Mesh optimization for singular axi-symmetric harmonic maps from the disc into the sphere (I)

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Abstract

We propose a new viewpoint for the numerical computation of singular solutions to PDEs. The idea is to mesh the graph of the singular solution. We apply this idea to the test case of minimizing axi-symmetric harmonic maps from B^2 into S^2. The formulation of the PDE in terms of a minimization problem gives the criterion for the optimal mesh of the graph.

In the first part we study the well-posedness of the continuous problem in the space of functions with bounded energy. We introduce a lower semi-continuous extension of the energy with respect to the weak convergence in BV and we prove that the extended minimization problem admits a unique singular solution.

In the second part we present the numerical results for various piecewise linear discretizations of the problem. The number of points in the mesh is fixed and the mesh is an unknown of the discrete minimization problem. This method allows us in particular to compute boundary layers of zero thickness.

Another paper, Part II (to be submitted to this Journal), will prove the convergence of the method.

I Introduction

The aim of this article is to propose a new approach for the numerical computation of singular solutions to PDEs. The idea is to mesh the *graph* of the singular solution rather than the space domain where it is posed. The solution and the adapted mesh are computed simultaneously by a moving-mesh method (see for instance [14]), in contrast to the classical mesh-refinement method wherein the two steps are decoupled.

We apply this idea in dimension 1 to the test case of minimizing axi-symmetric harmonic maps from B^2 into S^2. The singularity can be described as a "boundary layer with zero thickness" and the minimization problem is well posed by considering *the graph of the solution as the unknown.* The formulation of the PDE in terms of a minimization problem gives the criterion for the optimal mesh of the graph.

The exact solutions for the continuous problem can be described explicitly in terms of BV functions: this is the content of the first part. The second part presents the numerical results

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of our graph-meshing method, as well as computational considerations. The discretization is based on continuous piecewise linear (P_1) finite elements. What makes the numerical approach of particular interest is the possibility of extension to higher dimension.

A map $u : B^2 \to S^2$ defined on the unit disc B^2 and with values in the unit sphere S^2 is *harmonic* if it satisfies the following PDE:

$$-\Delta u = u|\nabla u|^2 \text{ in } \Omega. \tag{I.1}$$

A harmonic map is a critical point of the Dirichlet energy:

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx. \tag{I.2}$$

Harmonic maps between manifolds are a generalization of harmonic functions (for which S^2 is replaced by $\mathbb{R}$) and of geodesics (for which B^2 is replaced by $[0,1]$). They have been thorougly studied (see for instance the report by Hardt [10] and references therein). For numerical aspects the study of harmonic maps in dimension 3 is closely linked to the physics of *liquid crystals* [6]. Numerical algorithms have been proposed for instance by Alouges [1] and improved in [17].

When one tries to minimize the energy (I.2) among smooth S^2 valued maps with prescribed boundary condition, loss of compactness by concentration (*bubbling*) may appear [5, 20]. In order to catch this phenomenon, Giaquinta, Modica and Souček [7, 8] have proposed to consider the graph of the unknown as a current and to understand the convergence of minimizing currents. This led them to the notion of *cartesian currents* [7, 8].

Currents generalize the notion of manifolds in the same way that distributions generalize the notion of function. A *cartesian current* is a current which is limit of graphs of functions in the sense of currents.

A map $u : B^2 \to S^2$ from the unit disc into the unit sphere is *axially symmetric* if there exists an angle function $\theta : [0,1] \to \mathbb{R}$ (θ is the latitude) which depends only on $r := \sqrt{x^2+y^2}$ such that:

$$u(x,y) = \Big(\cos\theta(r)\frac{x}{r}, \cos\theta(r)\frac{y}{r}, \sin\theta(r)\Big) \quad \forall (x,y) \in B^2.$$

The axi-symmetric maps are the vector fields u which are invariant by rotation around the axis Oz and such that $u(x,y)$ is in the same line of longitude as (x,y).

If u is axi-symmetric with angle function $\theta(r)$:

$$E(u) := \frac{1}{2}\int_{B^2} |\nabla u|^2 dxdy = \pi\int_0^1 \frac{\cos^2\theta}{r} + r\theta'^2 dr =: E(\theta). \tag{I.3}$$

The Dirichlet condition on ∂B^2 is given by $\theta(1) = \alpha$. Notice that $E(\theta+\pi) = E(\theta)$ and $E(-\theta) = E(\theta)$ so we may choose $\alpha \in [-\frac{\pi}{2}, 0]$.

We define the space of bounded energy functions with boundary condition α:

$$\mathcal{E}_\alpha := \left\{\theta \in \mathcal{C}^0(]0,1],\mathbb{R}),\ \theta(1) = \alpha,\ \frac{\cos\theta}{\sqrt{r}} \in L^2(]0,1[) \text{ and } \sqrt{r}\theta' \in L^2(]0,1[)\right\}. \tag{I.4}$$

Let $\theta \in \mathcal{E}_\alpha$. The condition $E(\theta) < \infty$ implies that θ is continuous at $r = 0$ and that there exists $k \in \mathbb{Z}$ such that

$$\theta(0) = -\frac{\pi}{2} + k\pi. \tag{I.5}$$

The integer $k \in \mathbb{Z}$ given by (I.5) is called the $1d$ *degree of* θ [9]. It is the degree of a map from S^1 into S^1 which is canonically associated to θ by a process similar to the $2d$ case [20]. Notice that the $2d$ degree of the axi-symmetric map $u : B^2 \to S^2$ which corresponds to θ is necessarily 0 or 1 [16] : this means that the $1d$ degree is more precise than the $2d$ degree.

As a consequence we have a partition $\mathcal{E}_\alpha = \bigcup_{k\in\mathbb{Z}} \mathcal{E}_{\alpha,k}$, where

$$\mathcal{E}_{\alpha,k} := \{\theta \in \mathcal{E}_\alpha,\ \theta(0) = -\frac{\pi}{2} + k\pi\}.$$

Following the approach of Brézis and Coron [5] in the $2d$ case, we consider the following minimization problem:

$$\textbf{Minimize } \boldsymbol{E(\theta)} \textbf{ in } \boldsymbol{\mathcal{E}_{\alpha,k}}. \tag{I.6}$$

From the results of the theory for the $2d$ Dirichlet problem we expect an energy concentration at $r = 0$ for the $1d$ problem (I.6). This is interesting as a test case for the graph-meshing method. However, the $1d$ degree is more precise than the $2d$ degree and the existing theory does not describe completely the $1d$ Dirichlet problem (I.6). Therefore in the first part of this article we study the well-posedness of the $1d$ problem (I.6) by mimicking the $2d$ case. In particular we describe the singular solutions in a BV frame which simplifies the cartesian current approach [7].

In the second part of the article we introduce the graph-meshing method for the minimization problem (I.6). The idea is to consider the mesh with a fixed number of points N as a variable in the discrete minimization problem. For the computation we adapt classical optimization algorithms (such as gradient methods, alternate variable method or Newton algorithm) to the various discretized problem. We present the numerical results with three different continuous piecelinear (P_1) discretizations of E. Compared with a classical refinement method, the results are very satisfactory since we obtain a boundary layer with *exactly* zero thickness.

In part II of this article (to be submitted to this Journal; see also [2]) we prove the convergence of the method in terms of weak BV convergence. We also compute numerically the speed of convergence.

II The continuous problem

1 Notations

Let $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}[$. We remind that for $\theta \in \mathcal{E}_\alpha$ the energy is given by:

$$E(\theta) := \pi \int_0^1 \frac{\cos^2\theta}{r} + r\theta'^2 dr. \tag{II.7}$$

We point out that a distribution $\theta \in \mathcal{D}'(]0,1[)$ such that $\sqrt{r}\theta' \in L^2(]0,1[)$ is absolutely continuous on $]0,1]$: in the definition (I.4 of $\mathcal{E}_\alpha$ we thus have all the distributions with finite energy.

A function with bounded energy can only take discrete values at $r = 0$. This is the content of the following proposition.

Proposition II.1 *Let $\theta \in \mathcal{E}_\alpha$. Then θ is continuous on $[0,1]$ and there exists a (unique) integer $k \in \mathbb{Z}$ such that $\theta(0) = -\frac{\pi}{2} + k\pi$.*

Proof. For $0 < s \le t \le 1$, we have

$$\sin\theta(t) - \sin\theta(s) = \int_s^t \theta' \cos\theta dr.$$

Hence, by Cauchy-Swhwarz inequality,

$$|\sin\theta(t) - \sin\theta(s)| \le \sqrt{\int_s^t r\theta'^2 dr}\sqrt{\int_s^t \frac{\cos^2\theta}{r} dr}.$$

From this bound we deduce that $\sin\theta(r)$ satisfies the Cauchy criterium and thus admits a limit l as r tends to 0. Thus $\cos^2\theta(r)$ tends to $1 - l^2$ as r tends to 0 and since $\int_0^1 \frac{\cos^2\theta}{r} dr < \infty$, we have $l = \pm 1$. The continuity of θ on $]0,1]$ and the fact that $\sin\theta$ tends to ± 1 as r tends to 0 implie that θ tends to $-\frac{\pi}{2} + k\pi$ with $k \in \mathbb{Z}$ as r tends to 0. ■

Now we are able to define for all $k \in \mathbb{Z}$ the corresponding "homotopy class":

$$\mathcal{E}_{\alpha,k} := \{\theta \in \mathcal{E}_\alpha,\ \theta(0) = -\frac{\pi}{2} + k\pi\}, \tag{II.8}$$

and consider the following minimization problem:

$$\textbf{Minimize } \boldsymbol{E(\theta)} \textbf{ in } \boldsymbol{\mathcal{E}_{\alpha,k}}. \tag{II.9}$$

We will answer the related questions:

1. For which conditions α, k does there exist a minimizer, and what are these minimizers ?

2. What happens when there is no minimizer ?

2 Regular minimizers

In this section we find the conditions α, k for which the problem (II.9) admits a solution and give these solutions.

We first seek critical points by solving the Euler-Lagrange equation. Then, from the fact that the energy is bounded by below by the degree, we prove that these critical points are minimizers in their class.

A minimizer of E in $\mathcal{E}_{\alpha,k}$ is a critical point for variations with compact support (in the sense of the following definition II.1), because such a variation does not change the degree nor the boundary condition.

Definition II.1 (Euler-Lagrange equation) *We say that $\theta \in \mathcal{E}_\alpha$ is a critical point of E for variations with compact support if*

$$\frac{d}{dt} E(\theta + t\psi)|_{t=0} = 0 \textit{ for all } \psi \in \mathcal{C}_c^\infty(]0,1[).$$

This is equivalent to saying that θ satisfies the Euler-Lagrange equation:

$$(r\theta')' + \frac{\sin(2\theta)}{2r} = 0 \text{ in } \mathcal{D}'(]0,1[). \tag{II.10}$$

Proof. It suffices to see that $E(\theta + t\psi)$ is defined and to compute its derivative with respect to t. ■

A minimizer in $\mathcal{E}_{\alpha,k}$ satisfies equation (II.10). This ODE can be solved explicitly as in [11]. Some intermediate results will be useful so we sketch the proof.

The method is to multiply (II.10) by $r\theta'$ and to integrate. Using the bounded-energy condition we find that θ satisfies

$$r^2\theta'^2 = \cos^2\theta. \tag{II.11}$$

By carefully taking the square root in equation (II.11) we see that θ satisfies one of the following ODEs:

$$r\theta' = \cos\theta \text{ or } r\theta' = -\cos\theta. \tag{II.12}$$

These ODEs can be integrated, and using $\theta(1) = \alpha$ the following result is obtained:

Proposition II.2 (Jäger, Kaul [11])

1. *If $\alpha \equiv -\frac{\pi}{2} \pmod \pi$ then the unique solution of (II.10) in $\mathcal{E}_\alpha$ is the constant function $\theta \equiv \alpha$.*

2. *If $\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ there exist exactly two solutions $\theta_{\alpha,1}$ and $\theta_{\alpha,0}$ de (II.10) in $\mathcal{E}_\alpha$, with degree 1 and 0 respectively:*

$$\begin{aligned} \theta_{\alpha,1}(r) &= 2\arctan\left(\frac{\lambda - r}{\lambda + r}\right) \quad \forall r \in [0,1], \\ \theta_{\alpha,0}(r) &= -2\arctan\left(\frac{1 - \lambda r}{1 + \lambda r}\right) \quad \forall r \in [0,1], \end{aligned}$$

where $\lambda = \dfrac{1 + \tan(\alpha/2)}{1 - \tan(\alpha/2)} \in]0, +\infty[$.

Remark II.1 Point 1. of proposition II.2 is completely contained in Lemaire's theorem [12] in dimension 2, which states that the unique harmonic extension on the disc with values in S^2 of a constant boundary condition is the constant map.

2.1 Fundamental lower bound

The important consequence of proposition II.2 is that in general there is no minimizer of E in $\mathcal{E}_{\alpha,k}$. With the theory from the $2d$ case, we were of course expecting such a result.

In order to see that the critical points we found are minimizer, we need the following lemma, which states that the energy is bounded by below by the $1d$ degree.

Remark II.2 The $2d$ problem uses a similar lower bound with the $2d$ degree [5].

Lemma II.1 *Let* $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}[$, $k \in \mathbb{Z}$ *and* $\theta \in \mathcal{E}_{\alpha,k}$. *Then*

$$
\begin{aligned}
E(\theta) := \pi \int_0^1 \frac{\cos^2\theta}{r} + r\theta'^2 dr \quad &\geq \quad 2\pi \int_0^1 |\theta' \cos\theta| dr && \text{(II.13)}\\
&\geq \quad \begin{cases} 4\pi(k-1) + 2\pi(1-\sin\alpha) & \text{if } k \geq 1, \\ 4\pi|k| + 2\pi(1+\sin\alpha) & \text{if } k \leq 0. \end{cases} && \text{(II.14)}
\end{aligned}
$$

The left and right handsides are equal if and only if θ *is one of the critical points of* E *given by proposition II.2.*

Proof. The Cauchy-Schwarz inequality gives (II.13). We suppose $k \geq 1$ (the proof for $k \leq 0$ is similar). Since θ is a continuous path from $-\frac{\pi}{2} + k\pi \geq \frac{\pi}{2}$ to $\alpha < \frac{\pi}{2}$, there exists a sequence of reals $r_0 = 0 < r_1 < \ldots < r_k = 1$ such that $\theta(r_i) = -\frac{\pi}{2} + (k-i)\pi$ for $i \in \{0, \ldots, k-1\}$. We can decompose:

$$
\begin{aligned}
\int_0^1 |\theta' \cos\theta| dr = \sum_{i=0}^{k-1} \int_{r_i}^{r_{i+1}} |\theta' \cos\theta| dr \quad &\geq \quad \sum_{i=0}^{k-1} \left| \int_{r_i}^{r_{i+1}} \theta' \cos\theta dr \right| \\
&\geq \quad \sum_{i=0}^{k-1} |\sin\theta(r_{i+1}) - \sin\theta(r_i)| \\
&\geq \quad 2(k-1) + (1-\sin\alpha),
\end{aligned}
$$

and this is the second inequality (II.14).

