Bounded harmonic maps

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Abstract

The classical Fatou theorem identifies bounded harmonic functions on the unit disk with bounded measurable functions on the boundary circle. We extend this theorem to bounded harmonic maps.

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Contents

1 Introduction

The aim of this paper is to present an extension to harmonic maps of a classical theorem for harmonic functions due to Fatou around 1905-1910.

1.1 The Fatou theorem

We first recall the classical theorem of Fatou for bounded harmonic functions on the Euclidean disk.

This theorem deals with the unit open ball B and with the unit sphere $S = \partial B$ in the Euclidean space $\mathbb{R}^k$ with $k = 2$ or, more generally, with $k \geq 2$. It identifies the space $\mathcal{H}_b(B, \mathbb{R})$ of bounded harmonic functions $h : B \to \mathbb{R}$ on the ball with the space $L^\infty(\partial B, \mathbb{R})$ of bounded measurable functions on the boundary ∂B. We recall that a harmonic function on B is a C^2-function h that satisfies $\Delta_0 h = 0$, where Δ_0 is the Euclidean Laplacian. We denote by σ_0 the rotationally invariant probability measure on the sphere ∂B, and we refer to Section 1.3 for the definition of a non-tangential limit.

Fact 1.1. (Fatou) *a) Let $h : B \to \mathbb{R}$ be a bounded harmonic function. For σ_0-almost all ξ in ∂B, the function h admits a non-tangential limit $\varphi(\xi) := \operatorname*{NTlim}_{x \to \xi} h(x)$ at the point ξ.*
b) The map $h \mapsto \varphi$ is a bijection $\beta : \mathcal{H}_b(B, \mathbb{R}) \to L^\infty(\partial B, \mathbb{R})$ called the boundary transform.

The inverse of the map β is given by an explicit formula, the Poisson formula. For every $\varphi \in L^\infty(\partial B, \mathbb{R})$, one can indeed recover h as $h = P_0\varphi$ where $P_0\varphi$ is the bounded harmonic function defined on B by

$$P_0\varphi(x) := \int_{\partial B} \varphi(\xi)\, P_{0,\xi}(x)\, \mathrm{d}\sigma_0(\xi), \quad \text{where } \; P_{0,\xi}(x) = \frac{1 - |x|^2}{|x - \xi|^k}$$

is the Poisson kernel.

The proof of this fact can be found in Rudin's book [34, Chap. 11] or in [27] when $k = 2$, or in Armitage and Gardiner's book [5, Chap. 4] for $k \geq 2$.

Proving extensions of this fact has a long history that has already lasted for more than a century. Indeed, an important goal of Potential Theory is to understand to what extent this fact still holds for either harmonic or superharmonic functions, on more general spaces.

The aim of this paper is to extend Fatou's theorem to bounded harmonic maps. We will allow the target space Y to be any complete CAT(0) space, the first examples to have in mind being the hyperbolic spaces $\mathbb{H}^k$.

We will also allow more general source spaces. Since, in dimension $k = 2$, the harmonicity condition depends only on the conformal structure on the source space B, we can think of B as the hyperbolic plane. We will explain in Theorem 1.3 how to replace B by a **GGG** Riemannian manifold X. Later on, we will also explain in Corollary 5.6 how to replace B by any bounded Riemannian domain Ω with Lipschitz boundary.

1.2 Main result

We now state our main result, postponing the definitions to Section 1.3.

Definition 1.2. We will say that a Riemannian manifold X is **GGG** as a shortcut for **G**romov Hyperbolic with Bounded **G**eometry and Spectral **G**ap.

Theorem 1.3. *Let X be a* **GGG** *Riemannian manifold, and Y be a proper* CAT(0)*-space.*
a) Let $h\colon X\to Y$ be a bounded harmonic map. Then, for σ-almost all $\xi\in\partial X$, the map h admits a non-tangential limit $\varphi(\xi) := \operatorname{NTlim}_{x\to\xi} h(x)$ at the point ξ.
b) The map $h\mapsto\varphi$ is a bijection $\beta : \mathcal{H}_b(X,Y)\to L^\infty(\partial X, Y)$.

The main examples of **GGG** Riemannian manifolds X are the pinched Hadamard manifolds : those have negative curvature. The condition **GGG** allows a little bit of positive curvature on X. It also allows X to be non-contractible. For instance, the quotient of a pinched Hadamard manifold by a convex cocompact group of isometries is **GGG** .

1.3 Main definitions

Here are the definitions that are needed to understand our theorem 1.3. All our manifolds will be assumed to be connected and with dimension $k\geq 2$.

Definition 1.4. A Riemannian manifold X has *bounded geometry* if it is complete, with bounded sectional curvature $-K_{max}\leq K_X\leq K_{max}$, and if the injectivity radius has a uniform lower bound $\mathrm{inj}_X\geq r_{min}>0$.

As explained in [25, Section 1.1], one could replace in Definition 1.4 the bound on the sectional curvature by a bound on the Ricci curvature. Indeed, the important features of bounded geometry also hold if we just have a bound on the Ricci curvature.

Definition 1.5. The Riemannian manifold X is Gromov hyperbolic if there exists $\delta>0$ such that, for all o, x, y, z in X one has

$$(x|z)_o \geq \min((x|y)_o, (y|z)_o) - \delta. \tag{1.1}$$

Here $(x|y)_o := \frac{1}{2}(d(o,x)+d(o,y)-d(x,y))$ is the Gromov product of the points x and y seen from o.

In this case ∂X will denote the "Gromov boundary" of X and $\overline{X} = X\cup\partial X$ will be the "Gromov compactification" of X where ∂X is the set of geodesic rays on X, two geodesic rays being identified if they remain within bounded distance from each other. See [18].

Definition 1.6. A Riemannian manifold X has a *spectral gap*, or a *coercive Laplacian*, if the Rayleigh quotients admit a uniform lower bound

$$\lambda_1 := \inf_{\varphi\in C_c^\infty(X)} \frac{\int_X \|\nabla\varphi\|^2\,\mathrm{d}v_g}{\int_X \varphi^2\,\mathrm{d}v_g} \;>\; 0. \tag{1.2}$$

Note that the spectral gap implies that X is non-compact.

Definition 1.7. A pinched Hadamard manifold X is a complete simply-connected Riemannian manifold with dimension at least 2 whose sectional curvature is pinched between two negative constants : $-b^2 \leq K_X \leq -a^2 < 0$.

Examples are : hyperbolic spaces $\mathbb{H}^k$, rank one non-compact Riemannian symmetric spaces, any small perturbation of those...

Definition 1.8. A CAT(0) space Y is a geodesic metric space such that, for every geodesic triangle T in Y, there exists a 1-Lipschitz map $j : T_0 \to T$ where T_0 is the triangle the Euclidean plane with same side lengths as T and j sends each vertex of T_0 to the corresponding vertex of T. See [11].

Examples are : Hadamard manifolds (namely, complete and simply connected Riemannian manifolds with non positive curvature), Euclidean buidings, $\mathbb{R}$-trees, convex subsets in Hilbert spaces...

It is not restrictive to assume that Y is complete, since the metric completion of a CAT(0) space still is a CAT(0) space. Since the closed balls $B(y_0, R)$ in Y are also CAT(0) spaces, this will allow us to assume in the proof of Theorem 1.3 that Y is bounded. We will sometimes assume that Y is proper, i.e. that these closed balls $B(y_0, R)$ are compact.

Definition 1.9. A map $h : X \to Y$ is (energy minimizing) harmonic if it is locally Lipschitz continuous and if it is a minimum for the Korevaar-Schoen energy $E(h)$ with respect to variations of h with compact support $Z \subset X$.

When Y is a CAT(0) Riemannian manifold, the Korevaar-Schoen energy on Z coincides with the Dirichlet energy $E(h) = \int_Z |Dh(x)|^2 dv_g(x)$. In this case, the harmonicity condition can be expressed by a partial differential equation which is not linear any more, see [16], [21] or [23]. When Y is only a CAT(0) space, the energy of h on Z is the integral $E(h) = \int_Z e_h(x) dv_g(x)$ of the energy density e_h where, for a Lipschitz continuous map h, the energy density is given by $e_h(x) = \limsup_{\varepsilon \to 0} \varepsilon^{-2-k} v_k^{-1} \int_{B(x,\varepsilon)} d(h(x), h(x'))^2 dv_g(x')$, where v_k is the volume of the unit Euclidean ball and where this limit should be understood in a weak sense. See [28, Section 1.5] for a precise definition. See also [24].

