Convergence of a finite elements discretization for the Landau Lifshitz equations

François Alouges*and Pascal Jaisson†

Abstract

In this paper we propose a space and time finite elements discretization of the Landau Lifshitz equations and prove its convergence to a weak solution in the sense given in [1].

1 Introduction

The Landau Lifshitz equations govern the dynamic of ferromagnetic materials (see [3]). They may be written as a partial differential equation on the magnetization which is a three dimensional vectorfield constrained to be of constant magnitude through the sample (after a renormalization we may take this magnitude equal to 1). More precisely, calling u the magnetization, the Landau Lifshitz equations are written as

$$\partial_t u = u \times H + \lambda(H - (H \cdot u)u) \tag{1}$$

where H is the total magnetic field induced by several physical phenomena : exchange, stray-field, anisotropy, exterior field, etc, and $\lambda > 0$ is a constant which depends on the material.

Equivalently (at least for regular solutions), (1) may be written under the so-called Gilbert form

$$\partial_t u + \lambda u \times \partial_t u = \mu u \times H \tag{2}$$

where $\mu = 1 + \lambda^2$.

When the other contributions are considered as lower order terms compared to the exchange term, one has (up to a physical constant) :

$$H = \Delta u. \tag{3}$$

*Laboratoire de Mathématique, Bat. 425, Université d'Orsay, 91405 Orsay Cedex, and Centre de Mathématiques et de Leurs Applications, ENS de Cachan, 61 av. du Pdt Wilson, 94235, Cachan Cedex FRANCE

†Laboratoire de Mathématique, Bat. 425, Université d'Orsay, 91405 Orsay Cedex, and Laboratoire MAS, Ecole Centrale Paris, Grande voie des vignes, 92295 Chatenay-Mallabry, FRANCE

In that case, when one takes into account natural boundary conditions, as well as the constraint, the equations become

$$\begin{cases} \partial_t u = u \times \Delta u - \lambda u \times (u \times \Delta u) & \text{on } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u(x,0) = u_0(x) & \text{with } |u_0(x)| = 1 \text{ a.e. on } \Omega, u_0 \in (H^2(\Omega))^3, \\ u : Q_T \to \mathbb{R}^3,\ |u| = 1 \text{ a.e.} \end{cases} \tag{4}$$

where Ω is the three dimensional region occupied by the material, ν is the outside normal vector of $\partial\Omega$ and $Q_T = (0,T) \times \Omega$. The Gilbert form reads

$$\begin{cases} \partial_t u + \lambda u \times \partial_t u = \mu u \times \Delta u & \text{on } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \\ u(x,0) = u_0(x), & \text{with } |u_0(x)| = 1 \text{ a.e. on } \Omega, u_0 \in (H^2(\Omega))^3, \\ u : Q_T \to \mathbb{R}^3,\ |u| = 1 \text{ a.e.} \end{cases} \tag{5}$$

This latter form has been used in [1] to define a concept of weak solutions that we recall here. Existence of weak solutions has been proved in [1] or [5] using different approaches. In [1] an argument is given moreover for non uniqueness of such weak solutions.

Definition 1. *Let $u_0 \in H^1(\Omega)^3$ be such that $|u_0| = 1$ a.e., we say that u is a weak solution to (4) if for all $T > 0$*

1. *$u \in H^1(Q_T)^3$ ($Q_T = (0,T) \times \Omega$); $|u| = 1$ a.e.;*
2. *for all ϕ in $H^1(Q_T)^3$, there holds :*

$$\int_{Q_T} \frac{\partial u}{\partial t} \cdot \phi \, dx\, dt \quad + \quad \lambda \int_{Q_T} \left(u \times \frac{\partial u}{\partial t} \right) \cdot \phi \, dx\, dt$$
$$= \quad -(1+\lambda^2) \sum_{i=1}^{3} \int_{Q_T} \left(u \times \frac{\partial u}{\partial x_i} \right) \cdot \frac{\partial \phi}{\partial x_i} \, dx\, dt. \tag{6}$$

3. *$u(x,0) = u_0(x)$ in the trace sense ;*
4. *we have the energy inequality*

$$\frac{1}{2} \int_\Omega |\nabla u(T)|^2 \, dx + \frac{\lambda}{1+\lambda^2} \int_{Q_T} \left| \frac{\partial u}{\partial t} \right|^2 \, dx\, dt \le \frac{1}{2} \int_\Omega |\nabla u_0|^2 \, dx.$$

In this paper, we are interested in writing a finite elements approximation to (4) or (5) and proving its convergence to a weak solution when both the time step and the size of the elements of the mesh go to 0. This is not straightforward since the equation is nonlinear and the unknown must satisfy the pointwise constraint. In order to write a finite element formulation, we must use a kind of weak form for the laplacian and deal with the constraint on the nodes of the mesh.

The paper is divided as follows. In section 2, the discretization is proposed and shown to be well posed. Section 3 is devoted to the convergence of the scheme when first the time step and later on the space mesh tend to 0.

Adding to this reduced problem typical terms coming from more accurate description of the physic (stray field energy, anisotropy, etc.) does not change the method nor the result as remarked at the end of the paper.

