Efficient semi-parametric estimation of the periods in a superposition of periodic functions with unknown shape

Elisabeth Gassiat* Céline Lévy-Leduc†

September 5, 2003

Abstract

We consider the estimation of the periods in a sum of periodic functions of unknown shape corrupted by white noise. In the case of a single periodic signal, we propose a consistent and asymptotically efficient semi-parametric estimator of the period. In this case, the asymptotic variance of the estimation of the period is four times as big as it would be if the function was of known shape. We then study the case of a sum of two periodic signals of unknown shape with different periods. For a large class of signals, we propose semi-parametric estimators of the two periods that are consistent and asymptotically gaussian. When the ratio of the periods is not rational, the estimators are asymptotically independent and efficient.

1 Introduction.

Let us assume that we have n observations of a sampled signal corrupted by white noise:

$$X(j) = s_0(jt_n) + \varepsilon(j) \ ; \ 1 \leq j \leq n, \tag{1}$$

where s_0 is a real function (the signal), and the $\varepsilon(j)$ are independent normal random variables of unknown variance σ_0^2.
We consider the situation where the signal s_0 can be decomposed into a sum of periodic signals, and are interested in the estimation of the periods of the individual periodic components. Estimating the frequencies of periodic signals embedded in additive noise is a known problem that has been addressed mainly when the signal consists of a single periodic function or for signals of known shape. Many papers study this problem. For instance, Golubev (1988) proposes a consistent and asymptotically efficient estimator of the period in the white noise model (that is the continuous version of the model (1)), which is roughly a maximum likelihood estimator. Quinn and Thomson (1991) focus on the case where s_0 is a trigonometric polynomial. Hall et al. (2000) study the single period estimation problem with random design.
If now s_0 is the sum of two periodic signals, that is

$$s_0 = s_1 + s_2 \tag{2}$$

where s_1 and s_2 are periodic with distinct periods τ_1^0 and τ_2^0, then the problem appears to be harder. It is not difficult to see that the nature of the problem depends on whether the ratio τ_1^0/τ_2^0 is rational or not. Indeed, when this ratio is rational, the signal is still periodic, but when it is not rational this is no longer the case. At first glance one would think that to be able to infer two different periods, their ratio would need to be irrational. Of course, recovery (non parametric estimation) of the individual components in the sum is no longer possible in general when the ratio of the periods is rational. This is the problem considered by Hall and Yin (2003) with random design, see also De Cheveigné and Kawahra (1998). But if one knows that one has to search for

*Université Paris-Sud Orsay
†supported by the society Thalès Optronique (FRANCE)

two distinct periods, then the problem of estimating the two periods is solvable for a large class of pairs of signals (see section 4). In general, for a sum of periodic signals, without assumptions on the ratios of the periods, it is necessary to know the number of signals to be able to infer the different periods.
This problem appears in various applications. For instance, it can be a step towards target identification in laser vibrometry. Indeed, after emission of a continuous wave by a laser vibrometer, reflection on the illuminated target and demodulation, the received signal is of the form (1) with s_0 decomposed as a sum of periodic components. Thus, estimating the periods could be used to achieve long range target identification because they characterize the vibrations of the target, see Prenat (2001) and Lévy-Leduc (2003). It is also an interesting issue in astronomy, see Hall et al. (2000) and Gregory and Loredo (1992). As for the practical point of view, the proposed algorithms mainly deal with particular classes of signals. When s_0 is a trigonometric polynomial, see Quinn and Fernandes (1991), or Quinn and Hannan (2001). For another particular class of signals which is more general than the trigonometric polynomials, see the work of Carmona et al. (1998) who have developed a package in Splus to achieve instantaneous frequency estimation. A general algorithm, with no restriction on the class of signals, has been proposed by the second author jointly with M.Lavielle see Lavielle and Lévy-Leduc (2003).

To our knowledge, semi-parametric estimation of two (or more) distinct periods has not yet been solved. The goal of this paper is to give consistent and asymptotically efficient estimators of the periods in the semi-parametric case when the shape of the signals is unknown.
Let us now give an outline of the paper. In section 2, we give general ideas that lead to our estimators, some assumptions and some useful tools.
In section 3, we first study the case where s_0 is itself periodic. We begin by computing the efficient Fisher information matrix, which is obtained along a least favorable sub-model for the non parametric part, which is the shape of the signal. This is done in a classical way by using LAN theory, as explained in van der Vaart (1998) and McNeney and Wellner (2000). This makes us able to quantify the loss of information when the regression function is unknown: the asymptotic variance in the estimation of the period is four times as big as it would be if the function was known. Next, we define an estimator and prove its consistency and asymptotic efficiency.
In section 4, we solve the problem of efficient estimation of the two periods when s_0 is the sum of two periodic functions, this being the most interesting result of the paper. We first define the class of pairs of signals for which the two periods are identifiable, even when their ratio is rational. We then propose estimators that are consistent, asymptotically independent and asymptotically efficient, in particular when the ratio of the periods is not rational.
All technical proofs that are not needed for a comprehensive reading of the paper are given in section 5.

2 General ideas and assumptions.

If a signal s is periodic of period τ, and if its restriction to $[0\ ;\ \tau]$ is square integrable, then s can be expanded in Fourier series:

$$s(t) = \sum_{k\in\mathbb{Z}} c_k(s)\exp(\frac{2ik\pi t}{\tau}),\ c_k(s) = \frac{1}{\tau}\int_0^\tau \exp(-\frac{2ik\pi t}{\tau})s(t)dt, \forall k \in \mathbb{Z}.$$

Saying that the shape of the signal is unknown is the same as saying that the parameters $(c_k(s))_{k\in\mathbb{Z}}$ are unknown. A parametrization of periodic functions may thus be given as a period τ and a periodic function f with period 1, so that $s(t) = f(t/\tau)$. For such f one has $c_k(f) = c_k(s)$ for all k, and f is called the shape of the signal. Notice that to define the shape parameters (c_k) one has to know the period.
We shall consider smooth enough signals s, with Fourier coefficients $c_k(s)$ verifying

$$\textbf{(S1)} : \sum_{p\in\mathbb{Z}} |p|\,|c_p(s)| < +\infty$$

To ensure that a signal is periodic of period τ and not of period τ/m for some integer $m \geq 2$, we consider signals s such that

$$\textbf{(S2)}: \exists h > 0 \text{ such that } \forall m \geq 2, \sum_{k\geq 1} |c_{mk}(s)|^2 \leq (1-h) \sum_{k\geq 1} |c_k(s)|^2$$

2.1 A single periodic signal.

Consider first the case where the signal s_0 itself is periodic with period τ_0. Then, to be able to find the period even without noise, one needs to have observations on an interval which is larger than the period, and with asymptotically vanishing sampling parameter. We thus assume throughout the paper that

$$\textbf{(N1)}: \lim_{n\to+\infty} t_n = 0 \text{ and } \lim_{n\to+\infty} nt_n = +\infty. \tag{3}$$

The criterion to be maximized (it may be seen as an approximation of the maximum log-likelihood optimized with respect to the unknown parameters c_k):

$$\Lambda_n(\tau) = \sum_{k=1}^{m_n} \frac{1}{n^2} \left| \sum_{j=1}^{n} X(j) \exp(\frac{2ik\pi j t_n}{\tau}) \right|^2,$$

which is a cumulated periodogram. Here the cumulating parameter m_n tends to infinity in an appropriate way (see assumptions below).
A maximum likelihood estimator of the period τ_0 should be chosen as a maximizer of Λ_n over $\mathbb{R}$. We shall for technical reasons choose a parameter set which only asymptotically covers all positive numbers, that is we set

$$\Theta_n = [\delta_n, \Delta_n]$$

with

$$\textbf{(N2)}: \lim_{n\to+\infty} \delta_n = 0 \quad \lim_{n\to+\infty} \Delta_n = +\infty$$

in an appropriate way. More precisely, we will use:

$$\textbf{(N3)}: \quad \frac{m_n t_n}{\delta_n} \leq \frac{1}{8}, \lim_{n\to+\infty} m_n\delta_n = 0, \lim_{n\to+\infty} \frac{m_n}{\Delta_n} = 0, \lim_{n\to+\infty} \frac{m_n^{5/2}\Delta_n}{nt_n} = 0,$$
$$\lim_{n\to+\infty} \frac{m_n^4}{n} = 0, \lim_{n\to+\infty} \frac{m_n^2}{nt_n^2} = 0.$$

This will be enough to obtain asymptotically efficient estimators without any assumption on the range of possible values for the periods.
If we define $G_n(\tau)$ the deterministic part of Λ_n and $\eta_n(\tau)$ the random part of Λ_n, that is

$$G_n(\tau) = \sum_{k=1}^{m_n} \frac{1}{n^2} \left| \sum_{j=1}^{n} s_0(jt_n) \exp(\frac{2ik\pi j t_n}{\tau}) \right|^2, \ \eta_n(\tau) = \Lambda_n(\tau) - G_n(\tau),$$

we will see that η_n is uniformly $o_P(1)$, and that asymptotically the maximum values for $G_n(\tau)$ are obtained at any $\tau = k\tau_0$, k an integer, with maximum value of order $\sum_{k\geq 1} |c_k(s)|^2$. To obtain an asymptotically efficient semi-parametric estimator of the period, we follow the steps:

- If one maximizes Λ_n in a small ball around τ_0, one obtains an efficient linearized random variable $\bar{\tau}_n$ in the usual way, so that $n^{3/2}t_n(\bar{\tau}_n - \tau_0)$ is asymptotically gaussian.
- If one maximizes approximately Λ_n over Θ_n and chooses the smallest maximizer, one obtains an estimator $\tilde{\tau}_n$ of the period that may be proved to be consistent at some speed $O(1/nt_n)$.
- One may then define the estimator $\hat{\tau}_n$ as the maximizer of Λ_n on a ball centered at $\tilde{\tau}_n$ and with small radius so that with probability tending to 1, $\hat{\tau}_n = \bar{\tau}_n$ and is asymptotically efficient.

2.2 Sum of two periodic signals.

Consider now the case (2) where s_0 is the sum of two periodic signals. Λ_n can be seen as a criterion that computes the power spectrum of the signal along some frequency. Analogously, the criterion to be maximized will now be:

$$\mathcal{H}_n(\tau_1, \tau_2) = \Lambda_n(\tau_1) + \Lambda_n(\tau_2) - \Lambda_n^c(\tau_1, \tau_2) \tag{4}$$

where

$$\Lambda_n^c(\tau_1, \tau_2) = \frac{1}{n^2}\mathcal{R}e\left\{\sum_{(k,k')\in K_n(\tau_1,\tau_2)} \left(\sum_{j=1}^{n} e^{\frac{2ik\pi j t_n}{\tau_1}} X(j)\right)\left(\sum_{j=1}^{n} e^{-\frac{2ik'\pi j t_n}{\tau_2}} X(j)\right)\right\}$$

and

$$K_n(\tau_1, \tau_2) = \left\{(k_1, k_2) \in \{1, \dots, m_n\}^2,\ |\frac{\tau_1}{\tau_2} - \frac{k_1}{k_2}| \le \beta_n\right\}$$

for some β_n tending to 0. Indeed, $\mathcal{H}_n(\tau_1, \tau_2)$ computes the power spectrum of the signal as a sum of two periodic signals along some pair of frequencies: the substracted term takes into account the fact that the ratio is rational or not.
In the same way as for a single periodic signal, it will be seen that the maximum values for $\mathcal{H}_n$ are obtained around integer multiples of τ_1^0 and τ_2^0, and that the same steps may be followed.

In the study of the criterions, the function

$$\varphi_n(t) = \frac{1}{n}\sum_{k=1}^{n} \exp(2ik\pi t),\ t \in \left[-\frac{1}{2}\ ;\ \frac{1}{2}\right],$$

will be used, and we report some useful majorations. The roughest ones are:

$$|\varphi_n(t)| \le 1,\ \varphi'_n(t) \le \pi(n+1),\ \varphi''_n(t) \le 4\pi^2(n+1)^2/3. \tag{5}$$

But, by using the concavity of the sinus function on $\left[0, \frac{\pi}{2}\right]$ which provides:

$$|\sin(x)| \ge \frac{2}{\pi}|x|,\ \forall x \in \left[-\frac{\pi}{2}\ ;\ \frac{\pi}{2}\right]$$

we can deduce for some constant C that for all t in $\left[-\frac{1}{2}\ ;\ \frac{1}{2}\right]$ and $t \neq 0$:

$$|\varphi_n(t)| \le \frac{C}{n|t|},\ |\varphi'_n(t)| \le \frac{C}{|t|},\ |\varphi''_n(t)| \le \frac{Cn}{|t|}. \tag{6}$$

These inequalities come from

$$\varphi_n(t) = \frac{\exp(i\pi(n+1)t)}{n}\frac{\sin(n\pi t)}{\sin(\pi t)},\ \text{if } t \neq 0 \text{ and } \varphi_n(0) = 1,$$

straightforward computations and use of concavity of the sinus function.

3 The single period model.

We consider in this section model (1) where s_0 is periodic with period τ_0, and f_0 is its shape. Denote $\mathbb{P}_{n,(\tau,f,\sigma)}$ the distribution of $(X(1), \cdots, X(n))$ when the parameter is (τ, f, σ), and $\mathbb{P}_n^0$ when the parameter is (τ_0, f_0, σ_0).

3.1 LAN property and efficient Fisher information.

Let $L_n(\tau, f, \sigma)$ be the log-likelihood ratio, that is

$$L_n(\tau, f, \sigma) = \log \frac{d\mathbb{P}_{n,(\tau,f,\sigma)}(X(1), \cdots, X(n))}{d\mathbb{P}_{n,(\tau_0,f_0,\sigma_0)}(X(1), \cdots, X(n))}.$$

Define $\mathcal{T}$ the set of periodic functions f with period 1, with square integrable restriction on $[0,1]$ and such that $\sum_{k\in\mathbb{Z}} |c_k(f)| < +\infty$. We shall follow the semi-parametric analysis developed in McNeney and Wellner (2000).

