SURFACE ORDER LARGE DEVIATIONS FOR 2D FK-PERCOLATION AND POTTS MODELS

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10 March 2003

Abstract. By adapting the renormalization techniques of Pisztora, [32], we establish surface order large deviations estimates for FK-percolation on $\mathbb{Z}^2$ with parameter $q \geq 1$ and for the corresponding Potts models. Our results are valid up to the exponential decay threshold of dual connectivities which is widely believed to agree with the critical point.

1. Introduction and statement of results. In this paper we derive surface order large deviations for Bernoulli percolation, FK-percolation with parameter $q > 1$ and for the corresponding Potts models on the planar lattice $\mathbb{Z}^2$.

In dimension two, surface order large deviations behaviour and the Wulff construction has been established for the Ising model [15, 16, 23, 24, 25, 26, 30, 31, 33, 34, 35, 36], for independent percolation [3, 6] and for the random cluster model [4]. These works include also more precise results than large deviations for the Wulff shape. They are obtained by using the skeleton coarse graining technique to study dual contours which represent the interface. In higher dimensions other methods had to be used to achieve the Wulff construction, [9, 10, 11, 12], where one of the main tools that have been used was the blocks coarse graining of Pisztora [32]. This renormalization technique led to surface order large deviations estimates for FK-percolation and for the corresponding Potts models simultaneously. The results of [32], and thus the Wulff construction in higher dimensions, are valid up to the limit of the slab percolation thresholds. In the case of independent percolation, this threshold has been proved to agree with the critical point [21] and is believed to be so for all the FK-percolation models with parameter $q \geq 1$ in dimension greater than two.

Our aim is to adapt Pisztora's techniques to the two-dimensional lattice thereby opening the way to an other proof for the Wulff construction in dimension two. It is also worth noting that Pisztora's renormalization technique forms a building block that has been used to answer various other questions related to percolation [7, 8, 28, 29]. Thus, we expect that adapting [32] to the two-dimensional case will permit the use of this building block for other problems on the planar lattice. The main point

1991 *Mathematics Subject Classification.* 60F10, 60K35, 82B20, 82B43.

Key words and phrases. Large deviations, FK model, Potts model.

We would like to thank R. Cerf for suggesting the problem and for many helpful discussions.

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in our task is to get rid of the percolation in slabs which is specific to the higher dimensional case. For this we produce estimates analogue to those of theorem 3.1 in [32] relying on the hypothesis that the dual connectivities decay exponentially. This hypothesis is very natural in $\mathbb{Z}^2$, because it is possible to translate events from the supercritical regime to the subcritical regime by planar duality. For Bernoulli percolation, the exponential decay of the connectivities is known to hold in all the subcritical regime, see [17] and the references therein. For the random cluster model on $\mathbb{Z}^2$ with $q = 2$ the exponential decay follows from the exponential decay of the correlation function in the Ising model [13], and a proof has also been given when q is sufficiently large, see [19] and the references therein. Even if not proved, the exponential decay of the connectivities is widely believed to hold up to the critical point of all the FK-percolation models with $q \geq 1$. In addition to that, we use a property which is specific to the two dimensional case, namely the weak mixing property. This property has been proved to hold for all the random cluster models with $q \geq 1$ in the regime where the connectivities decay exponentially [1]. We need this property in order to use the exponential decay in finite boxes [2].

1.1. Statement of results. Our results concern asymptotics of FK-measures on finite boxes $B(n) = (-n/2, n/2]^2 \cap \mathbb{Z}^2$, where n is a positive integer. We will denote by $\mathcal{R}(p, q, B(n))$ the set of these FK-measures defined on $B(n)$ with parameters (p, q) and where we have identified some vertices of the boundary. For $q \geq 1$ and $0 < p \neq p_c(q) < 1$, it is known [20] that there is a unique infinite volume Gibbs measure that we will note $\Phi^{p,q}_\infty$. It is also known that $\Phi^{p,q}_\infty$ is translation invariant and ergodic. In the uniqueness region, we will denote by $\theta = \theta(p, q)$ the density of the infinite cluster. As the exponential-decay plays a crucial rule in our analysis, we will introduce the following threshold[1]

$$(1) \qquad p_g = \sup\{p : \exists c > 0,\ \forall\, x\ \forall\, y \in \mathbb{Z}^2,\ \Phi^{p,q}_\infty[x \leftrightarrow y] \leq \exp(-c|x - y|)\},$$

where $|x - y|$ is the $\mathcal{L}^1$ norm and $\{x \leftrightarrow y\}$ is the event that there exists an open path joining the vertex x to the vertex y.
By the results of [22], it is known that exponential decay holds as soon as the connectivities decay at a sufficient polynomial rate. We thus could replace (1) by

$$p_g = \sup\{p : \exists c > 0,\ \forall\, x\ \forall\, y \in \mathbb{Z}^2,\ \Phi^{p,q}_\infty[x \leftrightarrow y] \leq c/|x - y|)\}.$$

We introduce the point dual to p_g:

$$\widehat{p}_g = \frac{q(1 - p_g)}{p_g + q(1 - p_g)} \geq p_c(q),$$

which is conjectured to agree with the critical point $p_c(q)$.

Our result states that up to large deviations of surface order, there exists a unique biggest cluster in the box $B(n)$ with the same density than the infinite cluster, and that the set of clusters of intermediate size have a negligible volume. To be more precise, we say that a cluster in $B(n)$ is *crossing* if it intersects all the faces of $B(n)$. For $l \in \mathbb{N}$, we say that a cluster is *l-intermediate* if it is not of maximal volume and

[1] The notation p_g comes from [19].

its diameter does exceed l. We denote by $\mathbb{J}_l$ the set of l-intermediate clusters. Let us set the event

$$K(n,\varepsilon,l) = \Big\{\exists!\text{ open cluster } C_m \text{ in } B(n) \text{ of maximal volume, } C_m \text{ is crossing}\Big\}$$
$$\cap \Big\{n^{-2}|C_m| \in (\theta-\varepsilon,\theta+\varepsilon)\Big\} \cap \Big\{n^{-2}\sum_{C\in\mathbb{J}_l}|C| < \varepsilon\Big\},$$

Theorem 1. *Let $q \geq 1, 1 > p > \widehat{p}_g$ and $\varepsilon \in (0,\theta/2)$ be fixed. Then there exists a constant L such that*

$$-\infty < \liminf_{n\to\infty} \frac{1}{n} \log \inf_{\Phi\in\mathcal{R}(p,q,B(n))} \Phi[K(n,\varepsilon,L)^c]$$
$$\leq \limsup_{n\to\infty} \frac{1}{n} \log \inf_{\Phi\in\mathcal{R}(p,q,B(n))} \Phi[K(n,\varepsilon,L)^c] < 0.$$

This result, via the FK-representation, can be used as in [32] to deduce large deviations estimates for the magnetization of the Potts model. We will omit this as it would be an exact repetition of theorem 1.1 and theorem 5.4 in [32].

1.2. Organization of the paper. In the following section we introduce notation and give a summary of the FK model and of the duality in the plane. In section 4, we study connectivity properties of FK percolation in a large box $B(n)$ and establish estimates that will be crucial for the renormalization. In section 5, we introduce the renormalization and proof estimates on the N-block process. In section 6, we finally give the proof of theorem 1.

2. Preliminaries. In this section we introduce the notation used and the basic definitions.

Norm and the lattice. We will use the $\mathcal{L}^1$−norm on $\mathbb{Z}^2$, that is, $|x-y| = \sum_{i=1,2}|x_i - y_i|$ for any x, y in $\mathbb{Z}^2$. For every subset A of $\mathbb{Z}^2$ and $i = 1,2$ we define $\mathrm{diam}_i(A) = \sup\{|x_i - y_i| : x,\, y \in A\}$ and the diameter of A is $\mathrm{diam}(A) = \max(\mathrm{diam}_1(A), \mathrm{diam}_2(A))$. We turn $\mathbb{Z}^2$ into a graph $(\mathbb{Z}^2, \mathbb{E}^2)$ with vertex set $\mathbb{Z}^2$ and edge set $\mathbb{E}^2 = \{\{x,y\}; |x-y| = 1\}$. If x and y are nearest neighbors, we denote this relation by $x \sim y$.

