \ldots, f_m(x)\},$$

for all $x \in M$. On the other part the set of absolutely continuous paths which are tangent to Δ is not in general a manifold: its singularities correspond exactly to singular trajectories of the control system (22), which are intrinsic to the distribution Δ.

Existence of singular minimizing trajectories is closely related to subanalyticity of the sub-Riemannian distance, see [1, 3, 25]. Thus the following result is not surprising.

3.2 Statement of the result

Theorem 3.1. *Let Ω denote a bounded subanalytic open subset of $\mathbb{R}^n$, $m \geqslant 1$ be an integer, and let $H(x,p)$ be an Hamiltonian function on $\mathbb{R}^n \times \mathbb{R}^n$ defined as*

$$H(x,p) = \sum_{i=1}^{m} \langle p, f_i(x) \rangle^2, \tag{23}$$

where $f_1, \ldots, f_m$ are analytic vector fields on $\mathbb{R}^n$ satisfying the Hörmander condition

$$\forall x \in \mathbb{R}^n \quad \mathit{Lie}\ (f_1(x), \ldots, f_m(x)) = \mathbb{R}^n, \tag{24}$$

i.e. the Lie algebra spanned by the vector fields has maximal rank at all point. Let $d_{SR}(.,.)$ denote the sub-Riemannian distance associated to the m-uple of vector fields $(f_1, \ldots, f_m)$.

Then for all $y_0 \in I\!R^n$ the function $x \mapsto d_{SR}(x, y_0)$ is a viscosity solution of

$$H(x, \nabla v(x)) - 1 = 0 \text{ in } I\!R^n \backslash \{y_0\}, \ v(y_0) = 0. \tag{25}$$

Moreover under the additional assumption that the control system (22) has no nontrivial singular minimizer starting from y_0, then $d_{SR}(., y_0)$ is subanalytic on $I\!R^n \backslash \{y_0\}$.

Let g denote a subanalytic function on $\Sigma = \partial\Omega$, satisfying the compatibility condition

$$\forall x, y \in \Sigma \quad g(x) - g(y) \leqslant d_{SR}(x, y), \tag{26}$$

and for all $x \in \overline{\Omega}$ set

$$S(x) = \inf_{y \in \Sigma} (g(y) + d_{SR}(x, y)). \tag{27}$$

Then S is continuous on $\overline{\Omega}$ and is the unique viscosity solution of the Dirichlet problem

$$H(x, \nabla v(x)) - 1 = 0 \quad \text{in } \Omega, \ v_{|\Sigma} = g. \tag{28}$$

If morever the control system (22) has no nontrivial singular minimizer starting from Σ, then S is subanalytic on Ω.

Remark 3.2. The same comments as in Remark 2.2 and 2.3 hold again here.

Remark 3.3. If $m \geqslant n$ we are in the Riemannian situation and there exists no singular trajectory. In this case we are in the framework of Th. 2.1.

Remark 3.4. Denote by $\mathcal{F}_m$ the set of m-uples of linearly independent vector fields $(f_1, \ldots, f_m)$, endowed with the C^∞ Whitney topology. If $m \geqslant 3$, there exists an open dense subset of $\mathcal{F}_m$ such that any sub-Riemannian system associated to a m-uple of this subset admits no nontrivial singular minimizer, see [10, 11] (see also [3] for the existence of a dense set only). Hence generically the mapping S is subanalytic on Ω without assuming the absence of singular minimizers.

Remark 3.5. If there exist singular minimizing trajectories then the conclusion on subanalyticity of S may fail, see Section 3.3.

Example 3.1. Let Ω be a subanalytic bounded open set in $\mathbb{R}^3$ and $\Sigma = \partial\Omega$. There exists a unique viscosity solution of the Dirichlet problem

$$\left(\frac{\partial v}{\partial x_1} + x_2 \frac{\partial v}{\partial x_3}\right)^2 + \left(\frac{\partial v}{\partial x_2} - x_1 \frac{\partial v}{\partial x_3}\right)^2 - 1 = 0 \ \text{ in } \Omega, \ v_{|\Sigma} = 0, \tag{29}$$

which is continuous on $\overline{\Omega}$ and subanalytic on Ω.

Indeed it suffices to apply Th. 3.1 to the couple of vector fields in $\mathbb{R}^3$

$$f_1(x_1, x_2, x_3) = \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3}, \ f_2(x_1, x_2, x_3) = \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3}. \tag{30}$$

This case is well-known in sub-Riemannian geometry as the *Heisenberg case*.

Proof of Theorem 3.1. First of all notice that the sub-Riemannian problem is in fact equivalent to the time-optimal problem for the control system

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) f_i(x(t)), \ \sum_{i=1}^{m} u_i(t)^2 \leqslant 1,$$

see [7]. Moreover the sub-Riemannian distance $d_{SR}(x, y)$ is equal to the minimal time $T(x, y)$ needed to join x to y for this system. On the other part the Hörmander condition (24) implies that for each couple (x, y) of points of $\mathbb{R}^n$ there exists a minimizing curve joining x to y, that is $d_{SR}(x, y) < +\infty$ (it is a consequence of Chow's theorem, see [7]).

Let $y_0 \in \mathbb{R}^n$ be fixed. From [5, Prop. 2.3 p. 240] the function $x \mapsto T(x, y_0)$ is a viscosity solution of $H(x, \nabla u(x)) - 1 = 0$ on $\mathbb{R}^n \setminus \{y_0\}$. If moreover the sub-Riemannian system admits no nontrivial singular minimizer starting from y_0 then this function is subanalytic outside y_0, see [1, 3, 25].

Introduce S as in (27) and notice that for all $x, y \in \overline{\Omega}$

$$\begin{aligned} S(x) - S(y) &= \inf_{z \in \Sigma} (d_{SR}(x, z) + g(z)) - \inf_{z \in \Sigma} (d_{SR}(y, z) + g(z)) \\ &\leqslant \sup_{z \in \Sigma} (d_{SR}(x, z) - d_{SR}(x, y)) \\ &\leqslant d_{SR}(x, y), \end{aligned}$$

and thus S is continuous on $\overline{\Omega}$. The fact that S is a viscosity solution of (28) is again a consequence of [5]. Under the compatibility condition (26) uniqueness comes from [6, Th. 2.7 p. 37]. Finally, if there is no nontrivial singular minimizer, Proposition (1.1) implies the subanalyticity of S on Ω. □

3.3 A counterexample to subanalyticity

The counterexample that we are going to construct is based on the so-called *Martinet case* in sub-Riemannian geometry. We briefly recall the context, see [2, 9, 25]. Consider in $\mathbb{R}^3$ the two vector fields

$$f_1(x_1, x_2, x_3) = \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_3}, \ f_2(x_1, x_2, x_3) = \frac{\partial}{\partial x_2}.$$

There are only two singular trajectories starting at time $t = 0$ from the origin, namely $t \mapsto (t, 0, 0)$ and $t \mapsto (-t, 0, 0)$, which are moreover minimizing. The associated sub-Riemannian distance to the origin, namely the application $d_{SR}(0, .)$, can be proved to be not subanalytic along the axis $(0x_1)$, and moreover the shape of the sub-Riemannian spheres $\mathcal{S}_{SR}(0, r)$ near a point of this axis is well-known, see Fig. 2.

Here in the framework of Th. 3.1 we need rather consider a distance to a set. Let us explain the idea by analogy with the euclidian distance to a set in the plane: it is quite clear in this case how to construct a corner-shaped bounded open set Ω such that there exist a point $y \in \partial\Omega$ (at the corner), a point $x \in \Omega$

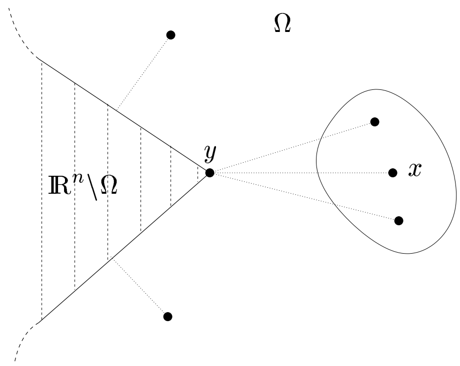

Figure 1: Analogy with the Euclidian distance.

and a neighborhood of x in which the distance to $\partial\Omega$ is equal to the distance to the point y, see Fig. 1.