Suppose now that $E(\theta)$ is equal to the right handside (II.14). Then θ is a minimizer in $\mathcal{E}_{\alpha,k}$ and thus a critical point of E. Conversely let θ be a critical point of E. If $\alpha = -\frac{\pi}{2}$, then $\theta \equiv -\frac{\pi}{2}$ and $E(\theta) = 0$. If $\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, then we can compute explicitly, using the fact that θ satisfies the ODEs (II.11) and (II.12):

$$
\begin{aligned}
E(\theta_{\alpha,1}) &= 2\pi(1-\sin\alpha), \\
E(\theta_{\alpha,0}) &= 2\pi(1+\sin\alpha),
\end{aligned}
$$

and the lemma is proved. ■

The results we obtained so far can be summarized in the following corollary:

Corollary II.1

1. *If* $\alpha = -\frac{\pi}{2}$, *the only class admitting a minimizer is* $\mathcal{E}_{\alpha,0}$ *and this minimizer is unique.*

2. *If* $\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, *the class* $\mathcal{E}_{\alpha,0}$ *admits a unique minimizer which is* $\theta_{\alpha,0}$, *the class* $\mathcal{E}_{\alpha,1}$ *admits a unique minimizer which is* $\theta_{\alpha,1}$, *and these are the only two classes that admit a minimizer.*

Remark II.3 The two solutions for $\alpha \in]-\frac{\pi}{2},\frac{\pi}{2}[$ are the ones found by Brézis and Coron [5] in the $2d$ case. One of the solutions is the absolute minimizer in $\mathcal{E}_\alpha$ and is the so-called "small solution" which covers a small portion of the sphere, and the other is the "large solution" which covers the complementary portion.

2.2 Values of the infima

We introduce a useful notation for the infimum:

$$\Lambda_{\alpha,k} := \inf_{\theta\in\mathcal{E}_{\alpha,k}} E(\theta). \tag{II.15}$$

The question we want to answer now is: *"Why aren't the values $\Lambda_{\alpha,k}$ reached in general ?"* or more explicitly: *"What happens to minimizing sequences ?"*.

In the end of this study of the regular minimizers we answer this question by showing that the energy gap observed at the limit corresponds to a loss of degree which occurs because of a concentration at $r=0$.

The lower bound of $\Lambda_{\alpha,k}$ given by lemma II.1 is actually optimal as shown by the following lemma.

Lemma II.2 *Let $k,l\in\mathbb{Z}$ and θ belong to $\mathcal{E}_{\alpha,l}$. For every $\epsilon>0$ there exists θ_ϵ which belongs to $\mathcal{E}_{\alpha,k}$ such that*

$$\theta_\epsilon \equiv \theta \text{ on } [\epsilon,1] \text{ and } E(\theta_\epsilon) \le E(\theta) + 4\pi|k-l| + C\epsilon,$$

where $C>0$ is a constant independant of ϵ.

Proof. We suppose $l=0$: the other constructions are made by simple translation. For $k=0$ we simply take $\theta_\epsilon\equiv\theta$ on $[0,1]$. Suppose now $k>0$. By continuity there exists $r_0>0$ with $r_0<\epsilon$ such that $|\theta(r_0)-\theta(0)|<\epsilon$. On $[r_0,1]$ we choose $\theta_\epsilon\equiv\theta$. On $[\frac{r_0}{2},r_0]$, θ_ϵ is the affine function such that $\theta_\epsilon(\frac{r_0}{2}) = -\frac{\pi}{2}+\epsilon$ and $\theta_\epsilon(r_0)=\theta(r_0)$. Let θ_1 be the solution of the Cauchy problem

$$\begin{cases} r\theta_1' &= -\cos\theta_1 \text{ on }]0,\frac{r_0}{2}],\\ \theta_1(\frac{r_0}{2}) &= -\frac{\pi}{2}+\epsilon. \end{cases}$$

This is actually one of the solutions $\theta_{\alpha,1}$ of the Euler-Lagrange equation (II.12) (for a different α); in particular θ_1 is of class $\mathcal{C}^1$ on $[0,\frac{r_0}{2}]$, stricly decreasing and $\theta_1(0)=+\frac{\pi}{2}$. If $k=1$ we choose $\theta_\epsilon\equiv\theta_1$ on $[0,\frac{r_0}{2}]$.

If $k>1$, we go on with the construction: let $r_1\in]0,\frac{r_0}{2}[$ such that $\theta_1(r_1)=\frac{\pi}{2}-\epsilon$. We choose $\theta_\epsilon\equiv\theta_1$ on $[r_1,\frac{r_0}{2}]$. On $[\frac{r_1}{2},r_1]$, θ_ϵ is the affine function such that $\theta_\epsilon(\frac{r_1}{2}) = \frac{\pi}{2}+\epsilon$ and

$\theta_\epsilon(r_1) = \dfrac{\pi}{2} - \epsilon$. We now use the solution of the Cauchy problem

$$\left\{\begin{array}{rcl} r\theta_2' & = & \cos\theta_2 \text{ on }]0, \frac{r_1}{2}], \\ \theta_2(\frac{r_1}{2}) & = & \frac{\pi}{2} + \epsilon. \end{array}\right.$$

Again θ_2 is defined and regular on $[0, \frac{r_1}{2}]$, strictly decreasing and $\theta_2(0) = \dfrac{3\pi}{2}$. If $k = 2$, we choose $\theta_\epsilon \equiv \theta_2$ on $[0, \frac{r_1}{2}]$. Otherwise we go on with the construction with $r_2 \in]0, \frac{r_1}{2}[$ such that $\theta_2(r_2) = \dfrac{3\pi}{2} - \epsilon$, etc. We finally obtain a sequence $0 < \dfrac{r_{k-1}}{2} < r_{k-1} < \dfrac{r_{k-2}}{2} \ldots < r_0$ and a function θ_ϵ which is solution of

$$r\theta_\epsilon' = \cos\theta_\epsilon \text{ or } r\theta_\epsilon' = -\cos\theta_\epsilon$$

on $[0, \frac{r_{k-1}}{2}]$ and on every $[r_i, \frac{r_{i-1}}{2}]$ and which is affine on every $[\frac{r_i}{2}, r_i]$.

We need now to estimate the error given by this construction. On an interval $[a, b] = [0, \frac{r_k}{2}]$ or $[a, b] = [r_i, \frac{r_{i-1}}{2}]$, this is the equality case for Cauchy-Schwarz, hence

$$\pi \int_a^b \frac{\cos^2\theta_\epsilon}{r} + r\theta_\epsilon'^2 dr = 2\pi \left| \int_a^b \theta_\epsilon' \cos\theta_\epsilon dr \right| \leq 4\pi.$$

On an interval $[\frac{r_i}{2}, r_i]$, θ_ϵ is a decreasing affine function equal to $\dfrac{\pi}{2}$ (mod π) up to ϵ so with a slope $a_i < 0$ such that $|a_i| \leq \dfrac{4\epsilon}{r_i}$. Hence

$$\int_{\frac{r_i}{2}}^{r_i} r\theta_\epsilon'^2 dr = \frac{3}{8} r_i^2 a_i^2 = 6\epsilon^2,$$

and

$$\int_{\frac{r_i}{2}}^{r_i} \frac{\cos^2\theta_\epsilon}{r} dr \leq \int_{\frac{r_i}{2}}^{r_i} \frac{\epsilon^2}{r} dr = \ln(2)\epsilon^2,$$

where we used

$$|\cos\theta_\epsilon| = |\sin(\theta_\epsilon \pm \frac{\pi}{2})| \leq |\theta_\epsilon - \frac{\pi}{2} \bmod (\pi)| \leq \epsilon.$$

Summing on all the intervals we obtain the desired estimate:

$$E(\theta_\epsilon) \leq 4\pi k + C(k)\epsilon + E(\theta).$$

For $k < 0$ we use a similar construction with the increasing solutions of the ODEs $r\theta_\epsilon' = \pm\cos\theta_\epsilon$. ∎

Using lemma II.1 and choosing θ to be one of the minimizers of corollary II.1 we obtain the following corollary:

Corollary II.2

1. *If* $\alpha = -\frac{\pi}{2}$ *then* $\Lambda_{\alpha,k} = 4\pi|k|$ *for every* $k \in \mathbb{Z}$.

2. *If* $\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$*:*

   - *for* $k \geq 1$, $\Lambda_{\alpha,k} = \Lambda_{\alpha,1} + 4\pi(k-1)$ *where* $\Lambda_{\alpha,1} = 2\pi(1-\sin\alpha)$,
   - *for* $k \leq 0$, $\Lambda_{\alpha,k} = \Lambda_{\alpha,0} + 4\pi|k|$ *where* $\Lambda_{\alpha,0} = 2\pi(1+\sin\alpha)$.

2.3 Energy gap by concentration

In the construction of lemma II.2 we added some degree by concentration at $r = 0$. This phenomenon is very similar to what happens for a minimizing sequence in $\mathcal{E}_{\alpha,k}$: at the limit there is a loss of degree (and of energy) by concentration.

In the $2d$ case, the gap in the energy occurs with formation of spheres and its value is 4π multiplied by the number of spheres. The $1d$ situation is similar, as shown by lemma II.3 and theorem II.1.

Lemma II.3 *Let* $k \in \mathbb{Z}$ *and* $(\theta_n)_n$ *be a sequence in* $\mathcal{E}_{\alpha,k}$ *such that* $\sup_n E(\theta_n) < \infty$.

Then there exist a subsequence $(\theta_{n_p})_p$*, an integer* $l \in \mathbb{Z}$ *and* $\theta \in \mathcal{E}_{\alpha,l}$ *such that:*

- $\theta_{n_p} \xrightarrow[p\to\infty]{} \theta$ *uniformly on* $[\epsilon, 1]$ *for all* $\epsilon > 0$,
- $\sqrt{r}\theta'_{n_p} \rightharpoonup \sqrt{r}\theta'$ *weakly in* $L^2(]0,1[)$,
- $\dfrac{\cos\theta_{n_p}}{\sqrt{r}} \rightharpoonup \dfrac{\cos\theta}{\sqrt{r}}$ *weakly in* $L^2(]0,1[)$.

Moreover

$$E(\theta) \leq \liminf_p E(\theta_{n_p}) - 4\pi|k-l|. \tag{II.16}$$

Proof. Since the sequence $(\sqrt{r}\theta'_n)_n$ is bounded in $L^2(]0,1[)$ it converges (up to a subsequence) weakly in L^2 to $\psi_1 \in L^2(]0,1[)$; the same holds for $\left(\dfrac{\cos^2\theta_n}{r}\right)_n$ which converges weakly to ψ_2. Now let $\epsilon > 0$: the sequence $(\theta_n)_n$ is bounded in $H^1(]\epsilon,1[)$ so again (up to a subsequence) it converges weakly in $H^1(]\epsilon,1[)$ and thus uniformly on $[\epsilon,1]$ by Rellich's theorem (cf. [4]). Since this is true for every $\epsilon > 0$ we find that the limit θ is continuous on $]0,1]$, that $\psi_1 = \sqrt{r}\theta'$ and that $\psi_2 = \dfrac{\cos^2\theta}{r}$. By semi-continuity of the L^2 norm:

$$\frac{E(\theta)}{\pi} \leq \liminf_n \int_0^1 r\theta_n'^2 dr + \liminf_n \int_0^1 \frac{\cos^2\theta_n}{r} dr \leq \frac{1}{\pi}\liminf_n E(\theta_n) < \infty,$$

and thus θ belongs to some $\mathcal{E}_{\alpha,l}$ (according to proposition II.1).

Now let us find the bound (II.16) on the energy. Since the limit θ is continuous on $[0,1]$ we can find a sequence $(r_p)_p$ of positive reals which tends to 0 and such that

$$\sup_{[0,r_p]} |\theta(r) - \theta(0)| \leq 1/p.$$

By uniform convergence on $[r_p, 1]$, we can extract a subsequence $(\theta_{n_p})_p$ such that

$$\sup_{[r_p,1]} |\theta_{n_p}(r) - \theta(r)| \leq \frac{1}{p}.$$

Let us divide the energy into three terms:

$$E(\theta_{n_p}) = \pi \int_0^{r_p} \left(\frac{\cos^2 \theta_{n_p}}{r} + r\theta'^2_{n_p} \right) dr + \pi \int_{r_p}^1 \frac{\cos^2 \theta_{n_p}}{r} dr + \pi \int_{r_p}^1 r\theta'^2_{n_p} dr.$$

By applying Fatou's lemma to the sequence $\left(\frac{\cos^2 \theta_{n_p}}{r} 1_{[r_p,1]} \right)_n$ which converges a.e. to $\frac{\cos^2 \theta_{n_p}}{r}$ on $[0,1]$ we obtain

$$\liminf_p \int_{r_p}^1 \frac{\cos^2 \theta_{n_p}}{r} dr \geq \int_0^1 \frac{\cos^2 \theta}{r} dr.$$

Then we notice that $(\sqrt{r}\theta'_{n_p} 1_{[r_p,1]})_n$ tends weakly to $\sqrt{r}\theta'$ in $L^2(]0,1[)$; by semi-continuity of the L^2 norm,

$$\liminf_p \int_{r_p}^1 r\theta'^2_{n_p} dr \geq \int_0^1 r\theta'^2 dr.$$

For the third term we have

$$\pi \int_0^{r_p} \frac{\cos^2 \theta_{n_p}}{r} + r\theta'^2_{n_p} dr \geq 4\pi|k-l| + \epsilon(p),$$

where $\epsilon(p) \to 0$ as $p \to \infty$. In order to obtain this last inequality we notice that $\theta(r_p) \to \theta(0) = -\frac{\pi}{2} + l\pi$, which allows for every p large enough to construct (as in the proof of lemma II.1) a subdivision $r'_0 = 0 < r'_1 < \ldots < r'_{|k-l|} = r_p$ of $[0, r_p]$ such that $r'_i = -\frac{\pi}{2} + (k-i)\pi$ for $i \in \{0, |k-l|-1\}$ and to get the lower bound

$$\int_0^{r_p} \frac{\cos^2 \theta_{n_p}}{r} + r\theta'^2_{n_p} dr \geq 2 \sum_{i=0}^{|k-l|-1} |\int_{r_i}^{r_{i+1}} \theta'_{n_p} \cos\theta_{n_p} dr| = 4|k-l| + 2(|\sin(r_p)| - 1).$$

Summing the three terms,

$$\liminf_p E(\theta_{n_p}) \geq \pi \int_0^1 \frac{\cos^2 \theta}{r} dr + \pi \int_0^1 r\theta'^2 dr + 4\pi|k-l|,$$

and the lemma is proved. ∎

In lemma II.3 we used convergences that are related to the energy. The following shortcuts will be useful:

Definition II.2 (E-convergence) *Let* $(\theta_n)_n$ *be a sequence in* $\mathcal{E}_{\alpha,k}$ *such that* $\sup_n E(\theta_n) < \infty$ *and let* $\theta \in \mathcal{E}_{\alpha,l}$.

We say that $(\theta_n)_n$ *converges weakly to* θ *in the E-sense if* $(\sqrt{r}\theta_n')_n$ *tends weakly to* $\sqrt{r}\theta'$ *in* $L^2(0,1)$.

We say that $(\theta_n)_n$ *converges strongly to* θ *in the E-sense if* $(\sqrt{r}\theta_n')_n$ *tends strongly to* $\sqrt{r}\theta'$ *in* $L^2(0,1)$ *and* $\left(\frac{\cos\theta_n}{\sqrt{r}}\right)_n$ *converges strongly to* $\frac{\cos\theta}{\sqrt{r}}$ *in* $L^2(0,1)$.

Remark II.4 By lemma II.3, if $(\theta_n)_n$ converges weakly to θ in the E-sense with $\sup_n E(\theta_n) < \infty$, then $(\theta_n)_n$ converges uniformly to θ on $[\epsilon, 1]$ for all $\epsilon > 0$ and $\left(\frac{\cos\theta_n}{\sqrt{r}}\right)_n$ converges weakly to $\frac{\cos\theta}{\sqrt{r}}$ in $L^2(0,1)$.

We can now state the theorem which answers the initial question:

Theorem II.1 *Let* $(\theta_n)_n$ *be a minimizing sequence in* $\mathcal{E}_{\alpha,k}$.