Geometric objects. A *box* Λ is a finite subset of $\mathbb{Z}^2$ of the form $\mathbb{Z}^2 \cap [a,b] \times [c,d]$. For $\underline{r} \in (0,\infty)^2$, we define a box centered at the origin by $B(\underline{r}) = \mathbb{Z}^2 \cap \Pi_{i=1,2}(-r_i/2, r_i/2]$. We say that the box is symmetric, if $r_1 = r_2 = r$, and we denote it by $B(r)$. For $t \in \mathbb{R}^+$, we note the set $\mathcal{H}_2(t) = \{\underline{r} \in \mathbb{R}^2 : r_i \in [t, 2t], i = 1,2\}$. The set of all boxes in $\mathbb{Z}^2$, which are congruent to a box $B(\underline{r})$ with $\underline{r} \in \mathcal{H}_2(t)$, will be denoted by $\mathcal{B}_2(t)$.

Discrete topology. Let A be a subset of $\mathbb{Z}^2$. We define two different boundaries:
- the inner vertex boundary: $\partial A = \{x \in A |\, \exists y \in A^c \text{ such that } y \sim x\}$;
- the edge boundary: $\partial^{\mathrm{edge}} A = \{\{x,y\} \in \mathbb{E}^2 \,|\, x \in A, y \in A^c\}$.

For a box Λ and for each $i = \pm1, \pm2$, we define the ith face $\partial_i\Lambda$ of Λ by $\partial_i\Lambda = \{x \in \Lambda |\; x_i \text{ is maximal}\}$ for i positive and $\partial_i\Lambda = \{x \in \Lambda |\; x_{|i|} \text{ is minimal}\}$ for i negative. A *path* γ is a finite or infinite sequence $x_1, x_2, \ldots$ of distinct nearest neighbors.

FK percolation.

Edge configurations. The basic probability space for the edge processes is given by $\Omega = \{0,1\}^{\mathbb{E}^2}$; its elements are called *edge configurations in* $\mathbb{Z}^2$. The natural projections are given by $\mathrm{pr}_e : \omega \in \Omega \mapsto \omega(e) \in \{0,1\}$, where $e \in \mathbb{E}^2$. An edge e is called open in the configuration ω if $\mathrm{pr}_e(\omega) = 1$, and closed otherwise.

For $E \subseteq \mathbb{E}^2$ with $E \neq \emptyset$, we write $\Omega(E)$ for the set $\{0,1\}^E$; its elements are called *configurations in* E. Note that there is a one-to-one correspondence between cylinder sets and configurations on finite sets $E \subset \mathbb{E}^2$, which is given by $\eta \in \Omega(E) \mapsto \{\eta\} := \{\omega \in \Omega \mid \omega(e) = \eta(e) \text{ for every } e \in E\}$. We will use the following convention: the set Ω is regarded as a cylinder (set) corresponding to the "empty configuration" (with the choice $E = \emptyset$.) We will sometimes identify cylinders with the corresponding configuration. For $A \subset \mathbb{Z}^2$, we set $\mathbb{E}(A) = \{(x,y) : \ x, y \in A, \ x \sim y\}$. And let Ω_A stand for the set of the configurations in $A : \{0,1\}^{\mathbb{E}(A)}$ and Ω^A for the set of the configurations *outside* $A : \{0,1\}^{\mathbb{E}^2 \setminus \mathbb{E}(A)}$. In general, for $A \subseteq B \subseteq \mathbb{Z}^2$, we set $\Omega_B^A = \{0,1\}^{\mathbb{E}(B) \setminus \mathbb{E}(A)}$. Given $\omega \in \Omega$ and $E \in \mathbb{E}^2$, we denote by $\omega(E)$ the restriction of ω to $\Omega(E)$. Analogously, ω_B^A stands for the restriction of ω to the set $\mathbb{E}(B) \setminus \mathbb{E}(A)$.

Given $\eta \in \Omega$, we denote by $\mathcal{O}(\eta)$ the set of the edges of $\mathbb{E}^2$ which are open in the configuration η. The connected components of the graph $(\mathbb{Z}^2, \mathcal{O}(\eta))$ are called *η-clusters.* The path $\gamma = (x_1, x_2, ...)$ is said to be η-open if all the edges $\{x_i, x_{i+1}\}$ belong to $\mathcal{O}(\eta)$. We write $\{A \leftrightarrow B\}$ for the event that there exists an open path joining some site in A with some site in B.

If $V \subseteq \mathbb{Z}^2$ and E consists of all the edges between vertices in V, the graph $G = (V, E) \subseteq (\mathbb{Z}^2, \mathbb{E}^2)$ is called the maximal subgraph of $(\mathbb{Z}^2, \mathbb{E}^2)$ on the vertices V. Let ω be an edge configuration in $\mathbb{Z}^2$ (or in a subgraph of $(\mathbb{Z}^2, \mathbb{E}^2)$). We can look at the *open clusters in* V or alternatively the open *V-clusters.* These clusters are simply the connected components of the random graph $(V, \mathcal{O}(\omega(E)))$, where $\omega(E)$ is th restriction of ω to E.

For $A \subseteq B \subseteq \mathbb{Z}^2$, we use the notation $\mathcal{F}_B^A$ for the σ-field generated by the finite-dimensional cylinders associated with configurations in Ω_B^A. If $A = \emptyset$ or $B = \mathbb{Z}^2$, then we omit them from the notation.

Stochastic domination. There is a partial order $\preceq$ in Ω given by $\omega \preceq \omega'$ iff $\omega(e) \leq \omega'(e)$ for every $e \in \mathbb{E}^2$. A function $f : \Omega \to \mathbb{R}$ is called *increasing* if $f(\omega) \leq f(\omega')$ whenever $\omega \preceq \omega'$. An event is called increasing if its characteristic function is increasing. Let $\mathcal{F}$ be a σ-field of subsets of Ω. For a pair of probability measures μ and ν on $(\Omega, \mathcal{F})$, we say that μ *(stochastically) dominates* ν if for any $\mathcal{F}$-measurable increasing function f the expectations satisfy $\mu(f) \geq \nu(f)$.

FK measures. Let $V \subseteq \mathbb{Z}^2$ be finite and $E = \mathbb{E}(V)$. We first introduce *(partially wired) boundary conditions* as follows. Consider a partition π of the set ∂V, say $\{B_1, ..., B_n\}$. (The sets B_i) are disjoint nonempty subsets of ∂V with $\bigcup_{i=1,...,n} B_i = \partial V$.) We say that $x, y \in \partial V$ are *π-wired,* if $x, y \in B_i$ for an $i \in \{1, ..., n\}$. Fix a configuration $\eta \in \Omega_V$. We want to count the η-clusters in V in such a way that π-wired sites are considered to be connected. This can be done in the following formal way. We introduce an equivalence relation on V: x and y are said to be *$\pi \cdot \eta$-wired* if they are both joined by η-open paths to (or identical with) sites $x', y' \in \partial V$ which are themselves π-wired. The new equivalence classes are called $\pi \cdot \eta$-clusters, or η-clusters in V with respect to the boundary condition π. The number of η-clusters in V with respect to the boundary condition π (i.e., the number of $\pi \cdot \eta$-clusters) is denoted by $\mathrm{cl}^\pi(\eta)$. (Note that cl^π is simply a random variable). For fixed $p \in [0,1]$ and $q \geq 1$, the *FK measure on the finite set $V \subset \mathbb{Z}^2$ with parameters (p,q) and boundary conditions* π is a probability measure on the σ-field $\mathcal{F}_V$, defined by the formula

$$(2) \qquad \forall \eta \in \Omega_V \qquad \Phi_V^{\pi,p,q}[\{\eta\}] = \frac{1}{Z_V^{\pi,p,q}} \left(\prod_{e \in E} p^{\eta(e)}(1-p)^{1-\eta(e)} \right) q^{\mathrm{cl}^\pi(\eta)},$$