Proposition 1. *If $f_0' \in \mathcal{T}$, under (N1), the sequence of statistical models $\left(\mathbb{P}_{n,(\tau,f,\sigma)}\right)$ is locally asymptotically normal at (τ_0, f_0, σ_0) with tangent space $\mathcal{H} = \mathbb{R} \times \mathcal{T} \times \mathbb{R}_+$ endowed with the inner product*

$$\langle (a,f,b),(c,g,d)\rangle_{\mathcal{H}} = \frac{1}{\sigma_0^2}\int_0^1 \left(f(t) - \frac{a}{2\tau_0^2} f_0'(t)\right)\left(g(t) - \frac{c}{2\tau_0^2} f_0'(t)\right) dt + \frac{ac}{12\sigma_0^2\tau_0^4}\int_0^1 f_0'^2(t)dt + 2\frac{bd}{\sigma_0^2},$$

that is

$$L_n(\tau_0 + \frac{a}{n^{3/2}t_n}, f_0 + \frac{f}{\sqrt{n}}, \sigma_0 + \frac{b}{\sqrt{n}}) = W_n(a,f,b) - \frac{1}{2} \parallel (a,f,b) \parallel_{\mathcal{H}}^2 + o_{\mathbb{P}_n^0}(1)$$

where

$$W_n(a,f,b) = \sum_{j=1}^n \left(\frac{1}{\sqrt{n}} f\left(\frac{jt_n}{\tau_0}\right) - \frac{a}{n^{3/2}t_n}\left(\frac{jt_n}{\tau_0^2}\right) f_0'\left(\frac{jt_n}{\tau_0}\right)\right)\frac{\varepsilon(j)}{\sigma_0^2} + \frac{1}{\sqrt{n}}\sum_{j=1}^n \frac{b}{\sigma_0}\left((\frac{\varepsilon(j)}{\sigma_0})^2 - 1\right),$$

and for any finite subset $h_1, \ldots h_d \in \mathcal{H}$, $(W_n(h_1), \ldots, W_n(h_d))$ converges in distribution under $\mathbb{P}_n^0$ to the centered gaussian with covariance $\langle h_i, h_j\rangle_{\mathcal{H}}$.
The parameter $\tau = \nu_n(\mathbb{P}_{n,(\tau,f,\sigma)})$ is differentiable with derivative $\dot\nu(a,f,b) = a$, and

$$\dot\nu^T(a) = \left(\frac{a}{I_*}, \frac{a}{2\tau_0^2 I_*} f_0', 0\right),$$

where I_* the efficient Fisher information for τ is given by

$$I_* = \frac{1}{12\sigma_0^2\tau_0^4}\int_0^1 f_0'^2(t)dt.$$

In particular if $\hat\tau$ satisfies:

$$n^{3/2}t_n(\hat\tau - \tau_0) = \frac{I_*^{-1}}{\sigma_0^2 n^{3/2} t_n}\sum_{j=1}^n \left(\frac{n}{2} - j\right)\frac{t_n}{\tau_0^2} f_0'\left(\frac{jt_n}{\tau_0}\right)\varepsilon(j) + o_{\mathbb{P}_n^0}(1) \tag{7}$$

then $\hat\tau$ is regular and efficient.

Proposition 1 is proved in section 5.
The fact that must be emphasized is that the best asymptotic variance is four times as big as the variance in the estimation of τ_0 when the shape of the regression function is known.

3.2 Consistent and efficient estimation of τ_0.

For any $R > 0$, define

$$B(\tau\ ;\ R) = \left\{\theta \in \Theta_n : \left|\frac{\tau}{\theta} - 1\right| < R\right\}.$$

Let ε_n be a sequence decreasing to 0 such that

$$\lim_{n\to+\infty} \frac{nt_n\varepsilon_n}{m_n} = +\infty,\ \lim_{n\to+\infty} \frac{\varepsilon_n}{n^{-1/5}} = +\infty.$$

Define

$$\mathcal{T}_n = \left\{ \theta \in \Theta_n \ : \ \Lambda_n(\theta) \geq (1-\varepsilon_n) \sup_{\Theta_n} \Lambda_n(\tau) \right\},$$

and $\tilde{\tau}_n = \inf \mathcal{T}_n$. The estimator $\hat{\tau}_n$ is defined as the maximizer of Λ_n over $B(\tilde{\tau}_n, 1/8)$.
Remark: this has to be compared with the estimator proposed by Golubev (1988), where one has to choose some $\varepsilon < h$ (see assumption (S2)). Here, one does not need the knowledge of h.

Theorem 1. *Under the assumptions (S1), (S2), (N1), (N2), (N3), $\hat{\tau}_n$ is asymptotically efficient.*

Proof. Let $\bar{\tau}_n$ be the maximizer of Λ_n over $B(\tau_0, 1/4)$. $\bar{\tau}_n$ will only be a theoretical tool since of course one needs to know τ_0 to compute it. To begin with, we prove that the random part of the criterion is asymptotically negligible in probability, and precise analysis of the deterministic part G_n will lead to convergence properties of $\tilde{\tau}_n$ and $\bar{\tau}_n$. All intermediate results are proved in section 5.

Lemma 1. *Under (S1), (N1), (N2), and (N3), for all x_n such as $\lim_{n\to+\infty} x_n n^{1/2-\beta} = +\infty$ for some $\beta > 0$,*

$$\lim_{n\to+\infty} P\left(\sup_{\tau\in\Theta_n} |\eta_n(\tau)| \geq x_n \right) = 0.$$

A consequence of this Lemma is that $\sup_{\tau\in\Theta_n} |\eta_n(\tau)| = o_P(n^{-1/5})$.

Let us state results on the deterministic part G_n.
For any τ,

$$G_n(\tau) = \sum_{k=1}^{m_n} \left| \sum_p c_p(s_0)\varphi_n\left(\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau}\right)\right|^2.$$

One has, uniformly for $\tau \in \Theta_n$, using (5),

$$\sum_{k=1}^{m_n} \left| \sum_{|p|\geq m_n} c_p(s_0)\varphi_n\left(\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau}\right)\right| \leq \sum_{|p|\geq m_n} |pc_p(s_0)| = o(1) \tag{8}$$

using (S1). Also, uniformly for $\tau \in \Theta_n$, for any $k \geq 1$, if $p \leq 0$, $|\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau}| \geq t_n/\Delta_n$, so that using (6),

$$\left| \sum_{-m_n\leq p\leq 0} c_p(s_0)\varphi_n\left(\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau}\right)\right| = O\left(\frac{\Delta_n}{nt_n}\right). \tag{9}$$

Assume now that α_n tends to 0 in such a way that

$$\lim_{n\to+\infty} \alpha_n m_n^2 = 0, \ \lim_{n\to+\infty} \frac{m_n}{(nt_n\alpha_n)^2} = 0 \tag{10}$$

(this is possible since $m_n = o((nt_n)^{2/5})$).

Then

$$\sup_{\tau\notin\left(\cup_{l\leq m_n,j} B(\frac{j\tau_0}{l},\alpha_n)\right)\cap\Theta_n} \left| \sum_{0<p\leq m_n} c_p(s_0)\varphi_n\left(\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau}\right)\right| = O(\frac{1}{nt_n\alpha_n}). \tag{11}$$

(8), (9) and (11) imply that

$$\sup\left\{ G_n(\tau) \ : \ \tau \notin \left(\cup_{l\leq m_n,j} B(\frac{j\tau_0}{l},\alpha_n)\right) \cap \Theta_n \right\} = o(1).$$

If now $l \le m_n$ and j is such that j and l are relatively prime, and $\frac{j\tau_0}{l} \in \Theta_n$, for any $\tau \in B(\frac{j\tau_0}{l}, \alpha_n)$, uniformly in l, j, τ,

$$\sum_p c_p(s_0)\varphi_n\left(\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau}\right) = \begin{cases} O\left(\frac{1}{nt_n}\right) & \text{if } j \nmid k \\ c_{k'l}(s_0)\varphi_n\left(\frac{k'lt_n}{\tau_0} - \frac{k'jt_n}{\tau}\right) + O\left(\frac{1}{nt_n}\right) & \text{if } k = jk' \end{cases} \tag{12}$$

where the notation $u \mid v$ for $v \in \mathbb{Z}$ and $u \in \mathbb{Z}^*$ means that there exists $k \in \mathbb{Z}$ such that $v = ku$ and $u \nmid v$ is the negation of $u \mid v$. Thus, uniformly

$$G_n(\tau) \le \sum_{k\ge1} |c_{kl}(s_o)|^2 + o(1) \le (1-h)\sum_{k\ge1}|c_k(s_0)|^2 + o(1) \tag{13}$$

when $l \neq 1$. This implies that with a probability tending to 1, the supremum of Λ_n is attained on $\underset{j}{\cup} B(j\tau_0, \alpha_n)$. But (12) and Lemma 1 lead to:

$$\sup_{\tau\in\cup_j B(j\tau_0,\alpha_n)} \Lambda_n(\tau) \le \sum_{1\le k\le m_n} |c_k(s_0)|^2 + O_P\left(\frac{m_n}{nt_n} + n^{-1/5}\right)$$

and

$$\Lambda_n(\tau_0) = \sum_{1\le k\le m_n} |c_k(s_0)|^2 + O_P\left(\frac{m_n}{nt_n} + n^{-1/5}\right),$$

so that with a probability tending to 1, $\tau_0 \in \mathcal{T}_n$. Straightforward consequences of these results together with Lemma 1 are that, with a probability tending to 1,

$$\mathcal{T}_n \subset \cup_{j\le\frac{\Delta_n}{\tau_0}} B(j\tau_0, \alpha_n),\ \tilde{\tau}_n \in B(\tau_0, \alpha_n),\ \bar{\tau}_n \in B(\tau_0, \alpha_n). \tag{14}$$

But for any $\rho > 0$, for any $\tau \in B(\tau_0, \alpha_n)$ and $\tau \notin B(\tau_0, \rho/nt_n)$ one has

$$\begin{aligned} G_n(\tau) - G_n(\tau_0) &= \sum_{k=1}^{m_n} |c_k(s_0)|^2 \left\{ \left|\varphi_n\left(\frac{kt_n}{\tau_0}(1-\frac{\tau_0}{\tau})\right)\right|^2 - 1 \right\} + o(1) \\ &= \sum_{k=1}^{m_n} |c_k(s_0)|^2 \left\{ \left(\frac{\sin\left[\frac{nkt_n}{\tau_0}(1-\frac{\tau_0}{\tau})\right]}{\frac{nkt_n}{\tau_0}(1-\frac{\tau_0}{\tau})} \right)^2 - 1 \right\} + o(1) \\ &\le -\sum_{k=1}^{m_n} |c_k(s_0)|^2 \int_0^1 \left(\sin\left[t\frac{nkt_n}{\tau_0}(1-\frac{\tau_0}{\tau})\right]\right)^2 dt + o(1) \\ &\le -\min\left\{C\frac{\rho^2}{\tau_0}; \frac{1}{2}\right\} \sum_{k\ge1} |c_k(s_0)|^2 + o(1) \end{aligned}$$

for some constant $C > 0$. One thus easily obtains that for any $\rho > 0$,

$$\lim_{n\to+\infty} P\left(\tilde{\tau}_n \notin B(\tau_0, \frac{\rho}{nt_n})\right) = 0, \tag{15}$$

and

$$\lim_{n\to+\infty} P\left(\bar{\tau}_n \notin B(\tau_0, \frac{\rho}{nt_n})\right) = 0,$$

which implies that there exists ρ_n tending to 0 such that,

$$\lim_{n\to+\infty} P\left(\bar{\tau}_n \notin B(\tau_0, \frac{\rho_n}{nt_n})\right) = 0. \tag{16}$$

Notice that the refinement of the speed of convergence of $\bar{\tau}_n$ to τ_0 as $o(1/nt_n)$ will be needed in the proof of the linearization below.
These results together with the definition of $\hat{\tau}_n$ leads to

$$\lim_{n\to+\infty} P\left(\hat{\tau}_n = \bar{\tau}_n\right) = 1. \tag{17}$$

We now may write:

$$\Lambda_n'(\bar{\tau}_n) = 0 = \Lambda_n'(\tau_0) + (\bar{\tau}_n - \tau_0)\,\Lambda_n''(\tau')$$

for some $\tau' \in B(\tau_0, \frac{\rho_n}{nt_n})$. We prove in section 5 that:

$$\Lambda_n'(\tau_0) = \left(\frac{1}{\tau_0^2}\sum_{j=1}^{n} t_n\left(\frac{1}{2}-\frac{j}{n}\right) f_0'\left(\frac{jt_n}{\tau_0}\right)\varepsilon(j)\right)(1+o_{\mathbb{P}_n^0}(1)), \tag{18}$$

and for any $\tilde{\tau} \in B(\tau_0, \frac{\rho_n}{nt_n})$

$$\Lambda_n''(\tilde{\tau}) = -n^2 t_n^2 \sigma_0^2 I_*(1+o_{\mathbb{P}_n^0}(1)). \tag{19}$$

But this leads to

$$n^{3/2}t_n(\bar{\tau}_n - \tau_0) = \frac{I_*{}^{-1}}{\sigma_0^2 n^{3/2} t_n}\sum_{j=1}^{n}\left(\frac{n}{2}-j\right)\frac{t_n}{\tau_0^2} f_0'\left(\frac{jt_n}{\tau_0}\right)\varepsilon(j) + o_p(1)$$

which by equation (17) and Proposition 1 concludes the proof. □

4 The two periods model.

We will focus now on the model (2), so that

$$X(j) = s_1^0(jt_n) + s_2^0(jt_n) + \varepsilon(j) = f_1^0\left(\frac{jt_n}{\tau_1^0}\right) + f_2^0\left(\frac{jt_n}{\tau_2^0}\right) + \varepsilon(j) \quad ; \quad 1 \le j \le n$$

where f_1^0 and f_2^0 are periodic functions of period 1,

$$s_1^0(t) = \sum_{k\in\mathbb{Z}} c_k^0 \exp(\frac{2ik\pi t}{\tau_1^0}),\ s_2^0(t) = \sum_{k\in\mathbb{Z}} d_k^0 \exp(\frac{2ik\pi t}{\tau_2^0}).$$

In this section, we assume that an upper bound and a lower bound are known for both periods, that is we set $\Theta_n = \Theta = [\delta, \Delta]$ for some fixed $\delta > 0$ and $\Delta > 0$.