2 Finite elements discretization of the Landau Lifshitz Gilbert equation.

2.1 Notations

We use hereafter the classical framework of finite elements. Namely, we consider a regular tetrahedrization $\mathcal{T}_h$ parametrized by the space step h. As a consequence (see [2]), $\mathcal{T}_h$ is a finite union of tetrahedra that share only a common vertex, edge or face and satisfy

$$\sup_{L\subset\mathcal{T}_h} d(L) \leq Ch$$

where C is independant of h and $d(L)$ denotes the diameter of L. The regularity of the family of tetrahedrizations $(\mathcal{T}_h)_{h>0}$ means that

$$\sup_{h>0}\sup_{L\subset\mathcal{T}_h} \frac{d(L)}{\rho(L)} < C$$

where $\rho(L)$ is the diameter of the biggest sphere incribed in L. For a given $\mathcal{T}_h$, we call $(\phi_i)_{1\leq i\leq n}$ the basis functions of the P^1 approximation on $\mathcal{T}_h$. Namely, calling $(x_i)_{1\leq i\leq n}$ the vertices of $\mathcal{T}_h$ (rigorously speaking, $(\phi_i)_{1\leq i\leq n}$ and $(x_i)_{1\leq i\leq n}$ depend on h, but we don't make it appear in a sake of clarity), the basis functions $(\phi_i)_{1\leq i\leq n}$ satisfy $\phi_i(x_j) = \delta_{ij}$, where δ_{ij} stands for the Kronecker symbol, and $(\phi_i)_{1\leq i\leq n}$ are assumed to be linear on each tetrahedron of $\mathcal{T}_h$. We call V_h the finite dimensional space of vector fields generated by $(\phi_i)_{1\leq i\leq n}$, therefore for any $v_h \in V_h$, we may write

$$\forall x\in\Omega, v_h(x) = \sum_{i=1}^{n} v_h(x_i)\phi_i(x),$$

with $v_h(x_i)\in\mathbb{R}^3$.

We denote by I_h the interpolation operator which is defined for continuous functions as

$$\begin{array}{rcl} I_h : \mathcal{C}^0(\bar\Omega,\mathbb{R}^3) & \longrightarrow & V_h \\ v & \longmapsto & \sum_{i=1}^{n} v(x_i)\phi_i(x). \end{array} \tag{7}$$

The magnetization will be sought in V_h where the constraint will be enforced on the nodes of $\mathcal{T}_h$, that is to say, we will discretize $u(x,k\delta t)$ by

$$u(x,k\delta t) \sim u_n^k(x) = \sum_{i=1}^{n} u_n^k(x_i)\phi_i(x)$$

with $|u_n^k(x_i)| = 1$ (here $\delta t = \frac{T}{K}$ for some integer K and $0 \leq k \leq K-1$).

We call $\mathcal{M}_h$ the subset of V_h defined by

$$\mathcal{M}_h = \{u \in V_h / \forall i \in \{1, \cdots, n\}, |u(x_i)| = 1\}.$$

For such a discrete magnetization $u_n^k \in \mathcal{M}_h$, we introduce the tangent space

$$F_n^k = \{w \in V_h / \forall i \in \{1, \cdots, n\}, w(x_i) \cdot u_n^k(x_i) = 0\}.$$

2.2 The Finite element scheme

We start with the following rewriting which comes from the definition of the weak solutions. Taking $\phi = u \times w$ in (6) where $w \in H^1(Q_T)^3 \cap L^\infty(Q_T)^3$ satisfies $w(x,t) \cdot u(x,t) = 0$ a.e., we get

$$\lambda \int_{Q_T} \frac{\partial u}{\partial t} \cdot w - \int_{Q_T} \left(u \times \frac{\partial u}{\partial t} \right) \cdot w = -\mu \int_{Q_T} \nabla u \cdot \nabla w, \tag{8}$$

from which the original weak formulation is reobtained by taking $w = u \times \phi$ for any $\phi \in \mathcal{C}_0^\infty(Q_T)^3$.

Therefore, a discretization of (5) could be obtained via the following Crank-Nicolson procedure which is directly derived from (8) : $u_n^k \in \mathcal{M}_h$ being given, find $u_n^{k+1} \in \mathcal{M}_h$ solution to

$$\forall w \in F_n^k, \lambda \int_\Omega \frac{u_n^{k+1} - u_n^k}{\delta t} \cdot w - \int_\Omega \left(u_n^k \times \frac{u_n^{k+1} - u_n^k}{\delta t}\right) \cdot w = -\mu \int_\Omega \nabla \left(\frac{u_n^k + u_n^{k+1}}{2} \right) \cdot \nabla w, \tag{9}$$

(where we have called δt the time step) initialized with a discrete version of u_0, called u_n^0 (for instance given by a linear interpolation of the values of u_0 on the nodes $(x_i)_{1 \leq i \leq n}$).

Unfortunately, this latter equation is very difficult to solve because of the constraint on u_n^{k+1}. We therefore solve only approximately this equation and reconstruct u_n^{k+1} in order to satisfy the constraint.

The governing idea is to consider that due to the constraint on u_n^k, $\dfrac{u_n^{k+1} - u_n^k}{\delta t}$ almost belongs to F_n^k. Therefore if we replace this term by an unknown v which belongs to F_n^k, we may approximate (9) by

$$\forall w \in F_n^k, \;\; \lambda \int_\Omega v \cdot w - \int_\Omega (u_n^k \times v) \cdot w = -\mu \int_\Omega \nabla u_n^k \cdot \nabla w. \tag{10}$$

This equation may be written into the abstract form

$$\forall w \in F_n^k, \;\; a(v,w) + b(v,w) = l(w), \tag{11}$$

where $a(v,w) = \lambda \int_\Omega v \cdot w$ defines a continuous positive definite symmetric bilinear form on F_n^k, $b(v,w) = -\int_\Omega (u_n^k \times v) \cdot w$ defines a skew-symmetric bilinear continuous form on

F_n^k and $l(w) = -\mu \int_\Omega \nabla u_n^k \cdot \nabla w$ stands for a continuous linear form on F_n^k (remember that F_n^k is a finite dimensional vectorspace on which all norms are equivalent). We therefore conclude that the equation (10) has a unique solution.