Analogously, in the Martinet case we claim that there exists an analytic open bounded set $\Omega \subset \mathbb{R}^n$ containing the origin in its interior, such that the function

$$S(x) = \inf_{y \in \partial\Omega} d_{SR}(x, y)$$

can be written, in some neighborhood of the origin, as

$$S(x) = d_{SR}(x, A),$$

where the point A is defined as $A = (r, 0, 0)$, $r \neq 0$, see Fig. 2.

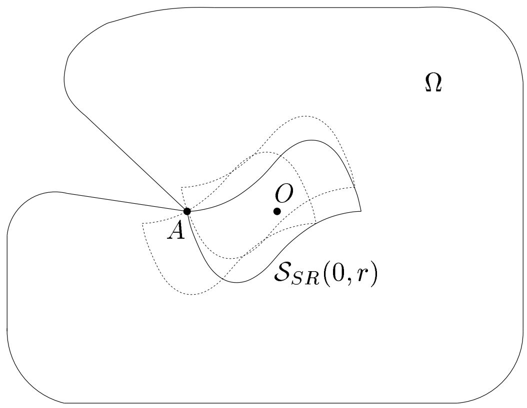

Figure 2: The Martinet case.

From Th. 3.1, S is the unique viscosity solution of the Dirichlet problem

$$\left(\frac{\partial v}{\partial x_1}+x_2^2\frac{\partial v}{\partial x_3}\right)^2+\left(\frac{\partial v}{\partial x_2}\right)^2-1=0 \text{ in } \Omega,\ v_{|\partial\Omega}=0.$$

The corresponding Hamiltonian

$$H(x,p)=(p_1+p_3x_2^2)^2+p_2^2$$

is analytic on $\mathbb{R}^n\times\mathbb{R}^n$. Anyway S is not subanalytic in a neighborhood of the origin, on the axis $(0x_1)$.

Remark 3.6. It is not difficult to construct explicitly the set Ω since we know precisely the asymptotics of the sub-Riemannian spheres with small radius, see [9].

In this example, all data are analytic and anyway the unique viscosity solution is not subanalytic, due to the existence of singular minimizing trajectories. This phenomenon is not exceptional and is, in a sense, generic, as explained in Rem. 3.8. In fact the Martinet case can be imbedded into generic distributions (see [3]).

3.4 Regularity on the boundary

Definition 3.1. Let $(f_1,\ldots,f_m)$ be a m-uple of smooth vector fields on $\mathbb{R}^n$. For all $x\in\mathbb{R}^n$ set

$$\begin{aligned}\Delta(x)&=\mathrm{Span}\{f_1(x),\ldots,f_m(x)\},\\ \Delta^2(x)&=\Delta(x)+\mathrm{Span}\{[f_i,f_j](x),\ 1\leqslant i<j\leqslant m\}.\end{aligned}$$

The m-uple $(f_1,\ldots,f_m)$ is said to be *medium-fat* at x if for any vector field $X\in\Delta(x)\setminus\{0\}$ there holds

$$\mathbb{R}^n=\Delta^2(x)+\mathrm{Span}\{[X,[f_i,f_j]](x),\ 1\leqslant i,j\leqslant m\}.$$

Proposition 3.2. *In the framework of Theorem 3.1, if the m-uple of vector fields $(f_1,\ldots,f_m)$ is moreover medium-fat at all point of Σ then S is subanalytic on the whole $\overline{\Omega}$.*

Indeed in this case and each mapping $x\mapsto d_{SR}(x,y_0)$ is moreover subanalytic at y_0, see [3].

Remark 3.7. If $n\leqslant m(m-1)+1$ then the m-uple of vector fields $(f_1,\ldots,f_m)$ is generically (in C^∞ Whitney topology) medium-fat, see [4].

Example 3.2. It can be easily checked that the system of vector fields (30) of Example 3.1 is medium-fat on $\mathbb{R}^3$, and hence the solution S of the Dirichlet problem (29) is moreover subanalytic on $\overline{\Omega}$.

Remark 3.8. The function S may happen not to be subanalytic on Σ. Indeed, a result from [3] asserts that for generic m-uples of vector fields $(f_1, \dots, f_m)$ in $\mathbb{R}^n$, if $m \geqslant 3$ and $n \leqslant (m-1)(\frac{m^2}{3} + \frac{5m}{6} + 1)$, then the sub-Riemannian distance at 0, namely the function $x \mapsto d_{SR}(0, x)$, is not subanalytic at 0, whereas sub-Riemannian spheres $\mathcal{S}_{SR}(0, r)$ with small radius $r > 0$ are subanalytic. For such data, let $B_{SR}(0, r)$ denote the sub-Riemannian ball centered at 0 and with radius r, and set

$$\Omega = B_{SR}(0, r) \setminus \{0\}.$$

In these conditions the boundary of Ω

$$\Sigma = \{0\} \cup \mathcal{S}_{SR}(0, r),$$

is subanalytic. For all $x \in \Omega$, define

$$S(x) = \inf\left(d_{SR}(0, x), d_{SR}(x, \mathcal{S}_{SR}(0, r))\right),$$

and for all $x \in \Sigma$ set $g(x) = 0$. Then S is the unique viscosity solution of (28) for the Hamiltonian function corresponding to $(f_1, \dots, f_m)$, and S is not subanalytic at 0.

4 Hamilton-Jacobi equations and affine control systems

4.1 A brief insight into optimal control theory

Consider a general control system in $\mathbb{R}^n$:

$$\dot{x}_u(t) = f(x_u(t), u(t)), \tag{31}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is smooth, and the controls u belong to an open subset of $L^\infty_{loc}(\mathbb{R}, \mathbb{R}^m)$. For all $T > 0$ denote by $\mathcal{U}_T$ the set of admissible controls on $[0, T]$, i.e. the set of controls such that the associated trajectory $x_u(.)$ is well-defined on $[0, T]$. It is an open subset of $L^\infty([0, T], \mathbb{R}^m)$. For all $x_0 \in \mathbb{R}^n$ define on $\mathcal{U}_T$ the *end-point mapping* $E_{x_0,T} : u \mapsto x_u(T)$, where $x_u(.)$ denotes the solution of (31) associated to the control $u \in \mathcal{U}_T$ and starting from x_0 at time $t = 0$. It is a smooth mapping. A trajectory $x_u(.)$ is said to be *singular* on $[0, T]$ if u is a singular point of the end-point mapping $E_{x_0,T}$.

Let now M_1 be a submanifold of $\mathbb{R}^n$, and consider the optimal control problem of determining, among all trajectories solutions of system (31) joining x_0 to M_1, a trajectory minimizing the *cost function*

$$C(t, u) = \int_0^t f^0(x_u(s), u(s))ds + g(x_u(t)), \tag{32}$$

where $f^0 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are smooth functions, and $x_u(t) \in M_1$. Set moreover

$$C_0(t, u) = \int_0^t f^0(x_u(s), u(s))ds.$$

If a control u, associated to a trajectory $x_u(.)$, is optimal on $[0,T]$, then there exists a nontrivial *Lagrange multiplier* $(\psi,\psi^0)\in\mathbb{R}^n\times\mathbb{R}$ such that

$$\psi.dE_{x_0,T}(u)=-\psi^0\frac{\partial C_0}{\partial u}(T,u).$$

Moreover $\psi-\psi^0\nabla g(x_u(T))\perp T_{x_u(T)}M_1$. This is a first-order necessary condition for optimality. The well-known *Pontryagin Maximum Principle* (see [21]) parametrizes this condition, and asserts that the trajectory $x_u(.)$ corresponding to this control is the projection of an *extremal*, i.e. a solution of the Hamiltonian system

$$\dot{x}_u=\frac{\partial H}{\partial p}(x_u,p_u,p_u^0,u),\ \dot{p}_u=-\frac{\partial H}{\partial x}(x_u,p_u,p_u^0,u),\ \frac{\partial H}{\partial u}(x_u,p_u,p_u^0,u)=0,$$

where $H(x,p,p^0,u)=\langle p,f(x,u)\rangle+p^0f^0(x,u)$ is the Hamiltonian of the system, $p_u(.):[0,T]\to\mathbb{R}^n$ is an absolutely continuous mapping called *adjoint vector*, and p_u^0 is a real nonpositive number. Moreover there holds at the final time, up to a multiplying scalar

$$(p_u(T),p_u^0)=(\psi,\psi^0).$$

If $p_u^0\neq 0$ the extremal is said to be *normal*, and in this case it is normalized to $p_u^0=-1/2$. If $p_u^0=0$ the extremal is said to be *abnormal*.