4.1 Identifiability.

For a pair of periodic functions (s_1, s_2), denote by $t_1(s_1)$ and $t_2(s_2)$ their respective periods. Let $\mathcal{T}(s_1, s_2)$ be the set of pairs of real numbers (t_1, t_2) such that there exists a pair of periodic functions $(\tilde{s}_1, \tilde{s}_2)$ with respective periods $t_1(\tilde{s}_1)$ and $t_2(\tilde{s}_2)$, $t_1(\tilde{s}_1) = t_1$, $t_2(\tilde{s}_2) = t_2$ and $s_1 + s_2 = \tilde{s}_1 + \tilde{s}_2$. Let now

$$T_2(s_1, s_2) = \inf\left\{\max(t_1, t_2)\ :\ (t_1, t_2) \in \mathcal{T}(s_1, s_2)\right\},$$

$$T_1(s_1, s_2) = \min\left\{t_1\ :\ (t_1, T_2(s_1, s_2)) \in \mathcal{T}(s_1, s_2)\right\}.$$

And let $\mathcal{S}$ be the set of pairs of periodic functions (s_1, s_2) such that:
$t_1(s_1) = T_1(s_1, s_2)$ and $t_2(s_2) = T_2(s_1, s_2)$. Then by construction:

Proposition 2. *As soon as $(s_1^0, s_2^0) \in \mathcal{S}$ or $(s_2^0, s_1^0) \in \mathcal{S}$, the periods are identifiable.*

Notice that when $(s_1, s_2) \in \mathcal{S}$, then $t_1(s_1) < t_2(s_2)$. Notice also that the class $\mathcal{S}$ contains almost all pairs of periodic signals. Indeed:

- If $\tau_1^0/\tau_2^0 \notin \mathbb{Q}$, $\mathcal{T}(s_1^0, s_2^0)$ reduces to $\{(\tau_1^0, \tau_2^0); (\tau_2^0, \tau_1^0)\}$ and all pairs with periods having non rational ratio have identifiable periods.
- If now $\tau_1^0/\tau_2^0 \in \mathbb{Q}$, saying that $(s_1^0, s_2^0) \in \mathcal{S}$ is just a condition on the spectra so that, in any same sum of two periodic signals, the couple of periods have: either the largest bigger than τ_2^0, else the lowest smaller than τ_1^0.

In case $\tau_1^0/\tau_2^0 \notin \mathbb{Q}$, the signal $s_0 = s_1 + s_2$ is no longer periodic. But in case $\tau_1^0/\tau_2^0 \in \mathbb{Q}$, say $\tau_1^0/\tau_2^0 = l/m$ where l and m are relatively prime integers, s_0 is periodic with period $\tau_0 = m\tau_1^0 = l\tau_2^0$. In this case, define

$$\left\{\begin{array}{rcl} g_0(x) & = & \sum\limits_{k\in\mathbb{Z}} (c_{lk}(f_1^0) + c_{mk}(f_2^0))e^{2ik\pi x}, \\ \tilde{f}_1(x) & = & \sum\limits_{k\in\mathbb{Z}, l\nmid k} c_k(f_1^0)e^{2ik\pi x}, \\ \tilde{f}_2(x) & = & \sum\limits_{k\in\mathbb{Z}, m\nmid k} c_k(f_2^0)e^{2ik\pi x}, \end{array}\right. \tag{20}$$

where again the notation $u \mid v$ for $v \in \mathbb{Z}$ and $u \in \mathbb{Z}^*$ means that there exists $k \in \mathbb{Z}$ such that $v = ku$ and $u \nmid v$ is the negation of $u \mid v$. Then

$$s_0(t) = \tilde{f}_1\left(\frac{t}{\tau_1^0}\right) + \tilde{f}_2\left(\frac{t}{\tau_2^0}\right) + g_0\left(\frac{t}{\tau_1^0/l}\right),$$

which shows that recovery of each individual signal s_1 and s_2 is impossible except in case $g_0 = 0$. The use of criterion $\mathcal{H}_n$ leads us to work with the following subset of $\mathcal{S}$. Define $\tilde{\mathcal{S}}$ the subset of $\mathcal{S}$ such that:
If $t_1(s_1)/t_2(s_2) = l/m$ where l and m are relatively prime integers, then there exists $h \in]0,1[$ such that for any integers $r, r' \geq 2$, for any set A of integers such that $\{k \in A, r' \mid k\} \cap \{k, r|k\} = \emptyset$, one has

$$\textbf{(I1)}: \sum_{k\geq 1, r|k} |c_{km}(s_2) + c_{kl}(s_1)|^2 + \sum_{k\geq 1, r'|k, k\in A} |c_{km}(s_2) + c_{kl}(s_1)|^2 \leq (1-h)\sum_{k\geq 1} |c_{km}(s_2) + c_{kl}(s_1)|^2,$$

$$\textbf{(I2)}: \sum_{k\geq 1, r|k, l\nmid k} |c_k(s_1)|^2 + \sum_{k\geq 1, r'|k, k\in A, l\nmid k} |c_k(s_1)|^2 \leq (1-h) \sum_{k\geq 1, l\nmid k} |c_k(s_1)|^2,$$

$$\textbf{(I3)}: \sum_{k\geq 1, r|k, m\nmid k} |c_k(s_2)|^2 + \sum_{k\geq 1, r'|k, k\in A, m\nmid k} |c_k(s_2)|^2 \leq (1-h) \sum_{k\geq 1, m\nmid k} |c_k(s_2)|^2,$$

and if the ratio of the periods is not rational, for some $h > 0$, (S2) holds for s_1 and s_2, and for any integers $r, r' \geq 2$,

$$\textbf{(I4)}: \sum_{k\geq 1, r\nmid kr'} |c_{kr'}(s_1)|^2 + \sum_{k\geq 1, r'\nmid kr} |c_{kr}(s_1)|^2 + \sum_{k\geq 1} |c_{krr'}(s_1)|^2 \leq (1-h)\left(\sum_{k\geq 1} |c_k(s_1)|^2 + |c_k(s_2)|^2\right),$$

$$\textbf{(I5)}: \sum_{k\geq 1, r\nmid kr'} |c_{kr'}(s_2)|^2 + \sum_{k\geq 1, r'\nmid kr} |c_{kr}(s_2)|^2 + \sum_{k\geq 1} |c_{krr'}(s_2)|^2 \leq (1-h)\left(\sum_{k\geq 1} |c_k(s_1)|^2 + |c_k(s_2)|^2\right).$$

4.2 LAN property and efficient Fisher information.

Let $\mathbb{P}_{n,(\tau_1,\tau_2,f_1,f_2,\sigma)}$ be the distribution of $(X(1), \cdots, X(n))$ when the parameter is $(\tau_1, \tau_2, f_1, f_2, \sigma)$, and $\mathbb{P}_n^0$ when the parameter is $(\tau_1^0, \tau_2^0, f_1^0, f_2^0, \sigma_0)$. Let $L_n(\cdot)$ be the log-likelihood ratio.
Let $\mathcal{H}$ be the quotient space of $\mathbb{R}^2 \times \mathcal{T}^2 \times \mathbb{R}_+$ by the equivalence relation

$$(a_1, b_1, f_1, g_1, c_1) \sim (a_2, b_2, f_2, g_2, c_2)$$

if and only if for all t

$$f_1(\frac{t}{\tau_1^0})+g_1(\frac{t}{\tau_2^0})-\frac{a_1}{2(\tau_1^0)^2}{f_1^0}'(\frac{t}{\tau_1^0})-\frac{b_1}{2(\tau_2^0)^2}{f_2^0}'(\frac{t}{\tau_2^0}) = f_2(\frac{t}{\tau_1^0})+g_2(\frac{t}{\tau_2^0})-\frac{a_2}{2(\tau_1^0)^2}{f_1^0}'(\frac{t}{\tau_1^0})-\frac{b_2}{2(\tau_2^0)^2}{f_2^0}'(\frac{t}{\tau_2^0}).$$

Proposition 3. *If ${f_1^0}'$, ${f_2^0}'$ are in $\mathcal{T}$, under (N1), the sequence of statistical models $\left(\mathbb{P}_{n,(\tau_1,\tau_2,f_1,f_2,\sigma)}\right)$ is locally asymptotically normal at $(\tau_1^0,\tau_2^0,f_1^0,f_2^0,\sigma_0)$ with tangent space $\mathcal{H}$ endowed with the inner product*

$$\begin{aligned}
&\langle (a_1,b_1,f_1,g_1,c_1),(a_2,b_2,f_2,g_2,c_2)\rangle_{\mathcal{H}} = \\
\tfrac{1}{\sigma_0^2}\lim_{T\to+\infty}\tfrac{1}{T}\int_{-T/2}^{T/2} &\left(f_1(\frac{t}{\tau_1^0})+g_1(\frac{t}{\tau_2^0})-\frac{a_1}{2(\tau_1^0)^2}{f_1^0}'(\frac{t}{\tau_1^0})-\frac{b_1}{2(\tau_2^0)^2}{f_2^0}'(\frac{t}{\tau_2^0})\right)\\
&\left(f_2(\frac{t}{\tau_1^0})+g_2(\frac{t}{\tau_2^0})-\frac{a_2}{2(\tau_1^0)^2}{f_1^0}'(\frac{t}{\tau_1^0})-\frac{b_2}{2(\tau_2^0)^2}{f_2^0}'(\frac{t}{\tau_2^0})\right)dt\\
+\tfrac{1}{12\sigma_0^2}\lim_{T\to+\infty}\tfrac{1}{T}\int_{-T/2}^{T/2} &\left(\frac{a_1}{(\tau_1^0)^2}{f_1^0}'(\frac{t}{\tau_1^0})+\frac{b_1}{(\tau_2^0)^2}{f_2^0}'(\frac{t}{\tau_2^0})\right)\left(\frac{a_2}{(\tau_1^0)^2}{f_1^0}'(\frac{t}{\tau_1^0})+\frac{b_2}{(\tau_2^0)^2}{f_2^0}'(\frac{t}{\tau_2^0})\right)dt+2\frac{c_1c_2}{\sigma_0^2},
\end{aligned}$$

that is

$$L_n(\tau_1^0+\frac{a}{n^{3/2}t_n},\tau_2^0+\frac{b}{n^{3/2}t_n},f_1^0+\frac{f}{\sqrt{n}},f_2^0+\frac{g}{\sqrt{n}}\sigma_0+\frac{c}{\sqrt{n}}) = W_n(a,b,f,g,c)-\frac{1}{2}\parallel (a,b,f,g,c)\parallel_{\mathcal{H}}^2+o_{\mathbb{P}_n^0}(1)$$

where

$$W_n(a,b,f,g,c)=\sum_{j=1}^{n}\frac{\varepsilon(j)}{\sigma_0^2}A_{j,n}+\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{c}{\sigma_0}\left((\frac{\varepsilon(j)}{\sigma_0})^2-1\right),$$

$$A_{j,n}=\left(\frac{1}{\sqrt{n}}f\left(\frac{jt_n}{\tau_1^0}\right)-\frac{a}{n^{3/2}t_n}\left(\frac{jt_n}{(\tau_1^0)^2}\right){f_1^0}'\left(\frac{jt_n}{\tau_1^0}\right)+\frac{1}{\sqrt{n}}g\left(\frac{jt_n}{\tau_2^0}\right)-\frac{b}{n^{3/2}t_n}\left(\frac{jt_n}{(\tau_2^0)^2}\right){f_2^0}'\left(\frac{jt_n}{\tau_2^0}\right)\right),$$

and for any finite subset $h_1,\dots h_d\in\mathcal{H}$, $(W_n(h_1),\dots,W_n(h_d))$ converges in distribution under $\mathbb{P}_n^0$ to the centered gaussian with covariance $\langle h_i,h_j\rangle_{\mathcal{H}}$.
The parameter $(\tau_1,\tau_2)=\nu_n(\mathbb{P}_{n,(\tau_1,\tau_2,f_1,f_2,\sigma)})$ is differentiable with derivative $\dot\nu(a,b,f,g,c)=(a,b)$, and

$$\dot\nu^T(a,b)=\left(\tilde a,\tilde b,\frac{\tilde a}{2(\tau_1^0)^2}{f_1^0}'(\frac{t}{\tau_1^0}),\frac{\tilde b}{2(\tau_2^0)^2}{f_2^0}'(\frac{t}{\tau_2^0}),0\right),\quad\begin{pmatrix}\tilde a\\ \tilde b\end{pmatrix}=J_*^{-1}\begin{pmatrix}a\\ b\end{pmatrix},$$

$$J_*=\frac{1}{12\sigma_0^2}\lim_{T\to+\infty}\frac{1}{T}\begin{pmatrix}\frac{1}{(\tau_1^0)^4}\int_{-T/2}^{T/2}({f_1^0}')^2(\frac{t}{\tau_1^0})dt & \frac{1}{(\tau_1^0)^2(\tau_2^0)^2}\int_{-T/2}^{T/2}{f_1^0}'(\frac{t}{\tau_1^0}){f_2^0}'(\frac{t}{\tau_2^0})dt\\ \frac{1}{(\tau_1^0)^2(\tau_2^0)^2}\int_{-T/2}^{T/2}{f_1^0}'(\frac{t}{\tau_1^0}){f_2^0}'(\frac{t}{\tau_2^0})dt & \frac{1}{(\tau_2^0)^4}\int_{-T/2}^{T/2}({f_2^0}')^2(\frac{t}{\tau_2^0})dt\end{pmatrix}.$$

In particular if $(\hat\tau_1,\hat\tau_2)$ satisfies:

$$n^{3/2}t_n\begin{pmatrix}\hat\tau_1-\tau_1^0\\ \hat\tau_2-\tau_2^0\end{pmatrix}=\frac{{J_*}^{-1}}{\sigma_0^2n^{3/2}t_n}\begin{pmatrix}\sum_{j=1}^{n}\left(\frac{n}{2}-j\right)\frac{t_n}{(\tau_1^0)^2}{f_1^0}'\left(\frac{jt_n}{\tau_1^0}\right)\varepsilon(j)\\ \sum_{j=1}^{n}\left(\frac{n}{2}-j\right)\frac{t_n}{(\tau_2^0)^2}{f_2^0}'\left(\frac{jt_n}{\tau_2^0}\right)\varepsilon(j)\end{pmatrix}+o_{\mathbb{P}_n^0}(1),$$

then $(\hat\tau_1,\hat\tau_2)$ is regular and efficient.