The measure σ refers to any finite Borel measure on ∂X which is equivalent to the harmonic measures on ∂X. The "σ-almost surely" means that the property holds except on a set of measure zero for the measure σ on ∂X. Note that, when X is a pinched Hadamard manifold, such a measure

σ is often found to be singular with respect to the Lebesgue measure on the sphere ∂X.

The set $\mathcal{H}_b(X,Y)$ is the set of bounded harmonic maps $h : X \to Y$, and the set $L^\infty(\partial X, Y)$ is the set of bounded measurable maps from ∂X to Y where two measurable maps are identified if they are σ-almost surely equal.

Definition 1.10. A function $h : X \to Y$ has a non-tangential limit y at a point $\xi \in \partial X$ (also called a conical limit), and we write $y = \operatorname{NTlim}_{x\to\xi} h(x)$, if $y = \lim_{n\to\infty} h(x_n)$ holds for any sequence (x_n) in X converging non-tangentially to ξ, i.e. such that $\sup_{n\geq 1} d(x_n, o\xi) < \infty$ where $o\xi$ is any geodesic ray from a point $o \in X$ to ξ.

1.4 Previous results

When $Y = \mathbb{R}$, we are dealing with harmonic functions. As we have already seen in Section 1.1, Theorem 1.3 for $X = B$ is the classical Fatou theorem. The extension to the case where X is a pinched Hadamard manifold appeared in the $80's$ and is due to Anderson and Schoen in [4]. The extension to the case where X is a **GGG** Riemannian manifold is due to Ancona in [3].

When Y is a CAT(0) Riemannian manifold and X is a pinched Hadamard manifold, Theorem 1.3.a is due to Aviles, Choi, Micallef in [6, Thm 5.1], and Theorem 1.3.b is conjectured to be true by these authors. Indeed, as a final observation in [6, Section 1] they write that such a theorem would be "a consequence of the solvability of the Dirichlet problem with L^∞ boundary condition". This solvability is one of the main technical issues in our paper (Proposition 1.18). Note that the solvability of the Dirichlet problem with continuous boundary condition is proven in [6, Thm 3.2]. The first case of Theorem 1.3.b that seems to be new is when both X and Y are the hyperbolic plane $\mathbb{H}^2$.

When Y is a CAT(0) space, the proof of Theorem 1.3 will rely on the solution of the Dirichlet problem for harmonic maps with values in a CAT(0) space under Lipschitz continuous boundary condition, due to Korevaar and Schoen in [28], a result that extends the Hamilton theorem in [21].

Remark 1.11. Note that we cannot assume Y to be only locally CAT(0). The fact that Y is simply connected will be important here. Indeed, it is not clear how to parametrize the set of harmonic maps from the unit disk to a compact hyperbolic surface. Similarly, it is not clear how to parametrize the set of all harmonic maps from the unit disk to the circle $\mathbb{R}/\mathbb{Z}$, because this is equivalent to parametrizing all the harmonic functions on the unit disk.

Remark 1.12. Theorem 1.3 is an analog of the theorems that parametrize unbounded harmonic maps between pinched Hadamard manifolds by their "quasi-symmetric" boundary condition at infinity. See the successive papers [32], [30], [8], [9], and [35]. that deal with an increasing level of generality.

1.5 Strategy of proof

We will split the statement of Theorem 1.3 into five propositions.

Proposition 1.13. (Construction of the boundary map) *Let X be a* **GGG** *Riemannian manifold and let Y be a proper* CAT(0)*-space. Let $h : X \to Y$ be a bounded harmonic map. Then, for σ-almost all $\xi \in \partial X$, the map h admits a non-tangential limit $\varphi(\xi) := \operatorname{NTlim}_{x\to\xi} h(x)$ at the point ξ.*

We denote by

$$\beta h := \varphi \in L^\infty(\partial X, Y) \tag{1.3}$$

the bounded measurable map from ∂X to Y given by Proposition 1.13. This map βh is called the boundary map of h, and the map

$$\beta : \mathcal{H}_b(X,Y) \to L^\infty(\partial X, Y)$$

is called the boundary transform.

Proposition 1.14. (Injectivity of the boundary transform) *Same notation. Two harmonic maps h, h' from X to Y with $\beta h = \beta h'$ are equal.*

In order to prove that the transformation β is onto, we will construct its inverse map P. We first rely on the theorem that solves the Dirichlet problem for harmonic maps with regular boundary data. It is due to Hamilton in [21] when the target is a manifold, and to Korevaar-Schoen in [28, Thm 2.2] when the target is a CAT(0) space.

Fact 1.15. (Hamilton, Korevaar and Schoen)
Let Ω be a bounded Lipschitz Riemannian domain and Y be a complete CAT(0) *space. Let $\varphi : \partial\Omega \to Y$ be a Lipschitz map. Then, there exists a unique harmonic map $h = P\varphi$ from Ω to Y that extends continuously φ.*

We then need to extend Fact 1.15 to continuous boundary data, and to deal with a boundary at infinity. This is included in the following proposition wich will be proven in Section 4.2. Note that when X is a pinched Hadamard manifold and Y is a CAT(0) Riemannian manifold, this proposition is already in [6, Thm 3.2 and 4.7].

Proposition 1.16. (Dirichlet problem with continuous data) *Let X be a* **GGG** *Riemannian manifold and Y be a complete* CAT(0)*-space. Let $\varphi : \partial X \to Y$ be a continuous map. Then, there exists a unique harmonic map $h = P\varphi$ from X to Y that extends continuously φ.*

The conclusion in Proposition 1.16 means that the map $\overline{h} : \overline{X} \to Y$ that is equal to h on X and to φ on ∂X is continuous. Idem for Fact 1.15.

The main result in this article, Theorem 1.3, extends Proposition 1.16 to more general boundary conditions φ. Indeed, it allows φ to be any bounded measurable map from ∂X to Y. As will be very clear in the next proposition, the proof of our main Theorem 1.3 relies on the Hamilton, Korevaar, Schoen theorem.

We endow the space $L^\infty(\partial X, Y)$ with "the topology of the convergence in probability", see (4.1). The subspace $C(\partial X, Y)$ of continuous maps is then dense in $L^\infty(\partial X, Y)$, see Lemma 4.2.

We also endow the space $\mathcal{H}_b(X, Y)$ of bounded harmonic maps $h : X \to Y$ with the topology of the uniform convergence on compact subsets of X.

Proposition 1.17. (Construction of the Poisson transform) *Let X be a* **GGG** *Riemannian manifold and Y be a bounded complete* CAT(0)*-space. The map*

$$P : C(\partial X, Y) \to \mathcal{H}_b(X, Y)$$

given by Proposition 1.16 *has a unique continuous extension*

$$P : L^\infty(\partial X, Y) \to \mathcal{H}_b(X, Y).$$

We still call the extended map P the Poisson transform.

Proposition 1.18. (Surjectivity of the boundary transform) *Let X be a* **GGG** *Riemannian manifold and Y be a compact* CAT(0)*-space. For all $\varphi \in L^\infty(\partial X, Y)$, one has $\varphi = \beta P\varphi$.*

1.6 Overview

In Chapter 2, we recall preliminary facts about harmonic, subharmonic and superharmonic functions u on a **GGG** Riemannian manifold. The key points that we will use are a control on the Poisson kernel in Proposition 2.8, upper bounds on the harmonic measures in Lemmas 2.15 and 2.17, and the existence of non-tangential limit for bounded Lipschitz superharmonic functions on X in Proposition 2.18.

In Chapter 3, we recall two facts about harmonic maps. The first one is the control, due to Cheng, of the Lipschitz constant of a harmonic map (Lemma 3.1). The second one is the subharmonicity of the distance function between two harmonic maps (Lemma 3.2). We use these two facts, together with Proposition 2.18, to prove the existence of non-tangential limit for our bounded harmonic map $h : X \to Y$ (Proposition 1.13). This provides the construction of the boundary map $\varphi = \beta h : \partial X \to Y$. These arguments also prove that the boundary transform $\beta : h \mapsto \varphi$ is injective (Proposition 1.14).