1. *Case* $\alpha = -\frac{\pi}{2}$ *:* $(\theta_n)_n$ *converges weakly to the global minimizer* $\theta_0 \equiv -\frac{\pi}{2}$ *in the E-sense. If* $k = 0$ *then the E-convergence is strong.*

2. *Case* $\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$*:*

   - *If* $k \geq 1$, $(\theta_n)_n$ *converges weakly to* $\theta_{\alpha,1}$ *(the minimizer in* $\mathcal{E}_{\alpha,1}$*) in the E-sense. If* $k = 1$ *then the E-convergence is strong.*
   - *If* $k \leq 0$, $(\theta_n)_n$ *converges weakly to* $\theta_{\alpha,0}$ *(the minimizer in* $\mathcal{E}_{\alpha,0}$*) in the E-sense. If* $k = 0$ *then the E-convergence is strong.*

Proof.

1. Case $\alpha = -\frac{\pi}{2}$: let $(\theta_n)_n$ be a minimizing sequence in $\mathcal{E}_{\alpha,k}$ with respect to $\Lambda_{\alpha,k}$; then by lemma II.3 $(\theta_n)_n$ converges (up to a subsequence) weakly in the E-sense to some $\theta \in \mathcal{E}_{\alpha,l}$ and
$$E(\theta) \leq \Lambda_{\alpha,k} - 4\pi|k-l| \leq 4\pi(|k| - |k-l|) \leq 4\pi|l| = \Lambda_{\alpha,l}.$$
So θ is a minimizer in $\mathcal{E}_{\alpha,l}$, and by corollary II.1, $\theta \equiv -\frac{\pi}{2}$. The uniqueness of the limit implies that the whole sequence converges to θ.

   If $k = 0$ then $\Lambda_{\alpha,0} = 0$ and
$$\liminf_n \pi \int_0^1 r\theta_n'^2 dr \leq \liminf_n E(\theta_n) = 0,$$

and the weak convergence added to this equality is equivalent to the strong convergence. The argument is the same for the sequence $\left(\frac{\cos\theta_n}{\sqrt{r}}\right)_n$.

2. Case $\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ with $k \geq 1$: let $(\theta_n)_n$ be a minimizing sequence in $\mathcal{E}_{\alpha,k}$ with respect to $\Lambda_{\alpha,k}$; then by lemma II.3, $(\theta_n)_n$ converges (up to a subsequence) weakly in the E-sense to some $\theta \in \mathcal{E}_{\alpha,l}$ and

$$E(\theta) \leq \Lambda_{\alpha,k} - 4\pi|k-l| \leq \Lambda_{\alpha,1} + 4\pi(k-1-|k-l|).$$

Since $\Lambda_{\alpha,1} < 4\pi$ and $E(\theta) \geq 0$ this inequality implies $l > 0$. Now if $l \geq 1$ this inequality added to the fact that $E(\theta) \geq \Lambda_{\alpha,l} = \Lambda_{\alpha,1} + 4\pi(l-1)$ implies first that $l \leq k$, then that $E(\theta) = \Lambda_{\alpha,1} + 4\pi(l-1) = \Lambda_{\alpha,l}$. So θ is a minimizer in $\mathcal{E}_{\alpha,l}$, that is $\theta \equiv \theta_{\alpha,1}$. The uniqueness of the limit implies that the whole sequence converges to θ.

If $k = 1$,

$$\begin{aligned}(\liminf_n \int_0^1 \frac{\cos^2\theta_n}{r} dr - \int_0^1 \frac{\cos^2\theta_1}{r} dr) \quad + \quad & (\liminf_n \int_0^1 r\theta_n'^2 dr - \int_0^1 r\theta_1'^2 dr) \\ & \leq \frac{1}{\pi}(\liminf_n E(\theta_n) - E(\theta)) = 0,\end{aligned}$$

which implies the strong convergence.

The case $k \leq 0$ is proved in the same way.

■

3 BV functions and relaxed energy

We want to close each class $\mathcal{E}_{\alpha,k}$ in order to obtain a minimizer.

For a minimizing sequence in a class without any minimizer, a part of the energy concentrates at $r = 0$; the sequence of the graphs of these functions forms at the limit a vertical part at $r = 0$ which contains a part of the energy. On the $2d$ problem the theory of cartesian currents describe this concentration by considering the limit of the graphs of the functions [7, 8].

Trying to mimick this theory in our $1d$ problem, we face the difficulty that the energy is not coercive since $E(\theta)$ does not bound the length of the graph of θ which is given by $\int_0^1 \sqrt{1+\theta'^2}dr$. By means of the lower bound from lemma II.1 and of the numerical results, we introduce a change of variable in order to solve this difficulty.

The second remark is good news: the codimension in this problem is 1 (scalar functions). Thus the convergence in the sense of graphs can be completely expressed by the weak convergence in the BV space [7].

Remark II.5 It is possible to construct a function with bounded energy and with a graph of infinite length: the function has oscillations near $r = 0$. We leave the proof to the reader.

3.1 Change of variable

The energy $E(\theta)$ does not bound the quantity $\int_0^1 |\theta'|dr$ (see remark II.5) but from lemma II.1 it bounds the quantity $\int_0^1 |\theta' cos\theta|dr$. Moreover, the angle function with values in S^1 is associated to a map with values in S^2: **the Lebesgue measure on S^2 induces the measure $|\cos\theta|d\theta$ on S^1**.

In order to consider the convergence in the sense of graphs, we therefore need to introduce the following change of variables which changes the measure $|\cos\theta|d\theta$ into the Lebesgue measure. Its properties are a direct consequence of the properties of the sinus function:

Definition II.3 (Change of variable F) *Let $F \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ be defined by:*

$$F(0) = 0 \text{ and } F'(\theta) = |\cos\theta| \quad \forall\theta \in \mathbb{R}.$$

Then $F(\theta + k\pi) = \sin(\theta) + 2k$ for all $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and for all $k \in \mathbb{Z}$. The function F is a homeorphism from $\mathbb{R}$ onto $\mathbb{R}$ which is uniformly continuous as well as F^{-1}.

The interest of F for our problem is the following:

Lemma II.4 *Let $\theta \in \mathcal{E}_{\alpha,k}$ and set $y := F \circ \theta$. Then*

$$y' = \theta'|\cos\theta| \text{ in } \mathcal{D}'(]0,1[). \tag{II.17}$$

In particular, $y' \in L^1(]0,1[)$ with $\displaystyle\int_0^1 |y'|dr \leq \frac{E(\theta)}{2\pi} < \infty$.

Proof. We notice that $y \in \mathcal{C}^1([0,1])$. For $\epsilon > 0$, θ belongs to $H^1(]\epsilon,1[)$ and F is $\mathcal{C}^1$ so $[F(\theta)]' = \theta' F'(\theta)$ in $\mathcal{D}'(]\epsilon,1[)$ (see for instance [4]). By letting $\epsilon \to 0$ we obtain (II.17). The L^1 bound is given by lemma II.1. ■

3.2 Weak convergence in BV

We use here the space of BV functions in order to describe the convergence of the graphs. In codimension 1, this approach is equivalent (but less technical) to the description using cartesian currents (see [7] for details).

Let I be an open bounded interval. We recall that

$$BV(I) := \{y \in L^1(I),\ M_I(y') < \infty\},$$

where for all $y \in L^1(I)$, the total mass of y' is given by:

$$M_I(y') := \sup\{\int y(r)\varphi'(r)dr \mid \varphi \in \mathcal{C}_c^\infty(I,\mathbb{R}),\ ||\varphi||_\infty \leq 1\}.$$

The space $BV(I)$ is equipped with the norm $||y||_{BV(I)} := ||y||_{L^1(I)} + M_I(y')$ for which it is a (non separable) Banach space.

We use the following definition of the weak convergence in BV (see[7]) :

Definition II.4 (Weak BV convergence) *Let $(y_n)_n$ be a sequence of functions in $BV(I)$ such that $\sup_n ||y_n'||_{BV(I)} < \infty$ and let $y \in BV(I)$. Then* ***$(y_n)_n$ converges weakly to y in $BV(I)$*** *if*

$$\begin{array}{ccl} y_n & \xrightarrow[n\to\infty]{} & y \text{ strongly in } L^1(I), \\ y_n' & \rightharpoonup & y' \text{ in } \mathcal{D}'(I). \end{array}$$

In this case, we have the following bound:

$$M_I(y') \leq \liminf_n M_I(y_n').$$

Remark II.6 The L^1 convergence implies the convergence in the sense of distributions; but the second line is here to remind that the derivatives converge in the sense of measures, by the uniform bound.

Remark II.7 The injection $BV(I) \subset L^1(I)$ is compact if I is bounded [7]. In this case any sequence $(y_n)_n$ in $BV(I)$ such that $\sup_n ||y_n||_{BV(I)} < \infty$ converges (up to a subsequence) weakly in $BV(I)$ to a $y \in BV(I)$.

Let us now come back to our minimization problem. In order to describe what happens at $r = 0$ we need to include $[0,1]$ in an open interval: this is done by symmetrization.

Definition II.5 *Let $y := F \circ \theta$ with $\theta \in \mathcal{E}_{\alpha,l}$; we define its extension of degree k by symmetry on $[-1,1]$:*

$$P_k(y)(r) := \left\{ \begin{array}{ccl} y(r) & \text{if} & r \in [0,1], \\ -y(-r) + 4k - 2 & \text{if} & r \in [-1,0[. \end{array} \right.$$

Remark II.8 If $\theta \in \mathcal{E}_{\alpha,k}$ then $P_k(F \circ \theta)$ is continuous on $[-1,1]$ and the extension P_k is canonically given by the axi-symmetric map $u : B^2 \to S^2$ which corresponds to θ.

More precisely, since $u \in \mathcal{C}^0(\overline{B^2}, S^2)$ and $u(0,0)$ is either the north pole or the south pole there is a continuous extension $\tilde{P}_k\theta$ of θ on $[-1,1]$ defined by:

$$u(x,0) = \left(\cos \tilde{P}_k\theta(x), 0, \sin \tilde{P}_k\theta(x)\right) \quad \forall x \in [-1,1]. \tag{II.18}$$

The extension P_k is given by $P_k \circ F = F \circ \tilde{P}_k$.

The following lemma completes lemma II.4. The notation δ_0 stands for the Dirac distribution at $r = 0$.

Lemma II.5 *Let $\theta \in \mathcal{E}_{\alpha,l}$ and $y := P_k(F \circ \theta)$ its extension of degree k through the change of variables F. Then*

$$y' = h + 4(l-k)\delta_0 \text{ in } \mathcal{D}'(]-1,1[),$$

where $h \in L^1(]-1,1[)$ is the function defined by $h(r) := \theta'(|r|)|\cos\theta(|r|)|$ ($r \in [-1,1]$).

In particular $y \in BV(]-1,1[)$ and $M_{]-1,1[}(y') \leq \dfrac{E(\theta)}{\pi} + 4|k-l|$.

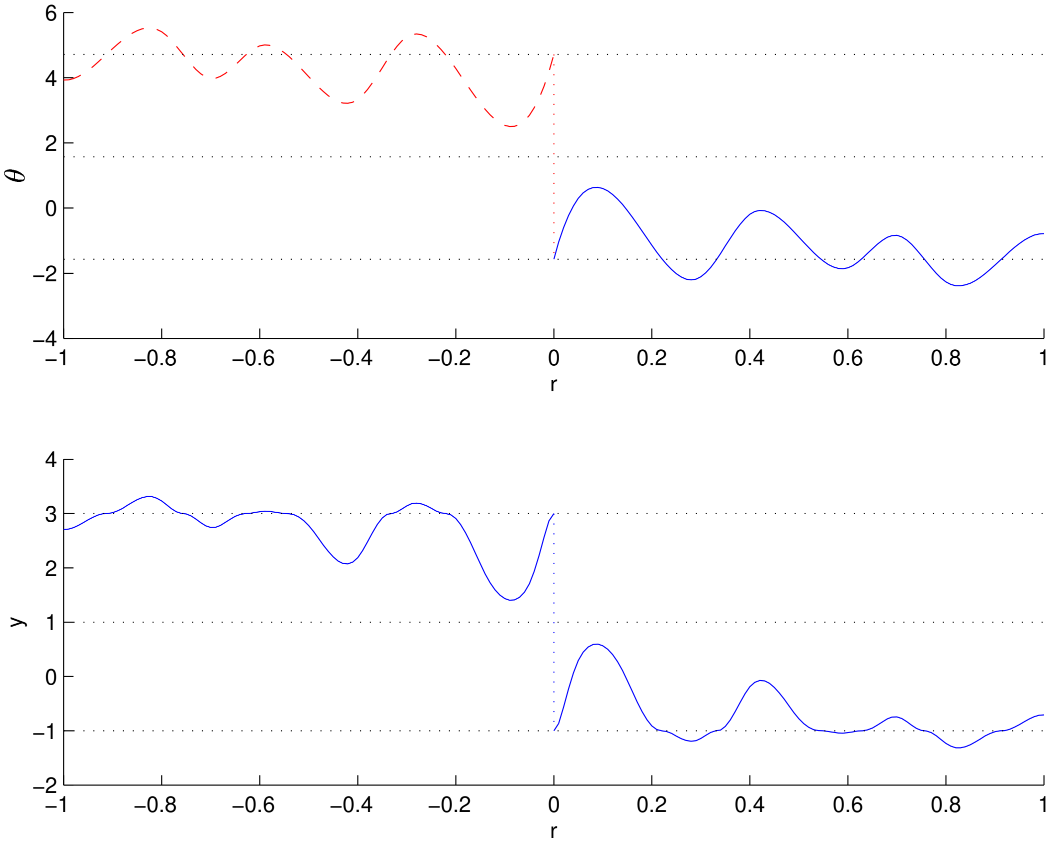

Figure 1: A function θ of degree 0 and its extension of degree 1 in variables y

Proof. We notice that $y \in L^\infty([-1,1])$. By lemma II.4 and definition II.5 we know that $y' = h$ in $\mathcal{D}'(]-1,0[\cup]0,1[)$ where h is the function in $L^1(]-1,1[)$ defined above. Let H be the primitive of h defined by $H(r) := y(1) - \int_r^1 h(s)ds$. Then $(y-H)' = 0$ in $\mathcal{D}'(]-1,0[\cup]0,1[)$ so $y - H$ is constant on $]-1,0[$ and on $]0,1[$ and its derivative in $\mathcal{D}'(]-1,1[)$ is δ_0 multiplied by the value of the jump at $r = 0$. Since H is continuous on $[-1,1]$ the value of the jump is given by

$$y(0^+) - y(0^-) = 4(l-k).$$

Now using $y = H + (y-H)$ we find $y' = H' + (y-H)'$ and the lemma is proved. ∎

We can now state:

Theorem II.2 *Let $(\theta_n)_n$ be a minimizing sequence in $\mathcal{E}_{\alpha,k}$ and let $\underline{\theta}$ denote the weak limit of (θ_n) given by theorem II.1. Then the sequence $(y_n)_n$ where $y_n := P_k(F \circ \theta_n)$ converges weakly in $BV(]-1,1[)$ to $\underline{y} := P_k(F \circ \underline{\theta})$. In particular, $\int_{-1}^1 |y_n - \underline{y}|dr \xrightarrow[n\to\infty]{} 0$.*

Remark II.9 The strong convergence in L^1 norm can be interpreted as the convergence of the graphs of the functions y_n to the graph of $\underline{y}$ in the "flat norm" (see [15]).

Proof. First we notice that the weak E-convergence and the uniform continuity of F implies that $(y_n)_n$ converges uniformly to $\underline{y} := P_k(F \circ \underline{\theta})$ on $[-1, -\epsilon] \cup [\epsilon, 1]$ for all $\epsilon > 0$.