The finite element scheme we propose is the following

- Start with an initial discrete magnetization u_n^0
- for $k = 0, \cdots$
    1. u_n^k being given, solve (10) and call v_n^k the solution
    2. Compute u_n^{k+1} by

$$\forall i \in \{1, \cdots, n\}, \;\; u_n^{k+1}(x_i) = \frac{u_n^k(x_i) + \delta t\, v(x_i)}{|u_n^k(x_i) + \delta t\, v(x_i)|}. \tag{12}$$

We emphasize that this procedure may be readily used in practice to write a finite element code to solve the Landau-Lifshitz equations. In particular it does not contain any penalization term that would be difficult to tune in practice.

The rest of the paper is devoted to the proof of the following theorem.

Theorem 1. *The sequence (u_n^k) defined by the preceding construction defines, after linear interpolation in time, a sequence of maps (u_n) that (weakly) converges when first the time step δt and then the space step h go to 0 to a weak solution of (4).*

Remark 1. *We remark that the preceding theorem also gives another construction of weak solutions to (4) in a different way than[1, 5]. In particular it proves the existence of weak solutions to (4).*

3 Proof of Theorem 1

3.1 Estimate on v

The first step is to find a uniform (independent of k) bound on v solution to (10) and to find the quality of approximation we have obtained.

From (11) we deduce

$$a(v, v) = l(v) \leq C||u_n^k||\,||v||,$$

where C does not depend on k, since both F_n^k and $\mathcal{M}_h$ may be imbedded into $\mathbb{R}^{3n}$ on which all norms are equivalent. The coercivity of a then leads the uniform bound on v with respect to k by taking for instance

$$||v|| = ||v||_\infty = \sup_{x \in \Omega} |v(x)|,$$

and realizing that for all k, $||u_n^k||_\infty = 1$.

Now, we write

$$u_n^{k+1}(x_i) = \frac{u_n^k(x_i) + \delta t\, v(x_i)}{|u_n^k(x_i) + \delta t\, v(x_i)|}$$

and set

$$r(x) = u_n^{k+1}(x) - u_n^k(x) - \delta t\, v(x). \tag{13}$$

This residual $r(x)$ is piecewise linear and is shown to be of second order in δt as expected :

$$\begin{aligned}
|r(x_i)| &= |u_n^{k+1}(x_i) - u_n^k(x_i) - \delta t\, v(x_i)| \\
&= \left| \frac{u_n^k(x_i) + \delta t\, v(x_i)}{|u_n^k(x_i) + \delta t\, v(x_i)|} - u_n^k(x_i) - \delta t\, v(x_i) \right| \\
&= |1 - |u_n^k(x_i) + \delta t\, v(x_i)||
\end{aligned}$$

Since $|u_n^k(x_i) + \delta t\, v(x_i)| = \sqrt{1 + \delta t^2 |v(x_i)|^2} \leq 1 + \frac{1}{2}\delta t^2 |v(x_i)|^2$, we get

$$|r(x_i)| \leq \frac{\delta t^2}{2} |v(x_i)|^2 \leq \frac{\delta t^2}{2} ||v||_\infty. \tag{14}$$

3.2 Approximate solution of the discretized equation

We show that the solution v to (10) allows us to find an approximate solution to (9). Indeed, using (13), we deduce

$$v = \frac{u_n^{k+1} - u_n^k}{\delta t} - \frac{r}{\delta t}, \tag{15}$$

and plugging this expression into (10) leads for all $w \in F_n^k$, to

$$\begin{aligned}
\lambda \int_\Omega \frac{u_n^{k+1} - u_n^k}{\delta t} \cdot w \,\mathrm{d}x - \int_\Omega \left(u_n^k \times \frac{u_n^{k+1} - u_n^k}{\delta t} \right) \cdot w \,\mathrm{d}x = -\mu \int_\Omega \nabla \left(\frac{u_n^k + u_n^{k+1}}{2} \right) \cdot \nabla w \,\mathrm{d}x \\
+ \delta t \int_\Omega (\lambda \frac{r(x)}{\delta t^2} \cdot w - (u_n^k \times \frac{r}{\delta t^2}) \cdot w + \frac{\mu}{2} \nabla v \cdot \nabla w + \frac{\mu}{2} \frac{\nabla r}{\delta t} \cdot \nabla w) \,\mathrm{d}x
\end{aligned} \tag{16}$$

(notice that $\dfrac{u_n^k + u_n^{k+1}}{2} = u_n^k + \dfrac{\delta t}{2} v + \dfrac{r}{2}$).

Bounding the right hand side gives, thanks to (14)

$$\begin{aligned}
\forall w \in F_n^k, \ \Bigg| \lambda \int_\Omega \frac{u_n^{k+1} - u_n^k}{\delta t} \cdot w \,\mathrm{d}x - \int_\Omega \left(u_n^k \times \frac{u_n^{k+1} - u_n^k}{\delta t} \right) \cdot w \,\mathrm{d}x \\
+ \mu \int_\Omega \nabla \left(\frac{u_n^k + u_n^{k+1}}{2} \right) \cdot \nabla w \,\mathrm{d}x \Bigg| \leq C \delta t ||w||_\infty,
\end{aligned} \tag{17}$$

where C does not depend on k or δt.

The energy inequality is obtained from (17) by taking $w = v = \dfrac{u_n^{k+1} - u_n^k}{\delta t} - \dfrac{r}{\delta t}$:

$$\left| \lambda \int_\Omega \left| \frac{u_n^{k+1} - u_n^k}{\delta t} \right|^2 - \lambda \int_\Omega (v\delta t + r) \cdot \frac{r}{\delta t^2} + \int_\Omega \left(u_n^k \times (v\delta t + r) \right) \cdot \frac{r}{\delta t^2} \right.$$
$$\left. + \mu \int_\Omega \frac{|\nabla u_n^{k+1}|^2 - |\nabla u_n^k|^2}{2\delta t} - \mu\delta t \int_\Omega \nabla \left(\frac{u_n^{k+1} + u_n^k}{2} \right) \cdot \nabla \frac{r}{\delta t^2} \right| \leq C\delta t ||v||_\infty.$$

And therefore

$$\lambda \int_\Omega \left| \frac{u_n^{k+1} - u_n^k}{\delta t} \right|^2 + \mu \int_\Omega \frac{|\nabla u_n^{k+1}|^2 - |\nabla u_n^k|^2}{2\delta t} \leq C\delta t, \tag{18}$$

where the constant C does not depend on k or δt.