In Chapter 4, we first construct the Poisson transform $P : \varphi \mapsto h$ when the boundary data φ is continuous (Proposition 1.16) by building on the Hamilton, Korevaar and Schoen theorem (Fact 1.15). We then extend this transform $P : \varphi \mapsto h$ to bounded measurable boundary data φ (Proposition 1.17). The key point is a suitable uniform continuity property of this transform $\varphi \mapsto h$.

In Chapter 5 we prove that the Poisson transform P is a right inverse for the boundary transform β, so that the boundary transform β is surjective (Proposition 1.18). The key point is an estimate on sequences of subharmonic functions (Lemma 5.3) that relies on the control of the Poisson kernel P_ξ in Proposition 2.8 and on the Lebesgue density theorem for a doubling measure on a compact quasi-metric space (Fact 5.2).

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2 Harmonic and subharmonic functions

In this second chapter, we gather a few results concerning harmonic and subharmonic functions on a **GGG** Riemannian manifold X that will be used in the proof ot Theorem 1.3.

We set g for the Riemannian metric, d for the Riemannian distance, Δ for the Laplace Beltrami operator and $k = \dim X$. For the Potential theory of a **GGG** Riemannian manifold, we refer to the seminal paper [3] and to its recent update [25].

2.1 The Harnack inequality and the Green function

In this section we present three classical Harnack inequalities for positive harmonic functions.

We recall that a function $u : X \to \mathbb{R}$ is superharmonic if it is lower semicontinuous, locally integrable, and if $\Delta u \leq 0$ holds in the weak sense. A function u is subharmonic if $-u$ is superharmonic. A function u is harmonic if it is both subharmonic and superharmonic

The Harnack inequality, which has been improved by Serrin and by S.T. Yau, gives a uniform control for positive harmonic functions on compact sets. See [31, Lemma 2.1] for a short proof, and also [20, Cor. 8.21].

Fact 2.1. (Harnack inequality) *Let X be a complete Riemannian manifold with bounded sectional curvature. There exists a constant $c_0 > 0$ such that, for any positive harmonic function u on a ball $B(x_0, r)$ with $r \leq 1$, one has*

$$\|\nabla \log u(x)\| \ \leq \ c_0/r \quad \text{for all } x \text{ in } B(x_0, r/2).$$

One then has

$$u(y) \ \leq \ e^{c_0} u(x) \quad \text{for all } x,\, y \text{ in } B(x_0, r/2).$$

The Green operator G is the "inverse" of the Laplacian. The spectral gap assumption ensures that the Green operator is bounded as an operator on $L^2(X)$. The Green kernel $G(x, y)$ is the kernel of the Green operator. It is is symmetric i.e. $G(x, y) = G(y, x)$. It is a positive C^∞-function on $X \times X \smallsetminus \Delta_X$ and, for each x in X, the function $G_x := G(x, .)$ satisfies $\Delta G_x = -\delta_x$. In particular the function G_x is harmonic outside $\{x\}$.

The bounded geometry assumption ensures the following control on the Green function. We set $\log_*(t) := \max(1, \log t)$.

Fact 2.2. (Green function) *Let X be a Riemannian manifold with bounded geometry and with spectral gap. There exist $C_0 > 1$ and $\varepsilon_0 > 0$ such that :*
a) For x, y in X with $d(x, y) \leq 1$, one has

$$\begin{array}{rl} C_0^{-1} d(x,y)^{2-k} \leq G(x,y) \leq C_0\, d(x,y)^{2-k} & \text{if } k \neq 2, \\ C_0^{-1} \log_*(1/d(x,y)) \leq G(x,y) \leq C_0 \log_*(1/d(x,y)) & \text{if } k = 2. \end{array}$$

b) For x, y in X with $d(x, y) \geq 1$, one has

$$G(x,y) \ \leq \ C_0\, e^{-\varepsilon_0\, d(x,y)} \tag{2.1}$$

See for instance [25, Prop. 2.7 and 2.12].

2.2 The Ancona Inequality

The Gromov hyperbolicity assumption ensures a much more precise control on the Green function due to Ancona.

Fact 2.3. (Ancona Inequality) *Let X be a* **GGG** *Riemannian manifold. Then, there exists $C_1 > 1$ such that for any point y on a geodesic segment $[x, z]$ in X such that $d(x, y) \geq 1$ and $d(y, z) \geq 1$ one has*

$$C_1^{-1}\, G(x, y)\, G(y, z) \ \leq\ G(x, z) \ \leq\ C_1\, G(x, y)\, G(y, z) \tag{2.2}$$

The boundary Harnack inequality compares the behavior of two positive harmonic functions near a piece of the boundary ∂X where they both go to zero. In order to state this inequality we need to introduce some notation.

We first recall the definition of the Gromov product for two points η_1, η_2 in $\overline{X} = X \cup \partial X$ seen from a point $o \in X$:

$$(\eta_1|\eta_2)_o := \limsup_{\substack{x_1 \to \eta_1 \\ x_2 \to \eta_2}} (x_1|x_2)_o.$$

This quantity is equal, up to a uniformly bounded error term, to the distance between o and a geodesic going from η_1 to η_2.

For x in $\overline{X}$, we introduce the sets

$$\mathcal{H}_o^t(x) = \{y \in X \mid (y|x)_o \geq t\},$$

$$\overline{\mathcal{H}}_o^t(x) = \{y \in \overline{X} \mid (y|x)_o \geq t\}.$$

Note that these sets are empty when $d(o, x) < t$.

We recall that the topology of $\overline{X}$ is the topology that extends the topology of X and such that a neighborhood basis of a point $\xi \in \partial X$ is given by the sets $\overline{\mathcal{H}}_o^t(\xi)$ with $t > 0$. See [18]. We will call them (rough) half-spaces.

We also recall that, since X is Gromov hyperbolic, we can choose $\delta > 1$ satisfying (1.1) and such that, for every geodesic triangle x, y, z in X, every point u on the edge xz is at distance at most δ of the union $xy \cup yz$ of the other edges.

We record a five properties of these half-spaces :

$$\overline{\mathcal{H}}_x^s(y) \cup \overline{\mathcal{H}}_y^t(x) \ = \ \overline{X} \ \text{ for all } x,\, y \text{ in } X \text{ with } d(x, y) \geq s + t, \tag{2.3}$$

$$\overline{\mathcal{H}}_x^s(y) \cap \overline{\mathcal{H}}_y^t(x) \ = \ \emptyset \ \text{ for all } x,\, y \text{ in } X \text{ with } d(x, y) < s + t. \tag{2.4}$$

$$\overline{\mathcal{H}}_x^t(y) \subset \overline{\mathcal{H}}_x^{t-\delta}(z) \ \text{ for all } x \text{ in } X,\ y,\, z \text{ in } \overline{X} \text{ with } z \in \overline{\mathcal{H}}_x^t(y), \tag{2.5}$$

$$\overline{\mathcal{H}}_x^{t+s}(y) \subset \overline{\mathcal{H}}_m^t(y) \ \text{ for all } m,\, x,\, y \text{ in } X \text{ with } d(x, m) = s\ , \tag{2.6}$$

$$\overline{\mathcal{H}}_m^t(y) \subset \overline{\mathcal{H}}_x^{t+s-\delta}(y) \ \text{ for } m \text{ in } xy \text{ with } d(x, m) = s \text{ and } t > \delta. \tag{2.7}$$

These properties explain why these sets are called half-spaces.

The first four ones are straightforward. For the last one, notice that, since $t > \delta$, for any z in $\overline{\mathcal{H}}^t_m(y)$, the distance between m and a geodesic yz is larger than δ and hence there exists a point m' on a geodesic xz such that $d(m, m') \leq \delta$.

We now state the strong boundary Harnack inequality for a **GGG** Riemannian manifold X. This inequality is actually equivalent to the Ancona Inequality in Fact 2.3.

Fact 2.4. (Strong Boundary Harnack inequality) *Let X be a* **GGG** *Riemannian manifold. There exists $C_2 > 0$ and $t_0 > 0$ such that for all $o \in X$, all $\xi \in \partial X$, all $t \geq 0$ and all positive continuous functions u, v on the half-space $\overline{\mathcal{H}} := \overline{\mathcal{H}}^t_o(\xi)$ which are zero on $\partial X \cap \overline{\mathcal{H}}$ and harmonic in the interior of $X \cap \overline{\mathcal{H}}$, one has*

$$\frac{u(y)}{v(y)} \leq C_2 \frac{u(x)}{v(x)} \quad \textit{for all} \quad x, y \in \mathcal{H}^{t+t_0}_o(\xi). \tag{2.8}$$

Remark 2.5. Fact 2.4 is due to Anderson and Schoen in [4, Corollary 5.2] when X is a pinched Hadamard manifold. It is due to Ancona in [3] when X is a **GGG** Riemannian manifold.