where $Z_V^{\pi,p,q}$ is the appropriate normalization factor. Since $\mathcal{F}_V$ is an atomic σ-field with atoms $\{\eta\}, \eta \in \Omega_V$, (2) determines a unique measure on $\mathcal{F}_V$. Note that every cylinder has nonzero probability. There are two extremal b.c.s: the free boundary condition corresponds to the partition f defined to have exactly $|\partial V|$ classes, and the wired b.c corresponds to the partition w with only one class. The set of all such measures called FK (or random cluster) measures corresponding to different b.c.s will be denoted by $\mathcal{R}(p,q,V)$. The stochastic process $(\mathrm{pr}_e)_{e \in \mathbb{E}(V)} : \Omega \to \Omega_V$ given on the probability space $(\Omega, \mathcal{F}, \Phi_V^{\pi,p,q})$ is called FK *percolation with boundary conditions* π. We will list some useful properties of FK measures with different b.c.s. There is a partial order on the set of partitions of ∂V. We say that π *dominates* $\pi', \pi \geq \pi'$, if x, y π'-wired implies that they are π-wired. We then have $\Phi_V^{\pi',p,q} \preceq \Phi_V^{\pi,p,q}$. This implies immediately that for each $\Phi \in \mathcal{R}(p,q,V)$, $\Phi_V^{f,p,q} \preceq \Phi \preceq \Phi_V^{w,p,q}$. Next we discuss properties of conditional FK measures. For given $U \subseteq V$ and $\omega \in \Omega$, we define a partition $W_V^U(\omega)$ of ∂U by declaring $x, y \in \partial U$ to be $W_V^U(\omega)$-wired if they are joined by an ω_V^U-open path. Fix a partition π of ∂V. We define a new partition of ∂U to be $\pi \cdot W_V^U(\omega)$-wired if they are both joined by ω_V^U-open paths to (or identical with) sites x', y', which are themselves π-wired. Then, for every $\mathcal{F}_U$-measurable function f,

$$(3) \qquad \Phi_V^{\pi,p,q}[f|\mathcal{F}_V^U](\omega) = \Phi_V^{\pi \cdot W_V^U(\omega),p,q}[f], \qquad \Phi_V^{\pi,p,q} \text{ a.s.}$$

Note that (3) can be interpreted as a kind of Markov property. A direct consequence is the *finite-energy property.* Fix an edge e of $\mathbb{E}(V)$ and denote by $\mathcal{F}_V^e$ the σ-algebra generated by the random variables $\{pr_b; b \in \mathbb{E}(V) \setminus \{e\}\}$. Then

$$(4) \qquad \Phi_V^{\pi,p,q}[e \text{ is open } |\mathcal{F}_V^e](\omega) = \begin{cases} p \text{ if the endpoints of } e \text{ are } \pi \cdot W_V^e\text{-wired,} \\ p/[p+q(1-p)] \text{ otherwise.} \end{cases}$$

The equality (3) leads to volume monotony for FK-measures. Let $U \subset V$, for every increasing function $g \in \mathcal{F}_U$ and $\Phi_V \in \mathcal{R}(p,q,V)$, we have that

$$\Phi_U^{f,p,q}[g] \leq \Phi_V[g \mid \mathcal{F}_V^U] \leq \Phi_U^{w,p,q}[g] \quad \Phi_V \text{ a.s. },$$
$$\Phi_U^{f,p,q}[g] \leq \Phi_V^{f,p,q}[g] \leq \Phi_V^{w,p,q}[g] \leq \Phi_U^{w,p,q}[g].$$

Planar duality for FK-measures. Let $n > 0$ and $\underline{n} \in \mathcal{H}_2(n)$. To the set $B(\underline{n}) \subset \mathbb{Z}^2$ we associate the set $\widehat{B}(\underline{n}) \subset \mathbb{Z}^2 + (1/2,1/2)$, which is defined as the smallest box of $\mathbb{Z}^2 + (1/2,1/2)$ containing $B(\underline{n})$, see the figure 1 below. Notice that if $\underline{n} \in \mathcal{H}_2(n)$ then $\widehat{B}(\underline{n}) \in [\mathcal{B}_2(n+1) \cup \mathcal{B}_2(n)] + (1/2,1/2)$. To each edge $e \in \mathbb{E}(B(\underline{n}))$ we associate the edge $\widehat{e} \in \mathbb{E}(\widehat{B}(\underline{n}))$ that crosses the edge e. According to [18], if we associate to each configuration $\omega \in \Omega_{B(\underline{n})}$ the dual configuration $\widehat{\omega}_\omega$:

$$\widehat{\omega}_\omega \in \Omega^{\partial \widehat{B}(\underline{n})}_{\widehat{B}(\underline{n})} \text{ such that } \forall e \in \mathbb{E}(B(\underline{n})),\ \widehat{\omega}_\omega(\widehat{e}) = 1 - \omega(e),$$

then we have that

$$\Phi^{f,p,q}_{B(\underline{n})}[\omega] = \Phi^{w,\widehat{p},q}_{\widehat{B}(\underline{n})}[\{\omega_d \in \Omega_{\widehat{B}(\underline{n})},\ \forall \widehat{e} \in \mathbb{E}(\widehat{B}(\underline{n})) \setminus \mathbb{E}(\partial \widehat{B}(\underline{n})) :\ \omega_d(\widehat{e}) = \widehat{\omega}_\omega(\widehat{e})\}],$$

where $\widehat{p}$ is the dual point of p : $\widehat{p} = q(1-p)/[p + q(1-p)]$.

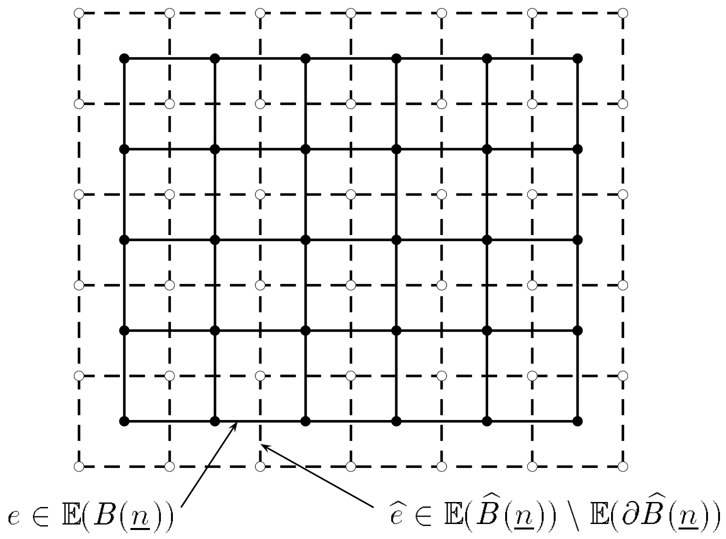

figure 1: A box and its dual

Thus, for each $\mathcal{F}_{B(\underline{n})}$ measurable event A we can associate a $\mathcal{F}^{\partial \widehat{B}(\underline{n})}_{\widehat{B}(\underline{n})}$ measurable event

$$\widehat{A} = \{\omega_d \in \Omega_{\widehat{B}(\underline{n})} :\ \exists\ \omega \in A, \forall \widehat{e} \in \mathbb{E}(\widehat{B}(\underline{n})) \setminus \mathbb{E}(\partial \widehat{B}(\underline{n})) :\ \omega_d(\widehat{e}) = \widehat{\omega}_\omega(\widehat{e})\},$$

which satisfies

$$\Phi^{f,p,q}_{B(\underline{n})}[A] = \Phi^{w,\widehat{p},q}_{\widehat{B}(\underline{n})}[\widehat{A}].$$

3. Connectivity in boxes. In this section we establish preliminary estimates on crossing events in boxes. We rely on the exponential decay of the connectivities in the dual subcritical model. The usual definition of the exponential decay is based on the infinite volume FK-measure $\Phi^{p,q}_\infty$. But we are concerned by asymptotics of finite volume measures and we would like to use the exponential decay in finite boxes. In order to translate the exponential decay to the finite volume measures we need a control on the effects of boundary conditions. As shown in [1], the infinite FK-measure on $\mathbb{Z}^2$ satisfies the weak mixing property as soon as the connectivities decay exponentially. That is to say for all events A, B which are respectively $\mathcal{F}_\Lambda$ measurable and $\mathcal{F}_\Gamma$ measurable with $\Lambda, \Gamma \subseteq \mathbb{Z}^2$ then $|\Phi^{p,q}_\infty[A \cap B] - \Phi^{p,q}_\infty[A]\Phi^{p,q}_\infty[B]|$ decreases exponentially in the distance between Λ and Γ. This weak mixing property implies, as proved in [2], that we have exponential decay in finite boxes as soon as the exponential decay for the infinite volume measure holds ($p < p_g$):