Proposition 3 is proved in section 5.
The limits involved in the Proposition exist. Indeed, define

$$\varphi(\tau)=\int_{-1/2}^{1/2}e^{2i\pi u\tau}du=\frac{\sin(\pi\tau)}{\pi\tau},\quad |\varphi(\tau)|\le C\min\left(1,1/|\tau|\right).$$

One has for any f_1 and f_2 in $\mathcal{T}$ and positive real numbers τ_1 and τ_2

$$\frac{1}{T}\int_{-T/2}^{T/2}f_1\left(\frac{t}{\tau_1}\right)f_2\left(\frac{t}{\tau_2}\right)dt=\sum_{k,l}c_k(f_1)c_l(f_2)\varphi\left[T\left(\frac{k}{\tau_1}+\frac{l}{\tau_2}\right)\right].\tag{21}$$

If τ_1/τ_2 is an irreducible fraction p/q,

$$\lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} f_1\left(\frac{t}{\tau_1}\right) f_2\left(\frac{t}{\tau_2}\right) dt = \sum_m c_{mp}(f_1)c_{-mq}(f_2).$$

If $\tau_1/\tau_2 \notin \mathbb{Q}$, it may be proved (see section 5) that

$$\lim_{T\to+\infty} \frac{1}{T}\int_{-T/2}^{T/2} f_1\left(\frac{t}{\tau_1}\right) f_2\left(\frac{t}{\tau_2}\right) dt = c_0(f_1)c_{-0}(f_2) = \int_0^1 f_1(t)dt \int_0^1 f_2(t)dt.$$

In particular, one sees that:

- If $\tau_1^0/\tau_2^0 \notin \mathbb{Q}$, J_* is diagonal and linearized efficient estimators of the periods are asymptotically independent,
- and if $\tau_1^0/\tau_2^0 \in \mathbb{Q}$, J_* is in general not diagonal and linearized efficient estimators of the periods are asymptotically correlated. But if $g_0 = 0$, one can see that J_* is diagonal, and in this case we will be able to estimate efficiently the two periods, see Theorem 2 below.

4.3 Consistent and efficient estimation of τ_1^0 and τ_2^0.

First of all, we shall assume with no loss of generality that $\tau_1^0 < \tau_2^0$. We shall also assume that $(s_1^0, s_2^0) \in \tilde{\mathcal{S}}$, and we choose some $\varepsilon \in]0; h[$.
In the criterion $\mathcal{H}_n$, we shall also use β_n verifying

$$(\mathbf{N4}): \lim_{n\to+\infty} m_n\beta_n = 0,\ \beta_n \geq \left(\frac{\Delta}{\delta}\right)\left(\frac{1+\alpha_n}{1-\alpha_n} - 1\right)$$

for the sequence α_n defined in section 3.2.
Define

$$\mathcal{U}_n = \left\{(\theta_1, \theta_2) \in \Theta^2 \ :\ \mathcal{H}_n(\theta_1,\theta_2) \geq (1-\varepsilon)\sup_{\Theta^2} \mathcal{H}_n(\tau_1,\tau_2)\right\}.$$

Notice that of course, $\mathcal{U}_n$ is symmetrical with respect to the first diagonal (that is $(\tau_1,\tau_2) \in \mathcal{U}_n$ if and only if $(\tau_2,\tau_1) \in \mathcal{U}_n$).
Let

$$\tilde{\tau}_2 = \inf\{\tau_2 \ : (\tau_1,\tau_2) \in \mathcal{U}_n,\ \tau_1 < \tau_2\}$$

and

$$\tilde{\tau}_1 = \inf\{\tau_1 \ : (\tau_1,\tilde{\tau}_2) \in \mathcal{U}_n\}.$$

The estimator $(\hat{\tau}_1, \hat{\tau}_2)$ is defined as the maximizer of $\mathcal{H}_n$ over $(B(\tilde{\tau}_1, 1/8) \times B(\tilde{\tau}_2, 1/8)) \cap \{\tau_1 < \tau_2\}$.

Theorem 2. *Assume (N1), (N3), (N4), $(s_1^0, s_2^0) \in \tilde{\mathcal{S}}$, $\tau_1^0 < \tau_2^0$, and s_1^0 and s_2^0 satisfy (S1).
Let g_0, $\tilde{f}_1$ and $\tilde{f}_2$ be defined by (20) if $\tau_1^0/\tau_2^0 = l/m$ and $\tau_* = \tau_1^0/l = \tau_2^0/m$, and $g_0 = 0$, $\tilde{f}_1 = f_1^0$, $\tilde{f}_2 = f_2^0$ if τ_1^0/τ_2^0 is not rational.
Then*

$$n^{3/2}t_n \begin{pmatrix} \hat{\tau}_1 - \tau_1^0 \\ \hat{\tau}_2 - \tau_2^0 \end{pmatrix} = \frac{{K_*}^{-1}}{n^{3/2}t_n} \begin{pmatrix} \sum_{j=1}^n \left(\frac{n}{2} - j\right) t_n \left(\frac{1}{(\tau_1^0)^2}\tilde{f}_1'\left(\frac{jt_n}{\tau_1^0}\right) + \frac{g_0'\left(\frac{jt_n}{\tau_*}\right)}{2\tau_1^0\tau_*}\right)\varepsilon(j) \\ \sum_{j=1}^n \left(\frac{n}{2} - j\right) t_n \left(\frac{1}{(\tau_2^0)^2}\tilde{f}_2'\left(\frac{jt_n}{\tau_2^0}\right) + \frac{g_0'\left(\frac{jt_n}{\tau_*}\right)}{2\tau_2^0\tau_*}\right)\varepsilon(j) \end{pmatrix} + o_{\mathbb{P}_n^0}(1),$$

where

$$K_* = \begin{pmatrix} \frac{1}{12}\left(\frac{1}{(\tau_1^0)^4}\int_0^1 {\tilde{f}_1'}^2(t)dt - \frac{1}{(\tau_1^0)^2(\tau_*)^2}\int_0^1 {g_0'}^2(t)dt\right) & \frac{1}{8(\tau_*)^2\tau_1^0\tau_2^0}\int_0^1 {g_0'}^2(t)dt \\ \frac{1}{8(\tau_*)^2\tau_1^0\tau_2^0}\int_0^1 {g_0'}^2(t)dt & \frac{1}{12}\left(\frac{1}{(\tau_2^0)^4}\int_0^1 {\tilde{f}_2'}^2(t)dt - \frac{1}{(\tau_2^0)^2(\tau_*)^2}\int_0^1 {g_0'}^2(t)dt\right) \end{pmatrix}.$$

Let

$$M_* = \frac{1}{12\sigma_0^2} \begin{pmatrix} \frac{1}{(\tau_1^0)^4} \int_0^1 \tilde{f}_1'^2(t)dt + \frac{1}{4(\tau_1^0)^2(\tau_*)^2} \int_0^1 g_0'^2(t)dt & \frac{1}{4(\tau_*)^2\tau_1^0\tau_2^0} \int_0^1 g_0'^2(t)dt \\ \frac{1}{4(\tau_*)^2\tau_1^0\tau_2^0} \int_0^1 g_0'^2(t)dt & \frac{1}{(\tau_2^0)^4} \int_0^1 \tilde{f}_2'^2(t)dt + \frac{1}{4(\tau_2^0)^2(\tau_*)^2} \int_0^1 g_0'^2(t)dt \end{pmatrix},$$

Then

$$n^{3/2}t_n \begin{pmatrix} \hat{\tau}_1 - \tau_1^0 \\ \hat{\tau}_2 - \tau_2^0 \end{pmatrix} \to \mathcal{N}\left(0, K_*^{-1}M_*(K_*^{-1})^T\right),$$

so that $(\hat{\tau}_1, \hat{\tau}_2)$ *is asymptotically efficient as soon as* $g_0 = 0$.

Comments: With our estimation method, the estimator $(\hat{\tau_1}, \hat{\tau_2})$ cannot be efficient when $g_0 \neq 0$ because it introduces a term which is a mixture of the signals s_1^0 and s_2^0 in the rational case which does not appear in the LAN. Then a natural question would be: is it possible to prove that an asymptotically efficient estimator of (τ_1^0, τ_2^0) does not exist and that $(\hat{\tau_1}, \hat{\tau_2})$ has the best asymptotic variance?

Proof. We shall use as intermediate tool $(\bar{\tau}_1, \bar{\tau}_2)$, the maximizer of $\mathcal{H}_n$ over $(B(\tau_1^0, 1/8) \times B(\tau_2^0, 1/8)) \cap \{\tau_1 < \tau_2\}$.

Denote by $H_n(\tau_1, \tau_2)$ the deterministic part of $\mathcal{H}_n(\tau_1, \tau_2)$ defined by

$$H_n(\tau_1, \tau_2) = G_n(\tau_1) + G_n(\tau_2) - G_n^c(\tau_1, \tau_2)$$

where

$$G_n^c(\tau_1, \tau_2) = \frac{1}{n^2} \mathcal{R}e \left\{ \sum_{(k,k') \in K_n(\tau_1,\tau_2)} \left(\sum_{j=1}^n e^{\frac{2ik\pi j t_n}{\tau_1}} (s_1^0 + s_2^0)(jt_n) \right) \left(\sum_{j=1}^n e^{-\frac{2ik'\pi j t_n}{\tau_2}} (s_1^0 + s_2^0)(jt_n) \right) \right\}$$

and by

$$R_n(\tau_1, \tau_2) = \mathcal{H}_n(\tau_1, \tau_2) - H_n(\tau_1, \tau_2) = R_{n,1}(\tau_1, \tau_2) + R_{n,2}(\tau_1, \tau_2)$$

the random part of $\mathcal{H}_n$, with

$$R_{n,2}(\tau_1, \tau_2) = \zeta_n(\tau_1) + \zeta_n(\tau_2) - \zeta_n^c(\tau_1, \tau_2)$$

where

$$\zeta_n^c(\tau_1, \tau_2) = \frac{1}{n^2} \mathcal{R}e \left\{ \sum_{(k,k') \in K_n(\tau_1,\tau_2} \left(\sum_{j=1}^n e^{\frac{2ik\pi j t_n}{\tau_1}} \varepsilon(j) \right) \left(\sum_{j=1}^n e^{-\frac{2ik'\pi j t_n}{\tau_2}} \varepsilon(j) \right) \right\}.$$

As for the single period case, the first point is to obtain consistency of $(\tilde{\tau}_1, \tilde{\tau}_2)$ and $(\bar{\tau}_1, \bar{\tau}_2)$ at some speed, and this point relies on precise analysis of the deterministic part. We thus begin to prove that the random part is negligible in probability.

Lemma 2. *Under the assumptions of Theorem 2, for all* x_n *such that* $\lim_{n \to +\infty} x_n n^{1/4-\beta} = +\infty$ *for some* $\beta > 0$,

$$\lim_{n \to +\infty} P\left(\sup_{(\tau_1,\tau_2) \in \Theta^2} |R_n(\tau_1, \tau_2)| \geq x_n \right) = 0$$

A consequence of this lemma is that $\sup_{(\tau_1,\tau_2) \in \Theta^2} |R_n(\tau_1, \tau_2)| = o_P(n^{-1/5})$.

Lemma 3. *There exists* ρ_n *tending to* 0 *such that with a probability tending to* 1,

$$(\tilde{\tau}_1, \tilde{\tau}_2) \in B(\tau_1^0, \frac{\rho_n}{nt_n}) \times B(\tau_2^0, \frac{\rho_n}{nt_n}),\ (\bar{\tau}_1, \bar{\tau}_2) \in B(\tau_1^0, \frac{\rho_n}{nt_n}) \times B(\tau_2^0, \frac{\rho_n}{nt_n}).$$

Proof. We shall consider separately the case of rational or irrational ratio of the two periods. Consider first the case where $\frac{\tau_1^0}{\tau_2^0} \in \mathbb{Q}$, say $\frac{\tau_1^0}{\tau_2^0} = \frac{l}{m}$.
Notice that for any integers j, r, $B(\frac{j\tau_1^0}{r}, \alpha_n) = B(\frac{jl\tau_2^0}{rm}, \alpha_n)$.
Let us notice the following simple facts concerning $K_n(\tau_1, \tau_2)$. Using $\lim_{n\to+\infty} m_n\beta_n = 0$, for big enough n,

$$K_n(\tau_1^0, \tau_2^0) = \left\{ (k_1, k_2) \in \{1, \ldots, m_n\}^2, \frac{k_1}{k_2} = \frac{l}{m} \right\}. \tag{22}$$

Also, one obtains that for any $r_i \leq m_n$ and j_i such that $(j_i\tau_i^0/r_i) \in \Theta$, $i = 1, 2$,

$$\forall \tau_1 \in B(\frac{j_1\tau_1^0}{r_1}, \alpha_n), \forall \tau_2 \in B(\frac{j_2\tau_2^0}{r_2}, \alpha_n), \left| \frac{\tau_1}{\tau_2} - \frac{j_1 r_2 l}{j_2 r_1 m} \right| \leq \left(\frac{\Delta}{\delta}\right) \left(\frac{1+\alpha_n}{1-\alpha_n} - 1 \right)$$

so that using (N4), for big enough n, for any $r_i \leq m_n$ and j_i such that $(j_i\tau_i^0/r_i) \in \Theta$, $i = 1, 2$, $\forall \tau_1 \in B(\frac{j_1\tau_1^0}{r_1}, \alpha_n), \forall \tau_2 \in B(\frac{j_2\tau_2^0}{r_2}, \alpha_n)$,

$$\left\{ (k_1, k_2) \in \{1, \ldots, m_n\}^2 : \frac{k_1}{k_2} = \frac{j_1 r_2 l}{j_2 r_1 m} \right\} \subset K_n(\tau_1, \tau_2). \tag{23}$$