The following statement is a Corollary of Facts 2.1 and 2.4.

Corollary 2.6. (Boundary Harnack inequality) *Let X be a* **GGG** *Riemannian manifold. Let $O \subset \overline{X}$ be a connected open subset and $K \subset O$ be a compact subset. There exists a constant $C = C_{K,O,X} > 0$ such that for all continuous functions u, v on O that are harmonic and positive on $O \cap X$ and zero on $O \cap \partial X$, one has*

$$\frac{u(y)}{v(y)} \leq C \frac{u(x)}{v(x)} \quad \textit{for all} \quad x, y \in K \cap X. \tag{2.9}$$

Note that Fact 2.4 is stronger than Corollary 2.6 since it requires the constant C_2 not to depend on the half-space $\overline{\mathcal{H}}$.

2.3 The Poisson kernel

We now recall the definition of the Poisson kernel, also called the Martin kernel.

The following fact is due to Anderson and Schoen in [4] for a pinched Hadamard manifold, and has been generalized by Ancona in [3] for a **GGG**

Riemannian manifold X. See also [25] for more general measured metric spaces. It describes all the positive harmonic functions on X and, more precisely, it describes the Martin boundary of X. The key point in the proof is the strong boundary Harnack inequality of Fact 2.4.

Fact 2.7. (Martin Boundary) *Let X be a* **GGG** *Riemannian manifold, and fix a point $o \in X$.*
a) For all ξ in ∂X, there exists a unique non-negative continuous function

$$x \mapsto P_\xi(x) = P_\xi(o, x)$$

on $\overline{X} \smallsetminus \{\xi\}$ which is harmonic on X, zero on $\partial X \smallsetminus \{\xi\}$ with $P_\xi(o) = 1$.
b) For $x \in X$ and $\xi \in \partial X$, $P_\xi(x)$ is obtained as the limit

$$P_\xi(x) = \lim_{y \to \xi} \frac{G(x, y)}{G(o, y)} \,. \tag{2.10}$$

c) Any positive harmonic function h on X can be written as

$$h(x) = \int_{\partial X} P_\xi(x) \, \mathrm{d}\mu(\xi)$$

for a unique positive finite Borel measure $\mu = \mu_h$ on ∂X.
d) The function $(x, \xi) \mapsto P_\xi(x)$ is continuous on $\overline{X} \times \partial X \smallsetminus \Delta_{\partial X}$ where $\Delta_{\partial X}$ denotes the diagonal in $\partial X \times \partial X$.

By the very definition of the Poisson functions, the following holds :

$$P_\xi(x, o) = P_\xi(o, x)^{-1}.$$

Those functions P_ξ are exactly the positive harmonic functions on X that are minimal, up to normalization: i.e. the only positive harmonic functions h on X that are bounded by P_ξ are the multiples $h = \alpha P_\xi$ with $\alpha \leq 1$.

In the proof of Proposition 1.18, we will need the following estimate for the Poisson functions.

Proposition 2.8. *Let X be a* **GGG** *Riemannian manifold. Then, there exists a constant $C_3 > 0$ such that for all o in X, all ξ in ∂X, all x on a geodesic ray $o\xi$, and for all η_1, η_2 in ∂X with*

$$|(\xi|\eta_1)_o - (\xi|\eta_2)_o| \leq 1 \,, \tag{2.11}$$

one has

$$C_3^{-1} \leq \frac{P_{\eta_1}(x)}{P_{\eta_2}(x)} \leq C_3 \,. \tag{2.12}$$

We begin by a lemma that extends Ancona's inequality to Poisson functions. Since X is Gromov hyperbolic, for every geodesic triangle with distinct vertices x, y, z in $\overline{X}$, there exists a point m whose distance to any of the three geodesic sides is at most δ. Such a point m is called a center of the triangle x, y, z. It is not unique, but the distance between two centers is at most 8δ. See [18].

Lemma 2.9. *Let X be a* **GGG** *Riemannian manifold. Then, there exists a constant $C_4 > 1$ such that for all ξ in ∂X, all points o, x in X one has*

$$C_4^{-1} \frac{G(m,x)}{G(m,o)} \;\leq\; P_\xi(o,x) \;\leq\; C_4 \frac{G(m,x)}{G(m,o)}, \tag{2.13}$$

where m is a center of a geodesic triangle with vertices o, x, ξ such that $d(o,m) \geq 1$ and $d(x,m) \geq 1$.

Note that it is possible to choose such a center m since X is Gromov hyperbolic with constant $\delta > 1$.

A particular instance of (2.13), when x is on a geodesic ray from o to ξ and $d(o,x) \geq 1$, reads as

$$C_4^{-1} G(o,x)^{-1} \;\leq\; P_\xi(o,x) \;\leq\; C_4\, G(o,x)^{-1}. \tag{2.14}$$

Proof. The center m is at a distance at most δ from both a geodesic ray $o\xi$ and a geodesic ray $x\xi$. Therefore when a point $y \in X$ is sufficiently near ξ, the point m is at distance at most 2δ from both the geodesic rays oy and xy. Therefore, using the Harnack inequality in Fact 2.1 and the Ancona inequality in Fact 2.3 one gets, with a constant c_4 depending only on X :

$$c_4^{-1}\, G(x,m)G(m,y) \;\leq\; G(x,y) \;\leq\; c_4\, G(x,m)G(m,y),$$
$$c_4^{-1}\, G(o,m)G(m,y) \;\leq\; G(o,y) \;\leq\; c_4\, G(o,m)G(m,y).$$

Taking the ratio of these estimates yields, with $C_4 = c_4^2$,

$$C_4^{-1} \frac{G(x,m)}{G(o,m)} \;\leq\; \frac{G(x,y)}{G(o,y)} \;\leq\; C_4 \frac{G(x,m)}{G(o,m)}.$$

One then gets (2.13) by letting y converge to ξ. □

Proof of Proposition 2.8. Let m_1 be a center of the triangle o, x, η_1 and m_2 be a center of the triangle o, x, η_2. Those points can be chosen so that $d(m_i, o) \geq 1$ and $d(m_i, x) \geq 1$. The assumption (2.11) tells us that the distance $d(m_1, m_2)$ is bounded by a constant depending only on δ. Therefore by the Harnack principle in Fact 2.1, there exists a constant $c_3 > 1$ depending only on X such that

$$c_3^{-1} \leq \frac{G(x,m_2)}{G(x,m_1)} \leq c_3 \text{ and } c_3^{-1} \leq \frac{G(o,m_2)}{G(o,m_1)} \leq c_3 \tag{2.15}$$

Taking the ratio of the bound (2.13) with $\xi = \eta_1$ by the bound (2.13) with $\xi = \eta_2$ one gets

$$C_4^{-2} \frac{G(x,m_1)}{G(x,m_2)} \frac{G(o,m_2)}{G(o,m_1)} \leq \frac{P_{\eta_1}(x)}{P_{\eta_2}(x)} \leq C_4^2 \frac{G(x,m_1)}{G(x,m_2)} \frac{G(o,m_2)}{G(o,m_1)}. \tag{2.16}$$

Combining (2.15) with (2.16), one obtains (2.12) with $C_3 = c_3^2 C_4^2$. □

Corollary 2.10. *Let X be a* **GGG** *Riemannian manifold. Then, there exists a constant $C_4 > 1$ such that for all ξ in ∂X, all points x in X and all point y on a ray $x\xi$ with $d(x,y) \geq 1$, one has*

$$C_4{}^{-1} G(x,y) P_\xi(y) \leq P_\xi(x) \leq C_4 G(x,y) P_\xi(y). \tag{2.17}$$

Proof. Since, by definition $P_\xi(x,y) = P_\xi(y)/P_\xi(x)$, inequalities (2.17) are nothing but a reformulation of (2.14). □

See also [4, Cor. 6.4] and [29] for other estimates on the Poisson kernel when X is a pinched Hadamard manifold.

We recall that, in Geometric Group Theory, the word "geodesic" means "minimizing geodesic". This is why the following lemma is non-trivial.