Proposition 2 (Theorem 1.2 of [2]). *Let $q \geq 1$ and $p < p_g$. There exists two positive constants c and λ such that for all boxes $\Lambda \subset \mathbb{Z}^2$ and for all x, y in Λ, we have that*

$$\Phi_\Lambda^{w,p,q}[x \leftrightarrow y \text{ in } \Lambda] \leq \lambda \exp(-c|x-y|).$$

In fact, theorem 1.2 of [2] is more general and applies to sets Λ which are not boxes and to general boundary conditions. From this result, we get that

Lemma 3. *Let $q \geq 1$ and $p < p_g$. There exists a positive constant c such that for all positive integers n and for l large enough, we have that*

$$\sup_{\underline{n} \in \mathcal{H}_2(n)} \Phi_{B(\underline{n})}^{w,p,q}[\exists \text{ an open path in } B(\underline{n}) \text{ of diameter } \geq l] \leq n^2 \exp(-cl).$$

Proof. Let us fix n and l, then we have

$$\begin{aligned}
&\sup_{\underline{n} \in \mathcal{H}_2(n)} \Phi_{B(\underline{n})}^{w,p,q}[\exists \text{ an open path in } B(\underline{n}) \text{ of diameter } \geq l] \\
\leq& 4n^2 \sup_{\underline{n} \in \mathcal{H}_2(n)} \sup_{x \in B(\underline{n})} \Phi_{B(\underline{n})}^{w,p,q}[x \leftrightarrow \partial B(x, 2l) \text{ in } B(\underline{n})] \\
\leq& 32 n^2 l \sup_{\underline{n} \in \mathcal{H}_2(n)} \sup_{x \in B(\underline{n})} \sup_{y \in \partial B(x,2l)} \Phi_{B(\underline{n})}^{w,p,q}[x \leftrightarrow y \text{ in } B(\underline{n})] \\
\leq& 32 \lambda n^2 l \exp(-cl),
\end{aligned}$$

where we used proposition 2 in the last line. The result follows by taking l large enough. $\square$

As a first consequence of the exponential decay in finite boxes, we obtain:

Lemma 4. *For $p > \widehat{p}_g$ we have,*

$$\lim_{n \to \infty} \Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(n)] = \theta(p,q).$$

Proof. Let $N < n$, then

$$\begin{aligned}
\Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(N)] - \Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(N),\ 0 \not\leftrightarrow \partial B(n)] =& \Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(n)] \\
\leq& \Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(N)].
\end{aligned} \tag{5}$$

Now we will estimate $\Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(N),\ 0 \not\leftrightarrow \partial B(n)]$: by symmetry,

$$\Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial B(N),\ 0 \not\leftrightarrow \partial B(n)] \leq 4 \Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial_1 B(N),\ 0 \not\leftrightarrow \partial B(n)].$$

Then for N large enough we have that

$$\begin{aligned}
\Phi_{B(n)}^{f,p,q}[0 \leftrightarrow \partial_1 B(N), 0 \not\leftrightarrow \partial B(n)] \leq& \Phi_{B(n)}^{f,p,q}\begin{bmatrix} \exists k > 0\ \exists j \geq N : \exists \text{ an open dual path} \\ \text{joining } (-k + \frac{1}{2}, \frac{1}{2}) \text{ to } (j + \frac{1}{2}, \frac{1}{2}) \end{bmatrix} \\
\leq& \sum_{k > 0,\ j \geq N} \exp(-c(k+j)) \\
\leq& \exp(-cN),
\end{aligned}$$

for a certain positive constant c.
By taking the limit $n \to \infty$ in (5) we get

$$\begin{aligned}\Phi^{p,q}_{\infty}[0 \leftrightarrow \partial B(N)] - 4e^{-dN} &\leq \liminf_{n\to\infty} \Phi^{f,p,q}_{B(n)}[0 \leftrightarrow \partial B(n)] \\ &\leq \limsup_{n\to\infty} \Phi^{f,p,q}_{B(n)}[0 \leftrightarrow \partial B(n)] \leq \Phi^{p,q}_{\infty}[0 \leftrightarrow \partial B(N)],\end{aligned}$$

finally by taking the limit $N \to \infty$, we get the desired result. $\square$

Next, we define events that will be crucial in the renormalization procedure. For this, we introduce the notion of *crossing*. Let $B \subset \mathbb{Z}^2$ be a finite box. For $i = 1, 2$ we say that a *i-crossing* occurs in B, if $\partial_{-i}B$ and $\partial_i B$ are joined by an open path in B. In addition to that, we say that a cluster C of B is crossing in B, if C contains a 1-crossing path and a 2-crossing path.
For $\underline{n} \in \mathcal{H}_2(n)$, we set

$$U(\underline{n}) = \{\exists! \text{ open cluster } C^* \text{ crossing } B(\underline{n})\}.$$

For a monotone, increasing function $g : \mathbb{N} \to [0, \infty)$ with $g(n) \leq n$, let us define

$$R^g(\underline{n}) = U(\underline{n}) \cap \left\{ \begin{array}{l} \text{every open path } \gamma \subset B(\underline{n}) \text{ with} \\ \operatorname{diam}(\gamma) \geq g(n) \text{ is contained in } C^* \end{array} \right\}.$$

And finally we set

$$O^g(\underline{n}) = R^g(\underline{n}) \cap \left\{ \begin{array}{l} C^* \text{ crosses every sub-box} \\ \mathcal{Q} \in \mathcal{B}_2(g(n)) \text{ contained in } B(\underline{n}) \end{array} \right\}.$$

The next theorem gives the desired estimates on the above mentioned events.

Theorem 5. *Assume $p > \widehat{p}_g$. We have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{\underline{n}\in\mathcal{H}_2(n)} \sup_{\Phi\in\mathcal{R}(p,q,B(\underline{n}))} \Phi[U(\underline{n})^c] < 0. \tag{6}$$

Also, there exists a constant $\kappa = \kappa(p, q) > 0$ such that $\liminf_{n\to\infty} g(n)/\log n > \kappa$ implies

$$\limsup_{n\to\infty} \frac{1}{g(n)} \log \sup_{n\in\mathcal{H}_2(n)} \sup_{\Phi\in\mathcal{R}(p,q,B(\underline{n}))} \Phi[R^g(\underline{n})^c] < 0. \tag{7}$$

There exists a constant $\kappa' = \kappa'(p, q) > 0$ such that $\liminf_{n\to\infty} g(n)/\log n > \kappa'$ implies

$$\limsup_{n\to\infty} \frac{1}{g(n)} \log \sup_{n\in\mathcal{H}_2(n)} \sup_{\Phi\in\mathcal{R}(p,q,B(\underline{n}))} \Phi[O^g(\underline{n})^c] < 0. \tag{8}$$

Remark. *Note that in dimension two, if there is a crossing cluster then it is unique.*

Proof. As $U(\underline{n})^c$ is decreasing we have for every $\Phi \in \mathcal{R}(p,q,B(\underline{n}))$ that

$$\begin{aligned}\Phi[U(\underline{n})^c] \leq & \Phi^{f,p,q}_{B(\underline{n})}[U(\underline{n})^c] \\ \leq & \Phi^{f,p,q}_{B(\underline{n})}[\nexists \text{ 1-crossing for } B(\underline{n})] + \Phi^{f,p,q}_{B(\underline{n})}[\nexists \text{ 2-crossing for } B(\underline{n})] \\ \leq & \sum_{i=1,2} \Phi^{f,p,q}_{B(\underline{n})}[\partial_{-i}\widehat{B}(\underline{n}) \leftrightarrow \partial_i \widehat{B}(\underline{n}) \text{ in } \widehat{B}(\underline{n}) \setminus \partial\widehat{B}(\underline{n})],\end{aligned}$$

the last inequality follows from planar duality: if there is no 1-crossing in the original lattice then $\partial_{-2}\widehat{B}(\underline{n}) \leftrightarrow \partial_2\widehat{B}(\underline{n})$ in $\widehat{B}(\underline{n}) \setminus \partial\widehat{B}(\underline{n})$ for the corresponding dual configuration. The same argument works for the 2-crossing. Thus, we have that

$$\Phi[U(\underline{n})^c] \leq 2\Phi^{w,\widehat{p},q}_{\widehat{B}(\underline{n})}[\exists \text{ an open path in } \widehat{B}(\underline{n}) \text{ of diameter } \geq n],$$

and (6) follows from lemma 3.
For the second inequality, let us note that

$$R^g(\underline{n})^c \subset U(\underline{n})^c \bigcup \left(U(\underline{n}) \cap \left\{ \begin{array}{l} \exists \text{ an open path } \gamma \text{ of } B(\underline{n}) \text{ with} \\ \operatorname{diam}(\gamma) \geq g(n) \text{ not contained in } C^* \end{array} \right\} \right).$$