3.3 Passing to the limit $\delta t \to 0$

In this section, the subscript n is kept fixed and we identify the limit as δt tends to 0 (or $K \to +\infty$). Following [4], we set for $t \in [k\delta t, (k+1)\delta t)$:

$$\begin{cases} u_n^{\delta t}(x,t) = u_n^k(x), \\ \bar{u}_n^{\delta t}(x,t) = \dfrac{u_n^{k+1}(x) + u_n^k(x)}{2}, \\ \bar{\bar{u}}_n^{\delta t}(x,t) = \dfrac{t - k\delta t}{\delta t} u_n^{k+1}(x) + \dfrac{(k+1)\delta t - t}{\delta t} u_n^k(x), \\ \bar{\bar{\bar{u}}}_n^{\delta t}(x,t) = u_n^{k+1}(x), \end{cases} \tag{19}$$

from which we have for $s \in [k\delta t, (k+1)\delta t)$,

$$\frac{\partial \bar{\bar{u}}_n^{\delta t}}{\partial t}(x,s) = \frac{u_n^{k+1}(x) - u_n^k(x)}{\delta t}.$$

Summing (18) from $k = 0$ to k' with $k' \leq K - 1$ gives

$$\sum_{k=0}^{k'} \left(\lambda \int_\Omega \left| \frac{u_n^{k+1} - u_n^k}{\delta t} \right|^2 + \mu \int_\Omega \frac{|\nabla u_n^{k+1}|^2 - |\nabla u_n^k|^2}{2\delta t} \right) \leq KC\delta t \tag{20}$$

which may be equivalently written as

$$\lambda \sum_{k=0}^{k'} \int_{k\delta t}^{(k+1)\delta t} \int_\Omega \left| \frac{\partial \bar{\bar{u}}_n^{\delta t}}{\partial t}(x,s) \right|^2 \mathrm{d}x\mathrm{d}s + \mu \int_\Omega \frac{|\nabla \bar{\bar{\bar{u}}}_n^{\delta t}(x,t)|^2}{2} \,\mathrm{d}x \leq \mu \int_\Omega \frac{|\nabla u_n^0(x)|^2}{2} \,\mathrm{d}x + KC\delta t^2$$
$$\leq \mu \int_\Omega \frac{|\nabla u_n^0(x)|^2}{2} \,\mathrm{d}x + CT\delta t \tag{21}$$

where $t \in [k'\delta t, (k'+1)\delta t)$.

This gives us the bound for all t $\in [0,T)$,

$$\lambda \int_0^t \int_\Omega \left| \frac{\partial \bar{\bar{u}}_n^{\delta t}}{\partial t}(x,s) \right|^2 \mathrm{d}x\mathrm{d}s + \mu \int_\Omega \frac{|\nabla \bar{\bar{u}}_n^{\delta t}(x,t)|^2}{2}\,\mathrm{d}x \leq \mu \int_\Omega \frac{|\nabla u_n^0(x)|^2}{2}\,\mathrm{d}x + CT\delta t \leq \tilde{C}. \quad (22)$$

Therefore

$$\left(\frac{\partial \bar{\bar{u}}_n^{\delta t}}{\partial t}\right) \text{ is bounded in } L^2(Q_T) \text{ as } \delta t \to 0, \quad (23)$$

$$\left(\nabla \bar{\bar{u}}_n^{\delta t}\right) \text{ is bounded in } L^2(Q_T) \text{ as } \delta t \to 0. \quad (24)$$

Moreover, from the definition of $\bar{u}_n^{\delta t}$, we get :

$$\forall t \in [k\delta t, (k+1)\delta t), \int_\Omega |\nabla \bar{u}_n^{\delta t}|^2 dx \leq 2(\int_\Omega |\nabla u_n^{k+1}|^2 + |\nabla u_n^k|^2) \leq C, \quad (25)$$

and $\left(\nabla \bar{u}_n^{\delta t}\right)$ is uniformly bounded in $L^2(Q_T)$, which from (23) allows us to deduce that $\left(\bar{\bar{u}}_n^{\delta t}\right)$ is bounded in $H^1(Q_T)$ as $\delta t \to 0$.
As far as $\bar{u}_n^{\delta t}$ is concerned, we get in the same manner that $\left(\nabla \bar{u}_n^{\delta t}\right)$ is bounded in $L^2(Q_T)$.