Lemma 2.11. *Let X be a* **GGG** *Riemannian manifold. Then there exists $C_5 > 1$ such that, for all x in X, there exists ξ, η in ∂X with $(\xi|\eta)_x \leq C_5$.*

We do not explicitely use this lemma. It illustrates the influence of the spectral gap condition on the geometry of X. It tells us there are no dead ends in X. Here is a sketch of proof.

Proof. Since X is Gromov hyperbolic, if there were dead ends, we would be able to find, for all $n \geq 1$, a ball $B_n := B(x_n, n)$ of radius n in X whose boundary $S_n := S(x_n, n)$ has diameter at most 2δ. Since X has bounded geometry, the volumes of the balls B_n go to infinity while the diameters of S_n are bounded. This contradicts the spectral gap. □

2.4 The harmonic measures

2.4.1 Harmonic measures for a bounded domain

We first recall the solution of the Dirichlet problem for harmonic functions on a Lipschitz bounded Riemannian domain Ω (see Section 5.5 for a precise definition). This can be found in [20, Chapter 6 and 8] when $\partial\Omega$ is smooth and in [2] when $\partial\Omega$ is Lipschitz continuous. It says:

Fact 2.12. (Dirichlet problem for functions on a bounded domain) *Let Ω be a Lipschitz bounded Riemannian domain. For every continuous function $\varphi \in C(\partial\Omega, \mathbb{R})$, there exists a unique continuous function $h : \overline{\Omega} \to \mathbb{R}$ which is harmonic on Ω and equal to φ on $\partial\Omega$.*

Moreover, for $x \in \Omega$ there exists a measure $\sigma_x = \sigma_x^\Omega$ on $\partial\Omega$, called the harmonic measure on $\partial\Omega$ seen from x, such that the harmonic extension h of φ is given by

$$h(x) := \int_{\partial\Omega} \varphi(\xi)\, \mathrm{d}\sigma_x(\xi). \tag{2.18}$$

By a theorem of Dahlberg in [14], the harmonic measure σ_x^Ω at each point $x \in \Omega$ is equivalent to the Riemannian measure on $\partial\Omega$.

The measure σ_x^Ω is a doubling measure on $\partial\Omega$. This means, see [12, Section 11.3], that there exists a constant $c = c_{\Omega,x}$ such that for all $r > 0$ and all ξ in $\partial\Omega$ one has

$$\sigma_x^\Omega(B(\xi, 2r)) \le c\, \sigma_x^\Omega(B(\xi, r)). \tag{2.19}$$

In this notation, we think of σ_x^Ω as a measure on X supported by $\partial\Omega$.

Remark 2.13. From a probabilistic point of view, the harmonic measure σ_x^Ω on $\partial\Omega$ is the exit probability measure of a Brownian motion on Ω starting at point x.

2.4.2 Harmonic measures on GGG Riemannian manifolds

We now recall the solution of the Dirichlet problem for harmonic functions on a **GGG** Riemannian manifold X. This is independently due to Anderson and Sullivan when X is a pinched Hadamard manifold. See [4] for a nice account. It is due to Ancona when X is a **GGG** Riemannian manifold, as a consequence of the description of the Martin boundary of X.

Fact 2.14. (Dirichlet problem for functions on GGG manifolds) *Let X be a* **GGG** *Riemannian manifold. For every continuous function*

$\varphi \in C(\partial X, \mathbb{R})$, there exists a unique continuous function $h : \overline{X} \to \mathbb{R}$ which is harmonic on X and is equal to φ on ∂X.

Moreover, for $x \in X$ there exists a measure $\sigma_x = \sigma_x^X$ on ∂X, called the harmonic measure on ∂X seen from x, such that the harmonic extension h of φ is given by

$$h(x) := \int_{\partial X} \varphi(\xi)\, \mathrm{d}\sigma_x(\xi). \tag{2.20}$$

For $x = o$, the probability measure σ_o on ∂X is the one that appears in the decomposition of the constant harmonic function $h = 1$ in Fact 2.7.*c*.

For every x in X, the positive measure σ_x^X is given by the formula

$$\mathrm{d}\sigma_x^X(\xi) = P_\xi(x)\, \mathrm{d}\sigma_o^X(\xi)$$

so that Equation (2.20) can be rewritten as

$$h(x) := \int_{\partial X} \varphi(\xi)\, P_\xi(x)\, \mathrm{d}\sigma_o^X(\xi). \tag{2.21}$$

When φ is continuous, the function h defined on X by (2.21) is harmonic and extends continuously φ. Indeed each function $\xi \mapsto P_\xi(x)$, for $x \in X$, is positive and satisfies $\int_{\partial X} P_\xi(x)\, \mathrm{d}\sigma_o^X(\xi) = 1$ and, when a sequence (x_n) converges to $\xi \in \partial X$, the sequence of probability measures $(P_\xi(x_n)\mathrm{d}\sigma_o^X(\xi))$ converges weakly to δ_ξ.

Note that, even when X is a pinched Hadamard manifold, the measure σ_o^X is not always equivalent to the "visual measure".

In order to have shorter notation, we will think of the harmonic measures σ_x^X as measures on $\overline{X}$ supported by ∂X.

2.4.3 Upper bound for the harmonic measures

We will need the following uniform control of the harmonic measures σ_o^Ω for bounded subdomains of X with Lipschitz boundary. By definition, the probability measure σ_o^Ω is supported by the boundary $\partial\Omega$. This control tells us that, seen from o, the measure of the part of $\partial\Omega$ cut out by a half space far away from o is uniformly small.

Lemma 2.15. *Let X be a* **GGG** *Riemannian manifold. For all $\varepsilon > 0$ there exists $\ell = \ell_\varepsilon > 0$ such that for all o in X, x in $\overline{X}$, one has*

$$\sigma_o^X(\overline{\mathcal{H}}_o^\ell(x)) \ \leq\ \varepsilon, \tag{2.22}$$

and, for all bounded Lipschitz subdomain $\Omega \subset X$ containing o, one has

$$\sigma_o^\Omega(\mathcal{H}_o^\ell(x)) \ \leq\ \varepsilon. \tag{2.23}$$

Proof. We first prove (2.23). We introduce the set

$$E := \mathcal{H}_o^\ell(x) \cap \partial\Omega,$$

where $\ell > \delta$ will be chosen later, and the open 1-neighborhood of E

$$U := \{y \in X \mid d(y, E) < 1\}.$$

We introduce then the reduced function $u := R_1^U$ of the constant function 1 to this open set U. By definition, u is the smallest positive superharmonic function on X which is larger than $\mathbf{1}_U$. This function is equal to 1 on U, it is harmonic on $X \smallsetminus \overline{U}$ and one has $0 \le u \le 1$. Since U is relatively compact this function u is a *potential* on X i.e. its largest harmonic minorant is 0.

The Riesz decomposition theorem tells us that every potential u on X can be written in a unique way as

$$u(x) = \int_X G(x, y)\, \mathrm{d}\lambda(y)\,, \tag{2.24}$$

where λ is a positive Radon measure on X called the *Riesz measure* of u.

In our case where u is the reduced function $u = R_1^U$ for a relatively compact open set U, the Riesz measure λ is a finite measure supported by the boundary ∂U.

Since u is a positive superharmonic function on X which is equal to 1 on E one has, for all $z \in \Omega$,

$$\sigma_z^\Omega(E) \le u(z). \tag{2.25}$$

We can assume that E is not empty, and hence that $d(o, x) \ge \ell$. Let m be a point on a geodesic segment from o to x with $d(o, m) = \ell$. We claim that there exists a constant $C > 0$ depending only on X such that

$$u(o) \le C\, G(o, m)\, u(m). \tag{2.26}$$

Indeed, for each y in $\mathcal{H}_o^\ell(x)$, any geodesic segment from o to y intersects the ball $B(m, \delta)$. Applying Harnack Inequality and Ancona Inequality, one finds a constant $C_6 > 0$ depending only on X such that, for all $t > 1$ and all y in ∂U, one has

$$G(o, y) \le C_6\, G(o, m)\, G(m, y).$$

Applying this inequality to each of the Green functions in the integral (2.24), one gets our claim (2.26).

Since u is bounded by 1, it follows from (2.25), (2.26) and (2.1) that

$$\sigma_o^\Omega(E) \le u(o) \le C_6\, G(o, m) \le C_6\, C_0\, e^{-\varepsilon_0 \ell} \le \varepsilon$$

if $\ell = \ell_\varepsilon$ is chosen large enough.