By (6), we have only to consider the second term.
In

$$U(\underline{n}) \cap \left\{ \begin{array}{l} \exists \text{ an open path } \gamma \text{ of } B(\underline{n}) \text{ with} \\ \operatorname{diam}(\gamma) \geq g(n) \text{ not contained in } C^* \end{array} \right\},$$

by proposition 11.2 of [17] and by considering all the edges of $\mathbb{E}(\partial\widehat{B}(\underline{n}))$ open, there is a unique innermost open dual circuit containing γ in its interior. From this dual circuit, we extract an open dual path living in the graph $(\widehat{B}(\underline{n}), \mathbb{E}(\widehat{B}(\underline{n}))\setminus\mathbb{E}(\partial\widehat{B}(\underline{n})))$ of diameter greater than $g(n)$: Without lost of generality, we can suppose that $\operatorname{diam}(\gamma) = \operatorname{diam}_1(\gamma)$ and that $\gamma \not\leftrightarrow \partial_2 B(\underline{n})$. Among the vertices of the dual circuit surrounding γ, let $\widehat{x}$ be the highest vertex among the most on the left, and let $\widehat{y}$ be the highest vertex among the most on the right. Then there is an arc joining $\widehat{x}$ and $\widehat{y}$ in $(\widehat{B}(\underline{n}), \mathbb{E}(\widehat{B}(\underline{n}))\setminus\mathbb{E}(\partial\widehat{B}(\underline{n})))$. This arc is of diameter larger than $g(n)$. Thus by lemma 3 there is a positive constant c such that for n large enough we have that

$$\Phi\left[U(\underline{n}) \cap \left\{ \begin{array}{l} \exists \text{ an open path } \gamma \text{ of } B(\underline{n}) \text{ with} \\ \operatorname{diam}(\gamma) \geq g(n) \text{ not contained in } C^* \end{array} \right\} \right] \leq n^2 \exp[-cg(n)].$$

Now, take an $\alpha > 0$ such that $\alpha c > 1$, then for g such that $g(n) > 2\alpha \log n/(\alpha c - 1)$ we have that

$$\limsup_{n\to\infty} \frac{1}{g(n)} \log(n^2 \exp[-cg(n)]) < -\frac{1}{\alpha},$$

which concludes the proof of (7).

To study $O^g(\underline{n})$, we remark that the number of boxes $\mathcal{Q}$ of $\mathcal{B}_2(g(n))$ contained in $B(\underline{n})$ is bounded by $16n^4$. This implies that for every $\Phi \in \mathcal{R}(p, q, B(\underline{n}))$ one gets

$$\begin{aligned}\Phi[O^g(\underline{n})^c] \leq& \Phi[R^g(\underline{n})^c] + 16n^4 \sup_{\mathcal{Q} \in \mathcal{B}_2(g(n))} \Phi[\nexists \text{ crossing in } \mathcal{Q}] \\ \leq& \Phi[R^g(\underline{n})^c] + 16n^4 \sup_{\mathcal{Q} \in \mathcal{B}_2(g(n))} \Phi^{f,p,q}_{B(\underline{n})}[\nexists \text{ crossing in } \mathcal{Q}] \\ \leq& \Phi[R^g(\underline{n})^c] + 16n^4 \sup_{\mathcal{Q} \in \mathcal{B}_2(g(n))} \Phi^{f,p,q}_{\mathcal{Q}}[\nexists \text{ crossing in } \mathcal{Q}].\end{aligned}$$

To deduce the last inequality, we notice that $\{\nexists \text{ crossing in } \mathcal{Q}\}$ is a decreasing event and that all the $\mathcal{Q} \in \mathcal{B}_2(g(n))$ are smaller than $B(n)$, thus for all $\mathcal{Q} \in \mathcal{B}_2(g(n))$ that are included in $B(n)$ we have that

$$\Phi^{f,p,q}_{B(\underline{n})}[\nexists \text{ crossing in } \mathcal{Q}] \leq \Phi^{f,p,q}_{\mathcal{Q}}[\nexists \text{ crossing in } \mathcal{Q}].$$

The first term in the r.h.s. has been treated previously. By (6) the second term is bounded by $n^4 \exp[-cg(n)]$ for a certain positive constant c and we conclude the proof as before. $\square$

4. Renormalization. In this section we adapt the renormalization procedure introduced in [32] to the two dimensional case. For this, let $N \geq 24$ be an integer. We say that a subset Λ of $\mathbb{Z}^2$ is a *N-large box* if Λ is a finite box containing a symmetric box of scale-length $3N$, i.e., if $\Lambda = \mathbb{Z}^2 \cap \prod_{i=1,2}(a_i, b_i]$ where $b_i - a_i \geq 3N$ for $i = 1, 2$. When Λ is a N-large box, one can partition it with blocks of $\mathcal{B}(N)$. We first define the *N-rescaled box* of Λ: $\Lambda^{(N)} = \{\mathbf{k} \in \mathbb{Z}^2 \mid T_{N\mathbf{k}}(-N/2, N/2]^2 \subseteq \Lambda\}$; where T_a is the translation in $\mathbb{Z}^2$ by a vector $a \in \mathbb{Z}^2$. We turn $\Lambda^{(N)}$ into a graph by endowing it with the set of edges $\mathbb{E}(\Lambda^{(N)})$. Then we define the partitioning blocks:

- If $\mathbf{k} \in \Lambda^{(N)} \setminus \partial\Lambda^{(N)}$ then $B_{\mathbf{k}} = T_{N\mathbf{k}}(-N/2, N/2]^2$.
- If $\mathbf{k} \in \partial\Lambda^{(N)}$ then some care is needed in order to get a partition. In this case we define the set

  $$\mathcal{M}(\mathbf{k}) = \{\mathbf{l} \in \mathbb{Z}^2 \mid \mathbf{l} \sim \mathbf{k}, T_{N\mathbf{l}}(-N/2, N/2]^2 \cap \Lambda \neq \emptyset,\ T_{N\mathbf{l}}(-N/2, N/2]^2 \cap \Lambda^c \neq \emptyset\},$$

  and the corresponding blocks become

  $$B_{\mathbf{k}} = T_{N\mathbf{k}}(-N/2, N/2]^2 \cup \bigcup_{\mathbf{l} \in \mathcal{M}(\mathbf{k})} \left(T_{N\mathbf{l}}(-N/2, N/2]^2 \cap \Lambda\right).$$

The collection of sets $\{B_{\mathbf{k}},\ \mathbf{k} \in \Lambda^{(N)}\}$ is a partition of Λ into blocks included in $\mathcal{B}(N)$, see figure 2.

In addition to the boxes $\{B_{\mathbf{k}},\ \mathbf{k} \in \Lambda^{(N)}\}$ we associate to each edge $(\mathbf{k}, \mathbf{l})$ of $\mathbb{E}(\Lambda^{(N)})$ the box $D_{(\mathbf{k},\mathbf{l})}$. More precisely, for $(\mathbf{k}, \mathbf{l}) \in \mathbb{E}(\Lambda^{(N)})$ such that $\sum_{j=1,2} |\mathbf{k}_j - \mathbf{l}_j| = \mathbf{k}_i - \mathbf{l}_i = 1$, we define $m(\mathbf{l}, \mathbf{k}) = T_{N\mathbf{l}}(\lfloor N/2 \rfloor e^{(i)})$, where $(e^{(1)}, e^{(2)})$ is the canonical orthonormal base of $\mathbb{Z}^2$ and $\lfloor r \rfloor$ denotes the integer part of r. The point $m(\mathbf{l}, \mathbf{k})$ represents the middle of the i-th face of $B_{\mathbf{l}}$. We then define the box $D_{(\mathbf{l},\mathbf{k})} = D_{(\mathbf{k},\mathbf{l})} = T_{m(\mathbf{l},\mathbf{k})}(B(\lfloor N/4 \rfloor))$.