Now, for any integer k and any τ,

$$\frac{1}{n^2} \sum_{j=1}^{n} e^{\frac{2ik\pi j t_n}{\tau}} (s_1^0 + s_2^0)(jt_n) = \sum_p c_p^0 \varphi_n \left(\frac{pt_n}{\tau_1^0} - \frac{kt_n}{\tau} \right) + \sum_p d_p^0 \varphi_n \left(\frac{pt_n}{\tau_2^0} - \frac{kt_n}{\tau} \right). \tag{24}$$

Using (22) and equation (12) repeatedly, one easily obtains

$$\mathcal{H}_n(\tau_1^0, \tau_2^0) = \sum_{k\geq 1} \left| c_{kl}^0 + d_{km}^0 \right|^2 + \sum_{k\geq 1, l\nmid k} \left| c_k^0 \right|^2 + \sum_{k\geq 1, m\nmid k} \left| d_k^0 \right|^2 + o_{\mathbb{P}_n^0}(1).$$

Also, using (23) and equation (12) repeatedly, one obtains that uniformly in $r_i \leq m_n$ and j_i such that $(j_i\tau_i^0/r_i) \in \Theta$, $i = 1, 2$, uniformly for $\tau_1 \in B(\frac{j_1\tau_1^0}{r_1}, \alpha_n)$ and $\tau_2 \in B(\frac{j_2\tau_2^0}{r_2}, \alpha_n)$,

$$\begin{aligned} H_n(\tau_1, \tau_2) \leq & \sum_{k\geq 1} \left| c_{kmr_3}(s_2^0) + c_{klr_3}(s_1^0) \right|^2 + \sum_{k\geq 1, b|k, l\nmid k} \left| c_{kr_3}(s_1^0) \right|^2 + \sum_{k\geq 1, m\nmid k, m_4|k} \left| c_{kr_3}(s_2^0) \right|^2 \\ & + \sum_{k\geq 1, r_5\nmid k} \left| c_{kmr_4}(s_2^0) + c_{klr_4}(s_1^0) \right|^2 + \sum_{k\geq 1, l_2|k, l\nmid k, l_2 r_7\nmid k} \left| c_{kr_4}(s_1^0) \right|^2 + \sum_{k\geq 1, m\nmid k, d|k, dr_{10}\nmid k} \left| c_{kr_3}(s_2^0) \right|^2 + o(1) \end{aligned}$$

where $(j_1; m) = a$ with $m = am_4$, $(r_1; l) = b$ with $l = bl_1$ and $r_1 = br_3$, $(m; r_2) = d$ with $r_2 = dr_4$, $(j_2; l) = c$ with $l = cl_2$, $(r_3; r_4) = e$ with $r_3 = er_5$, $(r_1; l_2 r_4) = f$ with $r_1 = fr_7$ and $(m_4 r_3; r_2) = g$ with $m_4 r_3 = gr_{10}$. Here, the notation $(a; b)$ for positive integers a and b refers to the biggest common divisor of a and b. This implies that,

$$\sup_{r_1\nmid j_1, r_2\nmid j_2} \sup_{\tau_1\in B(\frac{j_1\tau_1^0}{r_1}, \alpha_n), \tau_2\in B(\frac{j_2\tau_2^0}{r_2}, \alpha_n)} H_n(\tau_1, \tau_2) \leq (1-h) H_n(\tau_1^0, \tau_2^0) + o(1)$$

using the identifiability condition $(s_1^0, s_2^0) \in \tilde{\mathcal{S}}$. Same results hold exchanging τ_1 and τ_2.
Also, recalling that τ_0 is the global period of the signal,

$$\sup_{j_1, r_2\nmid j_2} \sup_{\tau_1\in B(j_1\tau_0, \alpha_n), \tau_2\in B(\frac{j_2\tau_2^0}{r_2}, \alpha_n)} \max\left\{ H_n(\tau_1, \tau_2), H_n(\tau_2, \tau_1) \right\} \leq H_n(\tau_1^0, \tau_2^0) + o(1)$$

and

$$\sup_{j_1, j_2} \sup_{\tau_1\in B(j_1\tau_1^0, \alpha_n), \tau_2\in B(j_2\tau_2^0, \alpha_n)} \max\left\{ H_n(\tau_1, \tau_2), H_n(\tau_2, \tau_1) \right\} \leq H_n(\tau_1^0, \tau_2^0) + o(1).$$

Now, using (8), (9) and (11),

$$\sup_{\tau_1,\tau_2\notin\left(\cup_{r\le m_n,j}B(j\tau_1^0/r,\alpha_n)\right)\cup\left(\cup_{r\le m_n,j}B(j\tau_2^0/r,\alpha_n)\right)} H_n(\tau_1,\tau_2)=o(1),$$

and if $r_1 \nmid j_1$,

$$\sup_{\tau_1\in B(j_1\tau_1^0/r_1,\alpha_n),\tau_2\notin\left(\cup_{r\le m_n,j}B(j\tau_1^0/r,\alpha_n)\right)\cup\left(\cup_{r\le m_n,j}B(j\tau_2^0/r,\alpha_n)\right)} H_n(\tau_1,\tau_2)\le(1-h)H_n(\tau_1^0,\tau_2^0)+o(1),$$

with similar inequality exchanging τ_1 and τ_2.
All these results imply that with a probability tending to 1,

$$\begin{aligned}\mathcal{U}_n \quad\subset\quad & \left(\cup_{j_1,j_2}B(j_1\tau_1^0,\alpha_n)\times B(j_2\tau_2^0,\alpha_n)\right)\cup\left(\cup_{j_1,j_2}B(j_2\tau_2^0,\alpha_n)\times B(j_1\tau_1^0,\alpha_n)\right)\\ & \cup\left(\cup_j B(j\tau_0,\alpha_n)\times\Theta\right)\cup\left(\Theta\times\cup_j B(j\tau_0,\alpha_n)\right).\end{aligned}$$

Moreover, since with a probability tending to 1 $(\tau_1^0,\tau_2^0)\in\mathcal{U}_n$, then with a probability tending to 1,

$$(\tilde\tau_1,\tilde\tau_2)\in B(\tau_1^0,\alpha_n)\times B(\tau_2^0,\alpha_n),\ (\bar\tau_1,\bar\tau_2)\in B(\tau_1^0,\alpha_n)\times B(\tau_2^0,\alpha_n).$$

Then, using computations similar than for (15) and (16), Lemma 3 follows for the case of rational ratio.
Consider now the case where $\frac{\tau_1^0}{\tau_2^0}\notin\mathbb{Q}$.
Let us notice the following simple facts concerning $K_n(\tau_1,\tau_2)$.
Using $\lim_{n\to+\infty}\beta_n=0$,

$$\text{if } \frac{\tau_1}{\tau_2}\notin\mathbb{Q},\ \inf\left\{k\ :\exists k',(k,k')\in K_n(\tau_1,\tau_2)\text{ or }(k',k)\in K_n(\tau_1,\tau_2)\right\}\to+\infty. \tag{25}$$

Let $num(n)$ and $den(n)$ be the smallest integers such that

$$\frac{num(n)}{den(n)}-2\left(\frac{\Delta}{\delta}\right)^3\beta_n\le\frac{\tau_1^0}{\tau_2^0}\le\frac{num(n)}{den(n)}+2\left(\frac{\Delta}{\delta}\right)^3\beta_n. \tag{26}$$

Then

$$\lim_{n\to+\infty}num(n)=+\infty,\ \lim_{n\to+\infty}den(n)=+\infty. \tag{27}$$

Now, for any fixed k, using (8), (9), (11) and (12), one obtains, uniformly in all parameters :

$$\begin{aligned}&\text{if } \tau\notin\left(\cup_{r\le m_n,j}B(j\tau_1^0/r,\alpha_n)\right)\cup\left(\cup_{r\le m_n,j}B(j\tau_2^0/r,\alpha_n)\right),\\ &\sum_p c_p^0\varphi_n\left(\frac{pt_n}{\tau_1^0}-\frac{kt_n}{\tau}\right)+\sum_p d_p^0\varphi_n\left(\frac{pt_n}{\tau_2^0}-\frac{kt_n}{\tau}\right)=o\left(\frac{1}{m_n}\right),\end{aligned}$$

$$\begin{aligned}&\text{if } \tau\in B(j_1\tau_1^0/r_1,\alpha_n)\text{ and }\tau\notin\left(\cup_{r\le m_n,j}B(j\tau_2^0/r,\alpha_n)\right),\\ &\sum_p c_p^0\varphi_n\left(\frac{pt_n}{\tau_1^0}-\frac{kt_n}{\tau}\right)+\sum_p d_p^0\varphi_n\left(\frac{pt_n}{\tau_2^0}-\frac{kt_n}{\tau}\right)=\begin{cases}o\left(\frac{1}{m_n}\right) & \text{if } j_1\nmid k\\ c_{k'r_1}^0+o\left(\frac{1}{m_n}\right) & \text{if } k=k'j_1.\end{cases}\end{aligned}$$

Notice now that,

$$\begin{aligned}&\text{if } B(\frac{j_1\tau_1^0}{r_1},\alpha_n)\cap B(\frac{j_2\tau_2^0}{r_2},\alpha_n)\neq\emptyset,\\ &\left(j_1\ge\sqrt{den(n)}\text{ and }r_1\ge\sqrt{num(n)}\right)\text{ or }\left(j_2\ge\sqrt{num(n)}\text{ and }r_2\ge\sqrt{den(n)}\right),\end{aligned}\tag{28}$$

so that, by using (27),

if $\tau \in B(j_1\tau_1^0/r_1, \alpha_n) \cap B(j_2\tau_2^0/r_2, \alpha_n)$,

$$\sum_p c_p^0 \varphi_n\left(\frac{pt_n}{\tau_1^0} - \frac{kt_n}{\tau}\right) + \sum_p d_p^0 \varphi_n\left(\frac{pt_n}{\tau_2^0} - \frac{kt_n}{\tau}\right) = \begin{cases} o\left(\frac{1}{m_n}\right) & \text{if } j_1 \nmid k \text{ and } j_2 \nmid k \\ c_{k'r_1}^0 + o\left(\frac{1}{m_n}\right) & \text{if } k = k'j_1 \text{ and } j_2 \nmid k \\ d_{k'r_2}^0 + o\left(\frac{1}{m_n}\right) & \text{if } k = k'j_2 \text{ and } j_1 \nmid k \\ c_{r_1k/j_1}^0 + d_{r_2k/j_2}^0 + o\left(\frac{1}{m_n}\right) & \text{if } j_1 \mid k \text{ and } j_2 \mid k. \end{cases}$$

First of all, we obtain

$$H_n(\tau_1^0, \tau_2^0) = \sum_{k\geq 1} |c_k^0|^2 + \sum_{k\geq 1} |d_k^0|^2 + o(1).$$

Then,

$$\sup_{\tau_1,\tau_2 \notin \left(\cup_{r\leq m_n,j} B(j\tau_1^0/r,\alpha_n)\right)\cup\left(\cup_{r\leq m_n,j} B(j\tau_2^0/r,\alpha_n)\right)} H_n(\tau_1, \tau_2) = o(1).$$

One has now to consider the cases where τ_1, τ_2 are or not in a ball around a fraction of τ_1^0 and are or not in a ball around a fraction of τ_2^0.
Notice that if $\tau_1 \in B(j_i\tau_i^0/r_i, \alpha_n)$ and $\tau_2 \in B(p_i\tau_i^0/q_i, \alpha_n)$, then

$$\left\{(k_1, k_2) \in \{1, \ldots, m_n\}^2 \; : \; \frac{k_1}{k_2} = \frac{j_i q_i}{r_i p_i}\right\} \subset K_n(\tau_1, \tau_2),$$

and if $\tau_1 \in B(j_1\tau_1^0/r_1, \alpha_n)$ and $\tau_2 \in B(p_2\tau_2^0/q_2, \alpha_n)$, then for any $(k_1, k_2) \in K_n(\tau_1, \tau_2)$,

$$\left|\frac{k_1 r_1 p_2}{k_2 j_1 q_2} - \frac{\tau_1^0}{\tau_2^0}\right| \leq 2\left(\frac{\Delta}{\delta}\right)^3 \beta_n,$$

so that

$$k_2 j_1 q_2 \geq den(n) \;\text{ and }\; k_1 r_1 p_2 \geq num(n)$$

and one may use (27).
All these tools together with the fact that $(s_1^0, s_2^0) \in \tilde{\mathcal{S}}$ lead to the fact that with a probability tending to 1,

$$\mathcal{U}_n \subset \left(\cup_{j,p} B(j\tau_1^0, \alpha_n) \times B(p\tau_2^0, \alpha_n)\right) \cup \left(\cup_{j,p} B(p\tau_2^0, \alpha_n) \times B(j\tau_1^0, \alpha_n)\right),$$

and that with a probability tending to 1,

$$(\tilde{\tau}_1, \tilde{\tau}_2) \in B(\tau_1^0, \alpha_n) \times B(\tau_2^0, \alpha_n),\; (\bar{\tau}_1, \bar{\tau}_2) \in B(\tau_1^0, \alpha_n) \times B(\tau_2^0, \alpha_n).$$

Now, for any $(\tau_1, \tau_2) \in B(\tau_1^0, \alpha_n) \times B(\tau_2^0, \alpha_n)$,

$$H_n(\tau_1, \tau_2) = \sum_k \left|c_k^0 \varphi_n\left(\frac{pt_n}{\tau_1^0} - \frac{kt_n}{\tau_1}\right)\right|^2 + \sum_k \left|d_k^0 \varphi_n\left(\frac{pt_n}{\tau_2^0} - \frac{kt_n}{\tau}\right)\right|^2 + o(1)$$

so that, using computations similar than for (15) and (16), Lemma 3 follows for the case of irrational ratio. □

Then,

$$\lim_{n\to+\infty} P\left((\hat{\tau}_1, \hat{\tau}_2) = (\bar{\tau}_1, \bar{\tau}_2)\right) = 1. \tag{29}$$