We now prove (2.22) with the same ε and ℓ_ε as in (2.23). If (2.22) were not true, there would exist a point $x \in \overline{X}$, a small constant $\alpha > 0$ and a continuous function $\varphi : \partial X \to [0, 1-\alpha]$ supported by an open subset of ∂X included in $\overline{\mathcal{H}}^\ell_o(x)$ whose harmonic extension $h : X \to [0, 1-\alpha]$ satisfies $h(o) > \varepsilon$. Since h is continuous, if Ω contains a sufficiently large ball $B(o, R)$, the restriction of h to the complement $\partial\Omega \smallsetminus \mathcal{H}^\ell_o(x)$ is bounded by $\alpha\varepsilon$. Therefore, applying formula (2.18) with $\varphi = h$ and using (2.23), we get

$$\varepsilon \ < \ h(0) \ \leq \ (1-\alpha)\, \sigma^\Omega_o(\mathcal{H}^\ell_o(x)) + \alpha\varepsilon \ \leq \ (1-\alpha)\, \varepsilon + \alpha\varepsilon \ = \ \varepsilon.$$

This contradiction proves that (2.22) is true. $\square$

2.4.4 Lower bound for the harmonic measure

In order to prove the doubling property of the harmonic measure on X, we will need the following uniform lower bound on the harmonic measure

Lemma 2.16. *Let X be a* **GGG** *Riemannian manifold. For all $\ell \geq 0$, there exists $\varepsilon_\ell > 0$ such that such that for all o in X and $\xi \in \partial X$, one has*

$$\sigma^X_o(\overline{\mathcal{H}}^\ell_o(\xi)) \ \geq \ \varepsilon_\ell. \tag{2.27}$$

Proof. Let $\ell_0 > 0$ be the length given by (2.22) with $\varepsilon = 1/2$. Let m be a point on a geodesic ray $o\xi$ such that $d(o,m) = \ell + \ell_0 + \delta$, so that, by (2.5) and (2.3),

$$\overline{\mathcal{H}}^\ell_o(\xi) \supset \overline{\mathcal{H}}^{\ell+\delta}_o(m) \supset \overline{X} \smallsetminus \overline{\mathcal{H}}^{\ell_0}_m(o).$$

By the Harnack inequality applied to the harmonic function $x \mapsto \sigma^X_x(\overline{\mathcal{H}}^\ell_o(\xi))$, there exists a constant $C_\ell > 0$ depending only on X and ℓ such that

$$\sigma^X_o(\overline{\mathcal{H}}^\ell_o(\xi)) \ \geq \ C_\ell^{-1} \sigma^X_m(\overline{\mathcal{H}}^\ell_o(\xi)) \ \geq \ C_\ell^{-1}\, (1 - \sigma^X_m(\overline{\mathcal{H}}^{\ell_0}_m(o))).$$

The choice of ℓ_0 implies $\sigma^X_m(\overline{\mathcal{H}}^{\ell_0}_m(o)) \leq 1/2$. This gives (2.27) with the constant $\varepsilon_\ell = 1/(2C_\ell)$. $\square$

2.4.5 Doubling for the harmonic measure

The following Lemma 2.17 tells us that the measure σ^X_o satisfies a doubling property. See [4, Lemma 7.4] when X is a pinched Hadamard manifold.

Lemma 2.17. *Let X be a* **GGG** *Riemannian manifold. There exists a constant $c = c_X$ such that for all o in X, $\xi \in \partial X$ and $t \geq 0$, one has*

$$\sigma_o^X(\overline{\mathcal{H}}_o^t(\xi)) \ \leq\ c\,\sigma_o^X(\overline{\mathcal{H}}_o^{t+1}(\xi)). \tag{2.28}$$

Proof. By Lemma 2.16, we may assume that $t \geq 1$.

We first claim that there exists a constant $C_7 > 1$ such that for all o in X, all ξ in ∂X, and all x_t on a geodesic ray $o\xi$ with $d(o, x_t) = t \geq 1$, one has

$$C_7^{-1}\, G(o, x_t) \ \leq\ \sigma_o^X(\overline{\mathcal{H}}_o^t(\xi)) \ \leq\ C_7\, G(o, x_t)\,. \tag{2.29}$$

In order to prove this claim, we introduce for each $t > 0$ the harmonic function

$$z \mapsto h_t(z) := \sigma_z^X(\overline{\mathcal{H}}_o^t(\xi)) = \int_{\overline{\mathcal{H}}_o^t(\xi)\cap\partial X} P_\eta(z)\,\mathrm{d}\sigma_o^X(\eta).$$

Integrating the inequalities (2.17), one finds a constant $C_4 > 1$ depending only on X such that

$$C_4{}^{-1} G(o, x_t) h_t(x_t) \ \leq\ h_t(o) \ \leq\ C_4\, G(o, x_t) h_t(x_t). \tag{2.30}$$

We recall from (2.7) that

$$\overline{\mathcal{H}}_{x_t}^{2\delta}(\xi) \ \subset\ \overline{\mathcal{H}}_o^t(\xi),$$

so that, using Lemma 2.16 with $\ell = 2\delta$, one gets a constant $\varepsilon_{2\delta} > 0$ such that

$$\varepsilon_{2\delta} \ \leq\ \sigma_{x_t}(\overline{\mathcal{H}}_{x_t}^{2\delta}(\xi)) \ \leq\ h_t(x_t) \ \leq\ 1. \tag{2.31}$$

Combining (2.30) with (2.31), we obtain our claim (2.29).

Now, the Harnack inequality in Fact 2.1 provides a constant C depending only on X such that, for $t \geq 1$,

$$G(o, x_t) \ \leq\ C\, G(o, x_{t+1}). \tag{2.32}$$

The bound (2.28) follows from (2.29) and (2.32). $\square$

2.5 Non-tangential limits

According to Fatou's theorem, every bounded harmonic function on the Euclidean ball $B \subset \mathbb{R}^k$ admits a non-tangential limit at σ_0-almost all points of the boundary sphere ∂B (see [5, Theorem 4.6.7]). This is not always true for a bounded superharmonic function u, see [36, p. 175].

Yet, according to Littlewood's theorem, every bounded superharmonic function u on B admits a radial limit at σ_0-almost all points of ∂B, see [5, Thm. 4.6.4, Cor. 4.6.8]. One needs an extra assumption on u to ensure that this radial limit is also a non-tangential limit. This condition is the "Lipschitz continuity of u for the hyperbolic metric on the ball B".

Proposition 2.18. *Let X be a* **GGG** *Riemannian manifold, $o \in X$ and let $\sigma = \sigma_o^X$. Let $u : X \to \mathbb{R}$ be a bounded Lipschitz superharmonic function.*
a) For σ-almost all $\xi \in \partial X$, the non-tangential limit $\psi(\xi) := \operatorname{NTlim}_{x \to \xi} u(x)$ exists.
b) If this limit $\psi(\xi)$ is σ-almost surely null, then one has $u \geq 0$.

Note that the Lipschitz continuity of u is true for all bounded harmonic functions, because of the Harnack inequality in Fact 2.1.

Proof of Proposition 2.18. : This follows from the Fatou–Naïm–Doob theorem and the Brelot-Doob trick, as they are explained by Ancona in [3].

a) It is proven in [3, Thm 1.8] that for any superharmonic function u on X which is bounded below, for σ-almost all ξ in ∂X, the minimal fine limit

$$\psi(\xi) := \operatorname{MFlim}_{x \to \xi} u(x)$$

exists. This means that the limit of $u(x)$ when $x \to \xi$ exists as soon as x avoids a subset $E = E_\xi$ which is minimally thin at ξ. We recall that a subset $E \subset X$ is minimally thin if the function $P_\xi \mathbf{1}_E$ is bounded by a potential on X. And we recall that a potential is a positive superharmonic function whose largest harmonic minorant is zero. Moreover, there is a formula for this limit :

$$\psi(\xi) = \frac{\mathrm{d}\mu_h}{\mathrm{d}\sigma}(\xi)$$

where μ_h is the trace measure on ∂X of the harmonic function h in the Riesz decomposition of u as a sum $u = p + h$ of a potential p and a harmonic function h.