Now we have all the needed geometric objects to construct our renormalized (dependent) site percolation process on $(\Lambda^{(N)}, \mathbb{E}(\Lambda^{(N)}))$. This process will depend

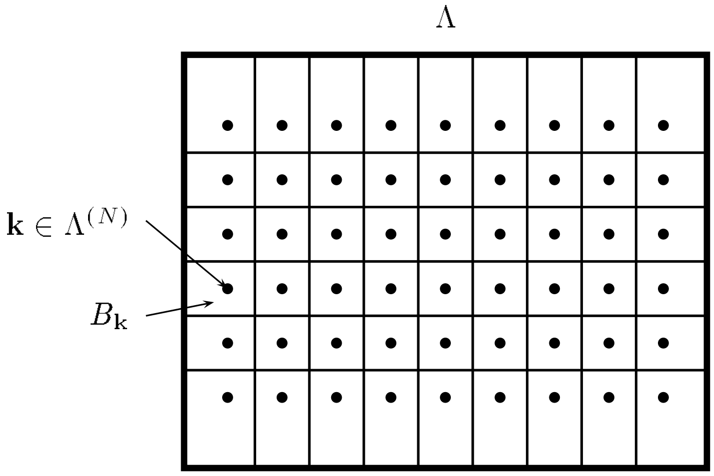

figure 2: The partition of Λ

on the original FK-percolation process only through a number of events defined in the boxes $(B_{\mathbf{k}})_{\mathbf{k}\in\Lambda^{(N)}}$ and $(D_e)_{e\in\mathbb{E}(\Lambda^{(N)})}$. These events are:

- For all $(\mathbf{k},\mathbf{l}) \in \mathbb{E}(\Lambda^{(N)})$ such that $\sum_{j=1,2} |\mathbf{k}_j - \mathbf{l}_j| = \mathbf{k}_i - \mathbf{l}_i = 1$, we define

$$K_{\mathbf{k},\mathbf{l}} = \{\exists\ i\text{-crossing in } D_{\mathbf{k},\mathbf{l}}\}, \qquad K_{\mathbf{k}} = \bigcap_{\mathbf{j}\in\Lambda^{(N)}:\mathbf{j}\sim\mathbf{k}} K_{\mathbf{k},\mathbf{j}}.$$

- For all $\mathbf{i} \in \Lambda^{(N)}$, we define

$$R_{\mathbf{i}} = \{\exists!\text{ a crossing cluster } C_{\mathbf{i}}^* \text{ in } B_{\mathbf{i}}\} \cap \left\{ \begin{array}{l} \text{every open path } \gamma \subset B_{\mathbf{i}} \text{ with} \\ \operatorname{diam}(\gamma) \geq \dfrac{\sqrt{N}}{10} \text{ is included in } C_{\mathbf{i}}^* \end{array} \right\}.$$

Finally our renormalized process is the indicator of the occurrence of the above mentioned events:

$$\forall \mathbf{k} \in \Lambda^{(N)} \quad X_{\mathbf{k}} = \begin{cases} 1 & \text{on} \qquad R_{\mathbf{k}} \cap K_{\mathbf{k}} \\ 0 & \text{otherwise} \end{cases}$$

We also call the process $\{X_{\mathbf{k}}, \mathbf{k} \in \Lambda^{(N)}\}$ the N-block process and whenever $X_{\mathbf{k}} = 1$, we say that the block $B_{\mathbf{k}}$ is occupied. As explained in [32], the N-block process has the following important geometrical property: if $C^{(N)}$ is a cluster of occupied blocks then there is a unique cluster C of the underlying microscopic FK-percolation process that crosses all the blocks $\{B_{\mathbf{k}},\ \mathbf{k} \in C^{(N)}\}$. Moreover, the events involved in the definition of the N-block process become more probable as the size of the blocks increases. This leads us to the following stochastic domination result:

Proposition 6. *Let $q \geq 1$ and $p > \widehat{p}_g$. Then for N large enough, every N-large box Λ and every measure $\Phi^\pi \in \mathcal{R}(p,q,\Lambda)$, the law of the N-block process $(X_{\mathbf{i}})_{\mathbf{i}\in\Lambda^{(N)}}$ under Φ^π, stochastically dominates independent site percolation on $\Lambda^{(N)}$ with parameter $p(N) = 1 - \exp(-C\sqrt{N})$, where C is a positive constant.*

Proof. According to [27], it is sufficient to establish that for N large enough and for all $\mathbf{i} \in \Lambda^{(N)}$ the following inequality holds:

$$\Phi^\pi[X_{\mathbf{i}} = 0 \mid \sigma(X_{\mathbf{j}} : |\mathbf{j} - \mathbf{i}| > 1)] \leq \exp(-C\sqrt{N}). \tag{9}$$

In what follows, we use the same notation for positive constants that may differ from one line to another. In order to prove (9), we will consider the set

$$E_{\mathbf{i}} = \bigcup_{|\mathbf{j}-\mathbf{i}|\leq 1} B_{\mathbf{j}} \setminus \bigcup_{\mathbf{j}\sim\mathbf{i}} \bigcup_{\mathbf{k}\sim\mathbf{j}, k\neq\mathbf{i}} D_{\mathbf{j},\mathbf{k}},$$

as drawn in figure 3.

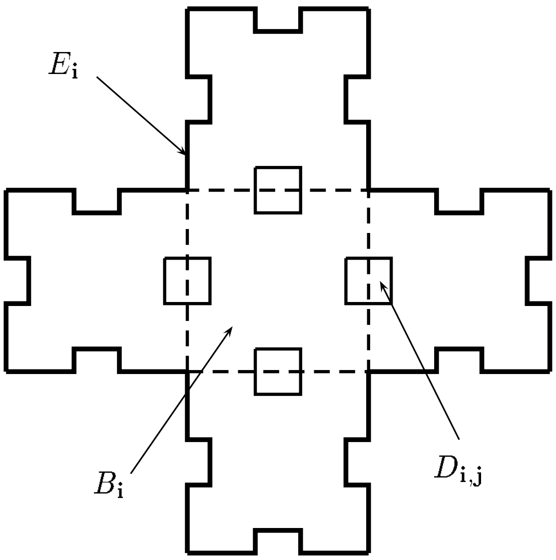

figure 3: The region $E_{\mathbf{i}}$

The σ-algebra $\mathcal{F}_\Lambda^{E_{\mathbf{i}}}$ is finer than $\sigma(X_{\mathbf{j}} : |\mathbf{j} - \mathbf{i}| > 1)$, thus it suffices to prove (9) for $\Phi^\pi[X_{\mathbf{i}} = 0 \mid \mathcal{F}_\Lambda^{E_{\mathbf{i}}}]$. Clearly $\mathcal{F}_\Lambda^{E_{\mathbf{i}}}$ is atomic and its atoms are of the form $\{\eta\}$, where $\eta \in \Omega_\Lambda^{E_{\mathbf{i}}}$. So let us consider such a $\eta \in \Omega_\Lambda^{E_{\mathbf{i}}}$, then we have that

$$\Phi^\pi[X_{\mathbf{i}} = 0 \mid \eta] \leq \sum_{\mathbf{j}\sim\mathbf{i}} \Phi^\pi[K_{\mathbf{i},\mathbf{j}}^c \mid \eta] + \Phi^\pi[R_{\mathbf{i}}^c \mid \eta]. \tag{10}$$

For each $\mathbf{i}, \mathbf{j} \in \Lambda^{(N)}$ such that $\mathbf{i} \sim \mathbf{j}$, let us fix $\eta' \in \Omega_{E_{\mathbf{i}}}^{B_{\mathbf{i}}}$, $\eta'' \in \Omega_{E_{\mathbf{i}}}^{D_{\mathbf{i},\mathbf{j}}}$ in order to construct $\eta\eta' \in \Omega_\Lambda^{B_{\mathbf{i}}}$ and $\eta\eta'' \in \Omega_\Lambda^{D_{\mathbf{i},\mathbf{j}}}$, which are the concatenation of η with η', respectively with η'':

$$\eta\eta'(e) = \eta'(e) \text{ for } e \in \mathbb{E}(E_{\mathbf{i}}) \setminus \mathbb{E}(B_{\mathbf{i}}) \qquad \eta\eta'(e) = \eta(e) \text{ for } e \in \mathbb{E}(\Lambda) \setminus \mathbb{E}(E_{\mathbf{i}});$$

and

$$\eta\eta''(e) = \eta''(e) \text{ for } e \in \mathbb{E}(E_{\mathbf{i}}) \setminus \mathbb{E}(D_{\mathbf{i},\mathbf{j}}) \qquad \eta\eta''(e) = \eta(e) \text{ for } e \in \mathbb{E}(\Lambda) \setminus \mathbb{E}(E_{\mathbf{i}}).$$