As soon as one proves that

$$\frac{\partial \mathcal{H}_n}{\partial \tau_1}(\tau_1^0, \tau_2^0) = \left(\sum_{j=1}^n \left(\frac{1}{2} - \frac{j}{n}\right) t_n \left(\frac{1}{(\tau_1^0)^2} \tilde{f}_1'\left(\frac{jt_n}{\tau_1^0}\right) + \frac{g_0'\left(\frac{jt_n}{\tau_*}\right)}{2\tau_1^0 \tau_*}\right) \varepsilon(j)\right)(1 + o_{\mathbb{P}_n^0}(1)),$$

$$\frac{\partial \mathcal{H}_n}{\partial \tau_2}(\tau_1^0,\tau_2^0) = \left(\sum_{j=1}^n \left(\frac{1}{2}-\frac{j}{n}\right) t_n \left(\frac{1}{(\tau_2^0)^2}\tilde{f}_2'\left(\frac{jt_n}{\tau_2^0}\right) + \frac{g_0'\left(\frac{jt_n}{\tau_*}\right)}{2\tau_2^0\tau_*}\right)\varepsilon(j)\right)(1+o_{\mathbb{P}_n^0}(1)),$$

and

$$\begin{pmatrix} \frac{\partial^2\mathcal{H}_n}{\partial\tau_1^2}(\tau_1,\tau_2) & \frac{\partial^2\mathcal{H}_n}{\partial\tau_1\partial\tau_2}(\tau_1,\tau_2) \\ \frac{\partial^2\mathcal{H}_n}{\partial\tau_1\partial\tau_2}(\tau_1,\tau_2) & \frac{\partial^2\mathcal{H}_n}{\partial\tau_2^2}(\tau_1,\tau_2)\end{pmatrix} = -n^2t_n^2\sigma_0^2K_*(1+o_{\mathbb{P}_n^0}(1)),$$

uniformly for τ_1 in $B(\tau_1^0,\rho_n/nt_n)$, τ_2 in $B(\tau_2^0,\rho_n/nt_n)$, one obtains Theorem 2 by using (29) and a Taylor expansion for $(\bar{\tau}_1,\bar{\tau}_2)$. But these results may be obtained by computations similar to those done for the single period model and by using arguments similar to those used in the proof of Lemma 3 to derive the asymptotic behaviour of all terms in the computation, in case of rational or irrational ratio of the periods. □

5 Proofs.

5.1 Proof of Proposition 1.

Easy computations lead to, with $u(j)=\varepsilon(j)/\sigma_0$,

$$\begin{aligned} L_n(\tau_0+\frac{a}{n^{3/2}t_n}, f_0+\frac{f}{\sqrt{n}}, \sigma_0+\frac{b}{\sqrt{n}}) &= \frac{1}{\sqrt{n}}\sum_{j=1}^n \frac{b}{\sigma_0}(u(j)^2-1)+\sum_{j=1}^n u(j)\frac{A_{j,n}}{\sigma_0} \\ &\quad -\frac{3}{2n}\frac{b^2}{\sigma_0^2}\sum_{j=1}^n\left(u(j)^2-\frac{1}{3}\right)-\frac{1}{2}\sum_{j=1}^n\left(\frac{A_{j,n}}{\sigma_0}\right)^2+o_{\mathbb{P}_n^0}(1)\end{aligned}$$

where:

$$A_{j,n}=\frac{1}{\sqrt{n}}f\left(\frac{jt_n}{\tau_0}\right)-\frac{a}{n^{3/2}t_n}\left(\frac{jt_n}{\tau_0^2}\right)f_0'\left(\frac{jt_n}{\tau_0}\right).$$

The result follows from the Lindeberg's theorem and that if f and g are in $\mathcal{H}$:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^n f\left(\frac{jt_n}{\tau_0}\right)g\left(\frac{jt_n}{\tau_0}\right)=\int_0^1 f(t)g(t)dt \tag{30}$$

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{j=1}^n j^2 f_0'^2\left(\frac{jt_n}{\tau_0}\right)=\frac{1}{3}\int_0^1 f_0'^2(t)dt \tag{31}$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{j=1}^n jf_0'\left(\frac{jt_n}{\tau_0}\right)f\left(\frac{jt_n}{\tau_0}\right)=\int_0^1 f_0'(t)f(t)dt \tag{32}$$

To prove (30), (31) and (32), write

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^n f\left(\frac{jt_n}{\tau_0}\right)g\left(\frac{jt_n}{\tau_0}\right) &= c_0(fg)+\sum_{k\in\mathbb{Z}^*}c_k(fg)\varphi_n(\frac{kt_n}{\tau_0}) \\ \frac{1}{n^3}\sum_{j=1}^n j^2 f_0'^2\left(\frac{jt_n}{\tau_0}\right) &= \frac{c_0(f_0'^2)}{3}+o(1)-\sum_{k\in\mathbb{Z}^*}c_k(f_0'^2)\frac{1}{4\pi^2k^2n^2t_n^2}\varphi_n''(\frac{kt_n}{\tau_0}) \\ \frac{1}{n^2}\sum_{j=1}^n jf_0'\left(\frac{jt_n}{\tau_0}\right)f\left(\frac{jt_n}{\tau_0}\right) &= \frac{c_0(f_0'f)}{2}+o(1)+\sum_{k\in\mathbb{Z}^*}c_k(f_0'f)\frac{1}{2\pi iknt_n}\varphi_n'(\frac{kt_n}{\tau_0}),\end{aligned}$$

use the fact that for $|k| \leq \tau_0/2t_n$, $|kt_n/\tau_0| \leq 1/2$, use (6) and

$$\sum_{|k|>\tau_0/2t_n} c_k(h) = o(1)$$

for $h = fg$ or $h = f_0'^2$ or $h = ff_0'$ since $t_n \to 0$.
$\dot{\nu}^T$ follows from direct computation, and efficiency of linearized estimators follows from McNeney and Wellner (2000).

5.2 Proof of Lemma 1.

Note that for all $\tau \in \Theta_n$, $\eta_n(\tau) = \zeta_n(\tau) + \xi_n(\tau)$ with:

$$\zeta_n(\tau) = \sum_{k=1}^{m_n} \frac{1}{n^2} \left| \sum_{j=1}^{n} \varepsilon(j) \exp\left(\frac{2ik\pi j t_n}{\tau}\right) \right|^2$$

and

$$\xi_n(\tau) = \sum_{j=1}^{n} \left(\frac{1}{n^2} \sum_{k=1}^{n} \left[\sum_{-m_n \leq l \leq m_n \ ; \ l \neq 0} e^{\frac{2i\pi l(j-k)t_n}{\tau}} \right] s_0(kt_n) \right) \varepsilon(j).$$

We use the following lemma due to Golubev (1988).

Lemma 4. ***Generalized Markov inequality***
For $\mathcal{L}$ stochastic process and $A = [\alpha; \beta]$, we have $\forall \lambda > 0$, $\forall R > 0$,

$$P\left(\sup_{\tau \in A} \mathcal{L}(\tau) > R\right) \leq \exp(-\lambda R) \sup_{\tau \in A} \left(\sqrt{E\left[e^{2\lambda \mathcal{L}(\tau)}\right]}\right) \left[1 + \lambda \int_{\tau \in A} \sqrt{E\left[|\mathcal{L}'(\tau)|^2\right]} d\tau\right].$$

Let us first control the Laplace transform of $\eta_n(\tau)$.

Lemma 5. *Under the assumptions of Lemma 1, for all $\lambda > 0$ such that $\lambda \sigma_0^2 m_n/n \leq 1/4$, for some constant $C > 0$,*

$$E\left[e^{\lambda \eta_n(\tau)}\right] \leq \exp\left[\frac{C\lambda^2}{n}\right].$$

By the Cauchy-Schwarz inequality, we just have to upper bound $E\left[e^{\lambda \xi_n(\tau)}\right]$ and $E\left[e^{\lambda \zeta_n(\tau)}\right]$.
Now, $E\left[e^{\lambda \xi_n(\tau)}\right] = e^{\frac{\lambda^2 V^2}{2}}$
where:

$$\begin{aligned}
V^2 &= \frac{\sigma_0^2}{n^2} \sum_{j=1}^{n} \left\{ \sum_{-m_n \leq l \leq m_n; l \neq 0} \left(\sum_{p \in \mathbb{Z}} c_p(s_0) \varphi_n\left(\frac{pt_n}{\tau_0} - \frac{lt_n}{\tau}\right)\right) e^{\frac{2i\pi l j t_n}{\tau}} \right\}^2 \\
&\leq \frac{\sigma_0^2}{n} \left\{ \sum_{-m_n \leq l \leq m_n; l \neq 0} \left| \sum_{p \in \mathbb{Z}} c_p(s_0) \varphi_n\left(\frac{pt_n}{\tau_0} - \frac{lt_n}{\tau}\right) \right| \right\}^2 \\
&\leq \frac{2\sigma_0^2}{n} G_n(\tau)(1 + o(1))
\end{aligned}$$

so that

$$E\left[e^{\lambda \xi_n(\tau)}\right] \leq \exp\left[\frac{C\lambda^2}{n}\right],$$

for some $C > 0$.
Now, $\zeta_n(\tau) =< A(\tau)\varepsilon \ ; \ \varepsilon >$ where $A(\tau)$ is the $n \times n$ matrix such that for $1 \leq j,k \leq n$:

$$A(\tau)_{j,k} = \frac{K_n\left[\frac{(j-k)t_n}{\tau}\right]}{2n^2}$$

where: $K_n(v) = \sum_{-m_n \leq l \leq m_n ; l \neq 0} e^{2i\pi l v}$. Thus A is real, symmetric and positive.
Furthermore,

$$K_n(v) = \begin{cases} \frac{2\sin[\pi v m_n]}{\sin(\pi v)} \cos[\pi v(m_n+1)], \ if \ v \neq 0, \\ 2m_n, \ if \ v = 0. \end{cases}$$

Now,

$$E\left[e^{\lambda\zeta_n(\tau)}\right] = E\left[e^{\lambda\sigma_0^2 \sum_{i=1}^n \mu_i \varepsilon_i'^2}\right]$$

where: μ_i are the eigenvalues of $A(\tau)$ and ε_i' are independent standard normal random variables, so that $E\left[e^{\lambda\zeta_n(\tau)}\right] = \exp\left[-\frac{1}{2}\sum_{i=1}^{n} \ln\left(1 - 2\lambda\sigma_0^2\mu_i\right)\right]$ as soon as for all i, $2\lambda\sigma_0^2\mu_i < 1/2$, which holds when $\lambda\sigma_0^2 m_n/n < 1/4$ since $tr(A(\tau)) = m_n/n$. We thus have

$$E\left[e^{\lambda\zeta_n(\tau)}\right] \leq \exp\left[\sigma_0^2\lambda\sum_{i=1}^n \mu_i - \sigma_0^4\lambda^2\sum_{i=1}^n \mu_i^2\right] \leq \exp(\frac{1}{4})$$

since $ln(1+u) \geq u - u^2$ if $u \geq -1/2$.

Lemma 6. *Under the assumptions of Lemma 1, for some $C > 0$,*

$$E\left[|\zeta_n'(\tau)|^2\right] \leq \frac{Cm_n^4 t_n^2}{\tau^4}, \tag{33}$$

$$E\left[|\xi_n'(\tau)|^2\right] \leq C\left(\frac{nm_n^2 t_n^2}{\tau^4} + \frac{nt_n^2}{\tau^2\tau_0^2}\right). \tag{34}$$

Thus, applying Lemma 4 one has for some constant $C_2 > 0$ and if $\lambda\sigma_0^2 m_n/n < 1/4$:

$$P\left(\sup_{\tau\in\Theta_n} \eta_n(\tau) > x_n\right) \leq e^{-\lambda x_n} e^{C\lambda^2/n}\left(1 + C_2\lambda\left[\frac{\sqrt{n}m_n t_n}{\delta_n} + \sqrt{n}t_n \ln(\frac{\Delta_n}{\delta_n})\right]\right).$$

If $x_n m_n$ tends to 0, then by (N3), taking $\lambda = nx_n/2C$, for some $C" > 0$,

$$P\left(\sup_{\tau\in\Theta_n} \eta_n(\tau) > x_n\right) \leq C" x_n n\sqrt{n}(\log n) e^{-Cnx_n^2/m_n},$$

tends to 0. If now $x_n m_n$ does not tend to 0, for big enough n, $x_n \geq y_n$ for a sequence y_n such that $y_n m_n$ tends to 0.
Lemma 1 follows since ζ_n is positive and $-\xi_n$ is normally distributed with the same mean and variance as ξ_n.
Proof of Lemma 6. We first prove the inequality (33).
Straightforward computations lead to,

$$E\left[\zeta_n'(\tau)^2\right] = \left(\frac{4\pi t_n}{n^2\tau^2}\right)^2 E\left[\left(\sum_{l=1}^{m_n} l\left[Z_1(l)Z_2(l) - Z_3(l)Z_4(l)\right]\right)^2\right]$$

where:

$$\begin{cases} Z_1(l) = \sum_{j=1}^{n} j \sin\left(\frac{2\pi l j t_n}{\tau}\right) \frac{\epsilon(j)}{\sigma_0}, \\ Z_2(l) = \sum_{k=1}^{n} \cos\left(\frac{2\pi l k t_n}{\tau}\right) \frac{\epsilon(k)}{\sigma_0}, \\ Z_3(l) = \sum_{j=1}^{n} j \cos\left(\frac{2\pi l j t_n}{\tau}\right) \frac{\epsilon(j)}{\sigma_0}, \\ Z_4(l) = \sum_{k=1}^{n} \sin\left(\frac{2\pi l k t_n}{\tau}\right) \frac{\epsilon(k)}{\sigma_0}. \end{cases}$$

So:

$$E\left[\zeta_n'(\tau)^2\right] = \left(\frac{4\pi t_n}{n^2\tau^2}\right)^2 E\left[\sum_{1\le l,k\le m_n} lU(l)U(k)\right]$$

where: $U(l) = Z_1(l)Z_2(l) - Z_3(l)Z_4(l)$.
Thus: $U(l)U(k) = Z_1(l)Z_2(l)Z_1(k)Z_2(k) - Z_1(l)Z_2(l)Z_3(k)Z_4(k) - Z_3(l)Z_4(l)Z_1(k)Z_2(k) + Z_3(l)Z_4(l)Z_3(k)Z_4(k)$.