It is also proven in [3, p.99-100] that for a Lipschitz continuous function u on X and a point ξ in ∂X, if the minimal fine limit $\ell := \operatorname{MFlim}_{x \to \xi} u(x)$ exists then the non-tangential limit $\operatorname{NTlim}_{x \to \xi} u(x)$ exists and is equal to ℓ.

b) Since u and hence its harmonic part h are bounded on X, the measure μ_h is absolutely continuous to σ. Hence, when the limit $\psi(\xi)$ is σ-almost surely zero, the trace measure μ_h is zero, and the harmonic function h is also 0. This tells us that u is a potential, so that one has in particular $u \geq 0$. □

3 The boundary transform

In this third chapter, we construct the boundary transform $\beta : \mathcal{H}_b(X, Y) \longrightarrow L^\infty(\partial X, Y)$ and prove that it is injective.

We recall that X is a **GGG** Riemannian manifold, that Y is a complete CAT(0) space, and that $\overline{X} = X \cup \partial X$.

3.1 Harmonic maps and subharmonic functions

The following lemmas relate harmonic maps $h : X \to Y$ with Lipschitz subharmonic functions u on X. They will allow us to apply the results on superharmonic functions from Chapter 2.

We begin by a useful bound for the Lipschitz constant of a harmonic map due to Cheng.

Lemma 3.1. *Let X be a Riemannian manifold with bounded geometry, and let Y be a bounded* CAT(0)*-space. There exists $L > 0$ such that for all x_0 in X, all $r \leq 1$ and any harmonic map $h : B(x_0, r) \to Y$, the restriction of h to the ball $B(x_0, r/2)$ is L/r-Lipschitz.*

Proof. When Y is a manifold, this is a simplified version of [13, Formula 2.9]. See also [19, Theorem 6]. When Y is a more general CAT(0) space, the extension of Cheng Lemma has been proven in [37, Theorem 1.4]. □

Lemma 3.2. *Let X be a complete Riemannian manifold with bounded sectional curvature, and let Y be a* CAT(0)*-space.*
a) Let $h : X \to Y$ be a bounded harmonic map and $y_0 \in Y$. Then the function $x \mapsto d(y_0, h(x))$ is a bounded Lipschitz subharmonic function on X.
b) Let $h, h' : X \to Y$ be two bounded harmonic maps. Then the function $x \mapsto d(h(x), h'(x))$ is also a bounded Lipschitz subharmonic function on X.

Proof. When Y is a manifold this is in[22, Lemmas 3.8.1 and 3.8.2].

a) We can assume that the CAT(0) space Y is bounded. Since Y is CAT(0), the function α on Y defined by $\alpha(y) := d(y_0, y)$ is convex. Therefore, by [23, Lemma 1.7.1] when Y is a manifold and [15, Lemma 10.2] in general, the function $u := \alpha \circ h$ is subharmonic on X. The Lipschitz continuity of u follows from the Cheng bound in Lemma 3.1.

b) The proof is as in *a*). Indeed, the map $(h, h') : X \to Y \times Y$ is harmonic, the product space $Y \times Y$ is CAT(0), and the function $(y, y') \to d(y, y')$ is a convex function. □

3.2 Construction of the boundary map

In this section, we prove Proposition 1.13.

Proof of Proposition 1.13. Remember that Y is here assumed to be proper. Fix $o \in X$ and set $\sigma = \sigma_o^X$. Let $h : X \to Y$ be a bounded harmonic map. We want to prove that, for σ-almost all $\xi \in \partial X$, the map h has a non-tangential limit $\varphi(\xi)$ at the point ξ.

Let $y \in Y$. By Lemma 3.2, the function $u_y : x \to d(y, h(x))$ is a bounded Lipschitz subharmonic function on X. Hence, by Proposition 2.18.a, there exists a subset F_y of full measure in ∂X such that the function u_y admits a non-tangential limit $\psi_y(\xi)$ at each point $\xi \in F_y$.

Let $Y_1 \subset Y$ be the closure of the convex hull of $h(X)$ in Y. This subspace Y_1 is a compact CAT(0) space. Let $D \subset Y_1$ be a countable dense subset of Y_1. The intersection $F \subset \partial X$ of all the sets F_y, for y in D, still has full σ-measure. Note that, for all y, y' in Y_1 and x in X, one has

$$d(u_y(x), u_{y'}(x)) \leq d(y, y').$$

Therefore, for all $\xi \in F$ and all y in Y_1, the function u_y has a non-tangential limit at the point ξ.

We introduce the map

$$\begin{array}{rcl} \Phi : Y_1 & \to & \mathrm{Lip}_1(Y_1, [0, \delta_{Y_1}]) \\ y' & \mapsto & (d(y, y'))_{y \in Y_1}. \end{array}$$

where δ_{Y_1} is the diameter of Y_1 and Lip_1 refers to the set of 1-Lipschitz functions endowed with the sup distance. This map Φ is an isometry onto its image $\Phi(Y_1)$ and, since Y_1 is compact, this image $\Phi(Y_1)$ is closed. Let $\xi \in F$. Since Y_1 is compact, what we have just seen tells us that the map $\Phi \circ h$ has a non-tangential limit at the point ξ. Therefore, the map h also has a non-tangential limit $\varphi(\xi) \in Y_1$ at the point ξ. □

3.3 Injectivity of the boundary transform

In this section, we prove Proposition 1.14.

Proof. Let h and h' be two harmonic maps from X to Y whose boundary maps βh and $\beta h'$ are σ-almost surely equal. We want to prove that $h = h'$.

By Lemma 3.2, the function $u : x \to d(h(x), h'(x))$ is a bounded Lipschitz subharmonic function on X. By assumption the non-tangential limit $\operatorname*{NTlim}_{x \to \xi} u(x)$ is zero for σ-almost all ξ in ∂X. Therefore, by Proposition 2.18.b, the function u must be non-positive. Since u is already non-negative, we must have $u = 0$, and hence $h = h'$. □

4 The Poisson transform

In this fourth chapter, we construct the Poisson transform.

4.1 Density of the Lipschitz maps

We first need a lemma on the density of Lipschitz maps on a compact manifold S inside the set of bounded measurable maps. We will apply it to the boundary $S = \partial\Omega$ of a bounded Lipschitz domain Ω. Such a boundary is bi-Lipschitz homeomorphic to a compact smooth manifold.

Lemma 4.1. *Let S be a compact manifold and Y be a CAT(0) space. Then, every continuous map $\varphi : S \to Y$ is a uniform limit of Lipschitz maps.*

We begin by recalling the classical construction of the weighted barycenter $\beta = \beta_\mu(y_0, \ldots, y_n)$ of $n+1$ points $(y_0, \ldots, y_n)$ in a CAT(0)-space Y. The weight μ belongs to the standard n-simplex

$$\Sigma_n := \{\mu = (\mu_1, \ldots, \mu_n) \mid \mu_i \geq 0 \text{ for all } i, \text{ and } \mu_0 + \cdots + \mu_n = 1\}.$$

We endow this simplex with the ℓ^1-distance. This barycenter β is the unique point where the strictly convex function on Y

$$y \mapsto \psi_\mu(y) := \sum_{0 \leq i \leq n} \mu_i \, d(y_i, y)^2$$

achieves its minimum. As a function of the weight, this barycenter map

$$\mu \mapsto \beta_\mu(y_0, \ldots, y_n)$$

is L-Lipschitz continuous where L is the diameter of the finite set $\{y_0, \ldots, y_n\}$. We refer to [26, Lemma 4.2] for these properties.

Proof of Lemma 4.1. Using a triangulation of S we can assume that S is a compact CW-complex. We endow each n-simplex Σ_0 of S with the ℓ^1-norm and we endow S with the corresponding length metric. This new metric is Lipschitz equivalent to the Riemannian metric on S. Each n-simplex Σ_0 of S can be decomposed as a union of 2^n half-size n-simplices. Iterating k times this process we obtain a decomposition of Σ_0 as a union of 2^{kn} n-simplices of level k whose size is 2^{-k} the size of Σ_0.

Fix $\varepsilon > 0$ and $\varphi \in C(S, Y)$. There exists an integer k such that, for each simplex Σ of level k, one can uniformly bound the diameter

$$\mathrm{diam}(\varphi(\Sigma)) \leq \varepsilon/2.$$

For each simplex Σ of level k, we denote by $f_\Sigma : \Sigma \to Y$ the barycenter map such that $f_\Sigma(s) = \varphi(s)$ for each vertex s of Σ. One then has

$$\mathrm{diam}(f_\Sigma(\Sigma)) \leq \varepsilon/2,$$

and $d(f_\Sigma(s), \varphi(s)) \leq \varepsilon$ holds for all s in Σ. These maps f_Σ are $2^k\varepsilon$-Lipschitz continuous.