Then, by theorem 5, there exist an integer $N_0 > 0$ and a real number $C > 0$ such that for all $N > N_0$

$$\Phi^\pi[R_{\mathbf{i}}^c \mid \eta\eta'] = \Phi^{\pi \cdot W_\Lambda^{B_{\mathbf{i}}}(\eta\eta')}[R_{\mathbf{i}}^c] \le \exp(-C\sqrt{N}),$$
$$\Phi^\pi[K_{\mathbf{i},\mathbf{j}}^c \mid \eta\eta''] = \Phi^{\pi \cdot W_\Lambda^{D_{\mathbf{i},\mathbf{j}}}(\eta\eta'')}[K_{\mathbf{i},\mathbf{j}}^c] \le \exp(-CN).$$

Finally, by averaging over all the η' and η'' we get from these estimates that

$$\begin{aligned}\Phi^\pi[X_{\mathbf{i}} = 0 \mid \eta] &\le 4\exp(-CN) + \exp(-C\sqrt{N})\\ &\le \exp(-CN^{1/2}),\end{aligned}$$

for N large enough. $\square$

We end this section by proving a useful estimates on the renormalized process. Let $B(n)$ be a N-large box, consider its N-partition and the corresponding N-block process. The rescaled box $B(n)^{(N)}$ will be denoted by $\mathbf{B}$. For $\delta > 0$ we consider the event

$$Z(n,\delta,N) = \left\{ \begin{array}{l} \exists! \text{ crossing cluster of blocks } \widetilde{\mathbf{C}} \\ \text{in } \mathbf{B} \text{ with } |\widetilde{\mathbf{C}}| \ge (1-\delta)|\mathbf{B}| \end{array} \right\}. \tag{11}$$

Remark 7. *The event $Z(n,\delta,N)$ has the following interesting property: the presence of the crossing cluster of blocks $\widetilde{\mathbf{C}}$ induces a set of clusters $\{\widetilde{C}_{\mathbf{i}}$ crossing for $B_{\mathbf{i}}$: $\mathbf{i} \in \widetilde{\mathbf{C}}\}$ in the original FK-percolation process. These clusters are connected and form a crossing cluster $\widetilde{C}$ for $B(n)$.*

Proposition 8. *Let $p > \widehat{p}_g$ and $q \ge 1$. Then for each $\delta > 0$ and $N > 0$ large enough*

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{\Phi \in \mathcal{R}(p,q,B(n))} \Phi\left[Z(n,\delta,N)^c\right] < 0.$$

Proof. By theorem 1.1 of [14], there exists $p_0 \in (0,1)$ such that for all $p > p_0$,

$$\limsup_{m\to\infty} \frac{1}{m} \log \sup_{\underline{m} \in \mathcal{H}_2(m)} P^{p,\,\mathrm{indpt}}_{B(\underline{m}),\mathrm{site}} \left[\begin{array}{l} \nexists \text{ crossing cluster } \widetilde{C} \text{ with} \\ |\widetilde{C}| \ge (1-\delta)|B(\underline{m})| \end{array} \right] < 0. \tag{12}$$

Now choose N such as in proposition 6 and such that $p(N) > p_0$. Then by proposition 6 and by (12) we have that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{\Phi \in \mathcal{R}(p,q,B(n))} \Phi \left[\begin{array}{l} \nexists \text{ crossing cluster of blocks } \widetilde{\mathbf{C}} \\ \text{in } \mathbf{B} \text{ with } |\widetilde{\mathbf{C}}| \ge (1-\delta)|\mathbf{B}| \end{array} \right]$$

$$\le \limsup_{n\to\infty} \frac{1}{n} \log P^{p,\,\mathrm{indpt}}_{\mathbf{B},\mathrm{site}} \left[\begin{array}{l} \nexists \text{ crossing cluster } \widetilde{\mathbf{C}} \text{ with} \\ |\widetilde{\mathbf{C}}| \ge (1-\delta)|\mathbf{B}| \end{array} \right] < 0. \qquad \square$$

5. Proof of the surface order large deviations. In this section we finally establish theorem 1. We begin by stating two lemmas. The first one deals with large deviations from above. Let $\mathbb{B}(n)$ denote the set of clusters in $B(n)$ intersecting $\partial B(n)$. Note that if the crossing cluster exists then it is in $\mathbb{B}(n)$.

Lemma 9. *Let $q \geq 1$ and $p \in [0,1]$. For $\delta > 0$, we have*

$$\limsup_{n\to\infty} \frac{1}{n^2} \log \sup_{\Phi \in \mathcal{R}(p,q,B(n))} \Phi\left[\sum_{C\in\mathbb{B}(n)} |C| > (\theta+\delta)n^2\right] < 0.$$

We omit the proof as it would be an exact repetition of Lemma 5.1 in [32].

The second lemma is about large deviations from below and is of surface order, in contrast to lemma 9. In section 3, we introduced the event $U(n) = \{\exists!\text{ open cluster } C^* \text{ crossing } B(n)\}$. For $\delta > 0$, let us define the event

$$V(n,\delta) = U(n) \cap \{|C^*| > (\theta-\delta)n^2\}.$$

Lemma 10. *Let $q \geq 1$ and $p > \widehat{p}_g$. Then for each $\delta > 0$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{\Phi\in\mathcal{R}(p,q,B(n))} \Phi[V(n,\delta)^c] < 0. \tag{13}$$

Proof. For $N > 0$, if we set $\mathcal{Q}(N) = \{x \in B(N), \mathrm{dist}(x, \partial B(N)) \geq \sqrt{N}\}$ then we have

$$\begin{aligned}
&\liminf_{n\to\infty} \Phi^f_{B(N)}\left[N^{-2} \sum_{C;\mathrm{diam}(C)\geq\sqrt{N}} |C|\right] \\
&\geq \liminf_{N\to\infty} N^{-2} \sum_{x\in\mathcal{Q}(N)} \Phi^f_{B(N)}[\mathrm{diam}(C_x) \geq \sqrt{N}] \\
&\geq \liminf_{N\to\infty} N^{-2} \sum_{x\in\mathcal{Q}(N)} \Phi^f_{B(x,\sqrt{N})}[x \leftrightarrow \partial B(x,\sqrt{N})] \\
&\geq \liminf_{N\to\infty} N^{-2} |\mathcal{Q}(N)| \Phi^f_{B(\sqrt{N})}[0 \leftrightarrow \partial B(\sqrt{N})] = \theta,
\end{aligned}$$

where the last equality follows from lemma 4.
Take N such that $\Phi^f_{B(N)}[\sum_{C;\mathrm{diam}(C)\geq\sqrt{N}} |C|] \geq (\theta - \delta/4)N^2$, let $B(n)$ be a N-large box and consider its N-partition and the corresponding N-block process. The rescaled box $B(n)^{(N)}$ will be denoted by $\mathbf{B}$. By proposition 8, it suffices to give an upper bound on the probability of the event

$$W(n) = Z(n,\delta/8,N) \cap \{|\widetilde{C}| \leq (\theta-\delta)n^2\},$$

where N is large enough and $Z(n,\delta/8,N)$ is defined in (11). By remark 7, on the event $Z(n,\delta/8,N)$ the crossing cluster $\widetilde{C}$ contains all the $B_{\mathbf{i}}$-crossing clusters $\widetilde{C}_{\mathbf{i}}$, where $\mathbf{i} \in \widetilde{\mathbf{C}}$ and $\{B_{\mathbf{i}},\ \mathbf{i} \in \mathbf{B}\}$ are the partitioning N-blocks. For each $\mathbf{i} \in \mathbf{B}$, set $Y_{\mathbf{i}} = \sum_{C;\mathrm{diam}\ C\geq N^{1/2}} |C|$, where C is a cluster of $B_{\mathbf{i}}$. Since for $\mathbf{i} \in \widetilde{\mathbf{C}}$, $Y_{\mathbf{i}} = |\widetilde{C}_{\mathbf{i}}|$, we obtain the following lower bound

$$|\widetilde{C}| \geq \sum_{\mathbf{i}\in\widetilde{\mathbf{C}}} Y_i \geq \sum_{\mathbf{i}\in\mathbf{B}} Y_{\mathbf{i}} - \sum_{\mathbf{i}\in\mathbf{B}\setminus\widetilde{\mathbf{C}}} |B_{\mathbf{i}}| \geq \sum_{\mathbf{i}\in\mathring{\mathbf{B}}} Y_{\mathbf{i}} - (\delta/2)n^2,$$

where $\overset{\circ}{\mathbf{B}} = \mathbf{B} \setminus \partial\mathbf{B}$. Hence on $W(n)$ we have that $\sum_{\mathbf{i}\in\overset{\circ}{\mathbf{B}}} Y_{\mathbf{i}} \le (\theta - \delta/2)n^2$.