By lemma II.5 with $l = k$ we have $\int_{-1}^{1} |y_n'| dr \leq E(\theta_n)$, so the sequence $(y_n)_n$ is uniformly bounded in $BV(]-1,1[)$. By compactness we can find a subsequence which converges weakly in $BV(]-1,1[)$ to some function $\tilde{y}$. Necessarily $\tilde{y} = \underline{y}$ in $\mathcal{D}'(]-1,0[\cup]0,1[)$ and thus $\tilde{y} = \underline{y}$ a.e. in $[-1,1]$. By uniqueness of the limit, the whole sequence $(y_n)_n$ converges weakly in $BV(]-1,1[)$ to $\underline{y}$, and this concludes the proof. ■

3.3 Closure of $\mathcal{E}_{\alpha,k}$

In order to close the classes $\mathcal{E}_{\alpha,k}$ in the weak BV convergence, we use the injection

$$\begin{array}{rcl} \mathcal{E}_\alpha & \to & BV(]-1,1[) \\ \theta & \to & y = P_k(F \circ \theta). \end{array}$$

Definition II.6 (Closure of $\mathcal{E}_{\alpha,k}$) *Let $\mathcal{Y}_{\alpha,k} := P_k \circ F(\mathcal{E}_{\alpha,k})$ be the class $\mathcal{E}_{\alpha,k}$ in coordinates y. We define the "closure" of $\mathcal{Y}_{\alpha,k}$ denoted $\overline{\mathcal{Y}}_{\alpha,k}$ as the subset of $BV(]-1,1[)$ given by*

$$\overline{\mathcal{Y}}_{\alpha,k} := P_k \circ F(\mathcal{E}_\alpha),$$

where $\mathcal{E}_\alpha = \cup_{l \in \mathbb{Z}} \mathcal{E}_{\alpha,l}$.

For every $y \in \overline{\mathcal{Y}}_{\alpha,k}$ there is a unique $\theta \in \mathcal{E}_\alpha$ such that $y = P_k(F(\theta))$ and we denote $y = y_{(\theta,k)}$.

Definition II.7 (Relaxed energy) *The **relaxed energy** of $y_{(\theta,k)} \in \overline{\mathcal{Y}}_{\alpha,k}$ is defined by*

$$\overline{E}(y_{(\theta,k)}) := E(\theta) + 4\pi|k - l|,$$

where l is the degree of θ given by $\theta(0) = -\dfrac{\pi}{2} + l\pi$.

Remark II.10 The singular part in the energy corresponds to the length of the vertical segment in $]-1,1[\times\mathbb{R}$ which joins $(0, y(0^-))$ to $(0, y(0^+))$. This segment fills the "hole" in the graph of y.

With these definitions we obtain the existence and uniqueness theorem:

Theorem II.3 *Let $(\alpha, k) \in \mathbb{R} \times \mathbb{Z}$. Then*

$$\inf_{\theta \in \mathcal{E}_\alpha} \overline{E}(y_{(\theta,k)}) = \inf_{\theta \in \mathcal{E}_{\alpha,k}} E(\theta),$$

and there exists a unique minimizer $y_{(\underline{\theta},k)}$ for $\overline{E}$ in $\overline{\mathcal{Y}}_{\alpha,k}$.

Moreover, if $(y_{(\theta_n,k)})_n$ is a minimizing sequence for $\overline{E}$ in $\overline{\mathcal{Y}}_{\alpha,k}$, then

$$\begin{array}{rcl} y_{(\theta_n,k)} & \underset{n\to\infty}{\rightharpoonup} & y_{(\underline{\theta},k)} \text{ weakly in } BV(]-1,1[) \text{ and} \\ \sqrt{r}\theta_n' & \rightharpoonup & \sqrt{r}\underline{\theta}' \text{ weakly in } L^2(]0,1[. \end{array}$$

Remark II.11 The variable y is not convenient to compute the energy (see definition II.3). The convergence of a minimizing sequence has an analytical part in coordinates θ (the E-convergence), and a geometrical part in coordinates y (the weak BV convergence). In $2d$ the same thing happens with the H^1 weak convergence and the convergence in the sense of currents.

Remark II.12 Compared to the $2d$ theorem of existence of singular solution [8] there is uniqueness. This is because the singularity occurs only at $r = 0$ (whereas in the $2d$ case, the sphere can take place anywhere in the disc) and because of the uniqueness of regular axi-symmetric harmonic maps (proposition II.2).

Proof. We suppose that $\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ and $k \geq 1$. The other cases can be proved in a similar way.

Let $y_{(\theta,k)} \in \overline{\mathcal{Y}}_{\alpha,k}$. Then θ belongs to $\mathcal{E}_{\alpha,l}$ for a certain $l \in \mathbb{Z}$; by lemma II.1,

$$\overline{E}(y_{(\theta,k)}) \geq \begin{cases} \Lambda_{\alpha,0} + 4\pi k + 4\pi|l| & \text{if } l \leq 0, \\ \Lambda_{\alpha,1} + 4\pi(k-1) & \text{if } 1 \leq l \leq k, \\ \Lambda_{\alpha,1} + 4\pi(l-1) & \text{if } l \geq k+1. \end{cases} \tag{II.19}$$

Since $0 < \Lambda_{\alpha,0}, \Lambda_{\alpha,1} < 4\pi$ in every possible case $\overline{E}(y_{(\theta,k)}) \geq \Lambda_{\alpha,1} + 4\pi(k-1)$ and if this is an equality then necessarily $1 \leq l \leq k$. Now with the values of the infima $\Lambda_{\alpha,l}$ (corollary II.2) and the equality case of lemma II.1, the preceding inequality is an equality if and only if $\theta = \theta_{\alpha,1}$, and the first part of the theorem is proved.

Now let $(y_{(\theta_n,k)})_n$ be a minimizing sequence in $\overline{\mathcal{Y}}_{\alpha,k}$, and denote by l_n the degree of θ_n. By the inequalities (II.19), for n large enough we have $1 \leq l_n \leq k$. Choose an $l \in \{1, \ldots, k\}$ and denote $(\theta_{n_p})_p$ the subsequence of $(\theta_n)_n$ formed by the elements of degree l (there is nothing to do if this subsequence is finite). Then the following inequalities

$$E(\theta_{n_p}) \leq \overline{E}(y_{(\theta_{n_p},k)}) - 4\pi(k-l) \leq \Lambda_{\alpha,1} + 4\pi(l-1) + o(1)$$

guarantee that $(\theta_{n_p})_p$ is a minimizing sequence in $\mathcal{E}_{\alpha,l}$. By theorem II.1 $(\theta_{n_p})_p$ converges weakly in the E-sense and the same argument as in the proof of theorem II.2 (uniform bound on the BV-norm of y_{n_p}) proves that $(y_{n_p})_p$ converges weakly in $BV(]-1,1[)$ to $y_{(\theta_{\alpha,1},k)} := P_k(F \circ \theta_{\alpha,1})$. This is true for every subsequence of $(y_{(\theta_n,k)})_n$ with a constant degree l_n: the uniqueness of the limit guarantees that the whole sequence $(y_{(\theta_n,k)})_n$ converges weakly to $y_{(\theta_{\alpha,1},k)}$ in $BV(]-1,1[)$, and that $(\theta_n)_n$ converges weakly in the E-sense to $\theta_{\alpha,1}$. This concludes the proof. ∎

3.4 Complements on the relaxed energy

We conclude this section by a few results which explain the structure of $\overline{\mathcal{Y}}_{\alpha,k}$ and the terms *"closure"* and *"relaxed energy"*.

In the following y (rather than $y_{(\theta,k)}$) denotes an element in $\overline{\mathcal{Y}}_{\alpha,k}$ and $\overline{E}(y)$ its energy.

Proposition II.3 (Compactness-closure) *Let $(y_n)_n$ be a sequence of $\overline{\mathcal{Y}}_{\alpha,k}$ such that*

$$\sup_n \overline{E}(y_n) < \infty.$$

Then there exist a subsequence $(y_{n_p})_p$ and $y \in \overline{\mathcal{Y}}_{\alpha,k}$ such that y_{n_p} converges weakly to y in $BV(]-1,1[)$.

Proof. Since the BV-norm is bounded by the energy (lemma II.5), we have $\sup_n ||y_n||_{BV(]-1,1[)} < \infty$, so by compactness there is a subsequence $(y_{n_p})_p$ and $y \in BV(]-1,1[)$ such that y_{n_p} converges weakly to y in $BV(]-1,1[)$. We need to prove that y belongs to $\overline{\mathcal{Y}}_{\alpha,k}$.

For every n let θ_n be the antecedent of y_n in $\mathcal{E}_\alpha$. Since $\sup_n E(\theta_n) \leq \sup_n \overline{E}(y_n) < \infty$ there is a subsequence still denoted $(\theta_{n_p})_p$ which converges weakly in the E-sense to some $\theta \in \mathcal{E}_\alpha$ (lemma II.3). The uniform convergence on $[-1,\epsilon] \cup [\epsilon, 1]$ for all $\epsilon > 0$ is enough to guarantee that $y = P_k(F \circ \theta)$, and the proposition is proved. ∎

In this BV frame, lemma II.3 gives a partial result of semi-continuity:

Lemma II.6 *Let $(y_n)_n$ be a sequence a functions in $\mathcal{Y}_{\alpha,k}$ and $y \in \overline{\mathcal{Y}}_{\alpha,k}$ such that* $\sup_n \overline{E}(y_n) < \infty$ *and $y_n \rightharpoonup y$ weakly in $BV(]-1,1[)$. Then*

$$\overline{E}(y) \leq \liminf_n \overline{E}(y_n).$$

Proof. Again the uniform bound on the energy gives a uniform bound on the BV-norm. For every y_n denote θ_n its antecedent in $\mathcal{E}_{\alpha,k}$. Since $\sup_n E(\theta_n) = \sup_n \overline{E}(y_n) < \infty$, $(\theta_n)_n$ converges (up to a subsequence) weakly in the E-sense to some $\theta \in \mathcal{E}_\alpha$; by lemma II.3 we obtain the following inequality where l is the degree of θ:

$$E(\theta) + 4\pi|k-l| \leq \liminf_n E(\theta_n). \tag{II.20}$$

Of course θ is the antecedent of y (use the uniform convergence for instance), and in terms of y (II.20) is the desired inequality. ∎

Lemma II.7 (Density) *Let $y \in \overline{\mathcal{Y}}_{\alpha,k}$. There is a subsequence $(y_n)_n$ in $\mathcal{Y}_{\alpha,k}$ such that $y_n \rightharpoonup y$ weakly in $BV(]-1,1[)$ and $\overline{E}(y_n) \to \overline{E}(y)$.*

Proof. Let $\theta \in \mathcal{E}_{\alpha,l}$ be the antecedent of y. For $n = 1, 2\ldots$ denote θ_n the function in $\mathcal{E}_{\alpha,k}$ given by lemma II.2 with respect to the value $\epsilon = \dfrac{1}{n}$. Then if we set $y_n := P_k(F \circ \theta_n)$, $(y_n)_n$ is a sequence in $\mathcal{Y}_{\alpha,k}$ which satisfies

$$E(y_n) \leq \overline{E}(\theta) + 4\pi|k-l| + \frac{1}{n},$$

and which converges uniformly to y on $[-1,\epsilon] \cup [\epsilon, 1]$ for all $\epsilon > 0$. Thus

$$\limsup_n \overline{E}(y_n) \leq E(\theta) + 4\pi|k-l|. \tag{II.21}$$

Moreover $\sup_n ||y_n||_{BV(]-1,1[)} < \infty$ (lemma II.5), thus by compactness $(y_n)_n$ converges weakly to y in $BV(]-1,1[)$. Lemma II.6 gives the complementary of inequality (II.21):

$$\overline{E}(y) = E(\theta) + 4\pi|k-l| \leq \liminf_n \overline{E}(y_n),$$

and this concludes the proof. ∎

Remark II.13 These two lemmas allow to consider $\overline{E}$ as a *relaxed energy* (i.e. a lower semi-continuous extension of E) in the following sense: for all $y \in \overline{\mathcal{Y}}_{\alpha,k}$,

$$\begin{aligned}\overline{E}(y) \quad &= \quad \inf\big\{ \liminf_n \overline{E}(y_n) \mid y_n \in \mathcal{Y}_{\alpha,k},\ \sup_n \overline{E}(y_n) < \infty, \\ &\qquad\qquad y_n \rightharpoonup y \text{ weakly in } BV(]-1,1[)\big\}. \end{aligned} \tag{II.22}$$

We can now state the semi-continuity of $\overline{E}$:

Proposition II.4 (Semi-continuity of $\overline{E}$) *Let $(y_n)_n$ be a sequence in $\overline{\mathcal{Y}}_{\alpha,k}$ and let $y \in \overline{\mathcal{Y}}_{\alpha,k}$ such that $\sup_n \overline{E}(y_n) < \infty$ and $y_n \rightharpoonup y$ weakly in $BV(]-1,1[)$. Then*

$$\overline{E}(y) \leq \liminf_n \overline{E}(y_n).$$

Proof. For every n by the density lemma II.7 we can find $\tilde{y}_n \in \mathcal{Y}_{\alpha,k}$ such that

$$||y_n - \tilde{y}_n||_{L^1(]-1,1[)} \leq \frac{1}{n} \quad \text{and} \quad |\overline{E}(\tilde{y}_n) - \overline{E}(y_n)| \leq \frac{1}{n}.$$

Then $(\tilde{y}_n)_n$ converges to y strongly in $L^1(]-1,1[)$ and:

$$\liminf_n \overline{E}(\tilde{y}_n) = \liminf_n \overline{E}(y_n).$$

So $(\tilde{y}_n)_n$ is a sequence with uniformly bounded energies and which converges weakly to y in $BV(]-1,1[)$; the semi-continuity lemma II.6 concludes the proof. ∎

We can now state the following result which is independent of the proof of theorem II.3:

Corollary II.3 *There exists a minimizer for $\overline{E}$ in $\overline{\mathcal{Y}}_{\alpha,k}$.*

Proof. Let $(y_n)_n$ be a minimizing sequence for $\overline{E}$ in $\overline{\mathcal{Y}}_{\alpha,k}$. Since $\sup_n \overline{E}(y_n) < \infty$ using the compactness-closure proposition II.3 we can find a subsequence still denoted $(y_n)_n$ and $\underline{y} \in \overline{\mathcal{Y}}_{\alpha,k}$ such that $y_n \rightharpoonup \underline{y}$ weakly in $BV(]-1,1[)$. The semi-continuity of $\overline{E}$ guarantees that

$$\overline{E}(\underline{y}) \leq \liminf_n \overline{E}(y_n) = \inf_{y \in \overline{\mathcal{Y}}_{\alpha,k}} \overline{E}(y),$$

so $\underline{y}$ is the desired minimizer. ∎

III Mesh optimization

1 Introduction

In this section we give a new theoritical and algorithmical approach for the computation of singular solutions of problem

$$\textit{Minimize } E(\theta) \textit{ in } \mathcal{E}_{\alpha,k}. \tag{III.23}$$

By theorem II.3 we know that the singular solution has a boundary layer with zero thickness. Therefore a classical refinement technique tends to add points indefinitely. Moreover the

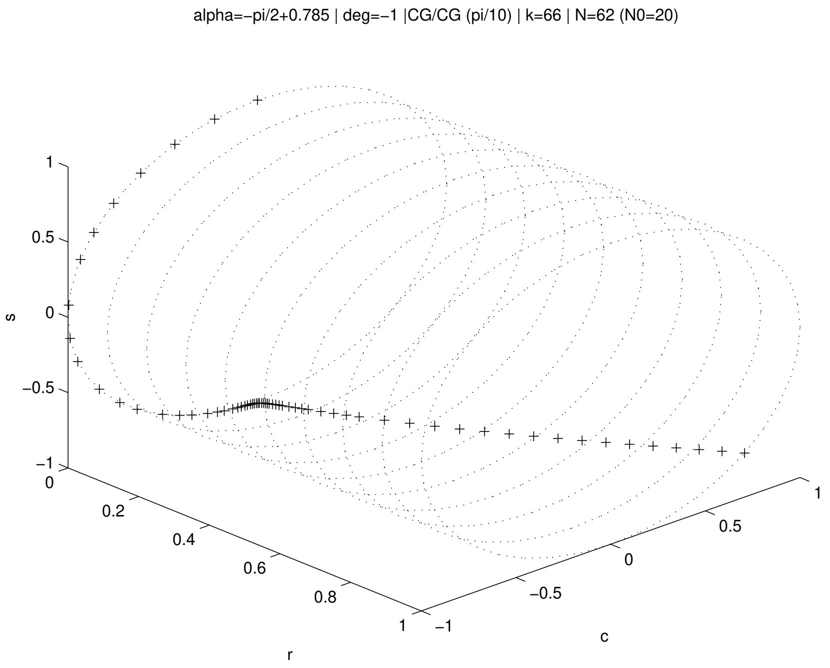

Figure 2: Refinement method for $\alpha = -\pi/4$ and $k = -1$

computed boundary layer always has a thickness $\epsilon > 0$ (with ϵ small). This is shown in figures 2 and 3 where the singular solution (in a S^1 formulation, see section 4 below) for $\alpha = -\frac{\pi}{4}$ and $k = -1$ has been computed with a classical mesh-refinement method (see [17, 16] for details).