These different terms will be noted respectively: $\boxed{1}, \boxed{2}, \boxed{3}, \boxed{4}$.
We will focus on: $E\left[\sum_{1\le k,l\le m_n} lk\boxed{1}\right]$ because the others can be treated in the same way.

$$E\left[\sum_{1\le k,l\le m_n} lk\boxed{1}\right]$$

$$= \sigma_0^4 \sum_{1\le l,k\le m_n} lk \left\{ \begin{array}{l} 3\sum_{j=1}^{n} j^2 \sin\left(\frac{2\pi l j t_n}{\tau}\right)\sin\left(\frac{2\pi k j t_n}{\tau}\right)\cos\left(\frac{2\pi l j t_n}{\tau}\right)\cos\left(\frac{2\pi k j t_n}{\tau}\right) \\ + \left(\sum_{j=1}^{n} j \sin\left(\frac{2\pi l j t_n}{\tau}\right)\cos\left(\frac{2\pi l j t_n}{\tau}\right)\right)\left(\sum_{m=1}^{n} m \sin\left(\frac{2\pi k m t_n}{\tau}\right)\cos\left(\frac{2\pi k m t_n}{\tau}\right)\right) \\ + \left(\sum_{j=1}^{n} j^2 \sin\left(\frac{2\pi l j t_n}{\tau}\right)\sin\left(\frac{2\pi k j t_n}{\tau}\right)\right)\left(\sum_{m=1}^{n} \cos\left(\frac{2\pi l m t_n}{\tau}\right)\cos\left(\frac{2\pi k m t_n}{\tau}\right)\right) \\ + \left(\sum_{j=1}^{n} j \sin\left(\frac{2\pi l j t_n}{\tau}\right)\cos\left(\frac{2\pi k j t_n}{\tau}\right)\right)\left(\sum_{m=1}^{n} m \cos\left(\frac{2\pi l m t_n}{\tau}\right)\sin\left(\frac{2\pi k m t_n}{\tau}\right)\right) \end{array} \right\}.$$

Now, the only thing that remains to do is to prove that each term into the braces is less than the expected quantity. This can be done by saying that the absolute values of the sinus and the cosinus are less than one.

For the inequality (34), one has after some straightforward computations,

$$\tau^4 E\left[\xi_n'(\tau)^2\right]$$
$$= \frac{\sigma_0^2}{n^2}\sum_{j=1}^{n}\left\{ \begin{array}{l} j \sum_{|l|\le m_n; l\neq 0} (2i\pi l t_n) e^{2i\pi l j t_n/\tau} \sum_{p\in\mathbb{Z}} c_p(s_0)\varphi_n\left[t_n\left(\frac{p}{\tau_0} - \frac{l}{\tau}\right)\right] \\ - \sum_{|l|\le m_n; l\neq 0} l t_n e^{2i\pi l j t_n/\tau} \sum_{p\in\mathbb{Z}} c_p(s_0)\varphi_n'\left[t_n\left(\frac{p}{\tau_0} - \frac{l}{\tau}\right)\right] \end{array} \right\}^2.$$

Focus on the first sum into the braces and cut $\sum_p$ into $\sum_{|p|>m_n}$ and $\sum_{|p|\le m_n}$.

$$\sum_{|l|\le m_n; l\neq 0} |l| t_n \sum_{|p|>m_n} |c_p(s_0)| \left|\varphi_n\left[t_n\left(\frac{p}{\tau_0} - \frac{l}{\tau}\right)\right]\right| \le m_n^2 t_n \frac{1}{m_n} \sum_{|p|>m_n} |p||c_p(s_0)|.$$

Thanks to **(N3)**,

$$\begin{aligned}
&\sum_{|l|\leq m_n; l\neq 0} |l| t_n \sum_{|p|\leq m_n} |c_p(s_0)| \left|\varphi_n\left[t_n\left(\frac{p}{\tau_0}-\frac{l}{\tau}\right)\right]\right| \\
= & \sum_{|l|\leq m_n; l\neq 0} |l| t_n \; c_{[l\tau_0/\tau]}(s_0) + m_n^2 t_n \sum_{|p|\leq m_n; p\neq [l\tau_0/\tau]} \frac{2\tau_0 |c_p(s_0)|}{n t_n} \\
\leq & \frac{t_n \tau}{\tau_0} \sum_{m\in\mathbb{Z}^*} |m+1||c_m(s_0)| + \frac{C\tau_0 m_n^2}{n} \leq \frac{C t_n \tau}{\tau_0} + \frac{C\tau_0 m_n^2}{n}
\end{aligned}$$

for some constant C. Thus the modulus of the first term into the braces is less than:

$$jC\left(m_n t_n + \frac{t_n\tau}{\tau_0} + \frac{\tau_0 m_n^2}{n}\right).$$

With almost the same arguments, the modulus of the second term into the braces is less than :

$$C\left(n m_n t_n + \frac{n t_n \tau}{\tau_0} + \tau_0 m_n^2\right).$$

5.3 Proof of equation (11).

If $0 < p \leq m_n$, for any $1 \leq k \leq m_n$, $\frac{k\tau_0}{p} \in \Theta_n$, so that if $\tau \notin \left(\cup_{l\leq m_n, j} B(\frac{j\tau_0}{l}, \alpha_n)\right) \cap \Theta_n$, then for any $1 \leq k \leq m_n$ and $0 < p \leq m_n$, $|p - \frac{k\tau_0}{\tau}| \geq |p|\alpha_n$. Use of (6) gives the result.

5.4 Proof of equation (12).

Write

$$\frac{pt_n}{\tau_0} - \frac{kt_n}{\tau} = \frac{pt_n}{\tau_0} - \frac{klt_n}{j\tau_0} + \frac{klt_n}{j\tau_0} - \frac{kt_n}{\tau}.$$

For $\tau \in B(\frac{j\tau_0}{l}, \alpha_n)$, $|\frac{klt_n}{j\tau_0} - \frac{kt_n}{\tau}| \leq \frac{\alpha_n t_n m_n^2}{j\tau_0}$ tends to 0.
For $k = jk'$ and $p \neq k'l$, $|p - k'l| \geq 1$, and for $k \neq jk'$, $|p - \frac{kl}{j}| \geq \frac{1}{j}$. Use of (6) leads to the results.

5.5 Proof of equation (18).

$$\Lambda_n'(\tau_0) = \xi_n'(\tau_0) + \zeta_n'(\tau_0) + G_n'(\tau_0).$$

First, after some straightforward computations:

$$\xi_n'(\tau_0) = \sum_{k=1}^{n} \varepsilon(k) \left(\frac{1}{n^2}\sum_{l=1}^{n} -\frac{(k-l)t_n}{(\tau_0)^2} K_n'\left(\frac{(k-l)t_n}{\tau_0}\right) s_0(lt_n)\right).$$

But,

$$\begin{aligned}
&\frac{1}{n}\sum_{l=1}^{n} -(k-l)t_n K_n'\left(\frac{(k-l)t_n}{\tau_0}\right) s_0(lt_n) \\
= & -\sum_{|q|\leq m_n}\sum_{p\in\mathbb{Z}} c_p(s_0) e^{\frac{2i\pi qkt_n}{\tau_0}} \left\{(2i\pi q)kt_n\varphi_n\left[\frac{t_n}{\tau_0}(p-q)\right] - t_n q \varphi_n{}'\left[\frac{t_n}{\tau_0}(p-q)\right]\right\} \\
= & \; t_n\left(k - \frac{n+1}{2}\right)\left(\sum_{|j|\leq m_n} 2i\pi j c_j(s_0) e^{\frac{2i\pi jkt_n}{\tau_0}}\right) \\
- & \sum_{|j|\leq m_n}\sum_{p\neq j} c_p(s_0) e^{\frac{2i\pi jkt_n}{\tau_0}} \left\{(2i\pi j)kt_n\varphi_n\left[\frac{t_n}{\tau_0}(p-j)\right] - t_n j \varphi_n{}'\left[\frac{t_n}{\tau_0}(p-j)\right]\right\}.
\end{aligned}$$

As $\sum_{p\in\mathbb{Z}} |c_p(s_0)| < +\infty$ the second and the third term of the previous equality are $O\left(\frac{km_n^2}{n}+m_n^2\right)$.
Thus:

$$\xi_n'(\tau_0) = \frac{1}{\tau_0^2}\sum_{k=1}^{n} t_n\left(\frac{1}{2}-\frac{k}{n}\right) f_0'\left(\frac{kt_n}{\tau_0}\right)\varepsilon(k) + O_{\mathbb{P}_n^0}(m_n^2).$$

Then, $\zeta_n'(\tau_0) =< B(\tau_0)\varepsilon\ ;\ \varepsilon >$ where

$$B_{j,k}(\tau) = -\frac{(j-k)t_n}{2n^2\tau^2}\sum_{|l|\le m(n)\ ;\ l\ne 0} 2i\pi l e^{2i\pi l t_n\left(\frac{j-k}{\tau}\right)}.$$

With this definition, $B(\tau)$ is real, symmetrical, $Tr(B(\tau)) = 0$ and one has, as usual,

$$\operatorname{Var}\left(\zeta_n'(\tau_0)\right) = 2\sigma_0^4 Tr\left(B(\tau_0)^2\right) = O\left(t_n^2 m_n^4\right),$$

so that

$$\zeta_n'(\tau_0) = o_{\mathbb{P}_n^0}(\sqrt{n}t_n).$$

At last,

$$\begin{aligned}
G_n'(\tau_0) &= \sum_{k=1}^{m_n}\sum_{p_1,p_2} c_{p_1}\overline{c_{p_2}}\left(\frac{t_n k}{(\tau_0)^2}\right)\left\{\begin{array}{c} \varphi_n'\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right] \\ +\varphi_n\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n'}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right]\end{array}\right\} \\
&= \sum_{k=1}^{m_n}\left[\begin{array}{c} |c_k|^2\left(\frac{t_n}{(\tau_0)^2}\right)\left(\varphi_n'(0)\overline{\varphi_n}(0)+\varphi_n(0)\overline{\varphi' n}(0)\right) \\ +\displaystyle\sum_{(p_1,p_2)\ne(k,k)} c_{p_1}\overline{c_{p_2}}\left(\frac{t_n k}{(\tau_0)^2}\right)\left\{\begin{array}{c} \varphi_n'\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right] \\ +\varphi_n\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n'}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right]\end{array}\right\}\end{array}\right] \\
&= O\left(\sum_k |k||c_k|\right) = O(1) = o(\sqrt{n}t_n).
\end{aligned}$$

5.6 Proof of equation (19).

First, $G_n''(\tilde\tau) = G_n''(\tau_0) + (\tilde\tau-\tau_0)G_n'''(\tau)$, for some $\tau \in B\left(\tau_0; \frac{\rho_n}{nt_n}\right)$.

$$\begin{aligned}
G_n''(\tau_0) &= \sum_{k=1}^{m_n}\sum_{p_1,p_2} c_{p_1}\overline{c_{p_2}}\left(-\frac{2t_n k}{(\tau_0)^3}\right)\left\{\begin{array}{c} \varphi_n'\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right] \\ +\varphi_n\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n'}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right]\end{array}\right\} \\
&+ \sum_{k=1}^{m_n}\sum_{p_1,p_2} c_{p_1}\overline{c_{p_2}}\left(\frac{t_n k}{(\tau_0)^2}\right)\left\{\begin{array}{c} \varphi_n''\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right] \\ +2\varphi_n'\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n'}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right] \\ +\varphi_n\left[t_n\left(\frac{p_1}{\tau_0}-\frac{k}{\tau_0}\right)\right]\overline{\varphi_n''}\left[t_n\left(\frac{p_2}{\tau_0}-\frac{k}{\tau_0}\right)\right]\end{array}\right\}.
\end{aligned}$$

By the same arguments as those used to compute $G_n'(\tau_0)$,

$$\begin{aligned}
G_n''(\tau_0) &= \sum_{k=1}^{m_n} k^2|c_k|^2\frac{t_n}{\tau_0^2}\left(-\frac{4\pi^2n^2}{3}+2\pi^2n^2-\frac{4\pi^2n^2}{3}\right)(1+o(1)) \\
&= -\frac{2n^2\pi^2}{3\tau_0^4}t_n^2\sum_{k=1}^{m_n} k^2|c_k|^2(1+o(1)) = -n^2t_n^2\sigma_0^2 I^*(1+o(1)).
\end{aligned}$$

In the same way, it can be proved that $G_n'''(\tau) = o(n^3t_n^3)$.
Finally, $G_n''(\tilde\tau) = -n^2t_n^2\sigma_0^2 I^*(1+o(1))$.