Denote by $E(n)$ the event that for each $\mathbf{i} \in \overset{\circ}{\mathbf{B}}$ every edge in $\partial^{\text{edge}} B_{\mathbf{i}}$ is closed. Observing that $\sum_{\mathbf{i}\in\overset{\circ}{\mathbf{B}}} Y_{\mathbf{i}}$ is an increasing function, we have for each $\Phi \in \mathcal{R}(p,q,B(n))$,

$$\Phi[W(n)] \le \Phi^f_{B(n)}\left[\sum_{\mathbf{i}\in\overset{\circ}{\mathbf{B}}} Y_{\mathbf{i}} < (\theta - \delta/2)n^2 \,\middle|\, E(n)\right].$$

The variables $(Y_{\mathbf{i}}, \mathbf{i} \in \overset{\circ}{\mathbf{B}})$ are i.i.d. with respect to the conditional measure, with an expected value larger than $(\theta - \delta/4)N^2$.
Cramér's large deviations theorem yields to

$$\Phi^f_{B(n)}\left[\frac{1}{n^2}\sum_{\mathbf{i}\in\overset{\circ}{\mathbf{B}}} Y_{\mathbf{i}} < \theta - \delta/2 \,\middle|\, E(n)\right] \le \exp(-C(\delta,\theta,N)n^2),$$

where $C(\delta,\theta,N)$ is a positive constant. This completes the proof. $\square$

Proof of Theorem 1. First we prove the upper bound. By lemma 9, we can replace the condition $n^{-2}|C_m| \in (\theta-\varepsilon, \theta+\varepsilon)$ in the definition of $K(n,\varepsilon,l)$ by $n^{-2}|C_m| > (\theta - \varepsilon)$ and denote the new but otherwise unchanged event by $K'(n,\varepsilon,l)$. Set

$$T(n,\varepsilon,N) = Z(n,\varepsilon/4,N) \cap \{|\widetilde{C}| > (\theta-\varepsilon)n^2\},$$

where $Z(n,\varepsilon/4,N)$ is defined by (11). Fix $\varepsilon < \theta/2$ and N such as in proposition 8 and such that $\sqrt{N} \ge 32/\varepsilon$.
Then by proposition 8 and by lemma 10, we have

$$\limsup_{n\to\infty} \sup_{\Phi\in\mathcal{R}(p,q,B(n))} \frac{1}{n} \log \Phi[T(n,\varepsilon,N)^c] < 0. \tag{14}$$

Set $n \ge 64N/\varepsilon$ and $L = 2N$, we claim that $T(n,\varepsilon,N) \subset K'(n,\varepsilon,L)$. This fact, together with (14), implies the upper bound. Therefore, to complete the upper bound we will proof that the cluster $\widetilde{C}$ of $T(n,\varepsilon,N)$, is the unique cluster with maximal volume and that the L-intermediate clusters have a negligible volume. So suppose that $T(n,\varepsilon,N)$ occurs. As $\varepsilon < \theta/2$ we have that $L^2 \le (\theta-\varepsilon)n^2$, thus the clusters of diameter less than L, have a smaller volume than $\widetilde{C}$. To control the size of the clusters different from $\widetilde{C}$ and of diameter greater than L, we define the following regions:

$$\forall\, \mathbf{i} \in \mathbf{B}: \quad G_{\mathbf{i}} = \{x \in B_{\mathbf{i}} \mid \text{dist}(x, \partial B_{\mathbf{i}}) \le \sqrt{N}\} \quad \text{and} \quad \mathcal{Q}_{\mathbf{i}} = B_{\mathbf{i}} \setminus G_{\mathbf{i}},$$
$$G = \bigcup_{\mathbf{i}\in\mathbf{B}} G_{\mathbf{i}},$$

as shown in figure 4 below:

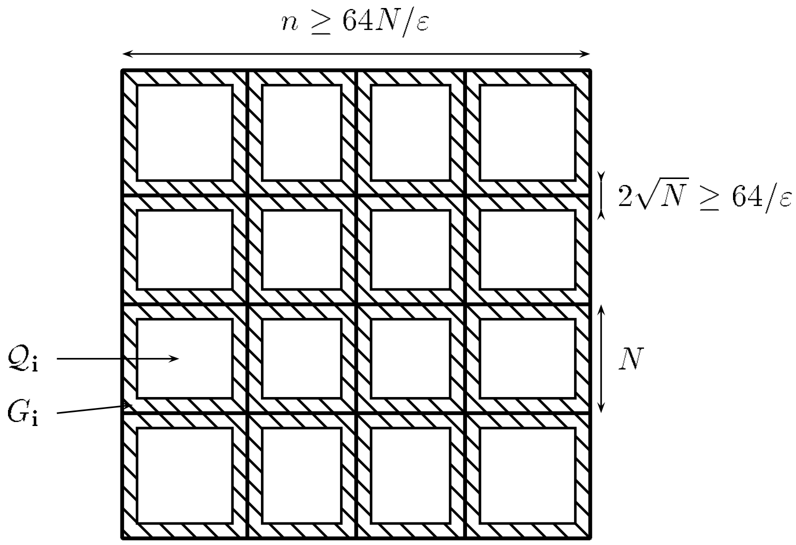

figure 4: The regions $G_{\mathbf{i}}$ and $\mathcal{Q}_{\mathbf{i}}$

Then, as $n \geq 64N/\varepsilon$, we have

$$\sum_{\mathbf{i} \in \partial \mathbf{B}} |B_{\mathbf{i}}| \leq 16nN \leq \frac{\varepsilon}{4} n^2,$$

and, as $\sqrt{N} \geq 32/\varepsilon$

$$|G| \leq 8 \frac{n^2}{\sqrt{N}} \leq \frac{\varepsilon}{4} n^2.$$

Take a cluster C of diameter greater than L and different from $\widetilde{C}$. Then C touches at least two blocks. However, it may not touch the set $\cup \mathcal{Q}_{\mathbf{i}}$ where $\mathbf{i}$ runs over $\widetilde{\mathbf{C}}$; otherwise we would have that $\operatorname{diam}(C \cap B_{\mathbf{i}}) \geq \sqrt{N}$ for an occupied block $B_{\mathbf{i}}$, and therefore we would have that $C = \widetilde{C}$. Hence all the clusters of diameter greater than L must lie in the set $G \cup (\cup_{\mathbf{i} \in \widetilde{\mathbf{C}}^c} B_{\mathbf{i}})$. Let us estimate the volume of this set:

$$|\bigcup_{\mathbf{i} \in \widetilde{\mathbf{C}}^c} B_{\mathbf{i}}| \leq \sum_{\mathbf{i} \in \partial \mathbf{B}} |B_{\mathbf{i}}| + N^2 |\widetilde{\mathbf{C}}^c| < \frac{\varepsilon}{2} n^2.$$

Thus

$$|G \cup (\bigcup_{\mathbf{i} \in \widetilde{\mathbf{C}}^c} B_{\mathbf{i}})| \leq \frac{3\varepsilon}{4} n^2.$$

Since $(3\varepsilon/4)n^2 < (\theta - \varepsilon)n^2$, $\widetilde{C}$ is the unique cluster of maximal volume and the L-intermediate class $\mathbb{J}_L$ has a total volume smaller than $(3\varepsilon/4)n^2$. This proves that $T(n, \varepsilon, L) \subset K'(n, \varepsilon, L)$ and completes the proof of the upper bound.

For the lower bound, it suffices to close all the horizontal edges in $B(n)$ intersecting the vertical line $x = 1/2$. This implies that there in no crossing cluster in $B(n)$. By (4) and FKG inequality, the probability of this event is bounded from below by $(1-p)^n$. $\square$

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