In the following we present the "optimal-mesh" method which solves these drawbacks. The idea here is to minimize the discrete energy **with a fixed number of points** in the mesh, but allowing these points to move (see for instance [14, 13] for examples of moving-meshes). The mesh is thus considered as a unknown of the minimization problem.

1.1 Notations

We use continuous P_1 elements to discretize the space $\mathcal{E}_{\alpha,k}$ (see [3]). In the following $r_0 = 0 < r_1 < \ldots < r_N = 1$ denotes a subdivision of $[0,1]$ in N intervals. In order to allow this subdivision $r_0 < r_1 < \ldots < r_N = 1$ to move, we define

$$\mathcal{E}_{\alpha,k}^N := \bigcup_{r_0=0<r_1<\ldots<r_N=1} \mathcal{E}_{\alpha,k}^{(r_i)_{0\leq i\leq N}},$$

where:

$$\begin{aligned}\mathcal{E}_{\alpha,k}^{(r_i)_{0\leq i\leq N}} \quad := \quad & \big\{\theta^N \in \mathcal{C}^0([0,1]),\, \theta^N(0) = -\frac{\pi}{2} + k\pi,\, \theta^N(1) = \alpha, \\ & \qquad \theta^N \text{ is affine on } [r_i, r_{i+1}]\, \forall i \in \{0, \ldots, N-1\}\big\}.\end{aligned}$$

Since $\mathcal{E}_{\alpha,k}^N \subset \mathcal{E}_{\alpha,k}$ the elements are *conformal*.

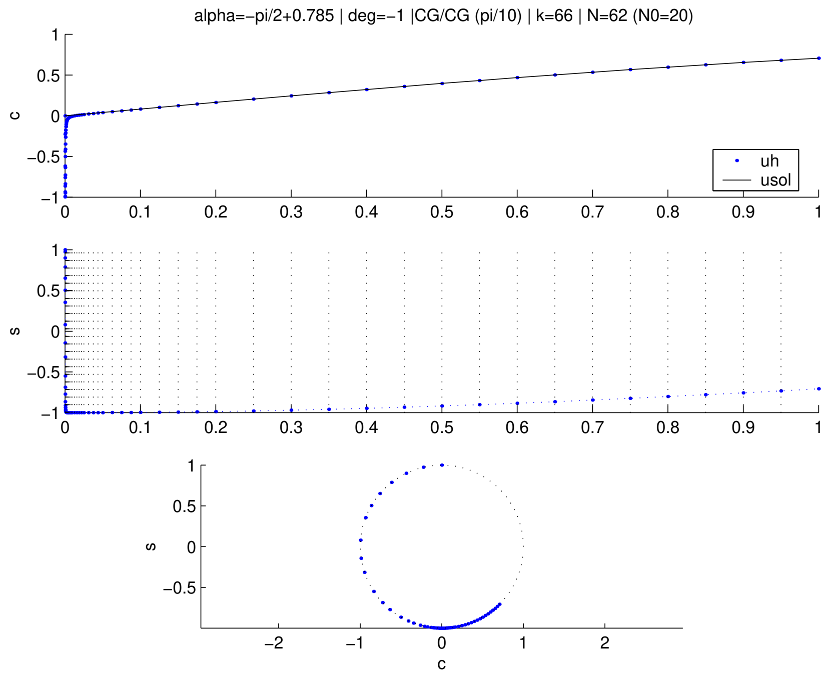

Figure 3: Refinement method for $\alpha = -\pi/4$ and $k = -1$ (projections)

In order to allow this subdivision $r_0 < r_1 < \ldots < r_N = 1$ to move, we define

$$\mathcal{E}^N_{\alpha,k} := \bigcup_{r_0=0<r_1<\ldots<r_N=1} \mathcal{E}^{(r_i)_{0\le i\le N}}_{\alpha,k}.$$

A function $\theta^N \in \mathcal{E}^N_{\alpha,k}$ is uniquely defined by the values $r_0 = 0 < r_1 < \ldots < r_N = 1$ and $\theta_i := \theta^N(r_i)$ for $i \in \{0, 1 \ldots, N\}$. In the following we make a constant use of the identification

$$\theta^N \simeq (r_0, r_1, \ldots, r_N, \theta_0, \ldots, \theta_N).$$

In particular, $\mathcal{E}^N_{\alpha,k} \simeq D^N$ where $D^N \subset \mathbb{R}^{2N+2}$ is defined by:

$$D^N := \left\{ r_0 = 0 < r_1 < \ldots < r_N = 1,\ \theta_0 = -\frac{\pi}{2} + k\pi, \theta_1, \ldots, \theta_N = \alpha \right\}. \tag{III.24}$$

1.2 Discrete relaxed energy

In the following, we give a discrete energy E^N which is a consistant approximation of the exact energy E. A numerical solution of problem (III.23) is a minimizer for E^N.

The energy $E^N(\theta^N) = E^N(r_0, \ldots, \theta_N)$ is a function of the variables r_i and θ_i which is continuous on D^N. Classiquely, the subdivision $r_0 < r_1 < \ldots < r_N = 1$ is given and E^N is minimized with respect to the variables (θ_i): only the *vertical variations* in the graph are allowed. Here, because we expect a boundary layer at $r = 0$, we also allow *horizontal variations* in the graph.

The (r_i) are ordered and our problem is to minimize $E^N(r_0, r_1, \ldots, r_N, \theta_0, \theta_1, \ldots, \theta_N)$ under the constraints $r_0 = 0 < r_1 < \ldots < r_N = 1$ (N is fixed). These are open constraints and in order to guarantee the existence of a minimizer it is necessary to close them and extend the energy on the closure: this is the discrete version of the relaxed energy (II.22).

The following proposition gives a natural way to build a lower semi-continuous extension of E^N (see for instance [7]):

Proposition III.1 (Discrete relaxed energy) *Let X be a metric space and let $D \subset X$. Suppose the function $E : D \to [0, \infty]$ is lower semi-continuous on D (i.e. for every $x_n \in D$ such that $x_n \xrightarrow[n\to\infty]{} x \in D$, $E(x) \leq \liminf_n E(x_n)$). Then the extension of E defined on the closure of D in X by:*

$$\begin{array}{rcl} \overline{E} : \overline{D} & \to & [0, \infty] \\ x & \to & \inf\{\liminf\limits_{n} E(x_n) \mid x_n \in D,\ x_n \to x\} \end{array}$$

is lower semi-continuous on $\overline{D}$.

In our problem (with exponents N to remind of the discrete) we extend $E^N(r_0, \ldots, \theta_N)$ on the closure

$$\overline{D^N} = \left\{ r_0 = 0 \leq r_1 \leq \ldots \leq r_N = 1,\ \theta_0 = -\frac{\pi}{2} + k\pi, \theta_1, \ldots, \theta_N = \alpha \right\}. \tag{III.25}$$

In other words the constraints are *relaxed.*

To every element $(r_0, \ldots, \theta_N) \in \overline{D^N}$ corresponds a "P_1 interpolate" y^N which can be seen as a BV function as in the continuous case. An equality $r_i = r_{i+1}$ corresponds to a *vertical part* in the graph of y^N. The definition of the relaxed energy is simpler than in the continuous case because of the finite dimension: all the norms on $\mathbb{R}^{2N+2}$ are equivalent and the weak topology is equivalent to the strong topology. By analogy with the continuous case an element in $\overline{D^N}$ can be called a **discrete cartesian current**. The extension $\overline{E^N}$ is the **discrete relaxed energy**.

1.3 Discrete minimizer

The discrete energy E^N is an approximation of the exact energy. Recall that for every P_1 element $\theta^N \in \mathcal{E}^N_{\alpha,k}$ which corresponds to $(r_0, \ldots, r_N, \theta_0, \ldots, \theta_N) \in D^N$, the exact energy is given by:

$$\begin{aligned} E(\theta^N) &:= \pi \int_0^1 \frac{\cos^2\theta}{r} + r\theta'^2 dr, \\ &= \pi \sum_{i=0}^{N-1} \left(\int_{r_i}^{r_{i+1}} \frac{\cos^2\theta^N}{r} dr \right) + \frac{(\theta_{i+1} - \theta_i)^2}{2} \left(\frac{r_{i+1} + r_i}{r_{i+1} - r_i} \right). \end{aligned} \tag{III.26}$$

In order to compute the integral with the cosinus we use in the following a *quadrature formula.* This gives a discrete energy E^N which is a continuous function on D^N.

The minimizer we wish to compute is actually a minimizer of the lower semi-continuous extension $\overline{E^N}$ of E^N on $\overline{D^N}$ given by proposition III.1. We state the discrete equivalent of corollary II.3:

Proposition III.2 *There exists a minimizer for $\overline{E^N}$ on $\overline{D^N}$.*

Proof. From definition (III.26), for all $(r_0, \ldots, \theta_N) \in D^N$:

$$E^N(r_0, \ldots, \theta_N) \geq \frac{\pi}{2} \sum_{i=0}^{N-1} (\theta_{i+1} - \theta_i)^2 \left(\frac{r_{i+1} + r_i}{r_{i+1} - r_i} \right), \tag{III.27}$$

$$E^N(r_0, \ldots, \theta_N) \geq \frac{\pi}{2} \sum_{i=0}^{N-1} (\theta_{i+1} - \theta_i)^2 . \tag{III.28}$$

By continuity the lower bound (III.28) is also true for $\overline{E^N}$ on $\overline{D^N}$. Since the expression $\left(\sum_{i=0}^{N-1} (\theta_{i+1} - \theta_i)^2 + \theta_N^2\right)^{\frac{1}{2}}$ is a norm on $\mathbb{R}^N$, we obtain that $\overline{E^N}$ tends to infinity at infinity on $\overline{D^N}$. The lower semi-continuity of $\overline{E^N}$ concludes the proof.

Remark III.1 Let $y^N \simeq (r_0, \ldots, \theta_N) \in \overline{D^N}$ and suppose there exists $i \in \{0, \ldots, N-1\}$ such that $r_{i+1} = r_i > 0$.

- if $\theta_{i+1} \neq \theta_i$ (i.e. the graph of y^N has a vertical part) then it is clear from (III.27) that $\overline{E^N} = +\infty$.
- if $\theta_{i+1} = \theta_i$ this corresponds to substracting one point in the graph ($E^N = E^{N-1}$).

As a consequence a vertical part in the graph of y^N can only appear at $r = 0$ (i.e. with $r_i = r_{i+1} = 0$) for every element y^N in $\overline{D^N}$ such that $\overline{E^N} < \infty$. Thus every y^N is caracterized as in the continuous case by its "regular part" θ^N and the value of the jump at $r = 0$. The regular part θ^N is a function on $[0, 1]$ which is a classical continuous P_1 element.

2 Midpoint formula

In order to compute the energy for a P_1 element, it is necessary to use a quadrature formula which introduces a *consistency error*. In the following we choose the midpoint formula.

For every $\theta^N \in \mathcal{E}^N_{\alpha,k}$ which corresponds to $(r_0, \ldots, \theta_N) \in D^N$ we define:

$$E^N(\theta^N) := 2\pi \sum_{i=0}^{N-1} E_i(\theta^N), \tag{III.29}$$

where for all $i \in \{0, \ldots, N-1\}$:

$$E_i(\theta^N) := \left[\cos^2\left(\frac{\theta_i + \theta_{i+1}}{2} \right) \left(\frac{r_{i+1} - r_i}{r_{i+1} + r_i} \right) + \left(\frac{\theta_{i+1} - \theta_i}{2} \right)^2 \left(\frac{r_{i+1} + r_i}{r_{i+1} - r_i} \right) \right]. \tag{III.30}$$

The function E_i is defined on $\mathcal{E}^N_{\alpha,k} \simeq D^N$ and depends only on θ_i, θ_{i+1} and the ratio $t_i = \dfrac{r_i}{r_{i+1}}$. Of course,

$$E_i(\theta^N) \approx \frac{1}{2} \int_{r_i}^{r_{i+1}} \frac{\cos^2 \theta^N}{r} + r (\theta^N)'^2 dr,$$

and the term with $(\theta^N)'$ is computed exactly.

2.1 Computational considerations

2.1.1 Differentiable expression for $\overline{E^N}$

Change of variable In order to have a computable expression of the discrete relaxed energy $\overline{E^N}$ we introduce the following change of variable.

Let $r_0 < r_1 < \ldots < r_N$. For all $i \in \{0, \ldots, N-1\}$ we set $t_i = \dfrac{r_i}{r_{i+1}}$ and

$$X_i := \frac{r_{i+1} - r_i}{r_{i+1} + r_i} = \frac{1 - t_i}{1 + t_i}.$$

Then

$$E_X^N(X_1, \ldots, \theta_{N-1}) = 2\pi \sum_{i=0}^{N-1} \cos^2\left(\frac{\theta_i + \theta_{i+1}}{2}\right) X_i + \left(\frac{\theta_{i+1} - \theta_i}{2}\right)^2 \frac{1}{X_i}. \qquad \text{(III.31)}$$

The new formula (III.31) is defined for every $X_i = 1$, so *we have a continuous (and differentiable) extension of E_X^N on*

$$D_X^N :=]0,1]^{N-1} \times \mathbb{R}^{N-1}. \qquad \text{(III.32)}$$

Discrete minimizer Let $\overline{E_X^N}$ be the lower semi-continuous extension of E_X^N on $\overline{D_X^N} = [0,1]^{N-1} \times \mathbb{R}^{N-1}$ given by proposition III.1. Then by the same arguments as in the proof of proposition III.2 there exists a minimizer for $\overline{E_X^N}$ on $\overline{D_X^N}$: *this is "the" minimizer we compute.*

If $(\underline{X}_1, \ldots, \underline{X}_{N-1}, \underline{\theta}_1, \ldots, \underline{\theta}_{N-1}) \in \overline{D_X^N}$ is such a minimizer then

- either $(\underline{X}_1, \ldots, \underline{\theta}_{N-1}) \in D_X^N$, in which case $\overline{E_X^N} = E_X^N$;
- either for all $i \in \{1, \ldots, N-1\}$ such that $\underline{X}_i = 0$ we have $\underline{\theta}_{i+1} = \underline{\theta}_i$ (otherwise $\overline{E_X^N} = +\infty$, see remark III.1): then $\overline{E_X^N} = \overline{E_X^M}$ for some $M < N$. This annoying possibility of loosing points during the minimization does not happen here because the X_i are independent (see the alternate variables algorithm section 2.1.4).

In conclusion the new formula (III.31) for E_X^N has two advantages: the constraints are affine and the relaxed energy is differentiable in every interesting cases.

2.1.2 Geometry of the steepest descent algorithm Suppose we want to compute a regular case. Then it is reasonable to assume that the minimizer for $\overline{E_X^N}$ belongs to $\overset{\circ}{D_X^N}$, and we can use the formula (III.29) for E^N with variables (r_i) to compute the minimizer.

The simplest optimization method is the *steepest descent algorithm (with fixed step-size).* This method is interesting here for its geometrical interpretation.

Starting from some $\theta^0 \in \mathcal{E}_{\alpha,k}^N$, the algorithm computes a minizing sequence $(\theta^p)_p$ in $\mathcal{E}_{\alpha,k}^N$ using the gradient of E^N. For every point (r_i^p, θ_i^p) which belongs to the graph of θ^p in $\mathbb{R}^2$, the gradient gives a vector

$$d_i := -\left(\frac{\partial E^N}{\partial r_i}, \frac{\partial E^N}{\partial \theta_i}\right)(\theta^p)$$

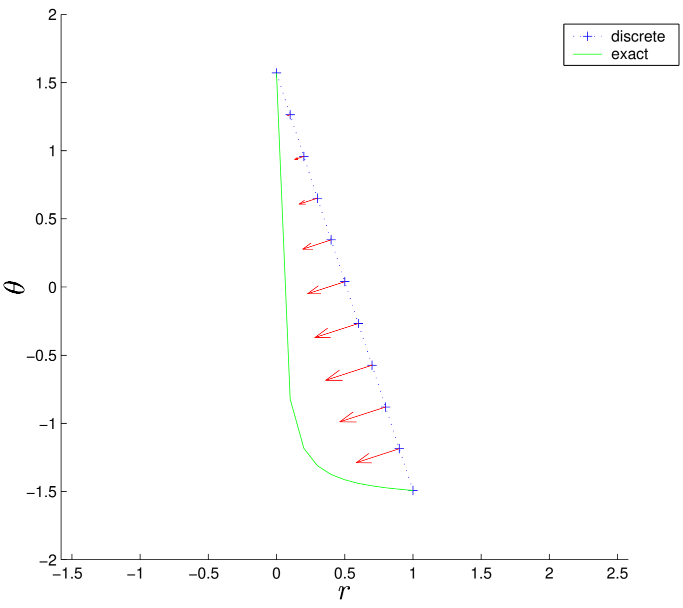

Figure 4: The gradient is normal to the graph

along which the decrease of E^N is the greatest (at first order).

Figure 4 concerns the regular case $k = 1$, $\alpha = -\dfrac{\pi}{2} + 0.0079$. We have represented the affine starting point $\theta^0 \in \mathcal{E}^N_{\alpha,k}$ (dashed line), the exact solution $\theta_{\alpha,1}$ (solide line) and every vector $0.3d_i$ pointing from (r_i, θ_i). It is remarkable to observe that d_i is orthogonal to the graph of θ^0. This is true in general for the current point θ^p, at least for the first values of p for which E^N decreases a lot.

This can be interpreted geometrically by the fact that *tangent movements along the graph do not change the energy* (at first order) since they do not change the curve. Only the normal movements change the graph and thus the energy.

Every minimization algorithm we used in this section has two phases. During the first phase the energy decreases a lot and the graph of θ^p changes a lot: these are the movements *normal* to the graph. During the second phase the energy decreases very slowly and the discrete graph practically does not change: the points $(r_i^p, \theta_i^p)_i$ move *tangently* along the graph.

2.1.3 Singularity of the hessian In order to improve the convergence of the steepest algorithm, we tried a Newton algorithm for the computation of some regular cases in variables $(r_i)_i$. However in each case the Newton algorithm failed: it exploded instead of converging.

This instability is not due to the choice of the starting point. The explosion is due to the singularity of the hessian of the energy of size $2(N-1)$. Table 1 represents values of the condition (in euclidian norm) of the hessian computed at the solution, with respect to various variables: the hessian has size $N-1$ for E^N considered as a function of the variables $(r_i)_{1\leq i<N}i$ (resp. $(X_i)_{1\leq i<N}$ or $(\theta_i)_{1\leq i<N}$) and size $N-1$ for E^N as a function of the variables $(X_i, \theta_i)_{1\leq i<N}$ (resp. $(r_i, \theta_i)_{1\leq i<N}$).

Table 1: Hessian condition for $\alpha = -\pi/3$, $k = 1$

N	$(\theta_i)_i$	$(X_i)_i$	$(r_i)_i$	$(X_i, \theta_i)_i$	$(r_i, \theta_i)_i$
10	200	10	560	6100	8200
20	860	54	2500	10000	600000
40	3500	310	11000	$1,7.10^6$	26.10^6

Table 2: Hessian condition for $k = 0$, $N = 10$

α	$(\theta_i)_i$	$(X_i)_i$	$(r_i)_i$	$(X_i, \theta_i)_i$	$(r_i, \theta_i)_i$
$-\pi/2 + 0.1$	43	253	43	$2,8.10^9$	$0,4.10^9$
$-\pi/2 + 0.01$	43	254	44	19.10^{12}	$3,9.10^{12}$
$-\pi/2 + 0.001$	43	254	43	23.10^{16}	$3,7.10^{16}$

We notice that the hessians of size $N-1$ have a good condition (although the problem is strongly non linear with respect to the variables $(r_i)_i$), whereas the hessians of size $2(N-1)$ have a very bad condition which means that the matrix is close to singular.

At the limit, for the case $\alpha = -\pi/2$ and $k = 0$ the discrete solution is constant: $\theta_i = -\pi/2$ for all i and the placement of the (r_i) does not change the energy. Thus $Hess((\underline{r}_i)_i) = 0$. This explains the values of the hessian in table 2 for $\alpha \approx -\pi/2$ and $k = 0$.

The singularity of the hessian is related to the geometrical interpretation of the steepest descent algorithm: *heuristically* since only the normal variations change the graph (and the energy), we minimize $E^N(r_0, \ldots, \theta_N)$ under the $N-1$ contraints of being normal to the graph. Thus the dimension of the problem is $N-1$, although we want to minimize with respect to $2(N-1)$ variables.

2.1.4 Alternate variables algorithm We want to minimize $\overline{E_X^N}$ given by (III.31) on $\overline{D_X^N}$. Let (X_i^p, θ_i^p) be the current minimization point. The important remark is the following: if the values $\theta_1^p, \ldots, \theta_{N-1}^p$ are fixed, then the minimization with respect to the variables $(X_1, \ldots, X_{N-1}) \in [0,1]^{N-1}$ is obvious because the X_i are "independent". Indeed,

$$E_X^N(X_1, \ldots, \theta_{N-1}) = 2\pi \sum_{i=0}^{N-1} \left(a_i^2 X_i + \frac{b_i^2}{X_i} \right),$$

where a_i and b_i depend only on the values $(\theta_1^p, \ldots, \theta_{N-1}^p)$ (see (III.31)). So

$$\inf_{(X_1,\ldots,X_{N-1}) \in)0,1]^N} E_X^N(X_1, \ldots, X_{N-1}, \theta_1^p, \ldots, \theta_{N-1}^p) = 2\pi \sum_{i=0}^{N-1} \inf_{X_i \in]0,1]} \left(a_i^2 X_i + \frac{b_i^2}{X_i} \right).$$

Moreover the value of the minimizing X_i is given for all $i \in \{1, \ldots, N-1\}$ by:

$$X_i^{p+1} = \min \left\{ 1, |\theta_{i+1}^p - \theta_i^p|/2 \cos\left(\frac{\theta_i^p + \theta_{i+1}^p}{2}\right) \right\}.$$

Now if the values $X_1^{p+1}, \ldots, X_{N-1}^{p+1}$ are fixed, the minimization with respect to the variables $\theta_1, \ldots, \theta_{N-1}$ can be done by a Newton algorithm: we obtain the values $(\theta_i^{p+1})_i$.

By repeating the process $(\theta_i^p)_i$ fixed, then $(X_i^{p+1})_i$ fixed, we obtain a minimization algorithm with respect to the variables $(X_i, \theta_i)_i$. The global convergence of this algorithm is not very good. More precisely the global convergence for the regular cases is asymptotically of the form

$$\left| \frac{\partial E}{\partial \theta}\left((X_i^{p+1}, \theta_i^p)_i\right) \right| = \frac{C_\theta}{p^\gamma} \text{ and } \left| \frac{\partial E}{\partial X}\left((X_i^{p+1}, \theta_i^{p+1})_i\right) \right| = \frac{C_X}{p^\gamma}.$$

Here $|\cdot|$ is the euclidian norm in $\mathbb{R}^{N-1}$.

Table 3: Convergence of the alternate variable algorithm for $k = 1$, $N = 10$

α	$-\frac{\pi}{2} + 0.1$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2} - 0.1$
γ	0.02	0.18	0.13	0.01	0.005	0.001	0.00001
C_X/C_θ	2.1	2.2	2.2	2.0	2.3	2.4	2.8

The values of γ and C_X/C_θ are given in table 3. The value $C_X/C_\theta \approx 2$ corresponds to an oscillation phenomenon due to the singularity of the hessian (see table 1). For $\alpha \approx \pi/2$, γ is small because this approaches the constant solution (here $k = 1$). For $\alpha \approx -\pi/2$, γ is small because this is close to the singular case.

2.1.5 Two remarks

- We used the MATLAB® software environment for the computation.
- In order to avoid numerical errors we computed exactly the gradients and the hessians that were needed.

2.2 Numerical results

2.2.1 Pure singular case Figure 5 represents the case $\alpha = -\frac{\pi}{2}$, $k = 1$ and $N = 10$. The discrete minimizer is in dashed line and the continuous minimizer in $\overline{\mathcal{Y}}_{\alpha,k}$ is in solid line. The left figure is the θ representation, and the right figure is the y representation (restricted to $[0, 1]$).

We obtain the **boundary layer** at $r = 0$ which is predicted by the theory. More precisely, $r_0 = r_1 = \ldots = r_{N-1} = 0$. The regular part of the graph of the discrete minimizer consists of only one line segment from $(0, \theta_{N-1})$ to $(1, \frac{\pi}{2})$ (in the θ coordinates).

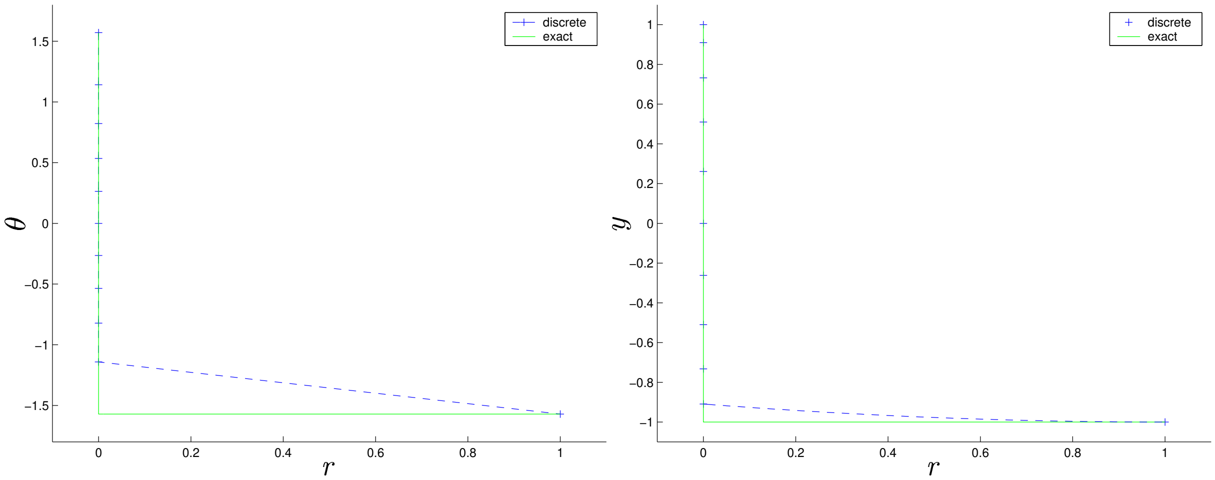

Figure 5: Case $\alpha = -\pi/2$, $k = 1$, $N = 10$

We notice that the points $(\theta_i)_i$ are not uniformly distributed along the singularity. Indeed, $|\theta_{i+1} - \theta_i|$ is large near $-\dfrac{\pi}{2}$ or $+\dfrac{\pi}{2}$ and small near 0. This remark is true for every degree and every N (see figure 6 for $k = 3$, $N = 22$). The distribution of the $(\theta_i)_i$ corresponds to a **uniform distribution with respect to the measure $|\cos\boldsymbol{\theta}|d\boldsymbol{\theta}$** predicted by the theory (see section 3.1). This is verified by the figure in the y coordinates where the distribution of the $(F(\theta_i))_i$ is uniform except the two boundary effects at -1 and $+1$.

Concerning the regular part of the discrete minimizer, we have $r_{N-1} = 0$ and $\theta_{N-1} = -\dfrac{\pi}{2} + \epsilon$ with $\epsilon > 0$, whereas the regular part of the continuous solution is the constant $-\dfrac{\pi}{2}$. This is due to the **consistency error** introduced by the midpoint formula. The element BV function y^N corresponding to $(r_0, \ldots, \theta_N)$ does not belong to $\overline{\mathcal{Y}}_{\alpha,k}$ (i.e. the domain of definition of the continuous relaxed energy): this element is **nonconforming**.

2.2.2 Case $\alpha \in]-\pi/2, \pi/2[$.

This time the continuous solution (in solid line) has a non-constant regular part. In figure 7 is the test case $\alpha = -\dfrac{\pi}{4}$, $k = 2$ and $N = 10$ (discrete solution in dashed line). The left part is the θ coordinates with a P_1 regular part and the right part is the y coordinates. We observe the same phenomenons as above.

The distribution of the $(\theta_i)_i$ is uniform with respect to the measure $|\cos\theta|d\theta$. This includes also the $(\theta_i)_i$ that belong to the regular part. In fact the distribution of the $(\theta_i)_i$ is independent of the $(r_i)_i$ whether they belong to the regular part or not. Indeed, from the remark used for the variable algorithm abovee, we find that the $(\theta_i)_i$ minimize the expression $\sum_{i=0}^{N-1} J(\theta_i, \theta_{i+1})$, where for all $i \in \{0, \ldots, N-1\}$:

$$J(\theta_i, \theta_{i+1}) := \inf_{X \in]0,1]} \cos^2\left(\frac{\theta_i + \theta_{i+1}}{2}\right) X + \left(\frac{\theta_{i+1} - \theta_i}{2}\right)^2 \frac{1}{X}. \tag{III.33}$$

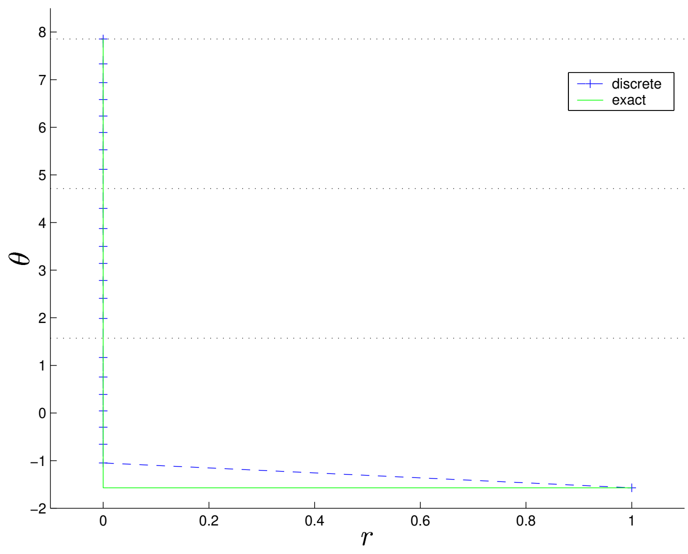

Figure 6: Case $\alpha = -\pi/2$, $k = 3$, $N = 22$

The last point in the singularity is badly positioned because of the consistency error: again this is an example of a nonconforming element (the same holds for the test case $\alpha = -\dfrac{\pi}{4}$, $k = -1$, $N = 13$ in figure 8).