Eventually, after the extraction of suitable subsequences, we may assume that

$$\left\{ \begin{array}{l} \bar{\bar{u}}_n^{\delta t} \rightharpoonup u_n \text{ weakly in } H^1(Q_T), \\ \bar{\bar{u}}_n^{\delta t} \to u_n \text{ strongly in } L^2(Q_T), \\ \nabla \bar{u}_n^{\delta t} \rightharpoonup \phi \text{ weakly in } L^2(Q_T). \end{array} \right. \quad (26)$$

It is easy to see that $\phi = \nabla u_n$ from the following inequalities,

$$\begin{aligned} |\bar{\bar{u}}_n^{\delta t} - \bar{u}_n^{\delta t}| &\leq \left| \frac{t-k\delta t}{\delta t} u_n^{k+1}(x) + \frac{(k+1)\delta t - t}{\delta t} u_n^k(x) - \frac{u_n^{k+1}+u_n^k}{2} \right| \\ &\leq \left| \frac{2t-(2k+1)\delta t}{2} \right| \left| \frac{u_n^{k+1}-u_n^k}{\delta t} \right| \\ &\leq \delta t \left| \frac{\partial \bar{\bar{u}}_n^{\delta t}}{\partial t} \right|. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{Q_T} |\bar{\bar{u}}_n^{\delta t} - \bar{u}_n^{\delta t}|^2 \,\mathrm{d}x\,\mathrm{d}t &\leq (\delta t)^2 \int_{Q_T} \left| \frac{\partial \bar{\bar{u}}_n^{\delta t}}{\partial t} \right|^2 \mathrm{d}x\,\mathrm{d}t \\ &\leq C(\delta t)^2. \end{aligned}$$

In the same manner, we can obtain that $\bar{\bar{u}}_n^{\delta t} \to u_n$ strongly in $L^2(Q_T)$.

We want to identify the equation that u_n solves. In other words we pass to the limit $\delta t \to 0$ in the weak inequality (17). In that aim, we take $f \in C_0^\infty$ a smooth function and we set

$f_i(t) = f(x_i,t)$, $F(t) = (f_1(t), \cdots, f_n(t)) \in \mathbb{R}^{3n}$ and $w(x,t) = \sum_{i=1}^n (u_n^{\delta t}(x_i,t) \times f_i(t))\phi_i(x)$. We get from (17):

$$\Big|\lambda \int_{Q_T} \frac{\partial \bar{\bar{u}}_n^{\delta t}}{\partial t}(x,t) \cdot \sum_{i=1}^n (u_n^{\delta t}(x_i,t) \times f_i)\, \phi_i(x)\, \mathrm{d}x\mathrm{d}t - \int_{Q_T} (u_n^{\delta t} \times \frac{\partial \bar{\bar{u}}_n^{\delta t}}{\partial t}(x,t)) \cdot \sum_{i=1}^n (u_n^{\delta t}(x_i,t) \times f_i)\phi_i(x)\, \mathrm{d}x\mathrm{d}t + \mu \int_{Q_T} \nabla \bar{u}_n^{\delta t} \cdot \nabla (\sum_{i=1}^n (u_n^{\delta t}(x_i) \times f_i)\phi_i(x))\, \mathrm{d}x\mathrm{d}t\Big| \leq C\delta t \int_0^T |F(t)|_{\mathbb{R}^{3n}} \mathrm{d}t \tag{27}$$

From the L^2 strong convergence of $(u_n^{\delta t})$ and the fact that $u_n^{\delta t}(x,t) = \sum_{i=1}^n u_n^{\delta t}(x_i,t)\phi_i(x)$, we may obtain the convergence (up to the extraction of a subsequence) of the components :

$$\forall i, u_n^{\delta t}(x_i,t) \longrightarrow u_n(x_i,t) \text{ strongly in } L^2((0,T)) \text{ and a.e. as } \delta t \to 0. \tag{28}$$

We now pass to the limit in each term. From (28), we easily find that

$$\sum_{i=1}^n (u_n^{\delta t}(x_i,t) \times f_i)\phi_i(x) \longrightarrow \sum_{i=1}^n (u_n(x_i,t) \times f_i)\phi_i(x) \text{ strongly in } L^2(Q_T), \tag{29}$$

and

$$\sum_{i=1}^n (u_n^{\delta t}(x_i,t) \times f_i(t))\nabla\phi_i(x) \longrightarrow \sum_{i=1}^n (u_n(x_i,t) \times f_i(t))\nabla\phi_i(x) \text{ strongly in } L^2(Q_T). \tag{30}$$

The first term poses no problem, and we deduce directly from (29) and (26) that

$$\int_{Q_T} \frac{\partial \bar{\bar{u}}_n^{\delta t}}{\partial t} \cdot \sum_{i=1}^n (u_n^{\delta t}(x_i,t) \times f_i)\phi_i\, \mathrm{d}x\mathrm{d}t \to \int_{Q_T} \frac{\partial u_n}{\partial t} \cdot \sum_{i=1}^n (u_n(x_i,t) \times f_i)\phi_i\, \mathrm{d}x\mathrm{d}t \tag{31}$$

as $\delta t \to 0$.

Next,

$$\Big|\int_{Q_T} u_n^{\delta t} \times \frac{\partial \bar{\bar{u}}_n^{\delta t}}{\partial t} \cdot \sum_{i=1}^n (u_n^{\delta t}(x_i,t) \times f_i(t))\phi_i(x) - \int_{Q_T} u_n \times \frac{\partial u_n}{\partial t} \cdot \sum_{i=1}^n (u_n(x_i,t) \times f_i(t))\phi_i(x)\Big| \leq \Big|\int_{Q_T} u_n^{\delta t} \times \frac{\partial \bar{\bar{u}}_n^{\delta t}}{\partial t} \cdot \sum_{i=1}^n ((u_n^{\delta t}(x_i,t) - u_n(x_i,t)) \times f_i(t))\phi_i(x)\Big| + \Big|\int_{Q_T} (u_n^{\delta t} \times \frac{\partial \bar{\bar{u}}_n^{\delta t}}{\partial t} - u_n \times \frac{\partial u_n}{\partial t}) \cdot \sum_{i=1}^n (u_n(x_i,t) \times f_i(t))\phi_i(x)\Big| \tag{32}$$

which tends to 0 as well as before using the convergence (26). Indeed for the first term of the right-hand side $u_n^{\delta t}\times\dfrac{\partial \bar{u}_n^{\delta t}}{\partial t}$ is bounded in $L^2(Q_T)$ and $\sum_{i=1}^n((u_n^{\delta t}(x_i,t)-u_n(x_i,t))\times f_i(t))\phi_i(x)$ tends to 0 in $L^2(Q_T)$ thanks to (29), and the convergence of the second term of the right-hand side comes directly from (26), and the fact that $\sum_{i=1}^n(u_n(x_i,t)\times f_i(t))\phi_i(x)\in L^\infty(Q_T)$.

At last,

$$
\begin{aligned}
&\left|\int_{Q_T}\nabla\bar{u}_n^{\delta t}\cdot\nabla\big(\sum_{i=1}^n u_n^{\delta t}(x_i,t)\times f_i(t)\phi_i(x)\big)-\int_{Q_T}\nabla u_n\cdot\nabla\big(\sum_{i=1}^n u_n(x_i,t)\times f_i(t)\phi_i(x)\big)\right|\\
&\qquad\le\left|\int_{Q_T}\nabla\bar{u}_n^{\delta t}\cdot\nabla\big(\sum_{i=1}^n(u_n^{\delta t}(x_i,t)-u_n(x_i,t))\times f_i(t)\phi_i(x)\big)\right|\\
&\qquad\qquad+\left|\int_{Q_T}(\nabla\bar{u}_n^{\delta t}-\nabla u_n).\nabla\big(\sum_{i=1}^n u_n(x_i,t)\times f_i(t)\phi_i(x)\big)\right|.
\end{aligned}
\tag{33}
$$

Due to (30), the first term tends to 0 exactly as before and the second term tends to 0 from the convergences (26). Passing to the limit in each term of the equation, we obtain that u_n satisfies

$$
\begin{aligned}
\lambda\int_{Q_T}\frac{\partial u_n}{\partial t}\cdot\sum_{i=1}^n(u_n(x_i,t)\times f_i)\phi_i(x)-&\int_{Q_T}(u_n\times\frac{\partial u_n}{\partial t})\cdot\sum_{i=1}^n(u_n(x_i,t)\times f_i)\phi_i(x)\\
&+\mu\int_{Q_T}\nabla u_n\cdot\nabla(\sum_{i=1}^n(u_n(x_i)\times f_i)\phi_i(x))=0.
\end{aligned}
\tag{34}
$$

Passing to the weak $H^1(Q_T)$ limit in the energy inequality (22) leads also to

$$
\lambda\int_0^t\int_\Omega\left|\frac{\partial u_n}{\partial t}(x,s)\right|^2\mathrm{d}x\mathrm{d}s+\mu\int_\Omega\frac{|\nabla u_n(x,t)|^2}{2}\,\mathrm{d}x\le\mu\int_\Omega\frac{|\nabla u_n^0(x)|^2}{2}\,\mathrm{d}x. \tag{35}
$$

Eventually, from the a.e. convergence in time of $u_n^{\delta t}(x_i,t)$, and since $|u_n^{\delta t}(x_i,t)|=1$ for all $0<t<T$ we get

$$
\forall i\in\{1,\cdots,n\},|u_n(x_i,t)|=1\text{ a.e. in }t. \tag{36}
$$

3.4 Passing to the limit $n\to+\infty$

We now want to pass to the limit as the mesh goes infinitely fine, in other words, as the mesh-step h goes to zero and equivalently n tends to $+\infty$. First, from the classical interpolation inequality (see [2] for instance) :

$$
\forall u_0\in H^2(\Omega),\|u_0-u_n^0\|_{H_1}\le Ch|u_0|_{H^2} \tag{37}
$$

where $|u|_{H_2} = \|\Delta u\|_{L^2}$. Therefore, $\|\nabla u_n^0\|_{L^2}$ is uniformly bounded when n tends to $+\infty$, and we deduce from (35) that (∇u_n) and $\left(\dfrac{\partial u_n}{\partial t}\right)$ are bounded in $L^2(Q_T)$. Moreover, (36) gives us that $|u_n(x,t)| \leq 1$ a.e. in x and t and then (u_n) is bounded in $L^2(Q_T)$.

Eventually, after the extraction of suitable subsequences, we may assume that

$$\left\{ \begin{array}{l} u_n \rightharpoonup u \text{ weakly in } H^1(Q_T), \\ u_n \to u \text{ strongly in } L^2(Q_T), \\ \nabla u_n \rightharpoonup \nabla u \text{ weakly in } L^2(Q_T). \end{array} \right. \tag{38}$$

We want to pass to the limit $n \to +\infty$ in the weak equation (34). In that aim, we take $f \in C_0^\infty(Q_T)$ a smooth function and we recall that $I_h f(x,t) = \sum_{i=1}^n f(x_i,t)\phi_i(x)$. This enables us to rewrite (34) as

$$\lambda \int_{Q_T} \frac{\partial u_n}{\partial t} \cdot I_h(u_n \times f) - \int_{Q_T} (u_n \times \frac{\partial u_n}{\partial t}) \cdot I_h(u_n \times f) + \mu \int_{Q_T} \nabla u_n \cdot \nabla(I_h(u_n \times f)) = 0. \tag{39}$$

We first show that $I_h(u_n \times f) \to u \times f$ strongly in $L^2(Q_T)$:

$$\|I_h(u_n \times f) - u \times f\|_{L^2(Q_T)} \leq \|I_h(u_n \times f) - u_n \times f\|_{L^2(Q_T)} + \|u_n \times f - u \times f\|_{L^2(Q_T)}.$$

But $\|u_n \times f - u \times f\|_{L^2(Q_T)} \to 0$ because $f \in C_0^\infty$ and $u_n \to u$ strongly in $L^2(Q_T)$. Moreover, $u_n \times f \in C^0 \cap H^1$ from which we get $\|I_h(u_n \times f) - u_n \times f\|_{L^2(Q_T)} \leq Ch \to 0$.