As for $\zeta_n''(\tilde{\tau})$, we prove that $\zeta_n''(\tilde{\tau}) = o_p(n^2 t_n^2)$.
For $\tau \in B\left(\tau_0; \frac{\rho_n}{nt_n}\right)$, $\zeta_n''(\tau) =< C(\tau)\varepsilon; \varepsilon >$ where

$$C_{j,k}(\tau) = \frac{1}{n^2\tau^3} \sum_{|l|\leq m_n; l\neq 0} 2i\pi l(j-k)t_n e^{\frac{2i\pi l(j-k)t_n}{\tau}} + \frac{1}{2n^2\tau^4} \sum_{|l|\leq m_n; l\neq 0} \left[2i\pi l(j-k)t_n\right]^2 e^{\frac{2i\pi l(j-k)t_n}{\tau}}.$$

By the generalized Markov inequality (lemma 4)

$$P\left(\sup_{\tau\in B\left(\tau_0;\frac{\rho_n}{nt_n}\right)} \zeta_n''(\tau) > n^2 t_n^2\right) \leq \exp(-\lambda n^2 t_n^2) E\left(e^{2\lambda\zeta_n''(\tau)}\right)\left(1 + \lambda\int_\tau \sqrt{E\left[|\zeta_n'''(\tau)|^2\right]} d\tau\right).$$

As usual,

$$E\left[e^{\lambda\zeta_n''(\tau)}\right] \leq \exp\left(2\lambda^2\sigma^4 \sum_{k=1}^n \mu_k^2\right)$$

if $\lambda = O\left(\frac{1}{nm_n^3 t_n^2}\right)$ (because the eigenvalues μ_k of the matrix C satisfy $\max_k |\mu_k| = O\left(\frac{m_n^2 t_n}{\tau^3} + \frac{nm_n^3 t_n^2}{\tau^4}\right)$).
With $Tr(C(\tau)^2) = O\left(\frac{m_n^4 t_n^2}{\tau^6} + \frac{n^2 m_n^6 t_n^4}{\tau^8}\right)$,

$$E\left[e^{\lambda\zeta_n''(\tau)}\right] = \exp\left[\frac{2\lambda^2{\sigma_0}^4 m_n^4 t_n^2}{\tau^6}\left(1 + \frac{m_n^2 t_n^2 n^2}{\tau^2}\right)\right].$$

With usual arguments,

$$E\left[\left|\zeta_n'''(\tau)\right|^2\right] = O\left(\frac{m_n^7 n^5 t_n^6}{\tau^{12}}\right)$$

so that,

$$P\left(\sup_{\tau\in B\left(\tau_0;\frac{\rho_n}{nt_n}\right)} \zeta_n''(\tau) > n^2 t_n^2\right) \leq C\exp(-\lambda n^2 t_n^2) exp(\lambda^2 m_n^6 t_n^4 n^2)\left[1 + \lambda\int_\tau O\left(\frac{m_n^7 n^5 t_n^6}{\tau^{12}}\right) d\tau\right].$$

And,

$$\lim_{n\to+\infty} P\left(\sup_{\tau\in B\left(\tau_0;\frac{\rho_n}{nt_n}\right)} \zeta_n''(\tau) > n^2 t_n^2\right) = 0.$$

Let us prove that $\xi_n''(\tilde{\tau}) = o_p(n^2 t_n^2)$.
Straightforward computations lead to

$$\begin{aligned}
\xi_n''(\tau) \quad &= \quad \frac{1}{n\tau^4} \sum_{|l|\leq m_n} \sum_{j=1}^n (2i\pi l j t_n)^2 \sum_{p\in\mathbb{Z}} c_p(s_0)\varphi_n\left[t_n\left(\frac{p}{\tau_0} - \frac{l}{\tau}\right)\right]\varepsilon(j) \\
&+ \quad \frac{1}{n\tau^4} \sum_{|l|\leq m_n} \sum_{j=1}^n (l t_n)^2 \sum_{p\in\mathbb{Z}} c_p(s_0)\varphi_n''\left[t_n\left(\frac{p}{\tau_0} - \frac{l}{\tau}\right)\right]\varepsilon(j) \\
&- \quad \frac{2}{n\tau^4} \sum_{|l|\leq m_n} \sum_{j=1}^n j l t_n \sum_{p\in\mathbb{Z}} c_p(s_0)\varphi_n'\left[t_n\left(\frac{p}{\tau_0} - \frac{l}{\tau}\right)\right]\varepsilon(j) \\
&- \quad \frac{2}{n\tau^3} \sum_{|l|\leq m_n} \sum_{j=1}^n 2i\pi l j t_n \sum_{p\in\mathbb{Z}} c_p(s_0)\varphi_n\left[t_n\left(\frac{p}{\tau_0} - \frac{l}{\tau}\right)\right]\varepsilon(j) \\
&+ \quad \frac{2}{n\tau^3} \sum_{|l|\leq m_n} \sum_{j=1}^n l t_n \sum_{p\in\mathbb{Z}} c_p(s_0)\varphi_n'\left[t_n\left(\frac{p}{\tau_0} - \frac{l}{\tau}\right)\right]\varepsilon(j).
\end{aligned}$$

So $E\left(e^{\lambda\xi''_n(\tau)}\right) = e^{\lambda^2\sigma^2/2}$ with $\sigma^2 = O\left(\frac{n^3t_n^4}{\tau^8}\right)$.
It can also be proved that

$$E\left[\xi'''_n(\tau)^2\right] = O\left[\frac{nm_n^4t_n^4}{\tau^{10}} + \frac{nt_n^4}{\tau^6} + \frac{m_n^6t_n^2}{n\tau^{10}} + \frac{nm_n^6t_n^6}{n\tau^{16}} + \frac{n^3t_n^6}{n\tau^{10}} + \frac{nm_n^6t_n^4}{\tau^{16}}\right]$$

so that using again the generalized Markov inequality (lemma 4),

$$P\left(\sup_{\tau\in B\left(\tau_0;\frac{\rho_n}{nt_n}\right)} \xi''_n(\tau) > n^2t_n^2\right) \le C\exp(-\lambda n^2t_n^2)\exp(\lambda^2n^3t_n^4)\left[1+\lambda\int_\tau \sqrt{E\left[\xi'''_n(\tau)^2\right]}d\tau\right].$$

The expected result follows with $\lambda = 1/(2nt_n^2)$.

5.7 Proof of Proposition 3.

The proof is similar to the one of Proposition 1 and one just needs to prove limits of sums of the form: $\frac{1}{n}\sum_{j=1}^n f_1\left(\frac{jt_n}{\tau_1^0}\right)f_2\left(\frac{jt_n}{\tau_2^0}\right)$, $\frac{1}{n^2}\sum_{j=1}^n jf_1\left(\frac{jt_n}{\tau_1^0}\right)f_2^{0\prime}\left(\frac{jt_n}{\tau_2^0}\right)$, $\frac{1}{n^2}\sum_{j=1}^n jf_2\left(\frac{jt_n}{\tau_2^0}\right)f_1^{0\prime}\left(\frac{jt_n}{\tau_1^0}\right)$, and $\frac{1}{n^3}\sum_{j=1}^n j^2f_1^{0\prime}\left(\frac{jt_n}{\tau_1^0}\right)f_2^{0\prime}\left(\frac{jt_n}{\tau_2^0}\right)$. Now

$$\frac{1}{n}\sum_{j=1}^n f_1\left(\frac{jt_n}{\tau_1^0}\right)f_2\left(\frac{jt_n}{\tau_2^0}\right) = \sum_{k,l} c_k(f_1)c_l(f_2)\varphi_n\left[t_n\left(\frac{k}{\tau_1^0}+\frac{l}{\tau_2^0}\right)\right], \tag{35}$$

so that, if they exist, the limit of (35) as $n\to+\infty$ and (21) as $T\to+\infty$ are the same.
If τ_1^0/τ_2^0 is an irreducible fraction p/q, this limit exists (and equals $\sum_m c_{mp}(f_1)c_{-mq}(f_2)$).
If $\tau_1^0/\tau_2^0 \notin \mathbb{Q}$, write

$$\sum_{k,l} c_k(f_1)c_l(f_2)\varphi\left[T\left(\frac{k}{\tau_1^0}+\frac{l}{\tau_2^0}\right)\right] = c_0(f_1)c_0(f_2) + \sum_{k\neq0,l\neq0} c_k(f_1)c_l(f_2)\varphi\left[\frac{Tl}{\tau_1^0}\left(\frac{k}{l}+\frac{\tau_1^0}{\tau_2^0}\right)\right] + o(1).$$

For any $l\neq0$, there exists $m(l)$ such that

$$\frac{m(l)}{l} < \frac{\tau_1^0}{\tau_2^0} < \frac{m(l)+1}{l},\ \alpha(l) = \min\left(\frac{\tau_1^0}{\tau_2^0} - \frac{m(l)}{l}; \frac{m(l)+1}{l} - \frac{\tau_1^0}{\tau_2^0}\right) > 0.$$

If $k < -m(l)-1$ or $k > -m(l)$, $|\frac{k}{l}+\frac{\tau_1^0}{\tau_2^0}| > \frac{1}{|l|}$, so that

$$\begin{aligned}\sum_{k,l} c_k(f_1)c_l(f_2)\varphi\left[T\left(\frac{k}{\tau_1^0}+\frac{l}{\tau_2^0}\right)\right] &= c_0(f_1)c_0(f_2) + o(1) + \sum_{l\neq0} c_{-m(l)}(f_1)c_l(f_2)\varphi\left[\frac{Tl}{\tau_1^0}\left(\frac{-m(l)}{l}+\frac{\tau_1^0}{\tau_2^0}\right)\right]\\ &\quad + \sum_{l\neq0} c_{-m(l)-1}(f_1)c_l(f_2)\varphi\left[\frac{Tl}{\tau_1^0}\left(\frac{-m(l)-1}{l}+\frac{\tau_1^0}{\tau_2^0}\right)\right].\end{aligned}$$

Now, since $\tau_1^0/\tau_2^0 \notin \mathbb{Q}$, $m(l)\to+\infty$ as $l\to+\infty$, and for any L such that, if $|l|\ge L$, $|c_{-m(l)-1}(f_1)| \le \epsilon/2$ and $|c_{-m(l)}(f_1)| \le \epsilon/2$:

$$\sum_{k,l} c_k(f_1)c_l(f_2)\varphi\left[T\left(\frac{k}{\tau_1^0}+\frac{l}{\tau_2^0}\right)\right] = c_0(f_1)c_0(f_2) + o(1) + O\left(\epsilon\sum_{|l|\ge L}|c_l(f_2)|\right) + o\left(\sum_{|l|<L,l\neq0}\frac{1}{l\alpha(l)}\right)$$

so that the limits of (35) as $n\to+\infty$ and (21) as $T\to+\infty$ exist (and equal $c_0(f_1)c_0(f_2)$).
In the same way:

$$\lim_{n\to+\infty}\frac{1}{n^2}\sum_{j=1}^n jf_1\left(\frac{jt_n}{\tau_1^0}\right)f_2'\left(\frac{jt_n}{\tau_2^0}\right) = \frac{1}{2}\lim_{T\to+\infty}\frac{1}{T}\int_{-T/2}^{T/2} f_1\left(\frac{t}{\tau_1^0}\right)f_2'\left(\frac{t}{\tau_2^0}\right)dt,$$

$$\lim_{n\to+\infty} \frac{1}{n^2} \sum_{j=1}^{n} j f_2\left(\frac{jt_n}{\tau_2^0}\right) {f'}_1\left(\frac{jt_n}{\tau_1^0}\right) = \frac{1}{2} \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} {f'}_1\left(\frac{t}{\tau_1^0}\right) f_2\left(\frac{t}{\tau_2^0}\right) dt,$$

$$\lim_{n\to+\infty} \frac{1}{n^3} \sum_{j=1}^{n} j^2 {f'}_1\left(\frac{jt_n}{\tau_1^0}\right) {f'}_2\left(\frac{jt_n}{\tau_2^0}\right) = \frac{1}{3} \lim_{T\to+\infty} \frac{1}{T} \int_{-T/2}^{T/2} {f'}_1\left(\frac{t}{\tau_1^0}\right) {f'}_2\left(\frac{t}{\tau_2^0}\right) dt.$$

5.8 Proof of Lemma 2.

First of all, using the fact that $\lim_{n\to+\infty} m_n\beta_n = 0$, one obtains that for any (τ_1, τ_2), for big enough n, for any $k \in \{1, \dots, m_n\}$, the sets $\{k_1 : (k_1, k) \in K_n(\tau_1, \tau_2)\}$ and $\{k_2 : (k, k_2) \in K_n(\tau_1, \tau_2)\}$ have at most one element. This allows to use Cauchy-Schwarz inequality to obtain

$$|R_{n,2}(\tau_1, \tau_2)| \le \zeta_n(\tau_1) + \zeta_n(\tau_2) + \sqrt{\zeta_n(\tau_1)}\sqrt{\zeta_n(\tau_2)},$$

$$|R_{n,1}(\tau_1, \tau_2)| \le 2\sqrt{\zeta_n(\tau_1)}\sqrt{G_n(\tau_1)} + 2\sqrt{\zeta_n(\tau_2)}\sqrt{G_n(\tau_2)} + 2\sqrt{\zeta_n(\tau_1)}\sqrt{G_n(\tau_2)} + 2\sqrt{\zeta_n(\tau_2)}\sqrt{G_n(\tau_1)},$$

so that

$$\sup_{\tau_1,\tau_2} |R_{n,2}(\tau_1, \tau_2)| \le 3 \sup_{\tau} \zeta_n(\tau)$$

and

$$\sup_{\tau_1,\tau_2} |R_{n,1}(\tau_1, \tau_2)| \le 8 \sup_{\tau} \sqrt{\zeta_n(\tau)} \sup_{\tau} \sqrt{G_n(\tau)}.$$

The Lemma follows from Lemma 1 and $\sup_{\tau} G_n(\tau) = O(1)$.

References

Carmona, R., Hwang, W.-L. and Torrésani, B. (1998). *Practical time-frequency analysis.* Wavelet analysis and its application, Academic Press.

De Cheveigné, A. and Kawahra, H. (1998). Multiple period estimation and pitch perception model. Unpublished, http://citeseer.nj.nec.com/284277.html.

Golubev, G. K. (1988). Estimation of the period of a signal with an unknown form against a white noise background. *Problemy Peredachi Informatsii* **24** 38–52.

Gregory, P. and Loredo, T. (1992). A new method for the detection of a periodic signal of unknown shape and period. Http://bayes.wustl.edu/gregory/gregory.html.

Hall, P., Reimann, J. and Rice, J. (2000). Nonparametric estimation of a periodic function. *Biometrika* **87** 545–557.

Hall, P. and Yin, J. (2003). Nonparametric methods for deconvolving multiperiodic functions. Unpublished.

Lavielle, M. and Lévy-Leduc, C. (2003). A model selection approach for estimating the period of periodic functions with unknown shape and applications in laser vibrometry. Submitted. Prépublication de l'université Paris-Sud(2003-21).

Lévy-Leduc, C. (2003). Algorithmes d'estimation de périodes de signaux périodiques inconnus et application à la vibrométrie laser. In *Proceedings of the 19th GRETSI Symposium on Signal and Image Processing, Paris September 8-11, 2003.*

McNeney, B. and Wellner, J. A. (2000). Application of convolution theorems in semiparametric models with non-i.i.d. data. *J. Statist. Plann. Inference* **91** 441–480.

Prenat, M. (2001). Vibration modes and laser vibrometry performance in noise. In *Proceedings of the Physics in Signal and Image Processing (PSIP'2001) conference, Marseille, France.*

Quinn, B. G. and Fernandes, J. M. (1991). A fast efficient technique for the estimation of frequency. *Biometrika* **78** 489–497.

Quinn, B. G. and Hannan, E. J. (2001). *The estimation and tracking of frequency.* Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge.

Quinn, B. G. and Thomson, P. J. (1991). Estimating the frequency of a periodic function. *Biometrika* **78** 65–74.

van der Vaart, A. W. (1998). *Asymptotic statistics.* Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge.