Law of large numbers and central limit theorem for randomly forced PDE's

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Abstract

We consider a class of dissipative PDE's perturbed by an external random force. Under the condition that the perturbation is sufficiently non-degenerate, a strong law of large numbers (SLLN) and a central limit theorem (CLT) for solutions are established and the corresponding rates of convergence are estimated. The proofs are based on the property of exponential mixing for the problem in question and some abstract SLLN and CLT for mixing-type Markov processes.

Contents

0 Introduction

This paper deals with a class of randomly forced PDE's arising in mathematical physics. To be precise, we confine ourselves in this introduction to the 2D Navier–Stokes system perturbed by an external force white in time and smooth in the space variables:

$$\dot{u} - \Delta u + (u, \nabla)u + \nabla p = \eta(t, x), \quad \operatorname{div} u = 0, \quad x \in D. \tag{0.1}$$

Here $D \subset \mathbb{R}^2$ is a bounded domain with C^1 boundary ∂D, $u = (u_1, u_2)$ is the velocity field of the fluid, p is the pressure, and η is a random force. Equation (0.1) is supplemented with Dirichlet boundary condition

$$u\big|_{\partial D} = 0. \tag{0.2}$$

Excluding the pressure, we can write the problem (0.1), (0.2) as an evolution equation in the space H of divergence-free vector fields $u \in L^2(D, \mathbb{R}^2)$ whose normal component vanishes at ∂D (see [22]):

$$\dot{u} + Lu + B(u, u) = \eta(t), \tag{0.3}$$

where L is the Stokes operator and B is a bilinear form resulting from the nonlinear term in (0.1). We assume that the right-hand side η, for which we retained the same notation as in the original equation, is a random process of the form

$$\eta(t) = \sum_{j=1}^{\infty} b_j \dot{\beta}_j(t) e_j, \tag{0.4}$$

where $b_j \geq 0$ are some constants such that $\sum_j b_j^2 < \infty$, $\{e_j\}$ is a complete set of normalised eigenfunctions of L, and $\{\beta_j\}$ is a sequence of independent standard Brownian motions. It was established in [15] (see also [6, 4, 2, 18]) that, if

$$b_j \neq 0 \quad \text{for} \quad j = 1, \ldots, N, \tag{0.5}$$

where the integer $N \geq 1$ is sufficiently large, then the family of Markov processes associated with Eq. (0.3) has a unique stationary measure μ and possesses a property of exponentially mixing. Namely, for a large class of functionals $f : H \to \mathbb{R}$ and any solution $u(t)$ of Eq. (0.3), we have

$$\big|\mathbb{E} f(u(t)) - (f, \mu)\big| \leq \text{const}\, e^{-\beta t}, \quad t \geq 0,$$

where $\mathbb{E}$ denotes the average in ensemble, (f, μ) is the mean value of f with respect to the measure μ, and $\beta > 0$ is a constant not depending on $u(t)$. Moreover, using the coupling of solutions constructed in [15] and some versions of the strong law of large numbers (SLLN) and of the central limit theorem (CLT) for stationary processes, it was shown in [11] that the following two assertions hold:

- with probability 1, we have

$$\frac{1}{t}\int_0^t f(u(s))\,ds \to (f,\mu) \quad \text{as} \quad t\to+\infty; \tag{0.6}$$

- if $(f,\mu)=0$, then there is $\sigma\ge 0$ such that

$$\frac{1}{\sqrt{t}}\int_0^t f(u(s))\,ds \to \mathcal{N}(0,\sigma) \quad \text{as} \quad t\to+\infty, \tag{0.7}$$

where $\mathcal{N}(0,\sigma)$ is the one-dimensional centred Gaussian distribution with variance σ, and the convergence holds in the sense of distribution.

We refer the reader to [15, 11] for a detailed discussion of the results on the NS system; see also [5, 7] for some related problems.

The aim of this article is to derive SLLN (0.6) and CLT (0.7) without using the coupling of solutions and to estimate the corresponding rates of convergence. To this end, we establish some abstract SLLN and CLT for mixing-type Markov processes and show that they apply to the problem under study.

The following theorem is a simplified version of the main results of this paper.

MAIN THEOREM. *Suppose that the non-degeneracy condition* (0.5) *is satisfied for a sufficiently large N. Then for any uniformly Lipschitz bounded functional $f: H\to\mathbb{R}$ and any solution $u(t)$ of Eq.* (0.3) *with deterministic initial condition the following statements hold.*

Law of large numbers: *For any $\varepsilon>0$ there is an a.s. finite random constant $T\ge 1$ such that*

$$\left|\frac{1}{t}\int_0^t f(u(s))\,ds-(f,\mu)\right|\le \text{const}\, t^{-\frac{1}{3}+\varepsilon} \quad \textit{for } t\ge T. \tag{0.8}$$

Central limit theorem: *If $(f,\mu)=0$, then there is a constant $\sigma\ge 0$ depending only on f such that*

$$\sup_{z\in\mathbb{R}}\left(\theta_\sigma(z)\left|\mathbb{P}\left\{\frac{1}{\sqrt{t}}\int_0^t f(u(s))\,ds\le z\right\}-\Phi_\sigma(z)\right|\right)\le \text{const}\, t^{-\frac{1}{5}} \quad \textit{for } t\ge 1, \tag{0.9}$$

where $\theta_\sigma\equiv 1$ for $\sigma>0$, $\theta_0(z)=1\wedge|z|$, and $\Phi_\sigma(z)$ is the centred Gaussian distribution function with variance σ.

We emphasize that (0.8) and (0.9) remain valid for a large class of Hölder continuous functionals on H with polynomial growth at infinity. Moreover, if we consider Eq. (0.1) on a 2D torus, similar results hold for functionals defined on a Sobolev space H^s with an arbitrarily large s, provided that the right-hand side is sufficiently smooth. We shall not give a precise formulation and a proof of this assertion, since they repeat almost literally the case of Dirichlet boundary condition.

Let us briefly describe the structure of the paper. In Section 1, the main results are presented. We consider the 2D Navier–Stokes system (0.1), (0.2), (0.4), as well as a class of dissipative PDE's perturbed by a random force of the form

$$\eta(t,x)=\sum_{k=1}^{\infty}\eta_k(x)\delta(t-k), \tag{0.10}$$

where $\delta(t)$ is the Dirac measure concentrated at zero and $\{\eta_k\}$ is a sequence of independent identically distributed (i.i.d.) random variables in an appropriate functional space. In Section 2, we establish an SLLN and a CLT for mixing-type Markov processes. Section 3 is devoted to the proof of the main results of this paper. In the Appendix, we have compiled some auxiliary assertions.

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Notation

Let H be a real Hilbert space with norm $|\cdot|$ and let $\alpha\in(0,1]$ be a constant. We shall use the following notation:

$B_H(R)$ is the closed ball in H of radius $R>0$ centred at zero;

$\mathcal{B}(H)$ is the Borel σ-algebra in H;

$\mathcal{P}(H)$ is the family of probability measures on $(H,\mathcal{B}(H))$;

$C(H)$ is the space of continuous functionals $f\colon H\to\mathbb{R}$;

$C_b(H)$ is the space of bounded functionals $f\in C(H)$ endowed with the norm $\|f\|_\infty:=\sup\limits_{u\in H}|f(u)|$.

$\mathcal{W}$ is the space of increasing continuous functions $w(r)>0$ defined for $r\geq 0$. The elements of $\mathcal{W}$ will be called *weight functions*. For the next two definitions, we fix an arbitrary $w\in\mathcal{W}$.

$C(H,w)$ is the space of continuous functionals $f\in C(H)$ such that

$$|f|_w:=\sup_{u\in H}\frac{|f(u)|}{w(|u|)}<\infty.$$

$C^\alpha(H,w)$ is the space of continuous functionals $f\in C(H)$ for which the following norm is finite:

$$|f|_{w,\alpha}:=|f|_w+\sup_{u\neq v}\frac{|f(u)-f(v)|}{|u-v|^\alpha\bigl(w(|u|)+w(|v|)\bigr)}.$$

If $f:H\to\mathbb{R}$ is a $\mathcal{B}(H)$-measurable functional and $\mu\in\mathcal{P}(H)$, then we denote by (f,μ) the integral of f over H with respect to μ.

C_i, $i=1,2,\dots$, stand for unessential positive constants.

1 Main results

1.1 Dissipative PDE's perturbed by random kicks

Let H be a real Hilbert space with norm $|\cdot|$ and orthonormal base $\{e_j\}$ and let $S: H \to H$ be a continuous operator such that $S(0) = 0$. We consider a discrete-time random dynamical system (RDS) in H:

$$u_k = S(u_{k-1}) + \eta_k, \tag{1.1}$$

where $k \geq 1$ and $\{\eta_k\}$ is a sequence of i.i.d. random variables in H. As was explained in [12, 13], a large class of dissipative PDE's perturbed by a random force of the form (0.10) reduces to the RDS (1.1), and in this case S is the time-one shift along trajectories of the unperturbed equation. We assume that the operator S satisfies the following three conditions introduced in [12, 13]:

(A) For any $R > r > 0$ there are positive constants $a = a(R,r) < 1$ and $C = C(R)$ and an integer $n_0 = n_0(R,r) \geq 1$ such that

$$|S(u_1) - S(u_2)| \leq C(R)|u_1 - u_2| \quad \text{for} \quad u_1, u_2 \in B_H(R),$$
$$|S^n(u)| \leq \max\{a|u|, r\} \quad \text{for} \quad u \in B_H(R), \quad n \geq n_0.$$

(B) For any compact set $\mathcal{K} \subset H$ and any bounded set $B \subset H$ there is a constant $R > 0$ such that the sets $\mathcal{A}_k(\mathcal{K}, B)$ defined recursively by the formulas $\mathcal{A}_0(\mathcal{K}, B) = B$ and $\mathcal{A}_k(\mathcal{K}, B) = S(\mathcal{A}_{k-1}(\mathcal{K}, B)) + \mathcal{K}$ are contained in the ball $B_H(R)$ for all $k \geq 0$.

(C) For any $R > 0$ there is an integer $N \geq 1$ such that

$$\left|\mathsf{Q}_N(S(u_1) - S(u_2))\right| \leq \tfrac{1}{2}|u_1 - u_2| \quad \text{for} \quad u_1, u_2 \in B_H(R),$$

where Q_N is the orthogonal projection onto the closed subspace spanned by $\{e_j, j \geq N+1\}$.

We note that the above conditions are satisfied for the resolving operators of the 2D Navier–Stokes system and the complex Ginzburg–Landau equation.

As for the random kicks η_k, we assume that they are i.i.d. random variables in H of the form

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j,$$

where $b_j \geq 0$ are some constants such that

$$B_0 := \sum_{j=1}^{\infty} b_j^2 < \infty, \tag{1.2}$$

and ξ_{jk} are independent scalar random variables satisfying the following condition:

(D) For any $j \geq 1$, the random variables ξ_{jk} have the same distribution $\pi_j(dr)$, which is absolutely continuous with respect to the Lebesgue measure. Moreover, the corresponding density $p_j(r)$ is a function of bounded total variation, is supported by the interval $[-1, 1]$, and satisfies the condition $\int_{|r|\leq\varepsilon} p_j(r)\,dr > 0$ for any $\varepsilon > 0$.

Let $(u_k, \mathbb{P}_u)$ be the family of Markov chains that is associated with the RDS (1.1) and is parametrized by the initial condition $u \in H$. We denote by $P_k(u, \Gamma)$ the corresponding transition function and by $\mathfrak{P}_k$ and $\mathfrak{P}_k^*$ the Markov operators generated by P_k:

$$\mathfrak{P}_k : C_b(H) \to C_b(H), \qquad \mathfrak{P}_k f(u) = \int_H P_k(u, dv) f(v),$$
$$\mathfrak{P}_k^* : \mathcal{P}(H) \to \mathcal{P}(H), \qquad \mathfrak{P}_k^* \mu(\Gamma) = \int_H P_k(u, \Gamma)\mu(du).$$

Recall that a measure $\mu \in \mathcal{P}(H)$ is said to be *stationary* for the family $(u_k, \mathbb{P}_u)$ if $\mathfrak{P}_1^*\mu = \mu$.

It was proved in [13, 14, 10] that if Hypotheses (A)–(D) are fulfilled together with the non-degeneracy condition (0.5), where $N \geq 1$ is sufficiently large, then the RDS (1.1) has a unique stationary measure μ, which is exponentially mixing in the following sense: for any $\alpha \in (0, 1]$ and $w \in \mathcal{W}$ there are positive constants C and β such that [1]

$$\big|\mathfrak{P}_k f(u) - (f, \mu)\big| \leq C|f|_{w,\alpha}\big(1 + |u|\big)\, e^{-\beta k}, \quad k \geq 0, \tag{1.3}$$

where $u \in H$ and $f \in C^\alpha(H, w)$ are arbitrary. (See Notation in the Introduction for the definition of the space $C^\alpha(H, w)$.)

The following theorem establishes an SLLN for the family $(u_k, \mathbb{P}_u)$ with an estimate of the rate of convergence.

Theorem 1.1. *Suppose that Hypotheses* (A) – (D) *and the non-degeneracy condition* (0.5) *are satisfied. Then for any* $\alpha \in (0, 1]$ *and* $w \in \mathcal{W}$ *there is a constant* $D > 0$ *such that, for any* $f \in C^\alpha(H, w)$ *and* $\varepsilon \in (0, \frac{1}{3})$, *the following statements hold:*

(i) *There is a random integer* $K(\omega) \geq 1$ *depending on* f *and* ε *such that*

$$\left|k^{-1}\sum_{l=0}^{k-1} f(u_l) - (f, \mu)\right| \leq D\,|f|_w\, k^{-\frac{1}{3}+\varepsilon} \quad \textit{for} \quad k \geq K(\omega).$$

(ii) *For any* $u \in H$, *the integer* K *is* $\mathbb{P}_u$*-a.s. finite. Moreover, if* $0 < r < 3\varepsilon$, *then*

$$\mathbb{E}_u K^r \leq \tfrac{D}{3\varepsilon - r}\, |f|_{w,\alpha}^2 (1 + |u|).$$

[1] In [13, 14, 10], inequality (1.3) is proved for uniformly Lipschitz bounded functionals on H, and the dependence of the right-hand side on u is not specified. However, the proofs given there remain valid for any functional that is uniformly Hölder continuous on bounded subsets of H, and the dependence on $|u|$ can easily be found using the fact that, for any $u \in H$ and sufficiently large k, the norm of u_k is $\mathbb{P}_u$-a.s. bounded by an absolute constant.

We now turn to the CLT. For any function $f \in C^\alpha(H, w)$ satisfying the condition $(f, \mu) = 0$, we set

$$g(u) = \sum_{l=0}^{\infty} \mathfrak{P}_k f(u), \quad u \in H.$$

The fact that $g(u)$ is well defined follows from the exponential mixing inequality (1.3). We introduce a non-negative constant σ_f such that

$$\sigma_f^2 = 2(gf, \mu) - (f^2, \mu). \tag{1.4}$$

The following relation, which can easily be verified with the help of (1.3) and the Markov property, shows that the right-hand side of (1.4) is indeed non-negative:

$$\sigma_f^2 = \lim_{k\to\infty} \mathbb{E}_\mu \left(\frac{1}{\sqrt{k}} \sum_{l=0}^{k-1} f(u_l) \right)^2, \tag{1.5}$$

where $\mathbb{E}_\mu$ is the expectation corresponding to $\mathbb{P}_\mu(\cdot) = \int_H \mathbb{P}_u(\cdot)\mu(du)$.

For any $\sigma > 0$, we denote by $\Phi_\sigma(r)$ the one-dimensional centred Gaussian distribution with variance σ:

$$\Phi_\sigma(r) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{r} e^{-s^2/2\sigma^2} ds.$$

Finally, for $\sigma = 0$, we set

$$\Phi_0(r) = \begin{cases} 1, & r \geq 0, \\ 0, & r < 0. \end{cases}$$

Theorem 1.2. *Suppose that Hypotheses* (A) – (D) *and the non-degeneracy condition* (0.5) *are satisfied. Then for any $\alpha \in (0, 1]$ and $w \in \mathcal{W}$ the following statements hold:*

(i) *For any $\bar\sigma > 0$ there is a function $h_{\bar\sigma}(r_1, r_2) \geq 0$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and increasing in both arguments such that, for any $f \in C^\alpha(H, w)$ satisfying the conditions $\sigma_f \geq \bar\sigma$ and $(f, \mu) = 0$, we have*

$$\sup_{z\in\mathbb{R}} \left| \mathbb{P}_u \left\{ k^{-\frac{1}{2}} \sum_{l=0}^{k-1} f(u_l) \leq z \right\} - \Phi_{\sigma_f}(z) \right| \leq h_{\bar\sigma}(|u|, |f|_{w,\alpha}) k^{-\frac{1}{5}},$$

where $k \geq 1$ and $u \in H$.

(ii) *There is a function $h(r_1, r_2) \geq 0$ defined for $\mathbb{R}_+ \times \mathbb{R}_+$ and increasing in both arguments such that, for any $f \in C^\alpha(H, w)$ satisfying the conditions $\sigma_f = 0$ and $(f, \mu) = 0$, we have*

$$\sup_{z\in\mathbb{R}} \left((|z| \wedge 1) \left| \mathbb{P}_u \left\{ k^{-\frac{1}{2}} \sum_{l=0}^{k-1} f(u_l) \leq z \right\} - \Phi_0(z) \right| \right) \leq h(|u|, |f|_{w,\alpha})\, k^{-\frac{1}{4}},$$

where $k \geq 1$ and $u \in H$.

1.2 Navier–Stokes system perturbed by white noise

We now turn to the problem (0.3), (0.4), where the right-hand side η satisfies the conditions formulated in the Introduction. Let H be the space of divergence-free vector fields $u \in L^2(D, \mathbb{R}^2)$ such that $(u, \nu)\big|_{\partial D} = 0$, where ν is the unit normal to ∂D (see [22] for details). It is well known (see [23, 3]) that the problem (0.3), (0.4) generates a family of Markov processes $(u_t, \mathbb{P}_u)$ in the space H. We denote by $P_t(u, \Gamma) = \mathbb{P}_u\{u_t \in \Gamma\}$ the corresponding transition function and by $\mathfrak{P}_t$ and $\mathfrak{P}_t^*$ the Markov operators associated with $P_t(u, \Gamma)$. As was shown in [15] (see also [4, 2, 18]), if

$$B_1 := \sum_j \alpha_j b_j^2 < \infty,$$

where α_j are the eigenvalues of the Stokes operator, and the non-degeneracy condition (0.5) holds with a sufficiently large $N \geq 1$, then the family $(u_t, \mathbb{P}_u)$ has a unique stationary measure $\mu \in \mathcal{P}(H)$. Moreover, as is shown in [16, Proposition 3.2], for any $\alpha \in (0, 1]$ and sufficiently small $\delta \geq 0$ there are positive constants β and C such that

$$\left|\mathfrak{P}_t f(u) - (f, \mu)\right| \leq C|f|_{v_\delta, \alpha} v_\delta(|u|)\, e^{-\beta t}, \quad t \geq 0, \tag{1.6}$$

where $u \in H$, $v_\delta(r) = e^{\delta r^2}$, and $f \in C^\alpha(H, w)$ is an arbitrary functional. The following result on SLLN is an analog of Theorem 1.1 for the case of the problem (0.3), (0.4).

Theorem 1.3. *Suppose that the non-degeneracy condition* (0.5) *is satisfied with sufficiently large $N \geq 1$. Then there are positive constants d and δ such that, for any $\varepsilon \in (0, \frac{1}{3})$ and $\alpha \in (0, 1]$, a sufficiently large $D > 0$, and an arbitrary functional $f \in C^\alpha(H, v_\delta)$, the following statements hold:*

- **(i)** *There is a random variable $T(\omega) \geq 1$ depending on α, ε, and f such that*

$$\left|t^{-1}\int_0^t f(u_s)\, ds - (f, \mu)\right| \leq D\,|f|_{v_\delta} t^{-\frac{1}{3}+\varepsilon} \quad \textit{for} \quad t \geq T(\omega).$$

- **(ii)** *For any $u \in H$, the variable T is $\mathbb{P}_u$-a.s. finite. Moreover, if $0 < r \leq 2\varepsilon$, then*

$$\mathbb{E}_u T^r \leq D\,|f|_{v_\delta, \alpha}^2 e^{d|u|^2}.$$

We now discuss the CLT for the problem (0.3), (0.4). Let us fix an arbitrary $p > 0$ and set $w_p(r) = (1 + r)^p$. For any $f \in C^\alpha(H, w_p)$ satisfying the condition $(f, \mu) = 0$ we introduce the function

$$g(u) = \int_0^\infty \mathfrak{P}_s f(u)\, ds, \quad u \in H. \tag{1.7}$$

Inequality (1.6) implies that g is well defined. Furthermore, we introduce a non-negative constant σ_f such that (cf. (1.4) and (1.5))

$$\sigma_f^2 = 2(gf, \mu) = \lim_{t \to +\infty} \mathbb{E}_\mu \left(t^{-\frac{1}{2}} \int_0^t f(u_s)\, ds\right)^2. \tag{1.8}$$

Theorem 1.4. *Suppose that the non-degeneracy condition* (0.5) *is satisfied with sufficiently large* $N \geq 1$. *Then for any* $\alpha \in (0,1]$ *and* $p \geq 0$ *the following statements hold:*

(i) *For any* $\bar{\sigma} > 0$ *there is a function* $h_{\bar{\sigma}}(r_1, r_2) \geq 0$ *defined on* $\mathbb{R}_+ \times \mathbb{R}_+$ *and increasing in both arguments such that, for any* $f \in C^\alpha(H, w_p)$ *satisfying the conditions* $\sigma_f \geq \bar{\sigma}$ *and* $(f, \mu) = 0$, *we have*

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P}_u \left\{ t^{-\frac{1}{2}} \int_0^t f(u_s)\, ds \leq z \right\} - \Phi_{\sigma_f}(z) \right| \leq h_{\bar{\sigma}}(|u|, |f|_{w_p,\alpha})\, t^{-\frac{1}{5}}, \tag{1.9}$$

where $t \geq 1$ *and* $u \in H$.

(ii) *There is a function* $h(r_1, r_2) \geq 0$ *defined on* $\mathbb{R}_+ \times \mathbb{R}_+$ *and increasing in both arguments such that, for any* $f \in C^\alpha(H, w_p)$ *satisfying the conditions* $\sigma_f = 0$ *and* $(f, \mu) = 0$, *we have*

$$\sup_{z \in \mathbb{R}} \left((|z| \wedge 1) \left| \mathbb{P}_u \left\{ t^{-\frac{1}{2}} \int_0^t f(u_s)\, ds \leq z \right\} - \Phi_0(z) \right| \right) \leq h(|u|, |f|_{w_p,\alpha})\, t^{-\frac{1}{4}}, \tag{1.10}$$

where $t \geq 1$ *and* $u \in H$.

2 LLN and CLT for mixing-type Markov processes

In this section, we establish some versions of SLLN and CLT for Markov processes possessing a mixing property. They are used in the next section to prove Theorems 1.3 and 1.4.

It should be noted that the results of this section may not be optimal. The LLN and CLT were the focus of attention for many researchers, and it is not excluded that stronger results are available (cf. [17, 9, 19]. However, we were not able to find theorems that apply to our problems, and therefore we give complete proofs of the results announced here.

2.1 Strong law of large numbers

Let H be a real Hilbert space with norm $|\cdot|$, let $(u_t, \mathbb{P}_u)$ a family of H-valued Markov processes, and let $\mathcal{F}_t$ be the σ-algebra generated by u_s, $0 \leq s \leq t$. We shall assume that, for any $u \in H$ and $\mathbb{P}_u$-a.e. $\omega \in \Omega$, the trajectory $u_t(\omega)$, $t \geq 0$, is continuous. Let $P_t(u, \Gamma) = \mathbb{P}_u\{u_t \in \Gamma\}$ be the transition function for $(u_t, \mathbb{P}_u)$ and let $\mathfrak{P}_t$ and $\mathfrak{P}_t^*$ be the corresponding Markov semi-groups (see Section 1.1).

Let us fix a constant $\alpha \in (0,1]$ and a weight function $w \in \mathcal{W}$ and assume that the operator $\mathfrak{P}_t$, which is defined originally on $C_b(H)$, can be extended to an operator from $C^\alpha(H, w)$ to $C(H)$. (See Notation for the definition of these spaces.) More precisely, for $f \in C^\alpha(H, w)$, we set

$$\mathfrak{P}_t f(u) = \mathbb{E}_u f(u_t), \quad u \in H, \tag{2.1}$$

assuming that the right-hand side of (2.1) is finite for all $u \in H$ and defines a continuous function on H.

Definition 2.1. We shall say that the family $(u_t, \mathbb{P}_u)$ is *uniformly mixing for the class* $C^\alpha(H,w)$ if it has a unique stationary measure $\mu \in \mathcal{P}(H)$ and there are non-negative continuous functions $\rho(r)$, $r \geq 0$, and $\gamma(t)$, $t \geq 0$, such that $\gamma(t) \to 0$ as $t \to \infty$ and, for any $f \in C^\alpha(H,w)$ and $u \in H$, we have [2]

$$\left|\mathfrak{P}_t f(u) - (f,\mu)\right| \leq \gamma(t)\rho(|u|)|f|_{w,\alpha}, \quad t \geq 0. \tag{2.2}$$

We now formulate the conditions imposed on the Markov family $(u_t, \mathbb{P}_u)$.

Condition 2.2. There are $\alpha \in (0,1]$ and $w \in \mathcal{W}$ such that the family $(u_t, \mathbb{P}_u)$ is uniformly mixing for the class $C^\alpha(H,w)$, and the function $\gamma(t)$ in (2.2) satisfies the condition

$$\bar\gamma := \int_0^\infty \gamma(t)\,dt < \infty. \tag{2.3}$$

Condition 2.3. There is a constant $q > 0$, a random time $T_0(\omega) \geq 1$, and a continuous function $\tau(r)$, $r \geq 0$, such that, for any $u \in H$, we have

$$\mathbb{P}_u\bigl\{|u_t(\omega)| \leq w^{-1}(t^q) \text{ for } t \geq T_0\bigr\} = 1, \tag{2.4}$$

$$\mathbb{E}_u T_0 \leq \tau(|u|), \tag{2.5}$$

where $w^{-1}(r)$ is the inverse functions of $w(r)$ (see Condition 2.2).

For any functional $f \in C(H)$, we set

$$S_t(f) = \int_0^t f(u_s)\,ds.$$

The following theorem shows that Conditions 2.2 and 2.3 imply a strong law of large numbers.

Theorem 2.4. *Let $(u_t, \mathbb{P}_u)$ be a family of Markov processes in H that satisfies Conditions 2.2 and 2.3. Suppose that $3q < 1$ and*

$$\mathbb{E}_u\left(\rho^2(|u_t|)\right) \leq \varkappa(|u|) \quad \textit{for all} \quad t \geq 0, \quad u \in H, \tag{2.6}$$

where $\varkappa(r)$, $r \geq 0$ is a continuous function. Then there is a constant $D > 0$ such that for any $f \in C^\alpha(H,w)$ and $\varepsilon \in (0, \frac{1}{3} - q)$ the following statements hold.

(i) *There is a random time $T(\omega) \geq 1$ depending on f and ε such that*

$$\left|t^{-1}S_t(f) - (f,\mu)\right| \leq D\,|f|_w t^{-\frac{1}{3}+q+\varepsilon} \quad \textit{for} \quad t \geq T(\omega). \tag{2.7}$$

(ii) *For any $u \in H$, we have $\mathbb{P}_u\{T < \infty\} = 1$. Moreover, if $0 < r < 3\varepsilon$, then*

$$\mathbb{E}_u K^r \leq \tfrac{D}{3\varepsilon - r}\,|f|_{w,\alpha}^2 \varkappa(|u|) + \tau(|u|). \tag{2.8}$$

[2] In particular, we assume that $(f,\mu) < \infty$ for any $f \in C^\alpha(H,w)$.

We note that if $w \in \mathcal{W}$ is bounded (and hence any functional in $C^\alpha(H,w)$ is bounded), then Condition 2.3 can be omitted. In this case, we should take $q=0$ and $\tau \equiv 0$ in the formulation of the theorem; see [21] for the case of discrete time.

Proof of Theorem 2.4. Let us fix an arbitrary function $f \in C^\alpha(H,w)$ and set

$$S_t = S_t(f), \quad s_t = t^{-1} S_t(f).$$

There is no loss of generality in assuming that $|f|_w \le 1$. Moreover, to simplify the notation, we shall assume that $(f,\mu)=0$; the proof in the general case is similar.

Step 1. We first show that

$$\mathbb{E}_u\, s_t^2 \le C_1 |f|_{w,\alpha}^2 \varkappa(|u|) t^{-1}, \quad t \ge 1. \tag{2.9}$$

Here and henceforth, we denote by C_i positive constants that do not depend on f, u, t and ε.

Let us note that

$$\begin{aligned} \mathbb{E}_u\, s_t^2 &= \frac{1}{t^2}\int_0^t\int_0^t \mathbb{E}_u\left(f(u_s)f(u_r)\right) dr ds \\ &\le \frac{2}{t^2} \iint\limits_{0\le s\le r\le t} \left|\mathbb{E}_u\left(f(u_s)f(u_r)\right)\right| dr ds. \end{aligned} \tag{2.10}$$

By the Markov property, for $s \le r$ we have

$$\mathbb{E}_u\left(f(u_s)f(u_r)\right) = \mathbb{E}_u\left(f(u_s)\mathbb{E}_u\left(f(u_r)\,\middle|\,\mathcal{F}_s\right)\right) = \mathbb{E}_u\left(f(u_s)(\mathfrak{P}_{r-s}f)(u_s)\right).$$

Furthermore, since $(f,\mu)=0$, inequality (2.2) implies that

$$\left|\mathfrak{P}_{r-s}f(u_s)\right| \le \gamma(r-s)\rho(|u_s|)|f|_{w,\alpha}.$$

Hence, using inequalities (2.2) and (2.6) with $t=0$ and $t=s$, respectively, we derive

$$\begin{aligned} \left|\mathbb{E}_u\left(f(u_s)f(u_r)\right)\right| &\le \gamma(r-s)|f|_{w,\alpha}\mathbb{E}_u\left(|f(u_s)|\rho(|u_s|)\right) \\ &\le \gamma(r-s)\gamma(0)\varkappa(|u|)|f|_{w,\alpha}^2. \end{aligned}$$

Substitution of this inequality into (2.10) results in (2.9) with $C_1 = 2\bar\gamma\gamma(0)$, where $\bar\gamma$ is the constant in (2.3).

Step 2. We now prove (2.7). To this end, we fix $\varepsilon \in (0, \frac13 - q)$ and, for any $n \ge 1$, set

$$t_n = n^{3+\beta}, \quad \beta = \frac{9\varepsilon}{1-3\varepsilon}.$$

Let us consider the events

$$G_n = \left\{\omega \in \Omega : |s_{t_n}| > n^{-1}\right\}, \quad n \ge 1.$$

Using (2.9) and the Chebyshev inequality, we derive

$$\mathbb{P}_u(G_n) \le n^2 \mathbb{E}_u |s_{t_n}|^2 \le C_1 |f|_{w,\alpha}^2 \varkappa(|u|)\, n^{-1-\beta}. \tag{2.11}$$

Let us define $m(\omega)$ as the smallest integer $m \ge 0$ such that

$$|s_{t_n}(\omega)| \le n^{-1} \quad \text{for} \quad n \ge m+1. \tag{2.12}$$

Inequality (2.11) and the Borel–Cantelli lemma imply that, for any $u \in H$, the random integer $m(\omega)$ is $\mathbb{P}_u$-a.s. finite. Moreover, it follows from the definition that, if $m(\omega) \ge 1$, then

$$|s_{t_m}(\omega)| > m^{-1}. \tag{2.13}$$

We now estimate $|s_t|$ for $t_n < t < t_{n+1}$. To this end, we note that

$$|s_t - s_{t_{n+1}}| \le \tfrac{1}{t} |S_t - S_{t_{n+1}}| + \left(\tfrac{1}{t} - \tfrac{1}{t_{n+1}}\right) |S_{t_{n+1}}|. \tag{2.14}$$

In view of (2.12), we have

$$\left(\tfrac{1}{t} - \tfrac{1}{t_{n+1}}\right) |S_{t_{n+1}}| \le \tfrac{t_{n+1}-t_n}{t_n} |s_{t_{n+1}}| \le \tfrac{t_{n+1}-t_n}{t_n} (n+1)^{-1}. \tag{2.15}$$

Furthermore, it follows from (2.4) that, for $t \ge T_0(\omega)$,

$$\begin{aligned}\frac{1}{t}|S_t - S_{t_{n+1}}| &\le \frac{1}{t}\int_t^{t_{n+1}} |f(u_s)|\, ds \le \frac{1}{t}\int_t^{t_{n+1}} w(|u_s|)\, ds \\ &\le \frac{1}{t}\int_t^{t_{n+1}} s^q\, ds \le C_2 \frac{t_{n+1}^{q+1} - t_n^{q+1}}{t_n}. \end{aligned} \tag{2.16}$$

The definition of t_n implies that, for any $r \in [1,2]$,

$$\frac{t_{n+1}^r - t_n^r}{t_n} \le C_3 (n+1)^{(3+\beta)(r-1)-1} = C_3\, t_{n+1}^{(r-1)-\frac{1}{3+\beta}}.$$

Combining this inequality with (2.14) – (2.16) and (2.12), for $t_n < t < t_{n+1}$, $n \ge m(\omega)$, and $t \ge T_0(\omega)$, we obtain

$$\begin{aligned}|s_t| \le |s_t - s_{t_{n+1}}| + |s_{t_{n+1}}| &\le C_2 C_3\, t_{n+1}^{q-\frac{1}{3+\beta}} + C_3\, t_{n+1}^{-\frac{2}{3+\beta}} + t_{n+1}^{-\frac{1}{3+\beta}} \\ &\le C_4\, t_{n+1}^{q-\frac{1}{3+\beta}} \le C_4\, t^{-\frac{1}{3}+q+\varepsilon}.\end{aligned}$$

Thus, inequality (2.7) holds with $T(\omega) = T_0(\omega) \vee m(\omega)^{3+\beta}$.

Step 3. It remains to establish (2.8). To this end, we note that, for $0 < d < \beta$,

$$\begin{aligned}\mathbb{E}_u m^d &= \sum_{n=1}^{\infty} \mathbb{P}_u\{m = n\} n^d \le \sum_{n=1}^{\infty} \mathbb{P}_u(G_n) n^d \\ &\le C_1\, \varkappa(|u|)\, |f|_{w,\alpha}^2 \sum_{n=1}^{\infty} n^{-1-\beta+d} \le \tfrac{C_5}{\beta-d}\, \varkappa(|u|)\, |f|_{w,\alpha}^2, \end{aligned} \tag{2.17}$$

where we used inequalities (2.11), (2.13) and the definition of $m(\omega)$ and G_n. We now fix r satisfying the condition $0 < r < 3\varepsilon$. It follows from (2.17) that

$$\mathbb{E}_u\, m^{(3+\beta)r} \le \tfrac{C_5}{3(3\varepsilon - r)}\, \varkappa(|u|)\, |f|_{w,\alpha}^2. \tag{2.18}$$

Furthermore, since $r < 3\varepsilon < 1$, we conclude from (2.5) that

$$\mathbb{E}_u T_0^r \le E_u T_0 \le \tau(|u|). \tag{2.19}$$

It remains to note that

$$\mathbb{E}_u T^r \le \mathbb{E}_u\, m^{(3+\beta)r} + \mathbb{E}_u T_0^r,$$

and to use (2.18) and (2.19) to estimate the right-hand side of this inequality. The proof of Theorem 2.4 is complete. $\square$

2.2 Central limit theorem

In this subsection, we show that the rate of convergence in CLT for uniformly mixing Markov processes can be expressed in terms of the conditional variance for an associated martingale. To formulate the corresponding result, we shall need some notation.

Let $(u_t, \mathbb{P}_u)$ be a family of Markov processes satisfying Condition 2.2 with some $\alpha > 0$ and $w \in \mathcal{W}$. As before, we denote by $\mathcal{F}_t$ the filtration generated by the Markov family. Let us fix $u \in H$ and an arbitrary function $f \in C^\alpha(H, w)$ such that $(f, \mu) = 0$ and set

$$M_t = \int_0^\infty \bigl\{\mathbb{E}_u\bigl(f(u_s)\,|\,\mathcal{F}_t\bigr) - \mathbb{E}_u\bigl(f(u_s)\,|\,\mathcal{F}_0\bigr)\bigr\}\,ds. \tag{2.20}$$

We claim that the random variables M_t, $t \ge 0$, are $\mathbb{P}_u$-a.s. finite and form a zero-mean martingale. Indeed, by the Markov property, with $\mathbb{P}_u$-probability 1, we have

$$\mathbb{E}_u\bigl(f(u_s)\,|\,\mathcal{F}_t\bigr) = \mathfrak{P}_{s-t} f(u_t) \quad \text{for} \quad s \ge t \ge 0. \tag{2.21}$$

Since $(u_t, \mathbb{P}_u)$ is uniformly mixing and $(f, \mu) = 0$, it follows that

$$\bigl|\mathfrak{P}_r f(u)\bigr| \le \gamma(r)\rho(|u|)|f|_{w,\alpha} \quad \text{for} \quad r \ge 0, \quad u \in H. \tag{2.22}$$

Using this inequality to estimate the right-hand side of (2.21), we derive

$$\int_0^\infty \bigl|\mathbb{E}_u\bigl(f(u_s)\,|\,\mathcal{F}_t\bigr)\bigr|\,ds \le \int_0^t \bigl|f(u_s)\bigr|\,ds + \bar\gamma\,|f|_{w,\alpha}\rho(|u_t|), \quad t \ge 0,$$

where $\bar\gamma$ is the constant in (2.3). Thus, integral (2.20) converges absolutely for $\mathbb{P}_u$-a.e. ω. The fact that M_t is a zero-mean martingale can easily be established by taking in (2.20) the (conditional) expectation.

For any integer $k \geq 1$, we define a conditional variance for M_k by the formula

$$V_k^2 = \sum_{l=1}^{k} E_u\big((M_l - M_{l-1})^2 \,|\, \mathcal{F}_{l-1}\big).$$

Given a random variable ζ and a non-negative constant σ, we set

$$\Delta_\sigma(\zeta, z) = \begin{cases} F_\zeta(z) - \varPhi_\sigma(z), & \sigma > 0, \\ (|z| \wedge 1)\big(F_\zeta(z) - \varPhi_0(z)\big), & \sigma = 0, \end{cases}$$

where $F_\zeta(z)$ is the distribution function of ζ.

The following theorem reduces the CLT for uniformly mixing Markov families to an LLN for the conditional variance (cf. [9, Theorem VIII.3.22]).

Theorem 2.5. *Let $(u_t, \mathbb{P}_u)$ be a family of Markov processes satisfying Condition 2.2 with some $\alpha \in (0,1]$ and $w \in \mathcal{W}$. Suppose that there is a constant $a > 0$ and a continuous function $\varkappa(r)$, $r \geq 0$, such that*

$$\mathbb{E}_u\Big(\sup_{t\in[k,k+1]} e^{\rho^a(|u_t|)}\Big) \leq \varkappa(|u|) \quad \textit{for} \quad k \geq 0, \quad u \in H. \tag{2.23}$$

Then the following statements hold:

(i) *For any $\bar\sigma > 0$ there is a non-negative continuous function $h_{\bar\sigma}(r_1, r_2)$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and increasing in both arguments such that, for any $\sigma \geq \bar\sigma$ and any function $f \in C^\alpha(H,p)$ with $(f,\mu) = 0$, we have*

$$\sup_{z\in\mathbb{R}}\big|\Delta_\sigma\big(t^{-\frac12}S_t(f), z\big)\big| \leq t^{-\frac15}\, h_{\bar\sigma}\big(|u|, |f|_{w,\alpha}\big) + \sigma^{-4}\hat t^{\frac45}\,\mathbb{E}_u\big|\hat t^{-1}V_{\hat t}^2 - \sigma^2\big|^2, \tag{2.24}$$

where $t \geq 1$ and $\hat t$ is the integer part of t.

(ii) *There is a non-negative continuous function $h(r_1, r_2)$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and increasing in both arguments such that, for any function $f \in C^\alpha(H,p)$ with $(f,\mu) = 0$ and any $t \geq 1$, we have*

$$\sup_{z\in\mathbb{R}}\big|\Delta_0\big(t^{-\frac12}S_t(f), z\big)\big| \leq t^{-\frac14}\, h\big(|u|, |f|_{w,\alpha}\big) + \hat t^{-\frac12}\big(\mathbb{E}_u V_{\hat t}^2\big)^{\frac12}. \tag{2.25}$$

We note that the conditions of this theorem are not optimal. However, they are sufficient for our purposes and enable us to give a rather simple proof of the conclusion.

Proof of Theorem 2.5. We first describe the idea of the proof. We fix an arbitrary function $f \in C^\alpha(H,w)$ and, following a well-known argument (cf. [17, 9]), represent $S_t = S_t(f)$ in the form

$$\begin{aligned} S_t &= M_{\hat t} + \int_{\hat t}^{t} f(u_s)\,ds - \int_{\hat t}^{\infty} \mathbb{E}_u\big(f(u_s)\,|\,\mathcal{F}_{\hat t}\big)\,ds + \int_0^\infty \mathbb{E}_u\big(f(u_s)\,|\,\mathcal{F}_0\big)\,ds \\ &= M_{\hat t} + \zeta_t - g(u_{\hat t}) + g(u_0), \end{aligned} \tag{2.26}$$

where $\hat{t}$ is the integer part of t, M_t is defined by (2.20), and

$$\zeta_t := \int_{\hat{t}}^{t} f(u_s)\,ds, \tag{2.27}$$

$$g(u) := \int_0^\infty \mathbb{E}_u f(u_s)\,ds = \int_0^\infty \mathfrak{P}_s f(u)\,ds. \tag{2.28}$$

It follows from (2.22) and (2.3) that the function $g(u)$ is well defined and satisfies the inequality

$$|g(u)| \le \bar{\gamma}\,\rho(|u|)|f|_{w,\alpha}, \quad u \in H. \tag{2.29}$$

Combining (2.26), (2.29), and (2.23), it is not difficult to show that the expression $\bigl|t^{-\frac12}S_t - \hat{t}^{-\frac12}M_{\hat{t}}\bigr|$ is small for $t \gg 1$. Therefore, to establish (2.24) and (2.25), it suffices to estimate the rate of convergence of $\hat{t}^{-\frac12}M_{\hat{t}}$ to a Gaussian distribution in terms of the conditional variance $V_{\hat{t}}$. This will be done by applying a result from [8].

The accurate proof is divided into three steps.

Step 1. Let us show that it suffices to establish inequalities (2.24) and (2.25) with $t^{-\frac12}S_t$ replaced by $\hat{t}^{-\frac12}M_{\hat{t}}$. We shall need the following simple lemma, whose proof is given in the Appendix (see Section 4.1).

Lemma 2.6. *Let ξ and η be real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then for any $\sigma \ge 0$ and $\varepsilon > 0$ we have*

$$\sup_{z\in\mathbb{R}} \bigl|\Delta_\sigma(\xi,z)\bigr| \le \sup_{z\in\mathbb{R}} \bigl|\Delta_\sigma(\eta,z)\bigr| + \mathbb{P}\{|\xi-\eta| > \varepsilon\} + c_\sigma\varepsilon, \tag{2.30}$$

where $c_\sigma = (\sigma\sqrt{2\pi})^{-1}$ for $\sigma > 0$ and $c_0 = 2$.

We wish to apply Lemma 2.6 with $\xi = t^{-\frac12}S_t$ and $\eta = \hat{t}^{-\frac12}M_{\hat{t}}$. To this end, we first estimate $\mathbb{E}_u|\xi-\eta|$. It follows from (2.26) that

$$\bigl|t^{-\frac12}S_t - \hat{t}^{-\frac12}M_{\hat{t}}\bigr| \le \bigl(t\,\hat{t}^{\frac12}\bigr)^{-1}|S_t| + \hat{t}^{-\frac12}\bigl|\zeta_t - g(u_{\hat{t}}) + g(u_0)\bigr|. \tag{2.31}$$

Inequality (2.9) implies that

$$\mathbb{E}_u|S_t| \le \bigl(\mathbb{E}_u S_t^2\bigr)^{\frac12} \le (C_1 t)^{\frac12}|f|_{w,\alpha}\varkappa^{\frac12}(|u|). \tag{2.32}$$

Using (2.2) and (2.3), we derive

$$\mathbb{E}_u|\zeta_t| \le \int_{\hat{t}}^{t} \bigl|\mathbb{E}_u|f(u_s)| - (|f|,\mu)\bigr|\,ds + (|f|,\mu) \le \bar{\gamma}\,\rho(|u|)|f|_{w,\alpha} + (|f|,\mu). \tag{2.33}$$

Furthermore, inequalities (2.29) and (2.23) imply that, for any $u \in H$ and $t \ge 1$, we have

$$\mathbb{E}_u\bigl|g(u_{\hat{t}}) - g(u_0)\bigr| \le \bar{\gamma}\,|f|_{w,\alpha}\mathbb{E}_u\bigl(\rho(|u_0|) + \rho(|u_{\hat{t}}|)\bigr) \le c\bar{\gamma}\,|f|_{w,\alpha}\varkappa(|u|), \tag{2.34}$$

where $c>0$ is a constant not depending on u and t. Combining (2.31) – (2.34), we see that

$$\mathbb{E}_u\left|t^{-\frac{1}{2}}S_t-\hat{t}^{-\frac{1}{2}}M_{\hat{t}}\right|\le t^{-\frac{1}{2}}d_1(|u|)|f|_{w,\alpha},\quad t\ge 1.$$

Here and henceforth, we denote by $d_i(r)$, $r\ge 0$, non-negative continuous functions. Applying now the Chebyshev inequality and using (2.30) with $\varepsilon=t^{-\frac{1}{4}}$, we see that

$$\sup_{z\in\mathbb{R}}\left|\Delta_\sigma(t^{-1}S_t,z)\right|\le\sup_{z\in\mathbb{R}}\left|\Delta_\sigma(\hat{t}^{-1}M_{\hat{t}},z)\right|+t^{-\frac{1}{4}}d_2(|u|)\,|f|_{w,\alpha}.$$

This implies the assertion formulated in the beginning of Step 1.

Step 2. We now prove (2.24) with $t^{-\frac{1}{2}}S_t$ replaced by $\hat{t}^{-\frac{1}{2}}M_{\hat{t}}$. We shall need the following proposition, which can be obtained by a slight modification of the proof of Theorem 3.7 in [8] (see Section 4.2 of the Appendix).

Proposition 2.7. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let M_k, $0\le k\le n$, be a zero-mean martingale with respect to σ-algebras $\mathcal{F}_k$. Suppose that, for some positive constants β and B, we have*

$$\mathbb{E}\exp\bigl(|M_k-M_{k-1}|^\beta\bigr)\le B,\quad 1\le k\le n. \tag{2.35}$$

Then for any $\bar{\sigma}>0$ there is a constant $A(\bar{\sigma})>0$ depending only on β and B such that

$$\sup_{z\in\mathbb{R}}\left|\Delta_\sigma(n^{-\frac{1}{2}}M_n,z)\right|\le A(\bar{\sigma})n^{-\frac{1}{5}}+\sigma^{-4}n^{\frac{4}{5}}\,\mathbb{E}\left|n^{-1}V_n^2-\sigma^2\right|^2, \tag{2.36}$$

where $\sigma\ge\bar{\sigma}$ is an arbitrary constant and V_n^2 is the conditional variance:

$$V_n^2=\sum_{k=1}^n\mathbb{E}\bigl((M_k-M_{k-1})^2\,|\,\mathcal{F}_{k-1}\bigr).$$

Let us fix $t\ge 1$. We wish to apply Proposition 2.7 with $n=\hat{t}$ to the zero-mean martingale M_k defined by (2.20). To this end, we shall show that condition (2.35) is satisfied with $\beta=\frac{a}{2}$ (see (2.23)) and some constant $B=B(|f|_{w,\alpha})$. This will imply the required assertion.

It follows from (2.20) and the Markov property that

$$M_k-M_{k-1}=\int_{k-1}^k f(u_s)\,ds+g(u_k)-g(u_{k-1}). \tag{2.37}$$

Therefore, using inequality (2.2) with $t=0$ and (2.29), we obtain

$$|M_k-M_{k-1}|^{\frac{a}{2}}\le\sup_{s\in[k-1,k]}\rho^a(|u_s|)+C(a)\,|f|_{w,\alpha}^a,$$

where $C(a)>0$ is a constant not depending f and ω. Inequality (2.35) follows now from (2.23).

Step 3. To prove inequality (2.25) with $t^{-\frac{1}{2}}S_t$ replaced by $\hat{t}^{-\frac{1}{2}}M_{\hat{t}}$, it suffices to show that

$$\sup_{z\in\mathbb{R}}\bigl|\Delta_0(k^{-\frac{1}{2}}M_k,z)\bigr| \le \bigl(k^{-1}\mathbb{E}_u V_k^2\bigr)^{\frac{1}{2}}, \quad k\ge 1. \tag{2.38}$$

It is a matter of direct verification to show that $\mathbb{E}_u M_k^2 = \mathbb{E}_u V_k^2$. Therefore, by the Chebyshev inequality, for any $z>0$ we have

$$\mathbb{P}_u\bigl\{|k^{-\frac{1}{2}}M_k|\ge z\bigr\} \le z^{-1}k^{-\frac{1}{2}}\,\mathbb{E}_u|M_k| \le z^{-1}k^{-\frac{1}{2}}\,\bigl(\mathbb{E}_u V_k^2\bigr)^{\frac{1}{2}}.$$

To prove (2.38), it remains to note that

$$\bigl|\Delta_0(k^{-\frac{1}{2}}M_k,z)\bigr| \le \bigl(|z|\wedge 1\bigr)\mathbb{P}_u\bigl\{|k^{-\frac{1}{2}}M_k|\ge |z|\bigr\}.$$

The proof of Theorem 2.5 is complete. □

3 Proof of the main results

We shall confine ourselves to the case of the Navier–Stokes system perturbed by the random force (0.4). The proof of the results on the discrete-time RDS (1.1) is similar and technically much simpler.

3.1 Proof of Theorem 1.3

We shall show that the family of Markov processes $(u_t,\mathbb{P}_u)$ associated with the problem (0.3), (0.4) satisfies Conditions 2.2 and 2.3, where $\alpha\in(0,1]$ and $q>0$ are arbitrary constants,

$$w(r)=\rho(r)=v_\delta(r), \quad \gamma(t)=\text{const}\,e^{-\beta t}. \tag{3.1}$$

This will imply that the hypotheses of Theorem 1.3 are fulfilled and that the constant q can be chosen arbitrarily small.

Inequalities (2.2) and (2.3) are straightforward consequences of (1.6). Furthermore, as is shown in [20, Remark 3.2] (see also [23, 1]), there is $\nu>0$ such that

$$\mathbb{E}_u\,e^{\nu|u_t|^2} \le C_1 e^{\nu|u|^2}, \quad t\ge 0, \quad u\in H. \tag{3.2}$$

Here and henceforth, we denote by C_i positive constants not depending on t and u. It follows from (3.1) and (3.2) that, if $2\delta\le\nu$, then inequality (2.6) holds with $\varkappa(r)=C_1e^{2\delta r^2}$. Thus, it remains to check Condition 2.3.

To this end, let us set

$$U_k=\sup_{t\in[k,k+1]}|u_t|^2, \tag{3.3}$$

and show that (cf. (3.2))

$$\mathbb{E}_u e^{\nu U_k} \le C_2 e^{\nu|u|^2}, \quad k \ge 0, \quad u \in H, \tag{3.4}$$

where $\nu > 0$ is sufficiently small. Indeed, it follows from the Markov property that

$$\mathbb{E}_u e^{\nu U_k} = \mathbb{E}_u \bigl\{\mathbb{E}_u \bigl(e^{\nu U_k} \,|\, \mathcal{F}_k\bigr)\bigr\} = \mathbb{E}_u \bigl(\mathbb{E}_{u_k} e^{\nu U_0}\bigr). \tag{3.5}$$

Hence, in view of (3.2), it suffices to prove (3.4) for $k = 0$.

As is shows in [15] (see Lemma 2.3 with $T = 1$), there is $c > 0$ such that

$$\mathbb{P}_u\bigl\{U_0 - B_0 - |u|^2 \ge z\bigr\} \le e^{-cz}, \quad z \in \mathbb{R}, \tag{3.6}$$

where B_0 is defined in (1.2). Setting $\Omega_u = \bigl\{U_0 \ge B_0 + |u|^2\bigr\}$ and denoting by I_{Ω_u} the indicator function of Ω_u, we derive

$$\begin{aligned} \mathbb{E}_u e^{\nu U_0} &\le \mathbb{E}_u \bigl(I_{\Omega_u} e^{\nu U_0}\bigr) + e^{\nu(B_0+|u|^2)} \\ &= \int_0^\infty \mathbb{P}_u\bigl\{I_{\Omega_u} e^{\nu U_0} > z\bigr\}\, dz + e^{\nu(B_0+|u|^2)}. \end{aligned} \tag{3.7}$$

Inequality (3.6) implies that, if $\nu \le \frac{c}{2}$, then the integral on the right-hand side of (3.7) does not exceed $e^{\nu(B_0+|u|^2)}$. This completes the proof of (3.4).

We now prove (2.4) and (2.5). Let us fix an arbitrary $q > 0$ and define a random integer by the formula

$$n(\omega) = \bigl\{m \ge 0 : U_k \le \tfrac{q \log k}{\delta} \text{ for } k \ge m+1\bigr\}.$$

Combining the Chebyshev inequality with (3.4), for $\delta < q\nu$ we derive

$$\sum_{k=1}^{\infty} \mathbb{P}_u\bigl\{U_k > \tfrac{q \log k}{\delta}\bigr\} \le \sum_{k=1}^{\infty} k^{-\frac{q\nu}{\delta}} \mathbb{E}_u e^{\nu U_k} \le C_3 e^{\nu|u|^2}.$$

Hence, by the Borel–Cantelli lemma, we have $\mathbb{P}_u\{n < \infty\} = 1$ for any $u \in H$. Moreover, if $2\delta < q\nu$, then

$$\mathbb{E}_u n = \sum_{k=1}^{\infty} k \mathbb{P}_u\{n = k\} \le \sum_{k=1}^{\infty} k \mathbb{P}_u\bigl\{U_k > \tfrac{q \log k}{\delta}\bigr\} \le \sum_{k=1}^{\infty} k^{1-\frac{q\nu}{\delta}} \mathbb{E}_u e^{\nu U_k} \le C_4 e^{\nu|u|^2}.$$

We now note that the definitions of $n(\omega)$ and $U_k(\omega)$ imply

$$|u_t(\omega)| \le \bigl(\tfrac{\log t^q}{\delta}\bigr)^{\frac{1}{2}} \quad \text{for} \quad t \ge n(\omega) + 1.$$

We have thus established (2.4) and (2.5) with $T_0 = n + 1$ and $\tau(r) = \text{const}\, e^{\nu r^2}$. The proof of Theorem 1.3 is complete.

3.2 Proof of Theorem 1.4

We first describe the scheme of the proof. The Markov family $(u_t, \mathbb{P}_u)$ associated with the NS system (0.3), (0.4) satisfies Condition 2.2 with any $\alpha \in (0,1]$ and $w = w_p$. Moreover, using (3.4), it is not difficult to prove that (2.23) is also valid. Therefore, by Theorem 2.5, inequalities (2.24) and (2.25) hold. To obtain the required assertion, we show that

$$V_k^2 = \sum_{l=0}^{k-1} \varphi(u_l), \tag{3.8}$$

where $\varphi \in C^\varepsilon(H, v_\delta)$ for some sufficiently small $\varepsilon \in (0,1]$ and $\delta > 0$. This will imply that $k^{-1}V_k^2$ converges (in an appropriate sense) to the mean value (φ, μ) as $k \to \infty$. It turns out that $(\varphi, \mu) = \sigma_f^2$. Therefore, if $\sigma_f > 0$, then taking $\sigma = \sigma_f$ in (2.24) and using (2.9), we prove assertion (i) of the theorem. If $\sigma_f = 0$, then a similar argument enables one to establish assertion (ii).

The accurate proof is divided into several steps.

Step 1. We first show that Condition 2.2 is satisfied, together with inequality (2.23). To this end, we shall need the following lemma.

Lemma 3.1. *Under the conditions of Theorem 1.4, for any $\alpha \in (0,1]$ and $p > 0$ there are positive constants C, β, and m such that*

$$\bigl|\mathfrak{P}_t f(u) - (f, \mu)\bigr| \le C\, e^{-\beta t} |f|_{w_p,\alpha} (1 + |u|)^m, \quad t \ge 0, \tag{3.9}$$

where $w_p(r) = (1+r)^p$, and $f \in C^\alpha(H, w_p)$ is an arbitrary functional. In particular, the family of Markov processes $(u_t, \mathbb{P}_u)$ satisfies Condition 2.2 with

$$w(r) = (1+r)^p, \quad \rho(r) = (1+r)^m, \quad \gamma(t) = C\, e^{-\beta t}. \tag{3.10}$$

Inequality (3.9) is an analogue of (1.6) for Hölder continuous functionals with polynomial growth at infinity. To prove (3.9), it suffices to repeat the scheme used in [16, Section 3] to establish (1.6); we shall not dwell on it.

Taking into account the second relation in (3.10), for $a = \frac{1}{m}$ we derive

$$\sup_{t \in [k,k+1]} \exp\bigl(\rho^a(|u_t|)\bigr) \le \exp\bigl(1 + U_k^{\frac{1}{2}}\bigr), \quad k \ge 0, \tag{3.11}$$

where the random variable U_k is defined by (3.3). Hence, using the Markov property, we obtain (cf. (3.5))

$$\mathbb{E}_u \Bigl(\sup_{t \in [k,k+1]} e^{\rho^a(|u_t|)} \Bigr) \le \mathbb{E}_u \exp\bigl(1 + U_k^{\frac{1}{2}}\bigr) = \mathbb{E}_u \bigl(\mathbb{E}_{u_k} \exp\bigl(1 + U_0^{\frac{1}{2}}\bigr)\bigr). \tag{3.12}$$

Furthermore, it follows from (3.6) that (cf. (3.7))

$$\mathbb{E}_u \exp\bigl(1 + U_0^{\frac{1}{2}}\bigr) \le C_5 e^{|u|}.$$

Combining this estimate with (3.12), we derive

$$\mathbb{E}_u\Bigl(\sup_{t\in[k,k+1]} e^{\rho^a(|u_t|)}\Bigr) \le C_6\mathbb{E}_u\, e^{|u_k|}, \quad k\ge 0, \quad u\in H. \tag{3.13}$$

Using (3.2) to estimate the right-hand side of (3.13), we obtain the required inequality (2.23). Thus, the conditions of Theorem 2.5 are fulfilled, and statements (i) and (ii) take place.

Step 2. Our next goal is to estimate the expectations on the right-hand sides of (2.24) and (2.25). To this end, we first establish (3.8). Namely, we shall show that there are small constants $\varepsilon\in(0,1]$ and $\delta>0$, depending only on α and p, and a functional $\varphi\in C^\varepsilon(H,v_\delta)$ such that, for any $u\in H$, relation (3.8) holds $\mathbb{P}_u$-a.s.

Let us recall that (see (2.37), (2.27), and (2.28))

$$M_l - M_{l-1} = \zeta_l^- + g(u_l) - g(u_{l-1}), \quad l=1,\dots,k,$$

where $\zeta_l^- = \int_{l-1}^l f(u_s)\,ds$. Therefore, by the Markov property, for any $u\in H$ we have

$$\mathbb{E}_u\bigl((M_l-M_{l-1})^2\,|\,\mathcal{F}_{l-1}\bigr) = \varphi(u_l) \quad \mathbb{P}_u\text{-a.s.}, \tag{3.14}$$

where we set

$$\varphi(u) = \mathbb{E}_u\, M_1^2 = \mathbb{E}_u\bigl(\zeta_1^- + g(u_1) - g(u_0)\bigr)^2. \tag{3.15}$$

Thus, relation (3.8) holds $\mathbb{P}_u$-a.s., and it remains to show that $\varphi\in C^\varepsilon(H,v_\delta)$.

We shall need the following lemma, whose proof is given in the Appendix (see Section 4.3).

Lemma 3.2. *Under the conditions of Theorem 1.4, the Markov semigroup $\mathfrak{P}_t$ associated with the problem* (0.3), (0.4) *possesses the following properties:*

(i) *For any $\alpha\in(0,1]$, $p>0$, and $\delta>0$ there is $\varepsilon>0$ such that the operator*

$$\mathfrak{P}_t : C^\alpha(H,w_p)\to C^\varepsilon(H,v_\delta), \quad t\ge 0, \tag{3.16}$$

is continuous, and its norm is uniformly bounded on any interval $[0,T]$. Moreover, if $f\in C^\alpha(H,w_p)$ and $(f,\mu)=0$, then the function $g(u)$ defined by (1.7) *belongs to $C^\varepsilon(H,v_\delta)$, and its norm can be estimated* const $|f|_{w_p,\alpha}$.

(ii) *For any $\alpha\in(0,1]$ and sufficiently small $\nu>0$ there are $\varepsilon>0$ and $\delta>0$ such that the operator*

$$\mathfrak{P}_t : C^\alpha(H,v_\nu)\to C^\varepsilon(H,v_\delta), \quad t\ge 0, \tag{3.17}$$

is continuous, and its norm is uniformly bounded on any interval $[0,T]$. Moreover, the constants ε and ν can be chosen in such a way that $\delta\to 0$ as $\nu\to 0$.

It follows from relation (3.15) that

$$\varphi(u)=\mathbb{E}_u(\zeta_1^-)^2+\mathbb{E}_u g^2(u_1)+\mathbb{E}_u g^2(u_0)+ \\ +2\,\mathbb{E}_u\bigl(\zeta_1^- g(u_1)\bigr)-2\,\mathbb{E}_u\bigl(\zeta_1^- g(u_0)\bigr)-2\,\mathbb{E}_u\bigl(g(u_1)g(u_0)\bigr). \quad (3.18)$$

Lemma 3.2 implies that the functions

$$\mathbb{E}_u g^2(u_0)=g^2(u), \qquad \mathbb{E}_u g^2(u_1)=\mathfrak{P}_1 g^2(u),$$
$$\mathbb{E}_u\bigl(g(u_0)g(u_1)\bigr)=g(u)\mathfrak{P}_1 g(u), \qquad \mathbb{E}_u\bigl(\zeta_1^- g(u_0)\bigr)=g(u)\int_0^1\mathfrak{P}_s f(u)\,ds$$

belong to the space $C^\varepsilon(H,v_\delta)$ with arbitrarily small ε and δ. Furthermore, using the Markov property, we write

$$\mathbb{E}_u\left(\int_0^1 f(u_s)\,ds\right)^2=\int_0^1\int_0^1\mathbb{E}_u\,\bigl(f(u_s)f(u_t)\bigr)\,dsdt$$
$$=2\int_0^1\int_0^t\mathfrak{P}_s\bigl(f\mathfrak{P}_{t-s}f\bigr)(u)\,dsdt,$$
$$\mathbb{E}_u\left(g(u_1)\int_0^1 f(u_s)\,ds\right)=\int_0^1\mathfrak{P}_s\bigl(f\mathfrak{P}_{1-s}g\bigr)(u)\,ds.$$

Applying again Lemma 3.2, we see that these two functions are also elements of $C^\varepsilon(H,v_\delta)$. What has been said implies that $\varphi\in C^\varepsilon(H,v_\delta)$. Moreover, it is easy to see that

$$|\varphi|_{v_\delta,\varepsilon}\le d\bigl(|f|_{w_p,\alpha}\bigr), \quad (3.19)$$

where $d(r)$, $r\ge 0$, is a non-negative continuous function.

Step 3. We now note that

$$\mathbb{E}_u\bigl|k^{-1}V_k^2-(\varphi,\mu)\bigr|^2\le C_7 d^2\bigl(|f|_{w_p,\alpha}\bigr)\,v_{2\delta}(|u|)\,k^{-1}, \quad k\ge 1. \quad (3.20)$$

Indeed, using inequality (1.6) with $f=\varphi$ and repeating the argument in the derivation of (2.9), we readily show that

$$\mathbb{E}_u\bigl|k^{-1}V_k^2-(\varphi,\mu)\bigr|^2\le C_7|\varphi|_{v_\delta,\varepsilon}^2\,v_{2\delta}(|u|)\,k^{-1},$$

Combining this with (3.19), we obtain (3.20).

Suppose that we have established the relation

$$(\varphi,\mu)=\sigma_f^2, \quad (3.21)$$

where σ_f is defined in (1.8). If $\sigma_f\ge\bar\sigma$, then substituting inequality (3.20) with $k=\hat t$ and relation (3.21) into (2.24), we obtain (1.9). Similarly, if $\sigma_f=0$, then applying the Cauchy inequality to estimate the second term on the right-hand side of (2.25) and using inequality (3.20) in which $(\varphi,\mu)=0$ and $k=\hat t$, we get (1.10). Thus, Theorem 1.4 will be established if we prove (3.21).