Remark 4.1. Since we did not set any constraint on the control, any singular trajectory is the projection of an abnormal extremal, and conversely.

Remark 4.2. As a consequence of the Maximum Principle, if a control u is singular on $[0,T]$ then it is singular on $[0,t]$, for all $t\in]0,T]$.

4.2 Affine Dirichlet problem

4.2.1 Statement of the results

Theorem 4.1. *Let Ω be a bounded subanalytic open subset of $\mathbb{R}^n$, $c>0$ be fixed, and $f_0,\dots,f_m$ be analytic vector fields on $\overline{\Omega}$. For all $x\in\Omega$ and $p\in\mathbb{R}^n$ set*

$$H(x,p)=-\langle p,f_0(x)\rangle+\frac{1}{4}\sum_{i=1}^m\langle p,f_i(x)\rangle^2-c.$$

Let $\Sigma=\partial\Omega$ and g be a subanalytic function on Σ. For all $x\in\overline{\Omega}$, consider the optimal control problem of steering x to Σ for the affine control system

$$\dot{x}_u(t)=f_0(x_u(t))+\sum_{i=1}^m u_i(t)f_i(x_u(t)),\tag{33}$$

and the cost

$$C(u)=\int_0^{t(x,u)}\Big(\sum_{i=1}^m u_i(s)^2+c\Big)\,ds+g(x_u(t(x,u))),\tag{34}$$

where $t(x,u)$ is the infimum of times t such that the solution $x_u(.)$ of the control system (33) associated to the control u steers the point $x \in \overline{\Omega}$ to Σ in time t. We make the following assumptions.

- *The boundary Σ is accessible from Ω, i.e. for any $x \in \Omega$ there exists a time t and a control on $[0,t]$ such that the solution of the system (33) associated to this control and starting from x at time 0, joins Σ in time t.* (35)

- *There exists no singular minimizing trajectory of the control system (33) for the cost (34), steering Ω to Σ.* (36)

Let $S(x)$ denote the value function associated to the optimal control problem (33), (34). Namely, if $\mathcal{S}$ denotes the set of solutions $(u(.),x(.))$ of (33) defined on various intervals $[0,t_1]$, such that $x(0) \in \overline{\Omega}$ and $x(t_1) \in \Sigma$, one has, for all $x \in \overline{\Omega}$,

$$S(x) = \inf \{C(u) \mid (u(.),x_u(.)) \in \mathcal{S},\ x_u(0) = x\}\,. \tag{37}$$

For all $x,z \in \Sigma$ define

$$L(x,z) = \inf \left\{ \int_0^t \Big(\sum_{i=1}^m u_i(s)^2 + c \Big)\, ds \mid x_u(.) \in \mathcal{S},\ x_u(0) = x,\ x_u(t) = z, \right.$$
$$\left. \forall s \in [0,t] \quad x_u(s) \in \overline{\Omega} \right\}, \tag{38}$$

and assume that g satisfies the compatibility condition

$$\forall x,z \in \Sigma \quad g(x) - g(z) \leqslant L(x,z). \tag{39}$$

Then S is well defined on $\overline{\Omega}$, is continuous and subanalytic on Ω, and is a viscosity solution of the Dirichlet problem

$$H(x, \nabla S(x)) = 0 \ \text{on}\ \Omega,\ S_{|\Sigma} = g. \tag{40}$$

Remark 4.3. Denote by $\mathcal{F}_m$ the set of $(m+1)$-uples of linearly independent vector fields $(f_0,\dots,f_m)$, endowed with the C^∞ Whitney topology. If $2 \leqslant m < n$, there exists an open dense subset of $\mathcal{F}_m$ such that any affine control system associated to a (m+1)-uple of this subset admits no singular minimizing trajectory. This is indeed an obvious adaptation of a result of [11] (see also [10, Cor. 3.4]).

As previously we can state further results concerning uniqueness and regularity of S on the whole $\overline{\Omega}$.

Proposition 4.2. *Under the assumptions of Theorem 4.1, if moreover*

$$\forall x \in \Sigma \quad Lie\ (f_1(x),\dots,f_m(x)) = \mathbb{R}^n, \tag{41}$$

then S is continuous on $\overline{\Omega}$. As a consequence S is the unique viscosity solution of the Dirichlet problem (40).

Proposition 4.3. *Under the assumptions of Theorem 4.1, if the m-uple of vector fields $(f_1, \ldots, f_m)$ is moreover medium-fat (see Def. 3.1) at all point of Σ, and if the compatibility inequality (39) is strict, then S is subanalytic on $\overline{\Omega}$.*

Remark 4.4. Assumption (41) actually implies that the affine control system (33) is STCΣ, i.e. small-time controllable near Σ (see [5] for a definition). For a proof of this fact we refer the reader for instance to [8, 19]. In [5] where the controls take their values in a compact subset of $\mathbb{R}^m$, this small-time controllability property is used to prove that S is upper semi-continuous. Their proof is however not adapted in our framework since our controls are not constrained (see also Rem. 4.10).

On the other part the compatibility condition of [5] is stronger than (39), and, coupled with small-time controllability, enables to prove that the value function S is continuous. Here the continuity of S is mainly due to the assumption (36) of the absence of singular minimizers.

Example 4.1. Let Ω be a subanalytic bounded open subset of $\mathbb{R}^3$ and $\Sigma = \partial\Omega$. For all $c > 0$ there exists a unique viscosity solution S of the Dirichlet problem

$$-\frac{\partial v}{\partial x_1} + \frac{1}{4}\left(\frac{\partial v}{\partial x_1} + x_2\frac{\partial v}{\partial x_3}\right)^2 + \frac{1}{4}\left(\frac{\partial v}{\partial x_2} - x_1\frac{\partial v}{\partial x_3}\right)^2 - c = 0 \text{ in } \Omega,\ v_{|\Sigma} = 0,$$

which is continuous and subanalytic on $\overline{\Omega}$. It is indeed an application of Th. 4.1, Prop. 4.2 and 4.3, with the vector fields

$$f_0 = \frac{\partial}{\partial x_1},\ f_1 = \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3},\ f_2 = \frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3}.$$

4.2.2 Proof of Theorem 4.1

First of all, assumption (35) implies that S is well-defined on $\overline{\Omega}$. Let us prove that the infimum in the definition (37) of S is actually a minimum. This is a consequence of the following lemma on the existence of optimal controls.