There is a difference in these cases with the pure singular case $\alpha = -\dfrac{\pi}{2}$ due to the convexity of regular part of the continuous solution. We notice that the distribution of the points $(r_i, \theta_i)_i$ on the regular part is such that the discrete graph is a very good approximation of the exact graph. In particular there are many points where the curvature is important (see also figure 9).

3 Gauss formula with two nodes

In order to reduce the consistency error from the midpoint formula, we use the Gaussian quadrature with two nodes [19]. We recall that the midpoint formula has order 1 (i.e. is exact for polynomes of degree 1) whereas the Gauss formula with two nodes has order 3 [19]. The notations are the same as in previous section.

Let $c_1 = \dfrac{1}{2} - \dfrac{\sqrt{3}}{6}$ and $c_2 = \dfrac{1}{2} + \dfrac{\sqrt{3}}{6}$ be the two nodes in $[0,1]$. For every $\theta^N \in \mathcal{E}^N_{\alpha,k}$ which corresponds to $(r_0, \dots, \theta_N) \in D^N$ we define:

$$E^N(\theta^N) := 2\pi \sum_{i=0}^{N-1} E_i(\theta^N), \tag{III.34}$$

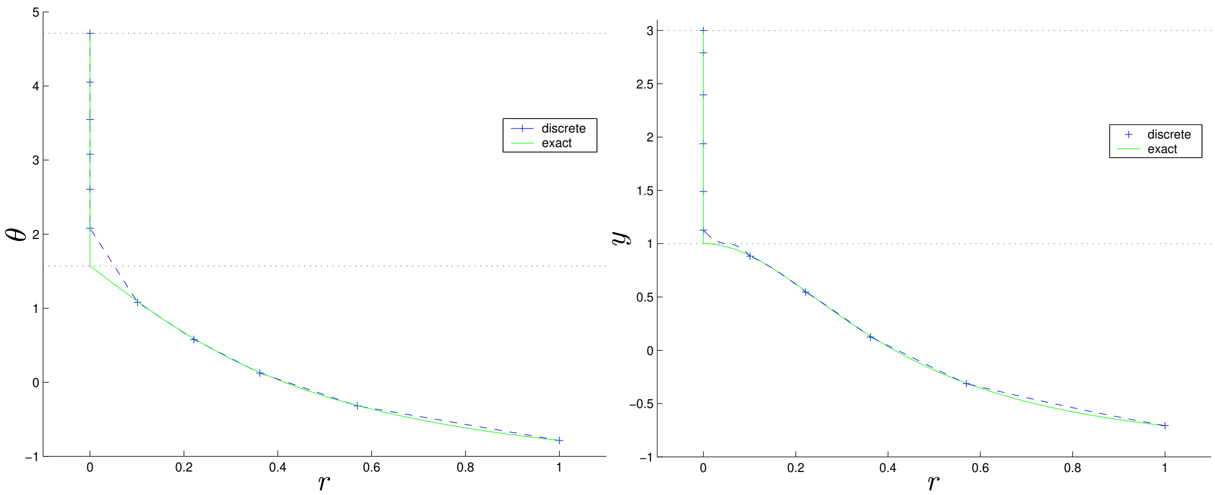

Figure 7: Case $\alpha = -\pi/4$, $k = 2$, $N = 10$

where for all $i \in \{0, \ldots, N-1\}$:

$$
\begin{aligned}
E_i(\theta^N) \quad &:= \quad \left[\frac{\cos^2(c_2\theta_i + c_1\theta_{i+1})}{c_2 r_i + c_1 r_{i+1}} + \frac{\cos^2(c_1\theta_i + c_2\theta_{i+1})}{c_1 r_i + c_2 r_{i+1}} \right] \frac{(r_{i+1} - r_i)}{4} \\
&\qquad + \left(\frac{\theta_{i+1} - \theta_i}{2} \right)^2 \left(\frac{r_{i+1} + r_i}{r_{i+1} - r_i} \right), \\
&= \quad \left[\frac{\cos^2(c_2\theta_i + c_1\theta_{i+1})}{c_2 t_i + c_1} + \frac{\cos^2(c_1\theta_i + c_2\theta_{i+1})}{c_1 t_i + c_2 2} \right] \frac{(1 - t_i)}{4} \\
&\qquad + \left(\frac{\theta_{i+1} - \theta_i}{2} \right)^2 \left(\frac{1 + t_i}{1 - t_i} \right).
\end{aligned}
$$

As for the midpoint formula the function E_i is defined on $\mathcal{E}^N_{\alpha,k} \simeq D^N$ and depends only on θ_i, θ_{i+1} and the ratio $t_i = \dfrac{r_i}{r_{i+1}}$.

The discrete energy E^N as a function of $(t_1, \ldots, t_{N-1}, \theta_1, \ldots, \theta_{n_1})$ is defined and continuously differentiable on $[0,1[^{N-1} \times \mathbb{R}^{N-1}$. We use this formulation for the computation with the assumption that the case $t_i = 1$ never happens (see section 2.1.1).

3.1 Algorithms

The alternate variable algorithm from the midpoint formula is not efficient here since the minimizer with respect to the variables $(t_1, \ldots, t_{N-1})$ for fixed $(\theta_1, \ldots, \theta_{N-1})$ is no longer obvious. We use projected gradient methods for the minimization [18].

3.1.1 Projected gradient algorithm A simple algorithm for the minimization with respect to $(t_i, \theta_i)_{1 \le i \le N-1}$ under the constraints $t_i \in [0,1[$ is the projected gradient algorithm.

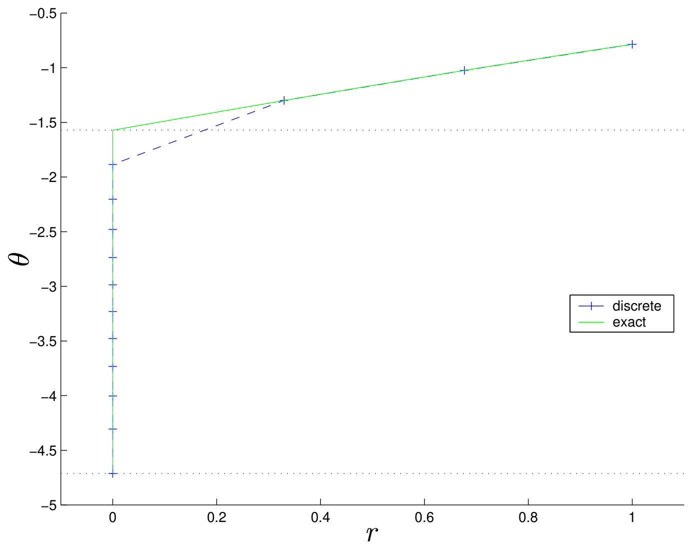

Figure 8: Case $\alpha = -\pi/4$, $k = -1$, $N = 13$

More precisely, let $(d_i)_{1\leq i\leq N-1} \in \mathbb{R}^{2(N-1)}$ be the descent direction as in the gradient algorith given by $d_i = -\left(\frac{\partial E^N}{\partial t_i}, \frac{\partial E^N}{\partial \theta_i}\right)$. Whenever the constraint $t_i = 0$ is active and $-\frac{\partial E}{\partial t_i} \leq 0$, we set $d_i^1 = 0$.

The optimal step-size is computed by a Newton line search or a simple dichotomy method. However the global convergence speed of the algorithm is (asymptotically) slow especially for the singular cases.

3.1.2 Projected CG with affine constraints

In order to improve the convergence of the previous algorithm we use a nonlinear conjugate gradient method. In order to take into account the constraints $t_i \in [0, 1[$ we adapt it using the same idea as previously.

Each time a constraint $t_i = 0$ becomes active, the algorithm is restarted with a dimension decreased by 1. The possibility of an active constraint $t_i = 0$ to become inactive is evaluated at each step by the sign of $d_i = -\left(\frac{\partial E^N}{\partial t_i}, \frac{\partial E^N}{\partial \theta_i}\right)$. If $d_i^1 \leq 0$ then $t_i = 0$ stays active and if $d_i^1 > 0$ then the algorithm is restarted.

The convergence speed of the algorithm is good. The gradient in euclidian norm presents oscillations similar to the oscillations observed in the alternate variable algorithm. For the mippoint formula, the average speed is similar to the average speed of the alternate variable algorithm (the average is taken over the period of the oscillations).

3.2 Numerical results

3.2.1 Nearly conforming elements

In figure 10 for $\alpha = -\frac{\pi}{2}$, $k = 1$ and $N = 10$ the discrete minimizer in $\overline{D^N}$ does not have a singular part: it belongs to D^N and corresponds to

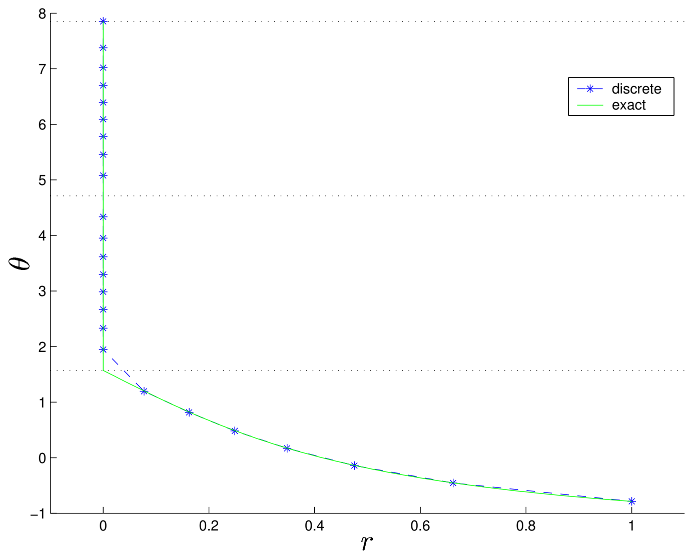

Figure 9: Case $\alpha = -\pi/4$, $k = 3$, $N = 23$

a *conforming* element $\theta^N \in \mathcal{E}^N_{\alpha,k}$.

Notice that θ^N is built like an optimal P_1 approximation of a minimizing sequence: there are many points where the curvature is important (see also figure 13).

With the exception of the case $\alpha = -\dfrac{\pi}{2}$, $k = 1$, the discrete minimizer of every singular test case has a boundary layer with exactly $r = 0$ (figures 11, 12 and 13). The last θ_i on the boundary is graphically close to $\dfrac{\pi}{2}$ (mod π), but not exactly. For instance with $\alpha = -\dfrac{\pi}{4}$, $k = 2$ and $N = 10$ this distance ϵ is approximately 0.07. The Gauss formula has reduced the consistency error.

The Gaussian discrete energy is more conform to the results that would be obtained with an exact discretization.

3.2.2 Boundary layer The distribution of the $(\theta_i)_i$ along the singularity is completely different than with the midpoint formula.

Since the variables t_i are independent (like the X_i with the midpoint formula), the values $(\theta_i)_i$ minimize a function independent of the t_i. The J function here is

$$J(\theta, \tilde{\theta}) := \inf_{t \in]0,1[} E_i(\theta, \tilde{\theta}, t), \tag{III.35}$$

and the $(\theta_i)_i$ minimize $\sum_{i=0}^{N-1} J(\theta_i, \theta_{i+1})$.

This explains why the distribution of the $(\theta_i)_i$ is the same in the singular part and the regular part. Moreover this J function (III.35) is different from the analogue J function (III.33) defined for the midpoint formula: this is why the distribution of the $(\theta_i)_i$ is different.

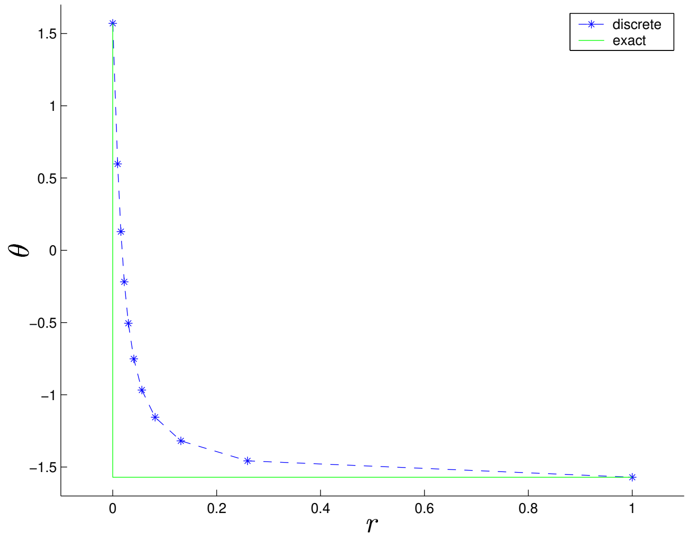

Figure 10: Two nodes for $\alpha = -\pi/2$, $k = 1$, $N = 10$

4 Midpoint formula for S^1

In view of the numerical applications and the possible extension to higher dimension we formulate the energy as a functionnal of S^1-valued maps. With this approach the energy E is quadratic and the constraint $|u(x)| = 1$ is the same as in dimension 2 or 3 [1]. The formulation of the Dirichlet problem (III.23) is similar to the $2d$ formulation [5].

4.1 The S^1 formulation

4.1.1 The continuous formulation Every angle function $\theta \in \mathcal{E}_{\alpha,k}$ can be seen as a S^1-valued map $u : [0,1] \to S^1 \subset \mathbb{R}^2$ defined by (see (II.18))

$$u(r) := (\cos\theta(r), \sin\theta(r)) =: (c(r), s(r)) \quad \forall r \in [0,1].$$

The Dirichlet energy with this formulation is given by:

$$E(u) := \pi \int_0^1 \frac{c^2(r)}{r} + r|u'(r)|^2 dr. \tag{III.36}$$

The energy (III.36) is quadratic: this allows to define the space $H^1_{axi} \subset \mathcal{C}^0([0,1],\mathbb{R}^2)$ of maps with bounded energy, which is a Hilbert space isomorphic to the subspace of $H^1(B^2,\mathbb{R}^3)$ made of axi-symmetric vector fields. The identification $\theta \simeq u$ allows to see $\mathcal{E}_{\alpha,k}$ as a subset of H^1_{axi}.