Then we show that $|\int_{Q_T} \nabla u_n \cdot \nabla I_h(u_n \times f) - \int_{Q_T} \nabla u \cdot \nabla(u \times f)| \to 0$.

Indeed,

$$\begin{aligned} \left| \int_{Q_T} \nabla u_n \cdot \nabla I_h(u_n \times f) - \int_{Q_T} \nabla u \cdot \nabla(u \times f) \right| & \\ \leq \left| \int_{Q_T} \nabla u_n \cdot \nabla I_h(u_n \times f) - \nabla u_n \cdot \nabla(u_n \times f) \right| & \\ + \left| \int_{Q_T} \nabla u_n \cdot \nabla(u_n \times f) - \int_{Q_T} \nabla u \cdot \nabla(u \times f) \right|, & \end{aligned} \tag{40}$$

and we first prove that the second term tends to 0 :

$$\left| \int_{Q_T} \nabla u_n \cdot \nabla(u_n \times f) - \int_{Q_T} \nabla u \cdot \nabla(u \times f) \right| = \left| \int_{Q_T} \nabla u_n \cdot (u_n \times \nabla f) - \int_{Q_T} \nabla u \cdot (u \times \nabla f) \right| \tag{41}$$

which tends to 0 as n tends to $+\infty$ since $u_n \to u$ strongly in $L^2(Q_T)$, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(Q_T)$ and $\nabla f \in C_0^\infty(Q_T)$.

Then we prove that $\left| \int_{Q_T} \nabla u_n \cdot \nabla I_h(u_n \times f) - \int_{Q_T} \nabla u_n \cdot \nabla(u_n \times f) \right|$ tends to 0.

$$\left| \int_{Q_T} \nabla u_n \cdot \nabla I_h(u_n \times f) - \int_{Q_T} \nabla u_n \cdot \nabla(u_n \times f) \right| \leq \|\nabla u_n\|_{L^2(Q_T)} \|\nabla I_h(u_n \times f) - \nabla(u_n \times f)\|_{L^2(Q_T)}.$$

As we know $\|\nabla u_n\|_{L^2(Q_T)}$ is bounded, it is sufficient to show that $\|\nabla I_h(u_n \times f) - \nabla(u_n \times f)\|_{L^2(Q_T)}$ tends to 0. In that aim, we consider a tetrahedron L of the tetrahedrization. Since $u_n \times f$ is C^∞ on L, we have

$$\begin{aligned}\int_L |\nabla I_h(u_n \times f) - \nabla(u_n \times f)|^2 \, \mathrm{d}x &= |u_n \times f - I_h(u_n \times f)|^2_{H^1(L)} \\ &\leq Ch^2 |u_n \times f|^2_{H^2(L)}.\end{aligned} \tag{42}$$

But

$$\begin{aligned}|u_n \times f|^2_{H^2(L)} &= \int_L |\Delta(u_n \times f)|^2 \\ &= \int_L |\Delta u_n \times f + 2\nabla u_n \times \nabla f + u_n \times \Delta f|^2 \\ &= \int_L |2\nabla u_n \times \nabla f + u_n \times \Delta f|^2\end{aligned} \tag{43}$$

because u_n is linear on L. Therefore we bound

$$\begin{aligned}|u_n \times f|^2_{H^2(L)} &\leq C \int_L |\nabla u_n|^2 |\nabla f|^2 + |u_n \times \Delta f|^2 \\ &\leq C(f) \|u_n\|^2_{H^1(L)}\end{aligned} \tag{44}$$

where $C(f)$ is a constant which only depends on f. Thus

$$\begin{aligned}\|\nabla I_h(u_n \times f) - \nabla(u_n \times f)\|^2_{L^2(\Omega)} &\leq \sum_L |u_n \times f|^2_{H^2(L)} h^2 \\ &\leq C \|u_n\|^2_{H^1(\Omega)} h^2.\end{aligned} \tag{45}$$

As (u_n) is uniformly bounded in $H^1(\Omega)$ with respect to n, we have proved that $\|\nabla I_h(u_n \times f) - \nabla(u_n \times f)\|_{L^2(\Omega)} \to 0$ when n tends to $+\infty$.
We may pass to the limit in (34) and in (35) (for the second term we have to used the same decomposition we have used in (32)) to obtain :

$$\lambda \int_{Q_T} \frac{\partial u}{\partial t} \cdot (u \times f) \, \mathrm{d}x\mathrm{d}t - \int_{Q_T} (u \times \frac{\partial u}{\partial t}) \cdot (u \times f) \, \mathrm{d}x\mathrm{d}t + \int_{Q_T} \nabla u \cdot \nabla(u \times f) \, \mathrm{d}x\mathrm{d}t = 0 \tag{46}$$

and

$$\lambda \int_0^t \int_\Omega \left| \frac{\partial u}{\partial t}(x,s) \right|^2 \mathrm{d}x\mathrm{d}s + \mu \int_\Omega \frac{|\nabla u(x,t)|^2}{2} \, \mathrm{d}x \leq \mu \int_\Omega \frac{|\nabla u^0(x)|^2}{2} \, \mathrm{d}x. \tag{47}$$

3.5 Magnitude of u and equation verified by u

We show in this section that the solution u to (46) obtained above verifies $|u(x,t)| = 1$ a.e. in x and t. We already have :

$$\begin{cases} |u_n(x_i,t)| = 1, \text{ for all } i, \text{ a.e. in } t \\ |u_n(x,t)| \leq 1, \text{ for all } x, \text{ a.e. in } t \text{ from the interpolation.} \end{cases} \tag{48}$$

We fix a time t_0 where (48) is verified. On one hand, we consider a tetrahedron L such that $|\nabla u_n(x,t_0)| \le \frac{1}{\sqrt{h}}$ on L (recall us that ∇u_n is constant on each tetrahedron). Thus, if x_i is a vertex of L, we have for $x \in L$,

$$\begin{aligned} |u_n(x,t_0)| &= |u_n(x_i,t_0) + \nabla u_n \cdot (x - x_i)| \\ &\ge |u_n(x_i,t_0)| - |\nabla u_n||x - x_i| \\ &\ge 1 - C\sqrt{h}, \end{aligned} \tag{49}$$

where C is a constant such that $\sup_L d(L) \le Ch$.