Lemma 4.4. *For any $x \in \overline{\Omega}$ there exists a control u minimizing the cost (34), such that the associated trajectory $x_u(.)$ satisfies $x_u(0) = x$, $x_u(t(x,u)) \in \Sigma$.*

Proof of Lemma 4.4. If $x \in \Sigma$ the conclusion is immediate. If $x \in \Omega$, consider a sequence of controls $(u_n)_{n\in\mathbb{N}}$, where $u_n \in L^2([0,t_n],\mathbb{R}^m)$, such that $C(u_n)$ converges towards the infimum $S(x)$. Here we denote $t_n = t(x,u_n)$. Assumption (H_1) implies that the sequence of real numbers $(C(u_n))_{n\in\mathbb{N}}$ is bounded. Since g is bounded on the compact set Σ and $c > 0$, one easily gets on the one part that the sequence of real numbers $(t_n)_{n\in\mathbb{N}}$ is bounded, and on the other part that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T],\mathbb{R}^m)$, where T is a real greater than each t_n, and where $u_n(.)$ is extended by 0 on $]t_n,T]$. Hence, up to a subsequence, we can assert that t_n tends to a real number $t > 0$ and u_n tends to a control $u \in L^2([0,t],\mathbb{R}^m)$ in the weak L^2-topology. Since the control system (33) is affine, it is not difficult to prove that the sequence $(x_{u_n}(.))_{n\in\mathbb{N}}$

converges uniformly towards $x_u(.)$ on $[0,t]$ (see [24] for details), where $x_u(.)$ is the trajectory associated to the control u. In particular $x_u(s) \in \overline{\Omega}$ for all $s \in [0,t]$, and since $x_{u_n}(t_n) \in \Sigma$ we infer that $x_u(t) \in \Sigma$.

On the other part, using the weak convergence of u_n towards u, one has

$$\int_0^t (\sum_{i=1}^m u_i(s)^2 + c)ds + g(x_u(t)) \ \leqslant\ \liminf C(u_n) = S(x),$$

and since $S(x)$ is an infimum this inequality is actually an equality. Thus the control u steers the system from x to Σ in time t, with a cost $S(x)$. By definition of $t(x,u)$ it is clear that $t(x,u) \leqslant t$, anyway equality does not hold necessarily. At this stage we need the compatibility assumption (39), which yields the inequality

$$g(x_u(t(x,u))) - g(x_u(t)) \leqslant L(x_u(t(x,u)), x_u(t)).$$

Moreover from the definition of L we get

$$L(x_u(t(x,u)), x_u(t)) \leqslant \int_{t(x,u)}^t \Big(\sum_{i=1}^m u_i(s)^2 + c\Big)\, ds,$$

Hence

$$\int_0^{t(x,u)} \Big(\sum_{i=1}^m u_i(s)^2 + c\Big)\, ds + g(x_u(t(x,u))) \ \leqslant\ \int_0^t \Big(\sum_{i=1}^m u_i(s)^2 + c\Big)\, ds + g(x_u(t)) = S(x),$$

and as previously this inequality is actually an equality, i.e.

$$S(x) = \int_0^{t(x,u)} \Big(\sum_{i=1}^m u_i(s)^2 + c\Big)\, ds + g(x_u(t(x,u))),$$

which proves that the control u is minimizing. □

Lemma 4.5. *The function S is continuous on Ω.*

Proof of Lemma 4.5. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points of Ω converging towards $x \in \Omega$, and let us prove that $S(x_n)$ tends to $S(x)$. First of all, let us show that the sequence $(S(x_n))_{n\in\mathbb{N}}$ is bounded. From Lemma 4.4, there exists an optimal control u steering the system from x to Σ in time $t(x,u)$, and from assumption (H_2) this control cannot be singular. Hence the end-point mapping $E_{x,t(x,u)}$ is a submersion at the point u, and consequently the equation

$$E_{y,t(x,u)}(v) = x_u(t(x,u))$$

may be solved in v in a L^2-neighborhood of u, and for all $y \in \Omega$ close to x. In particular if n is large enough there exists a control u_n close to u in L^2-topology such that

$$E_{x_n,t(x,u)}(u_n) = x_u(t(x,u)).$$

From the previous equality, u_n is a control steering x_n to Σ in time $t(x,u)$, and since u_n is close to u in L^2-topology it is not difficult to infer that the sequence $(S(x_n))_{n\in\mathbb{N}}$ is bounded.

Let now a be a cluster point of this sequence. To finish the proof we shall show that $S(x)=a$, i.e. a is the unique cluster point of $(S(x_n))_{n\in\mathbb{N}}$. First, we can assert that up to a subsequence $S(x_n)$ tends to a. From Lemma 4.4 there exists for any integer n a minimizing control u_n such that $S(x_n)=C(u_n)$. A reasoning similar to the proof of Lemma 4.4, using the compatibility condition (39), yields the inequality

$$S(x)\leqslant a.$$

Conversely, similarly as to prove that the sequence $(S(x_n))_{n\in\mathbb{N}}$ is bounded, and using again (39), one states

$$a=\liminf S(x_n)\leqslant C(u)=S(x),$$

which ends the proof. $\square$

Lemma 4.6. *The function S is subanalytic on Ω.*

Proof of Lemma 4.6. Let us prove that S is subanalytic on any subanalytic compact subset K of Ω. For all $x\in K$, there exists from Lemma 4.4 a minimizing control u such that $S(x)=C(u)$, which cannot be singular by assumption. From the Pontryagin maximum principle, the associated trajectory $x_u(.)$ is the projection of a normal extremal, i.e. there exists an adjoint vector $p_u(.)$ on $[0,t(x,u)]$, each $p_u(t)$ being identified to a row vector, and a constant $p_u^0=-1/2$ such that there holds

$$\begin{aligned}
\dot{x}_u(t)&=f_0(x_u(t))+\sum_{i=1}^m u_i(t)f_i(x_u(t)),\\
\dot{p}_u(t)&=-p_u(t)\frac{\partial f_0}{\partial x}(x_u(t))-\sum_{i=1}^m u_i(t)p_u(t)\frac{\partial f_i}{\partial x}(x_u(t)),
\end{aligned}$$

almost everywhere on $[0,t(x,u)]$, with

$$u_i(t)=p_u(t)f_i(x_u(t)).$$

Hence normal extremals are solutions of the differential system

$$\dot{x}(t)=\frac{\partial H_1}{\partial p}(x(t),p(t)),\ \dot{p}(t)=-\frac{\partial H_1}{\partial x}(x(t),p(t)),$$

where

$$H_1(x,p)=\langle p,f_0(x)\rangle+\frac{1}{2}\sum_{i=1}^m\langle p,f_i(x)\rangle^2.$$

This is an analytic differential system in (x,p) parametrized by the initial condition $p(0)=p_0$. In particular normal extremals $(x(.),p(.))$ such that $x(0)=x_0$

depend *analytically* on the initial condition p_0. Notice moreover that the mapping

$$\begin{array}{rcl} \Phi : \mathbb{R}^n & \longrightarrow & L^2([0,T],\mathbb{R}^m) \\ p_0 & \longmapsto & u_{p_0}\ , \end{array} \tag{42}$$

needed further, where u_{p_0} denotes the corresponding normal control, is *analytic*.

On the other part we have at the final time $t(x,u)$ the Lagrange multipliers identity

$$p_u(t(x,u))dE_{x,t(x,u)}(u) = \frac{1}{2}\frac{\partial C_0}{\partial u}(t(x,u),u), \tag{43}$$

where

$$C_0(t,u) = \int_0^t \Big(\sum_{i=1}^m u_i(s)^2 + c\Big)\, ds.$$

It is clear that for all control $u \in L^2([0,T],\mathbb{R}^m)$ the differential of C_0 with respect to u writes

$$\forall v \in L^2([0,T],\mathbb{R}^m) \quad \frac{\partial C_0}{\partial u}(T,u).v = 2\int_0^T \sum_{i=1}^m u_i(t)v_i(t)dt.$$

Hence, identifying L^2 with its dual space, equation (43) leads to the equality in $L^2([0,t(x,u)],\mathbb{R}^m)$

$$p_u(t(x,u))dE_{x,t(x,u)}(u) = u. \tag{44}$$

Let us now prove the following fact.

Fact 1. The set $\{p_u(t(x,u)) \mid x \in K \text{ and } S(x) = C(u)\}$ is compact in $\mathbb{R}^n$.

If not, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points of K converging towards $x \in K$, such that

$$\|p_{u_n}(t_n)\| \to +\infty,$$

where u_n is an optimal (necessarily non singular) control steering x_n to Σ in time $t_n = t(x_n,u_n)$. Since S is continuous on K, the sequence $(S(x_n))_{n\in\mathbb{N}}$ is bounded, and thus up to a subsequence we can assume that t_n tends to a real t and u_n tends to $u \in L^2([0,t],\mathbb{R}^m)$ in weak L^2-topology. From (44) there holds for all integer n

$$p_{u_n}(t_n)dE_{x_n,t_n}(u_n) = u_n. \tag{45}$$

Up to a subsequence we may assume that

$$\frac{p_{u_n}(t_n)}{\|p_{u_n}(t_n)\|} \to \psi,$$

where $\psi \in \mathbb{R}^n \setminus \{0\}$. Taking the limit in (45), we infer

$$\psi dE_{x,t}(u) = 0. \tag{46}$$

Indeed, for an affine control system, the end-point mapping and its differential can be easily proved to be continuous with respect to the weak L^2-topology, see

[24] for a proof and for a similar reasoning. On the other part, since x_n tends to x, we get

$$E_{x,t}(u) \in \Sigma. \tag{47}$$

As a consequence of (46) and (47), u is a singular control on $[0,t]$ steering x to Σ in time t. It is still singular on $[0,t(x,u)]$; let us prove that it is optimal on this interval. Indeed, each control u_n is optimal, i.e. $C(u_n) = S(x_n)$. Since t_n tends to t and u_n tends to u in weak L^2-topology we get

$$C_0(t,u) + g(x_u(t)) \leqslant \liminf S(x_n) = S(x),$$

for S is continuous. Moreover using the compatibility condition (39) leads to

$$C(u) = C_0(t(x,u),u) + g(x_u(t(x,u))) \leqslant C_0(t,u) + g(x_u(t)) \leqslant S(x),$$

thus this inequality is actually an equality, i.e. u is optimal. This contradiction with assumption (H_2) ends the proof of Fact 1.

Fact 2. The set $P_0 = \{p_u(0) \mid x \in K \text{ and } S(x) = C(u)\}$ is compact in $\mathbb{R}^n$.

The proof of this fact is a consequence of the continuity of the extremal flow (for more details see [24, Lemma 4.9]).

Let us now end the proof of Lemma 4.6. Let A be a subanalytic compact subset of $\mathbb{R}^n$ containing the set P_0. Using the mapping Φ defined as (42) we can express for all $x \in K$ the final time $t(x,u) = \inf\{t \mid E_{x,t}(u) \in \Sigma\}$ restricted to minimizing controls as a function of p_0

$$\tau(x,p_0) = t(x,\Phi(p_0)) = \inf\{t \mid (E_{x,t} \circ \Phi)(p_0) \in \Sigma\},$$

where $p_0 \in A$. It is clear from Prop. 1.1 that this mapping is subanalytic on $K \times A$. Let us further express in this way the value function

$$S(x) = \inf\{C(u) \mid E_{x,t(x,u)}(u) \in \Sigma\}.$$

To this aim set

$$\mathcal{C}(x,p_0) = C(\Phi(p_0)).$$

The function $\mathcal{C}$ is clearly subanalytic on $K \times A$. We have

$$S(x) = \inf\{\mathcal{C}(x,p_0) \mid p_0 \in A \cap (E_{x,\tau(x,p_0)} \circ \Phi)^{-1}(\Sigma)\},$$

and Prop. 1.1 implies that S is subanalytic on K. □

To finish the proof of Theorem 4.1, it remains to prove that S is a viscosity solution of the Dirichlet problem (40). But since S is continuous from Lemma 4.5, this is a consequence of [5, Prop. 3.12 p. 255].

4.2.3 Proof of Proposition 4.2

Let us prove that S is upper and lower semi-continuous on $\overline{\Omega}$.

Lemma 4.7. *The function S is upper semi-continuous on $\overline{\Omega}$.*

Proof. Let $z \in \Sigma$ be fixed. We have to prove that

$$\limsup_{x\in\Omega, x\to z} S(x) \leqslant S(z) = g(z).$$

For all $x \in \Omega$, we have by definition

$$S(x) = \inf\left\{ \int_0^{t(x,u)} \Big(\sum_{i=1}^m u_i(s)^2 + c\Big)\, ds + g(x_u(t(x,u))) \mid x_u(0) = x \right\},$$

and on the other part the compatibility condition (39) implies that for all control u and all $t > t(x,u)$ such that $x_u(t) \in \Sigma$ and $x_u(s) \in \overline{\Omega}$, $s \in [0,t]$, there holds

$$\int_0^{t(x,u)} \Big(\sum_{i=1}^m u_i(s)^2 + c\Big)\, ds + g(x_u(t(x,u))) \leqslant \int_0^{t} \Big(\sum_{i=1}^m u_i(s)^2 + c\Big)\, ds + g(x_u(t)).$$

Hence

$$S(x) = \inf\left\{ \int_0^{t} \Big(\sum_{i=1}^m u_i(s)^2 + c\Big)\, ds + g(x_u(t)) \mid t > 0, x_u(0) = x, x_u(t) \in \Sigma, \right.$$
$$\left. \forall s \in [0,t] \quad x_u(s) \in \overline{\Omega} \right\}.$$

For all $x \in \overline{\Omega}$, $y \in \Sigma$, and $t > 0$, define

$$S(t,x,y) = \inf\left\{ \int_0^t \sum_{i=1}^m u_i(s)^2 ds \mid x_u(0) = x, x_u(t) = y \right\}$$

(with the agreement that $S(t,x,y) = +\infty$ if there does not exist any trajectory $x_u(.)$ joining x to y), so that

$$S(x) = \inf\left\{ S(t,x,y) + ct + g(y) \mid t > 0, y \in \Sigma \right\}.$$

In particular we have for all $t > 0$

$$S(x) \leqslant S(t,x,z) + ct + g(z). \tag{48}$$

On the other part, we claim that there exist a neighborhood of z in $\overline{\Omega}$ and positive real numbers α_1, α_2, such that if $t > 0$ is small enough then

$$S(t,x,z) \leqslant \frac{d_{SR}(x,z)^2}{t} + \alpha_1 d_{SR}(x,z) + \alpha_2 t, \tag{49}$$

where $d_{SR}(.,.)$ denotes the sub-Riemannian distance associated to the m-uple of vector fields $(f_1,\dots,f_m)$. This inequality is proved apart in the next lemma. From (48) and (49) we infer easily

$$\limsup_{x\in\Omega, x\to z} S(x) \leqslant g(z).$$

□

Lemma 4.8. *For all $z\in\Sigma$ there exist a neighborhood V of z and positive real numbers $\varepsilon,\alpha_1,\alpha_2$, such that for all $x\in V\cap\overline{\Omega}$ and $t\in]0,\varepsilon[$ there holds*

$$S(t,x,z) = \frac{d_{SR}(x,z)^2}{t} + R(t,x),$$

with

$$|R(t,x)| \leqslant \alpha_1 d_{SR}(x,z) + \alpha_2 t.$$

Proof. Let $z\in\Sigma$ be fixed. First of all, it was already noticed in Remark 4.4 that assumption (41) implies that the control affine system (33) is small-time locally controllable near Σ, and thus if x is close enough to z and $t>0$ is small enough then $S(t,x,z)<+\infty$. On the other part, under assumption (41) the sub-Riemannian distance $x\mapsto d_{SR}(x,z)$ associated to the m-uple of vector fields $(f_1,\dots,f_m)$ is well-defined in a neighborhood of z.

For all solution $x_u(.)$ of the affine system (33) defined on $[0,t]$ and steering a point $x\in V$ to $z\in\Sigma$, set

$$\forall s\in[0,1]\quad \tilde{x}(s) = x_u(st) \quad\text{and}\quad \tilde{u}_i(s) = tu_i(ts),\ i=1,\dots,m.$$

Then we have

$$\tilde{x}'(s) = tf_0(\tilde{x}(s)) + \sum_{i=1}^m \tilde{u}_i(s) f_i(\tilde{x}(s)) \quad \text{a.e. on } [0,1], \tag{50}$$

and $\tilde{x}(0)=x, \tilde{x}(1)=z$. If the parameter $t>0$ is small, the control system (50) can be considered as a perturbation of the sub-Riemannian system associated to $(f_1,\dots,f_m)$.

By assumption, minimizing trajectories solutions of (50), steering a point $x\in V\cap\Omega$ to $z\in\Sigma$, cannot be singular, and thus are associated to normal controls. Hence from the Pontryagin Maximum Principle we have for all $i\in\{1,\dots,m\}$ and $s\in[0,1]$

$$\tilde{u}_i(s) = \langle \tilde{p}(s), f_i(\tilde{x}(s))\rangle,$$

where $\tilde{p}(.):[0,1]\to\mathbb{R}^n$ is an absolutely continuous function which satisfies

$$\tilde{p}'(s) = -\tilde{p}(s)\left(t\frac{\partial f_0}{\partial x}(\tilde{x}(s)) + \sum_{i=1}^m \tilde{u}_i(s)\frac{\partial f_i}{\partial x}(\tilde{x}(s))\right) \quad \text{a.e. on } [0,1]. \tag{51}$$

When $t=0$, the system (50), (51), still makes sense, and parametrizes normal extremals steering x to z for the sub-Riemannian structure associated to the m-uple $(f_1,\dots,f_m)$. As this system depends analytically on the parameter t and on the initial condition x, up to reducing V we can write, for all $i\in\{1,\dots,m\}$

$$\tilde{u}_i(s)=\tilde{u}_i^0(s)+t\,r_i(s),$$

where $r_i(.)$ is uniformly bounded as $s\in[0,1]$, $t\in[0,\varepsilon]$ and $x\in V$. Hence we get

$$\begin{aligned} t\int_0^t\sum_{i=1}^m u_i(\tau)^2d\tau &= \int_0^1\sum_{i=1}^m\tilde{u}_i(s)^2ds \\ &= \int_0^1\sum_{i=1}^m\tilde{u}_i^0(s)^2ds+2t\int_0^1\sum_{i=1}^m\tilde{u}_i^0(s)r_i(s)ds+t^2\int_0^1\sum_{i=1}^m r_i(s)^2ds, \end{aligned}$$

from which it is not difficult to infer that

$$tS(t,x,z)=d_{SR}(x,z)^2+tR(t,x),$$

where $R(t,x)$ satisfies the inequality announced in the statement of the lemma. $\square$

Lemma 4.9. *The function S is lower semi-continuous on $\overline{\Omega}$.*

Proof. We have to prove that for all $x\in\Sigma$ and all sequence $(x_n)_{n\in\mathbb{N}}$ of points of Ω converging towards x there holds

$$S(x)\leqslant\liminf S(x_n).$$

Let us notice again that assumption (41) implies that the affine control system (33) is locally controllable near Σ, hence clearly the sequence $(S(x_n))_{n\in\mathbb{N}}$ is bounded. A reasoning similar to proof of Lemma 4.5 permits to conclude. $\square$

Lemmas 4.7 and 4.9 prove the continuity of S on the whole $\overline{\Omega}$. Uniqueness is then a consequence of [6, Th. 2.7 p. 37] or [5, Prop. 3.13 p. 256].

4.2.4 Proof of Proposition 4.3

For all $x\in\Omega$ there exists a minimizing control u (not necessarily unique) steering x to Σ in time $t(x,u)$. By the choice axiom, we contruct a function $\tau(.)$ on Ω which to $x\in\Omega$ associates $\tau(x)=t(x,u)$ for a choice of such a control u. The following lemma is a consequence of the strict compatibility inequality which is now further assumed.

Lemma 4.10. *Under the strict compatibility assumption, the time $\tau(x)$ tends to 0 as $x\in\Omega$ tends to Σ.*

Proof. If not there exists $x \in \Sigma$ and a sequence $(x_n)_{n\in\mathbb{N}}$ of points of Ω converging to x such that $\tau(x_n)$ does not tend to 0. Since S is continuous on $\overline{\Omega}$ from Prop. 4.2, the sequence $(\tau(x_n))_{n\in\mathbb{N}}$ is clearly bounded, and hence up to a subsequence it converges to a real $t > 0$. For all integer n let u_n be a minimizing control steering x_n to Σ in time $t(x_n, u_n) = \tau(x_n)$, and let $x_{u_n}(.)$ denote the associated trajectory. As in the proof of Lemma 4.6, we may assume that u_n tends to a control u in weak L^2-topology, and this yields the inequality

$$\int_0^t \Big(\sum_{i=1}^m u_i(s)^2 + c\Big)\, ds + g(x_u(t)) \leqslant \liminf S(x_n) = S(x) = g(x). \tag{52}$$

Moreover $x_{u_n}(.)$ tends uniformly to $x_u(.)$, and hence $x_u(.)$ is a trajectory contained in $\overline{\Omega}$ steering $x \in \Sigma$ to $x_u(t) \in \Sigma$. According to the strict compatibility assumption there must hold

$$g(x) - g(x_u(t)) < \int_0^t \Big(\sum_{i=1}^m u_i(s)^2 + c\Big)\, ds,$$

and this contradicts inequality (52). $\square$

For all $x \in \Omega$ there exists a minimizing control u steering x to Σ in time $\tau(x)$, and the associated trajectory $x_u(.)$ satisfies

$$\dot{x}_u(t) = f_0(x_u(t)) + \sum_{i=1}^m u_i(t) f_i(x_u(t)) \quad \text{a.e. on } [0, \tau(x)].$$

As in the proof of Lemma 4.8, let us reparametrize this control system on $[0,1]$, by setting, for all $s \in [0,1]$,

$$\tilde{x}(s) = x_u(s\tau(x)) \quad \text{and} \quad \tilde{u}_i(s) = \tau(x) u_i(s\tau(x)),\ i = 1, \dots, m,$$

so that all extremal is solution of the system

$$\begin{aligned}
\tilde{x}'(s) &= \tau(x) f_0(\tilde{x}(s)) + \sum_{i=1}^m \tilde{u}_i(s) f_i(\tilde{x}(s)),\\
\tilde{p}'(s) &= -\tilde{p}(s)\left(\tau(x)\frac{\partial f_0}{\partial x}(\tilde{x}(s)) + \sum_{i=1}^m \tilde{u}_i(s)\frac{\partial f_i}{\partial x}(\tilde{x}(s))\right),
\end{aligned}$$

with

$$\tilde{u}_i(s) = \tilde{p}(s) f_i(\tilde{x}(s)),\ i = 1, \dots, m.$$

One should prove that the set of vectors $\tilde{p}(0)$ such that the associated trajectory steers x to Σ, is compact as x varies in $\overline{\Omega}$. Of course a problem arises when the point $x \in \Omega$ tends to Σ, and in this case $\tau(x) \to 0$. At the limit one recovers a sub-Riemannian structure associated to the m-uple $f_1, \dots, f_m$), which is medium-fat by assumption. In this case we know from [3, Proof of Th. 6] (see also [1, Th. 5]) that the set of initial adjoint vectors is compact. Since our system is a perturbation of this sub-Riemannian structure as $\tau(x)$ tends to 0, we infer the desired compactness property.

The end of the proof is then strictly similar to the proof of Lemma 4.6.

4.3 The Cauchy problem

4.3.1 Statement of the results

We first investigate Cauchy problems in the whole $\mathbb{R}^n$.

Theorem 4.11. *Let $f_0, \ldots, f_m$ be analytic vector fields on $\mathbb{R}^n$, and let $H(x,p)$ be the Hamiltonian function defined on $\mathbb{R}^n \times \mathbb{R}^n$ as*

$$H(x,p) = \langle p, f_0(x) \rangle + \sum_{i=1}^{m} \langle p, f_i(x) \rangle^2.$$

Let $T > 0$ be fixed, and let g be a subanalytic function on $\mathbb{R}^n$. For all $x \in \mathbb{R}^n$ and $t \in [0,T]$, consider the problem of determining a solution of the affine control system

$$x_u'(s) = f_0(x_u(s)) + \sum_{i=1}^{m} u_i(s) f_i(x_u(s)) \quad a.e. \text{ on } [0,t], \tag{53}$$

such that

$$x_u(t) = x, \tag{54}$$

minimizing the cost

$$C(t,u) = \int_0^t \sum_{i=1}^{m} u_i(s)^2 ds + g(x_u(0)). \tag{55}$$

We assume that for all $x \in \mathbb{R}^n$ there exists a control $u = (u_1, \ldots, u_m) \in L^2([0,T], \mathbb{R}^m)$ such that the associated trajectory $x_u(.)$ is well defined on $[0,T]$, and satisfies $x_u(T) = x$.

Let $S(t,x)$ denote the value function associated to the optimal control problem (53), (54), (55). Namely, if $\mathcal{S}_t$ denotes the set of couples $(x_u(.), u(.))$ solutions of (53), one has, for all $t \in [0,T]$ and $x \in \mathbb{R}^n$,

$$S(t,x) = \inf \left\{ C(t,u) \mid (x_u(.), u(.)) \in \mathcal{S}_t,\ x_u(t) = x \right\}. \tag{56}$$

If g is proper and if for all $x \in \mathbb{R}^n$ and $t \in]0,T]$ there exists no singular minimizing trajectory of the optimal control problem (53), (54), (55), then S is continuous and subanalytic on $]0,T] \times \mathbb{R}^n$, and is solution of the Cauchy problem

$$\begin{aligned} &\frac{\partial S}{\partial t} + H(x, \frac{\partial S}{\partial x}) = 0 \quad a.e. \text{ in }]0,T] \times \mathbb{R}^n, \\ &S(0,.) = g(.). \end{aligned} \tag{57}$$

Remark 4.5. Since S is subanalytic on $]0,T] \times \mathbb{R}^n$, it is almost everywhere differentiable, and thus (57) makes sense.

Remark 4.6. There is a priori no guarantee that the function S be a viscosity solution on the one part, and that S be the unique solution on the other part.

Remark 4.7. Assumption (H) is satisfied if for instance the vector field f_0 is complete.

Remark 4.8. Remark 4.3 on genericity holds again here.

Proposition 4.12. *Under the assumptions of Th. 4.11, if there holds moreover*

$$\text{Lie}\ (f_1(x), \ldots, f_m(x)) = I\!R^n$$

for all $x \in I\!R^n$, then S is continuous and subanalytic on $[0,T] \times I\!R^n$.

We then investigate Cauchy-Dirichlet problems on a subset Ω of $\mathbb{R}^n$.

Theorem 4.13. *Let Ω be a bounded analytic open subset of $I\!R^n$. We consider the Hamiltonian function on $\Omega \times I\!R^n$ defined as*

$$H(x,p) = \langle p, f_0(x)\rangle + \sum_{i=1}^{m} \langle p, f_i(x)\rangle^2,$$

where $f_0, \ldots, f_m$ are analytic vector fields on $I\!R^n$. Let $\Sigma = \partial\Omega$, $T > 0$ be fixed and g be a subanalytic function on $[0,T] \times \Sigma$. Consider the affine control system

$$x'_u(s) = f_0(x_u(s)) + \sum_{i=1}^{m} u_i(s) f_i(x_u(s)) \tag{58}$$

and the cost

$$C(t,u) = \int_0^t \sum_{i=1}^{m} u_i(s)^2 ds + g(t, x_u(t)), \tag{59}$$

and assume that for all $t \in]0,T]$:

1. *The boundary Σ is accessible from Ω in time t, i.e. for all time $t \in]0,T]$ and all $x \in \Omega$ there exists a control u on $[0,t]$ such that the associated solution $x_u(.)$ of (58) satisfies $x_u(0) = x$ and $x_u(t) \in \Sigma$.*

2. *There exists no singular minimizing trajectory for the optimal control problem (58), (59), steering Ω to Σ in time t.*

For all $t \in]0,T]$ and $x \in \overline{\Omega}$, let $S(t,x)$ be the value function associated to the optimal control problem of determining a trajectory solution of the control system (58) on $[0,t]$, minimizing the cost (59), and such that $x_u(0) = x$ and $x_u(t) \in \Sigma$. Namely, if $\mathcal{S}$ denotes the set of couples $(u(.), x_u(.))$ solutions of the control system (58), one has

$$S(t,x) = \inf \Big\{ C(t,u) \mid (x_u(.), u(.)) \in \mathcal{S},\ x_u(0) = x,\ x_u(t) \in \Sigma \Big\}.$$

For all $s, t \in [0,T]$ such that $s < t$ and all $x \in \Sigma$, $y \in \overline{\Omega}$, set

$$S(t,x,s,y) = \inf \Big\{ \int_s^t \sum_{i=1}^{m} u_i(\tau)^2 d\tau \mid (x_u(.), u(.)) \in \mathcal{S},\ x_u(s) = y,\ x_u(t) = x \Big\},$$

and assume that g satisfies the compatibility condition

$$\begin{aligned}\forall (s,y)\in \big(\{0\}\times\overline{\Omega}\big)\cup([0,T[\times\partial\Omega) \quad \forall (t,x)\in]s,T]\times\partial\Omega \\ g(t,x)-g(s,y)\leqslant S(t,x,s,y).\end{aligned} \tag{60}$$

Then S is continuous and subanalytic on $]0,T]\times\Omega$, and is a viscosity solution of the Cauchy-Dirichlet problem

$$\begin{aligned}\frac{\partial S}{\partial t}+H(x,\frac{\partial S}{\partial x})=0 \quad &\text{in }]0,T]\times\Omega,\\ S=g \quad &\text{on }]0,T]\times\partial\Omega,\\ \lim_{t\to 0} S(t,x)=g(0,x) \quad &\text{in } \Omega.\end{aligned} \tag{61}$$

Remark 4.9. Remark 4.3 on genericity holds again here.

Proposition 4.14. *Under the assumptions of Th. 4.13, if there holds moreover*

$$\text{Lie }(f_1(x),\ldots,f_m(x))=\mathbb{R}^n \tag{62}$$

for all $x\in\overline{\Omega}$, then S is continuous on $[0,T]\times\overline{\Omega}$. As a consequence, S is the unique viscosity solution of the Cauchy-Dirichlet problem (61).

Proposition 4.15. *Under the assumptions of Th. 4.13, if the m-uple of vector fields $(f_1,\ldots,f_m)$ is moreover medium-fat on $\overline{\Omega}$, then S is subanalytic on $[0,T]\times\overline{\Omega}$.*

4.3.2 Proof of Th. 4.11

The proof is similar to the proof of Th. 4.1 and is only sketched. We proceed in four steps.

Step 1. The infimum of formula (56) is a minimum, i.e. there exist minimizing trajectories for the optimal control problem (53), (54), (55).

Indeed, let $((u_i^n)_{1\leqslant i\leqslant n},x_0^n)_{n\in\mathbb{N}}$ be a sequence converging to the infimum, and for each n let $x_n(.)$ denote the corresponding trajectory, with $x_n(0)=x_0^n$. Since g is proper we infer that the sequence $(x_0^n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^n$ (and thus, resp., $(u_i^n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,t],\mathbb{R})$, for all $i\in\{1,\ldots,m\}$), and hence up to a subsequence it converges to a point $x_0\in\mathbb{R}^n$ (resp. converges weakly to a control $u_i\in L^2([0,t],\mathbb{R})$, for all $i\in\{1,\ldots,m\}$). Therefore the sequence of curves $x_n(.)$ converges uniformly to $x_u(.)$ on $[0,t]$, and in particular $x_u(0)=x_0$ and $x_u(t)=x$. Moreover, one has

$$\int_0^t\sum_{i=1}^m u_i(s)^2ds+g(x_0)\leqslant\liminf_{n\to+\infty}\int_0^t\sum_{i=1}^m u_i^n(s)^2ds+g(x_0^n)=S(t,x),$$

and this inequality is necessarily an equality. Hence $x_u(.)$ is minimizing.

Step 2. Let us write $S(t,x)$ so that Prop. 1.1 shall be applied.

In the same way as in the proof of Th. 4.1, application of the Pontryagin Maximum Principle leads to write

$$S(t,x) = \inf \{\psi(t,x_0,p_0) \mid \varphi(t,x_0,p_0) = x\}, \tag{63}$$

where φ and ψ are analytic mappings on $]0,T] \times \mathbb{R}^n \times \mathbb{R}^n$.

Step 3. The value function S is continuous on $]0,T] \times \mathbb{R}^n$.

The proof is similar to Lemma 4.5 and we skip it.

Step 4. S is subanalytic on $]0,T] \times \mathbb{R}^n$.

Indeed, let K be a compact subanalytic subset of $]0,T] \times \mathbb{R}^n$. To prove that S is subanalytic on K, similarly to Lemma 4.6, it suffices to show that the set

$$P_0 = \{(x_0,p_0) \in \mathbb{R}^n \times \mathbb{R}^n \mid \exists (t,x) \in K \quad \varphi(t,x_0,p_0) = x\}$$

is compact.

On the one part, since the initial point is not fixed, all extremals of the problem can be chosen to be normal, and moreover

$$p(0) = -\frac{1}{2}\nabla g(x_0).$$

On the other part, S is continuous and thus bounded on K. Since g is proper the compactness of P_0 follows easily.

Application of Prop. 1.1 ends the proof.

Remark 4.10. The assumption of nonexistence of singular minimizing trajectories cannot be skipped in Th. 4.11. Indeed it permits to prove that the value function S is continuous (actually that S is upper semi-continuous ; it is indeed always lower semi-continuous), and this fact is essential in the step 4 to assert that S has an upper bound on the compact K.

4.3.3 Proof of Prop. 4.12

Regarding Remark 4.10 it suffices to show that S is continuous as t tends to 0. Similarly to Lemma 4.8, one can prove for $t > 0$ small enough that

$$S(t,x) = \inf_{y \in \mathbb{R}^n} \left(\frac{d_{SR}(x,y)^2}{t} + R(t,x) + g(y) \right), \tag{64}$$

where R is uniformly bounded, and $d_{SR}(.,.)$ is the sub-Riemannian distance associated to the m-uple of vector fields $(f_1,\dots,f_m)$. It follows easily from (64) that

$$\lim_{t \to 0} S(t,x) = g(x).$$

4.3.4 Proof of Th. 4.13

The proof is similar (and simpler, for the final time is fixed) to the proof of Th. 4.1, and we skip it.

4.3.5 Proof of Prop. 4.14

The proof of the continuity of S as t tends to 0 is the same as in Prop. 4.12. Outside $t=0$ the proof is a little more intricate. What follows is quite analogous to [20, pp. 219–220]. Let $(t_0,x_0)\in]0,T]\times\Sigma$; let us prove that $S(t,x)\rightarrow S(t_0,x_0)=g(t_0,x_0)$ as $(t,x)\rightarrow(t_0,x_0)$. First of all, notice that

$$S(t,x)=\inf\left\{S(t,x,s,y)+g(s,y)\mid (s,y)\in\left(\{0\}\times\overline{\Omega}\right)\cup([0,t[\times\Sigma)\right\}. \tag{65}$$

We proceed in two steps.

First step: S is upper semi-continuous at (t_0,x_0).

Indeed, since $x_0\in\Sigma$ one has

$$S(t,x)\leqslant g(s,x_0)+S(t,x,s,x_0)$$

for all $s\in[0,t_0[$. Moreover, similarly to Lemma 4.8 one can prove

$$S(t,x,s,x_0)\leqslant\frac{d_{SR}(x_0,x)^2}{t-s}+\alpha_1 d_{SR}(x_0,x)+\alpha_2(t-s)$$

for all $x\in\overline{\Omega}$, if $t-s>0$ is small enough, where $\alpha_1,\alpha_2\geqslant 0$. We then infer easily that

$$\limsup_{(t,x)\rightarrow(t_0,x_0)} S(t,x)\leqslant g(t_0,x_0).$$

Second step: S is lower semi-continuous at (t_0,x_0).

Indeed, let $(t^n,x^n)_{n\in\mathbb{N}}$ be a sequence of $]0,T]\times\Omega$ converging to (t_0,x_0). From (65) there exists a sequence $(s^n,y^n)_{n\in\mathbb{N}}$ of $\left(\{0\}\times\overline{\Omega}\right)\cup([0,T]\times\Sigma)$ such that $s^n<t^n$ for all n, and

$$S(t^n,x^n)-g(s^n,y^n)-S(t^n,x^n,s^n,y^n)\underset{n\rightarrow+\infty}{\longrightarrow}0.$$

Up to a subsequence we can suppose that (s^n,y^n) tends to $(s_0,y_0)\in\left(\{0\}\times\overline{\Omega}\right)\cup([0,t_0]\times\Sigma)$. Three cases occur:

1. If $s_0<t_0$: using the assumption of the absence of singular minimizers, it is easy to prove that

$$S(t^n,x^n,s^n,y^n)\underset{n\rightarrow+\infty}{\longrightarrow}S(t_0,x_0,s_0,y_0)$$

(see [24] for details, or previously in the paper). Hence

$$\liminf_{n\rightarrow+\infty}S(t^n,x^n)\geqslant g(s_0,y_0)+S(t_0,x_0,s_0,y_0),$$

and applying the compatibility condition (60) we infer

$$\liminf_{n\to+\infty} S(t^n, x^n) \geqslant g(t_0, x_0).$$

2. If $s_0 = t_0$ and $x_0 = y_0$: we have obviously

$$\liminf_{n\to+\infty} S(t^n, x^n) \geqslant g(t_0, x_0)$$

since $S(t^n, x^n, s^n, y^n) \geqslant 0$ for all n.

3. If $s_0 = t_0$ and $x_0 \neq y_0$: in this case $t^n - s^n$ tends to 0, and similarly to Lemma 4.8 one can prove the existence of nonnegative real numbers α_1, α_2 such that

$$S(t^n, x^n, s^n, y^n) \geqslant \frac{d_{SR}(x^n, y^n)^2}{t^n - s^n} - \alpha_1 d_{SR}(x^n, y^n) - \alpha_2(t^n - s^n)$$

for all integer n. Hence $S(t^n, x^n, s^n, y^n) \underset{n\to+\infty}{\longrightarrow} +\infty$ for $d_{SR}(x_0, y_0) \neq 0$. In particular $S(t^n, x^n) \underset{n\to+\infty}{\longrightarrow} +\infty$. On the other part notice that due to the assumption (62) one can easily prove that the value function S is bounded on each $[\varepsilon, T] \times \overline{\Omega}$, where $\varepsilon > 0$. This is a contradiction.

4.4 Further comments

It is rather clear how to extend the previous results to more general situations. Indeed notice the three following facts:

1. First of all, using notations of Section 4.1, the author gives in [25] some general conditions on the control system (31) and on the cost (32) ensuring that the associated value function is subanalytic. The main assumption is the absence of singular minimizing trajectories.

2. On the other part, it is well-known that, under some general assumptions, the previous value function is a viscosity solution of the Hamilton-Jacobi equation
$$\frac{\partial v}{\partial t} + H_1(x, \frac{\partial v}{\partial x}) = 0, \tag{66}$$
where $H_1(x,p) = \max_u H(x,p,u)$. Notice that all comments here also hold in the Dirichlet case where the value function does not depend on t.

3. Finally, in [18, 22], the authors prove that under general assumptions on the Hamiltonian H_1, there exists an optimal control problem such that the associated value function is exactly the viscosity solution of (66) (*inverse optimal control problem*). Moreover their proof can be quite readily extended to the subanalytic case.

Gathering these facts leads to a general statement ensuring that the unique viscosity solution of an Hamilton-Jacobi equation is subanalytic, provided that the associated optimal control problem would not admit any singular minimizing trajectory.

However the proof of the third fact, mainly based Kakutani Fixed Point Theorem, is not constructive. Hence in general it may be difficult to check whether or not an underlying optimal control problem admits some singular minimizing trajectories.

We conclude with the following question: given an analytic function $H_1(x,p)$, convex in p (but not strictly convex), is it possible to set some conditions ensuring that an associated subanalytic optimal control problem does not admit any singular minimizing trajectory ?

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