Step 4. Let us recall that $\mathbb{E}_\mu(\cdot) = \int_H \mathbb{E}_v(\cdot)\,\mu(dv)$. Therefore, in view of (3.15), we have

$$(\varphi,\mu) = \mathbb{E}_\mu\left(\zeta_1^- + g(u_1) - g(u_0)\right)^2. \tag{3.22}$$

On multiplying out the brackets, we represent the right-hand side of (3.22) as a sum of six terms (cf. relation (3.18) with $\mathbb{E}_u$ replaced by $\mathbb{E}_\mu$). Let us calculate each of them. Using the Markov property and the stationarity of μ, we derive

$$\begin{aligned}
\mathbb{E}_\mu\left(\zeta_1^-\right)^2 &= \int_0^1\int_0^1 \mathbb{E}_\mu\left(f(u_s)f(u_t)\right)dsdt = 2\int_0^1\int_0^t \mathbb{E}_\mu\left(f(u_s)\mathbb{E}_\mu\left(f(u_t)\,|\,\mathcal{F}_s\right)\right)dsdt\\
&= 2\int_0^1\int_0^t \mathbb{E}_\mu\left(f(u_s)\mathfrak{P}_{t-s}f(u_s)\right)dsdt = 2\int_0^1\int_0^t (f\mathfrak{P}_{t-s}f,\mu)\,dsdt\\
&= 2\int_0^1 (1-s)(f\mathfrak{P}_s f,\mu)\,ds.
\end{aligned}\tag{3.23}$$

Similar arguments combined with the relation

$$\mathfrak{P}_s g = \int_0^\infty \mathfrak{P}_{s+t} f\,dt = g - \int_0^s \mathfrak{P}_t f\,dt$$

enable one to show that

$$\begin{aligned}
\mathbb{E}_\mu g^2(u_0) &= \mathbb{E}_\mu g^2(u_1) = (g^2,\mu),\\
\mathbb{E}_\mu\left(g(u_0)g(u_1)\right) &= (g\mathfrak{P}_1 g,\mu) = (g^2,\mu) - \int_0^1 (g\mathfrak{P}_s f,\mu)\,ds,\\
\mathbb{E}_\mu\left(\zeta_1^- g(u_0)\right) &= \int_0^1 (g\mathfrak{P}_s f,\mu)\,ds,\\
\mathbb{E}_\mu\left(\zeta_1^- g(u_1)\right) &= \int_0^1 (f\mathfrak{P}_s g,\mu)\,ds = (fg,\mu) - \int_0^1 (1-s)(f\mathfrak{P}_s f,\mu)\,ds.
\end{aligned}$$

Substituting these relations together with (3.23) into (3.22), we derive

$$(\varphi,\mu) = 2(fg,\mu).$$

Recalling the definition of σ_f (see (1.8)), we see that this relation coincides with (3.21). The proof of Theorem 1.4 is complete.

4 Appendix

4.1 Proof of Lemma 2.6

We confine ourselves to the case $\sigma > 0$, since the proof for $\sigma = 0$ is similar. For any $\varepsilon > 0$ and $z \in \mathbb{R}$, we have

$$F_\xi(z) \le \mathbb{P}\{\xi \le z, |\xi-\eta| \le \varepsilon\} + \mathbb{P}\{|\xi-\eta| > \varepsilon\} \le F_\eta(z+\varepsilon) + \mathbb{P}\{|\xi-\eta| > \varepsilon\}.$$

It follows that

$$\begin{aligned}\Delta_\sigma(\xi, z) &\le \Delta_\sigma(\eta, z+\varepsilon) + \Phi_\sigma(z+\varepsilon) - \Phi_\sigma(z) + \mathbb{P}\{|\xi-\eta| > \varepsilon\}\\ &\le \Delta_\sigma(\eta, z+\varepsilon) + \frac{\varepsilon}{\sigma\sqrt{2\pi}} + \mathbb{P}\{|\xi-\eta| > \varepsilon\}.\end{aligned} \tag{4.1}$$

Interchanging the roles of ξ and η and replacing z by $z-\varepsilon$, we obtain

$$\Delta_\sigma(\xi, z) \ge \Delta_\sigma(\eta, z-\varepsilon) - \frac{\varepsilon}{\sigma\sqrt{2\pi}} - \mathbb{P}\{|\xi-\eta| > \varepsilon\}. \tag{4.2}$$

Combining (4.1) and (4.2), we derive (2.30).

4.2 Proof of Proposition 2.7

As is shown in [8] (see inequality (3.74) with $\Delta = n^{-2/5}$ and $p=2$),

$$\sup_{z\in\mathbb{R}} \left|\Delta_1(n^{-\frac{1}{2}} M_n, z)\right| \le A\, n^{-\frac{1}{5}} + \mathbb{P}\left\{|n^{-1}V_n^2 - 1| > n^{-\frac{2}{5}}\right\}, \tag{4.3}$$

where $A > 0$ is a constant depending only on B and β. Applying the Chebyshev inequality to the second term on the right-hand side of (4.3), we obtain (2.36) with $\sigma = 1$.

To prove (2.36) for an arbitrary $\sigma > 0$, let us note that

$$\Delta_\sigma(n^{-\frac{1}{2}} M_n, z) = \Delta_1(n^{-\frac{1}{2}} M_n(\sigma), \sigma^{-1} z),$$

where we set $M_n(\sigma) = M_n/\sigma$. It follows from the estimate

$$|M_n(\sigma)|^{\frac{\beta}{2}} \le |M_n|^\beta + \frac{1}{4\sigma^\beta}$$

that the zero-mean martingale $M_n(\sigma)$ satisfies inequality (2.35) with β and B replaced by $\frac{\beta}{2}$ and $B_\sigma := B\exp(\frac{1}{4\sigma^\beta})$, respectively.

Let us fix $\bar\sigma > 0$. The constants B_σ are uniformly bounded for $\sigma \ge \bar\sigma$. Hence, by inequality (2.36) with $\sigma = 1$, there is $A(\bar\sigma) > 0$ such that

$$\begin{aligned}\sup_{z\in\mathbb{R}} \left|\Delta_\sigma(n^{-\frac{1}{2}} M_n, z)\right| &= \sup_{z\in\mathbb{R}} \left|\Delta_1(n^{-\frac{1}{2}} M_n(\sigma), \sigma^{-1} z)\right|\\ &\le A(\bar\sigma) n^{-\frac{1}{5}} + n^{\frac{4}{5}} \mathbb{E}\left|n^{-1} V_n^2(\sigma) - 1\right|^2,\end{aligned}$$

where $V_n^2(\sigma)$ is the conditional variance for $M_n(\sigma)$. It remains to note that $V_n^2(\sigma) = \sigma^{-2} V_n^2$. The proof is complete.

4.3 Proof of Lemma 3.2

We shall confine ourselves to the proof of (i), since assertion (ii) can be established using similar ideas.

Let us fix arbitrary constants $\alpha \in (0,1]$, $p > 0$, $\delta > 0$ and a functional $f \in C^\alpha(H, w_p)$ with norm $|f|_{w_p,\alpha} \le 1$. The continuity of operator (3.16) and uniform boundedness of its norm will be established if we show that, for some appropriate positive constants ε, c, C and any $u, v \in H$, we have

$$\big|\mathfrak{P}_t f(u)\big| \le C\, e^{\delta|u|^2}, \quad t \ge 0, \tag{4.4}$$

$$\big|\mathfrak{P}_t f(u) - \mathfrak{P}_t f(v)\big| \le C\,|u-v|^{\varepsilon}\, e^{ct+\delta(|u|^2+|v|^2)}, \quad t \ge 0. \tag{4.5}$$

Inequality (4.4) follows immediately from Lemma 3.1. To prove (4.5), let us denote by u_t and v_t the solutions of the problem (0.3), (0.4) that correspond to the initial functions u and v, respectively. Then the difference $u_t - v_t$ satisfies the inequality (see [22])

$$|u_t - v_t| \le |u-v|\, \exp\Big(C_1 \int_0^t \|u_s\|^2 ds\Big), \quad t \ge 0. \tag{4.6}$$

Now note that, for any $\varepsilon \in (0, \alpha]$ we have

$$\begin{aligned}\big|f(u_t) - f(v_t)\big| &\le |u_t - v_t|^{\alpha}\big(w_p(|u_t|) + w_p(|v_t|)\big)\\ &\le C_2 |u_t - v_t|^{\varepsilon}\big(1 + |u_t| + |v_t|\big)^{p+1}.\end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7), we derive

$$\begin{aligned}\big|\mathfrak{P}_t f(u) - \mathfrak{P}_t f(v)\big| &\le \mathbb{E}\big|f(u_t) - f(v_t)\big|\\ &\le C_3\,|u-v|^{\varepsilon}\, \mathbb{E}\exp\Big(\varepsilon C_1 \int_0^t \|u_s\|^2 ds + \varepsilon(|u_t|^2 + |v_t|^2)\Big).\end{aligned} \tag{4.8}$$

Using Lemma 2.3 in [15] and repeating the arguments used in the proof of (3.4), it is easy to show that, if $\varepsilon > 0$ is sufficiently small, then the expectation on the right-hand side of (4.8) does not exceed $e^{ct+\delta(|u|^2+|v|^2)}$. This completes the proof of (4.5).

We now fix a functional $f \in C^{\alpha}(H, w_p)$ such that $|f|_{w_p,\alpha} \le 1$ and $(f, \mu) = 0$ and consider the function $g(u)$. We have (cf. (2.29))

$$|g(u)| \le C_4\big(1 + |u|\big)^m, \quad u \in H. \tag{4.9}$$

Furthermore, it follows from (3.9) and (4.5) that, for any $u, v \in H$ and $T > 0$,

$$\begin{aligned}\big|g(u) - g(v)\big| &\le \int_0^T \big|\mathfrak{P}_t f(u) - \mathfrak{P}_t f(v)\big|\, dt + \int_T^{\infty} \big(|\mathfrak{P}_t f(u)| + |\mathfrak{P}_t f(v)|\big)\, dt\\ &\le C_5\,|u-v|^{\varepsilon}\, e^{cT+\delta(|u|^2+|v|^2)} + C_6 e^{-\beta T}(1 + |u| + |v|)^m.\end{aligned} \tag{4.10}$$

Minimizing the right-hand side of (4.10) with respect to T, we obtain

$$\big|g(u) - g(v)\big| \le C_7\,|u-v|^{\nu}\, e^{\delta(|u|^2+|v|^2)}.$$

where $\nu = \frac{\beta\varepsilon}{\beta+c}$. This completes the proof of assertion (i).

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