4.1.2 Midpoint formula We use a discretization based on P_1 elements in H^1_{axi}. The integer N is fixed. Let $r_0 = 0 < r_1 < \ldots < r_N = 1$ be a subdivision of $[0,1]$ and let $(u_i)_{0\le i\le N} \in (S^1)^{N+1}$ represent a P_1 element in H^1_{axi}. For all $i \in \{0,\ldots,N\}$ we denote $u_i = (c_i, s_i) \in \mathbb{R}^2$ where $c_i^2 + s_i^2 = 1$. The discrete energy is given by:

$$E^N(r_0,\ldots,r_N,u_0,\ldots,u_N) := 2\pi \sum_{i=0}^{N-1} E_i(r_0,\ldots,r_N,u_0,\ldots,u_N), \tag{III.37}$$

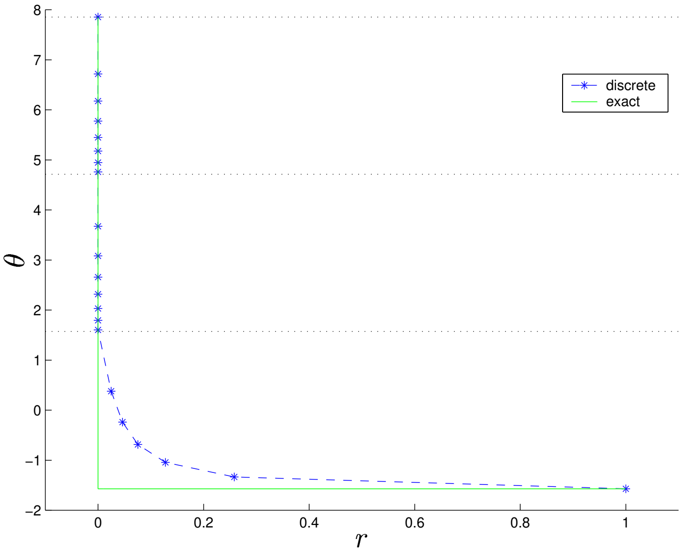

Figure 11: Two nodes for $\alpha = -\pi/2$, $k = 3$, $N = 20$

where for all $i \in \{0, \dots, N\}$:

$$E_i(r_0, \dots, u_N) := \frac{(c_i + c_{i+1})^2}{|u_i + u_{i+1}|^2}\left(\frac{r_{i+1} - r_i}{r_{i+1} + r_i}\right) + \frac{|u_i - u_{i+1}|^2}{|u_i + u_{i+1}|^2}\left(\frac{r_{i+1} + r_i}{r_{i+1} - r_i}\right).$$

Here $|\cdot|$ is the euclidian norm in $\mathbb{R}^2$.

Necessarily $u_{i+1} \neq -u_i$. More precisely the function E_i is defined on

$$\begin{aligned} D_u^N \quad := \quad & \{r_0 = 0 < r_1 < \dots < r_N = 1\} \times \\ & \left\{(u_i)_{0 \leq i \leq N} \in (S^1)^{N+1},\ u_i \neq u_{i+1}\ \forall i \in \{0, \dots, N-1\}\right\}. \end{aligned} \tag{III.38}$$

4.1.3 Differentiable expression for $\overline{E^N}$

As previously we set $X_i := \dfrac{r_{i+1} - r_i}{r_{i+1} + r_i}$, so:

$$E_X^N(X_1, \dots, u_{N-1}) = 2\pi \sum_{i=0}^{N-1} \frac{(c_i + c_{i+1})^2}{|u_i + u_{i+1}|^2} X_i + \frac{|u_{i+1} - u_i|^2}{|u_{i+1} + u_i|^2}\frac{1}{X_i}. \tag{III.39}$$

This expression is continuously differentiable on

$$D_{u,X}^N :=]0, 1]^{N-1} \times \left\{(u_i)_{1 \leq i \leq N-1} \in (S^1)^{N-1}, u_{i+1} \neq u_i\right\}.$$

4.2 Algorithms

Since the minimization with respect to the variables $(X_i)_i$ is obvious, the *alternate variable algorithm* (section 2.1.4) could be used. However, because of the constraint $|u(r)| = 1$ there is not efficient algorithm for the minimization with respect to the variables $(u_i)_i$. We use a projected gradient with fixed step-size λ.

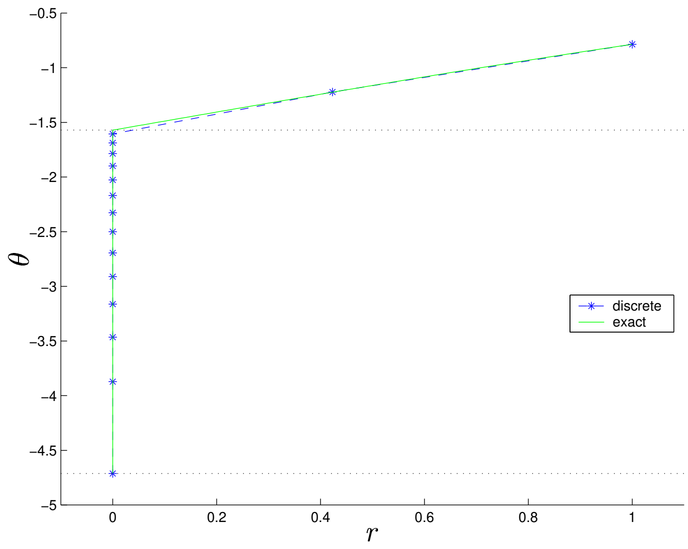

Figure 12: Two nodes for $\alpha = -\pi/4$, $k = -1$, $N = 15$

Another method generalizable to higher dimension is the *twice projected gradient*. Indeed we have inequality constraints $X_i \in]0,1]$ and equality constraints $|u_i| = 1$. The first step is the computation of $d_i := -(\frac{\partial E}{\partial X_i}, \frac{\partial E}{\partial c_i}, \frac{\partial E}{\partial s_i})$. For every active constraint $X_i = 1$ with $d_i^1 \leq 0$ we set $d_i = 0$. After the computation of $(X_i, c_i, s_i)_i + \lambda(d_i)_i$ we project on S^1.

4.3 Numerical results

4.3.1 Nonconforming elements The results are comparable to those with the midpoint formula in the θ formulation. We obtain the boundary layer with thickness 0 exactly. (figures 14, 15 and 16) Because of the consistency error the elements are nonconforming: there is a distance $\epsilon > 0$ between the last u_i on the boundary layer and the north pole (or south pole).

Figure 14 shows the pure singular case in $[0,1] \times S^1 \subset \mathbb{R}^3$ with the projections on the three coordinate planes. Again the boundary layer is present and there is no subdvision in the regular part ($r_{N-1} = 0$).

4.3.2 Boundary layer The distribution of the $(u_i)_i$ along the singularity has a certain pattern. This distribution is close to the result with the midpoint formula in the θ representation (see figure 14 in the sin coordinate or the distribution on the circle in figures 15 and 16). As in the previous cases this distribution is the same whether u_i belongs to the regular part or the singular part. Since the X_i are independent, the distribution $(u_i)_i$ in S^1 minimizes the expression $\sum_{i=0}^{N-1} J(u_i, u_{i+1})$ where

$$J(u, \tilde{u}) := \inf_{X \in]0,1[} E_i(X, u, \tilde{u}).$$

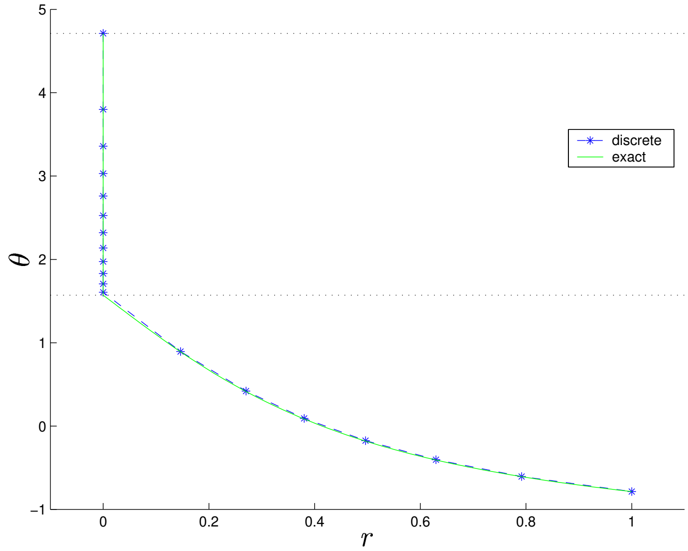

Figure 13: Two nodes for $\alpha = -\pi/4$, $k = 2$, $N = 18$

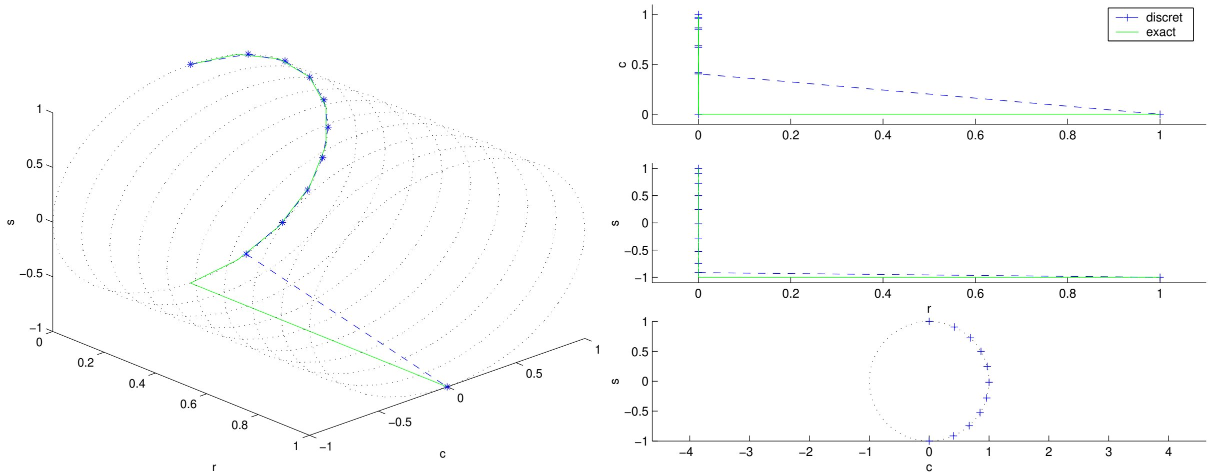

Figure 14: Midpoint for S^1 with $\alpha = -\pi/2$, $k = 1$, $N = 10$

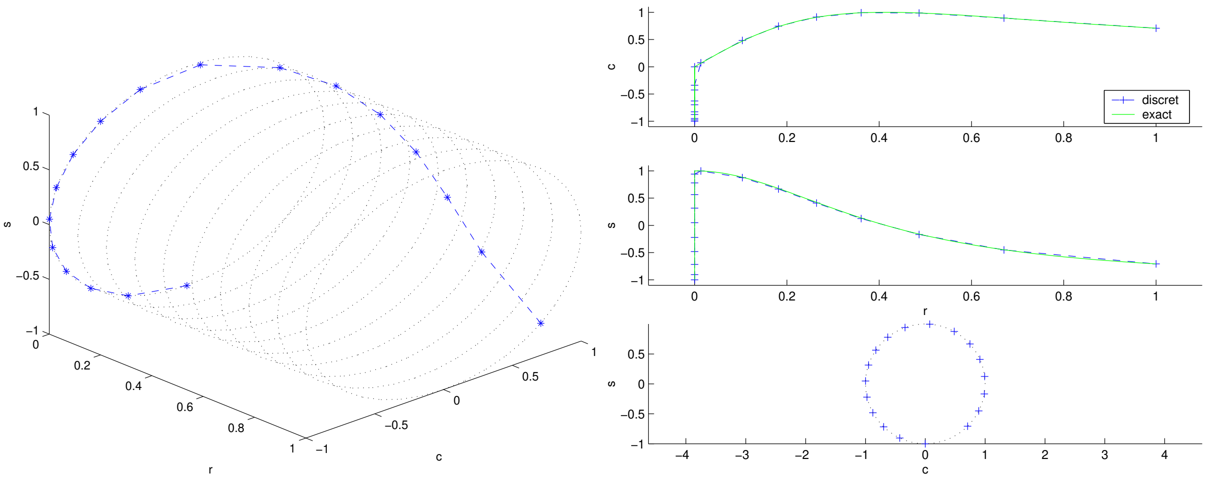

Figure 15: Midpoint for S^1 with $\alpha = -\pi/4$, $k = 2$, $N = 17$

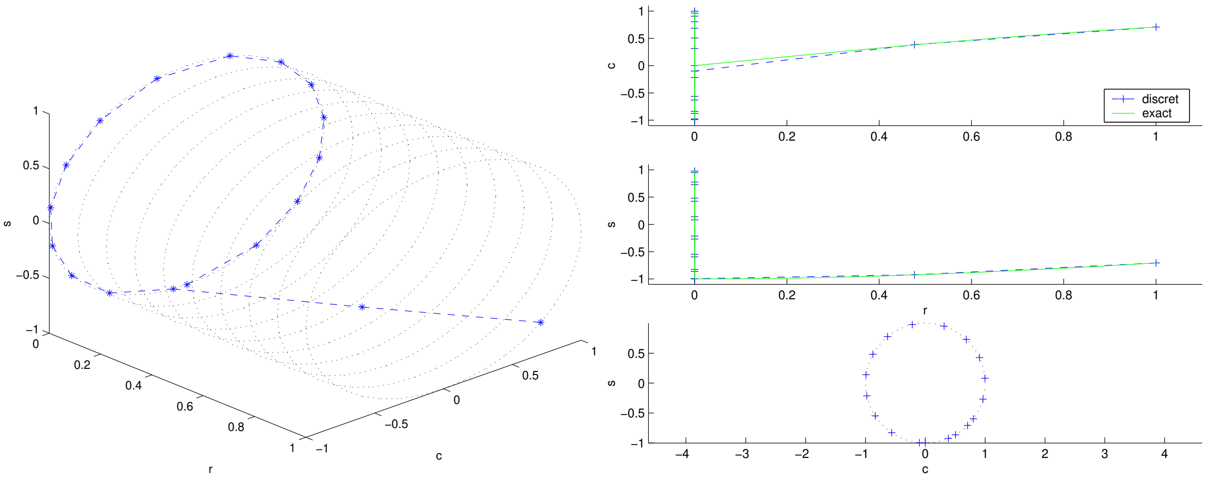

Figure 16: Midpoint for S^1 with $\alpha = -\pi/4$, $k = -2$, $N = 17$

IV Conclusion

Our new "graph-meshing" approach is very satisfactory for the computation of singularities of the type "thin boundary layer". For the various test cases and discretizations, we managed to compute boundary layers of zero thickness with very few points. The distribution of the points along the singularity corresponds to a uniform distribution of the energy. This discrete approach mimicks the properties of the continuous problem.

Because of the consistency error introduced in the discretization of the energy, the discrete minimizer in non-conforming. This raises the question of the convergence of the method: this is the aim of the second part II of this article. In part II, we explain the non-conformity and we prove the convergence of the method in terms of weak BV convergence. We also compute numerically the speed of convergence (see [2]).

The idea of the "mesh as an unknown of the problem" is very general in a finite element approach of a minimizing problem. It is directly applicable to problems of higher dimensions (and to finite elements of higher order as well). A direct extension of the problem studied in this article could concern for instance computations for the general $2d$ Dirichlet problem of minimizing harmonic maps from the disc into the sphere with prescribed degree and boundary value [5]. However the algorithmical difficulties would be much greater for the following reasons.

- The constraints on the $2d$ mesh: the open constraint is to have triangles of positive area in the abscisse mesh; the closing condition is to deal with triangles of non-negative area. Directly related to this is the expression of the relaxed energy on the closure. In the axi-symmetric case we found a change of variable ($t_i = x_i/x_{i+1}$) that gives a differentiable expression of the relaxed energy even for $x_i = x_{i+1} = 0$. For the $2d$ problem this seems difficult.

- One algorithmical difficulty is the possibility of loosing points during the minimization. In our problem the particularity is that the t_i are independent so we manage to keep the number of points fixed (cf. the alternate variable algorithm in section 2.1.4). Loosing points during the minimization corresponds to computing a local minimizer of the relaxed energy. In this case the discrete solution obtained has a lower energy than the starting point and can be an acceptable approximation of the continuous solution. The difficulty is to still have enough points at the end (and not only one !)

- In higher dimension the number of variables for the mesh is a computational difficulty, as well as the strong non-linearity of the minimization problem with respect to the (abscisse) mesh variables. However this difficulty is not overwhelming: we gave an example of computations in a $2d$ case in [16]

Finally, the original geometric idea of "graph-meshing" is interesting for problems for which the singularities are of the type thin boundary layer or shock. The idea is to link the problem of the mesh adaptation and the problem of the computation of the solution. The difficulties are of course both theorical (such as considering the continuous problem from a graph point of view) and algorithmical (deal with surface, volume meshing in $\mathbb{R}^n$ with $n > 3$).

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