On the other hand, if L is a tetrahedron such that $|\nabla u_n(x,t_0)| > \dfrac{1}{\sqrt{h}}$ on L, then

$$\int_L |\nabla u_n(x,t_0)|^2 \,\mathrm{d}x > \frac{1}{h}\mathrm{vol}(L) \tag{50}$$

but if we call $L_n = \bigcup_{L/|\nabla u_n(x,t_0)| > \frac{1}{\sqrt{h}} \text{ on } L} L$,

$$\frac{1}{h}\mathrm{vol}(L_n) \le \sum_{L/|\nabla u_n(x,t_0)| > \frac{1}{\sqrt{h}} \text{ on } L} \int_L |\nabla u_n(x,t_0)|^2 \,\mathrm{d}x \le \int_\Omega |\nabla u_n(x,t_0)|^2 \,\mathrm{d}x \tag{51}$$

Thus thanks to (35), the right member is bounded and $\dfrac{1}{h}\mathrm{vol}(L_n) \le C$. Then thanks to (49), we obtain :

$$\begin{aligned} \int_\Omega |u_n(x,t_0)|^2 \,\mathrm{d}x \ge \int_{L/|\nabla u_n(x,t_0)| \le \frac{1}{\sqrt{h}} \text{ on } L} (1 - C\sqrt{h})^2 \,\mathrm{d}x &= (1 - C\sqrt{h})^2(\mathrm{vol}(\Omega) - \mathrm{vol}(L_n)) \\ &\ge (1 - C\sqrt{h})^2(\mathrm{vol}(\Omega) - Ch). \end{aligned} \tag{52}$$

Therefore

$$\int_{Q_T} |u_n(x,t)|^2 \,\mathrm{d}x\mathrm{d}t \ge T(1 - C\sqrt{h})^2(\mathrm{vol}(\Omega) - Ch). \tag{53}$$

When n tends to $+\infty$, we find $\int_{Q_T} |u(x,t)|^2 \,\mathrm{d}x\mathrm{d}t \ge \mathrm{vol}(Q_T)$. Since we know that $|u(x,t)| \le 1$ a.e. in x and in t, we get

$$\int_{Q_T} |u(x,t)|^2 \,\mathrm{d}x\mathrm{d}t = \mathrm{vol}(Q_T) \tag{54}$$

and

$$|u(x,t)| = 1 \text{ a.e.} \tag{55}$$

Finally, we have

$$\lambda \int_{Q_T} \frac{\partial u}{\partial t} \cdot (u \times f) \,\mathrm{d}x\mathrm{d}t = -\lambda \int_{Q_T} u \times \frac{\partial u}{\partial t} \cdot f \tag{56}$$

and

$$\int_{Q_T} (u \times \frac{\partial u}{\partial t}) \cdot (u \times f)\,\mathrm{d}x\mathrm{d}t = -\int_{Q_T} (u \times (u \times f)) \cdot \partial_t u$$
$$= -\int_{Q_T} (u \cdot f)(u \cdot \partial_t u) - \partial_t u \cdot f \tag{57}$$
$$= \int_{Q_T} \partial_t u \cdot f$$

(indeed $u \cdot \partial_t u = 0$ because $|u| = 1$).
Then we obtain (5) thanks to (46) and (55).

Remark 2. *The above study can be generalized when we want to take account other energy terms. For instance the stray-field energy may be added to the formulation by replacing* (9) *by*

$$\forall w \in F_n^k, \lambda \int_\Omega \frac{u_n^{k+1} - u_n^k}{\delta t} \cdot w - \int_\Omega \left(u_n^k \times \frac{u_n^{k+1}}{\delta t}\right) \cdot w$$
$$= -\mu \int_\Omega \nabla \left(\frac{u_n^k + u_n^{k+1}}{2}\right) \cdot \nabla w + \mu \int_\Omega H_s(\frac{u_n^k + u_n^{k+1}}{2}) \cdot w \tag{58}$$

and (10) *by*

$$\forall w \in F_n^k, \lambda \int_\Omega v \cdot w - \int_\Omega (u_n^k \times v) \cdot w = -\mu \int_\Omega \nabla u_n^k \cdot \nabla w + \mu \int_\Omega H_s(u_n^k) \cdot w, \tag{59}$$

where we have denoted by $H_s(u)$ the stray-field induced by the magnetization u. We refer the reader to [3] for the definition of $H_s(u)$. We here only need that $H_s(u)$ is the solution to

$$\begin{cases} div(H_s(u) + u) = 0 \\ curl(H_s(u)) = 0 \end{cases} \tag{60}$$

in $\mathcal{D}'(\mathbb{R}^3)$, decaying at infinity. Therefore, $H_s(u)$ may be viewed as the L^2 orthogonal projection of $-u$ on gradient fields along curls. We also have

$$\|H_s(u)\|_{L^2(\mathbb{R}^3)} \leq \|u\|_{L^2(\Omega)} \tag{61}$$

which gives the L^2 continuity of H_s. This is sufficient to use the technique explained before.

4 Conclusion

We have proposed a finite elements discretization of the Landau Lifshitz Gilbert equations. This formulation may be used in practice for numerical computations. Although it does not contain any penalization term to enforce the constraint, we have been able to prove the convergence to a weak solution of the continuous model.

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