These maps f_Σ being compatible, each f_Σ is the restriction to Σ of a map $f : S \to Y$. This map f is also $2^k\varepsilon$-Lipschitz continuous, and one has $d(f(s), \varphi(s)) \leq \varepsilon$ for all s in S. $\square$

Lemma 4.2. *Let S be a compact metric space, σ be a Borel probability measure on S, and Y be a CAT(0) space. Then the set $C(S, Y)$ of continuous maps $f : S \to Y$ is dense in the set $L^\infty(S, Y)$ of bounded measurable maps $\varphi : S \to Y$.*

We recall that $L^\infty(S, Y)$ is endowed with the "topology of the convergence in probability". The distance between two maps φ, φ' in $L^\infty(S, Y)$ is given by

$$d(\varphi, \varphi') := \inf\{\delta \geq 0 \mid \sigma(\{\xi \in S \mid d(\varphi(\xi), \varphi'(\xi)) \geq \delta\}) \leq \delta\}\,. \tag{4.1}$$

The space $L^\infty(S, Y)$ and its topology do not depend on the choice of the measure σ inside its equivalence class of measures.

Proof of Lemma 4.2. Let $\varphi : S \to Y$ be a bounded measurable map. Let $\varepsilon > 0$. By Lusin's theorem, there exists a compact subset $K \subset S$ such that the complement K^c satisfies $\sigma(K^c) \leq \varepsilon$ and such that the restriction $\varphi|_K$ is continuous. Since a CAT(0) space Y is an absolute retract metric space, see [33, Lemma 1.1], and since an absolute retract metric space is an absolute extension metric space, there exists a continuous function $f : S \to Y$ whose restriction to K is equal to $\varphi|_K$, so that $d(\varphi, f) \leq \varepsilon$.

Lusin's theorem is usually stated under the assumption that the metric target space Y is separable. Here we do not need this assumption since S is a standard Borel space endowed with a Radon measure σ. Indeed, in this case, all measurable maps $\varphi : S \to Y$ are strongly measurable. This means that φ is an almost sure limit of a sequence of measurable step functions φ_n or equivalently that there exists a conull subset $S' \subset S$ such that the image $\varphi(S')$ is separable. See [17, Thm 2.B] for more details. $\square$

4.2 The continuous Dirichlet problem

In this section we prove Proposition 1.16.

We first deal with bounded domains.

Proposition 4.3. *Let Ω be a bounded Lipschitz Riemannian domain, Y a complete* CAT(0) *space and $\varphi : \partial\Omega \to Y$ a continuous map, then there exists a unique harmonic map $h : \Omega \to Y$ which is a continuous extension of φ.*

Proof. By Lemma 4.1, we can choose a sequence $\varphi_n \in \mathrm{Lip}(\partial\Omega, Y)$ that converges uniformly to φ. It suffices to prove that the sequence of their harmonic extensions $h_n := P\varphi_n$ given by Fact 1.15 converge uniformly. We introduce the subharmonic functions on Ω given by

$$u_{m,n}(x) := d(h_m(x), h_n(x)).$$

They extend the continuous functions on $\partial\Omega$ given by

$$\psi_{m,n}(\xi) = d(\varphi_m(\xi), \varphi_n(\xi)).$$

By the maximum principle, the supremum of $u_{m,n}$ on Ω is equal to the supremum of $\psi_{m,n}$ on $\partial\Omega$. Hence it goes to zero when $m, n \to \infty$. Therefore the sequence h_n converges uniformly to a map h which is harmonic and which extends continuously φ.

This harmonic extension is unique because if h' is another harmonic extension, the positive function

$$x \mapsto u(x) := d(h(x), h'(x))$$

is subharmonic on Ω and goes to zero near the boundary. Hence $u = 0$ and $h = h'$. $\square$

We now deal with a **GGG** Riemannian manifold X.

Proof of Proposition 1.16 . We fix o in X. We can assume that the diameter δ_Y of Y is finite. Indeed we can always replace Y by a closed ball $B(o, R)$ that contains the image of φ, since such a ball is also a complete CAT(0) space.

As we have seen in the proof of Lemma 4.2, since the compactification $\overline{X}$ is a metrizable compact space and Y is a CAT(0) space, there exists a continuous function

$$\psi : \overline{X} \to Y \ \text{ such that } \ \psi|_{\partial X} = \varphi.$$

We choose an increasing sequence of bounded Lipschitz domain $\Omega_N \subset X$ such that $o \in \Omega_0$ and Ω_N contains the 1 neighborhood of Ω_{N-1}. We denote by $h_N : \Omega_N \to Y$ the harmonic extension of the function $\psi_N := \psi|_{\partial\Omega_N}$ given by Proposition 4.3. We claim that

$$\forall \varepsilon > 0,\ \exists n_0 > 0,\ \forall N > n > n_0,\ \forall x \in \Omega_n,\ \ d(h_N(x), h_n(x)) \leq \varepsilon\,. \tag{4.2}$$

Since the function $x \mapsto d(h_n(x), h_N(x))$ is subharmonic, by the maximum principle it is enough to check (4.2) for x in $\partial\Omega_n$, that is :

$$\forall \varepsilon > 0, \exists n_0 > 0, \forall N > n > n_0,\ \forall x \in \partial\Omega_n,\ \ d(h_N(x), \psi(x)) \leq \varepsilon\,. \tag{4.3}$$

Let $\varepsilon > 0$. According to Lemma 2.15, there exists $t_0 > 0$ such that for all x in Ω_N,

$$\sigma_x^{\Omega_N}(\mathcal{H}_x^{t_0}(o)) \leq \varepsilon/(2\delta_Y). \tag{4.4}$$

By uniform continuity of ψ there exists $t_1 > 0$ such that, for all x in $\overline{X}$:

$$d(\psi(x), \psi(y)) \leq \varepsilon/2 \ \text{ for all } y \text{ in } \mathcal{H}_o^{t_1}(x). \tag{4.5}$$

We choose $n_0 \geq t_0{+}t_1$ and let $N \geq n \geq n_0$. We fix x in $\partial\Omega_n$ and introduce the subharmonic function $z \mapsto u(z) := d(h_N(z), \psi(x))$ on Ω_N. We want to prove that $u(x) \leq \varepsilon$. We observe that

$$u(x) \ \leq \ \int_{\partial\Omega_N} u(y)\, \mathrm{d}\sigma_x^{\Omega_N}(y). \tag{4.6}$$

Since $d(o, x) \geq n \geq t_0 + t_1$, by (2.3), one has

$$X = \mathcal{H}_x^{t_0}(o) \cup \mathcal{H}_o^{t_1}(x),$$

and we can bound this integral (4.6) by the sum $I' + I''$ where :
- I' is the integral on the half-space $\mathcal{H}_x^{t_0}(o)$, which by (4.4) has small volume,
- I'' is the integral on $\mathcal{H}_o^{t_1}(x)$ on which by (4.5) the function u is small. Hence

$$u(x) \ \leq \ \delta_Y\ \sigma_x^{\Omega_N}(\mathcal{H}_x^{t_0}(o)) \ + \ \frac{\varepsilon}{2}\, \sigma_x^{\Omega_N}(\mathcal{H}_o^{t_1}(x)) \ \leq \ \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ = \ \varepsilon\,,$$

which proves our claim (4.2).

Now the claim (4.2) proves that the sequence of maps (h_N) converges uniformly to a harmonic map $h : X \to Y$ that extends continuously φ.

The proof of uniqueness is as for Proposition 4.3. □

4.3 Construction of the Poisson transform

The construction uses the following classical "continuous extension theorem".

Lemma 4.4. *Let E be a metric space, $D \subset E$ a dense subset and F a complete metric space. Then every uniformly continuous map $P : D \to F$ admits a unique continuous extension $P : E \to F$.*

Proof of Lemma 4.4. This is classical. □

Proof of Proposition 1.17. Remember that Y is assumed to be bounded. We use Lemma 4.4 with $E = L^\infty(\partial X, Y)$, $D = C(\partial X, Y)$ and $F = \mathcal{H}_b(X, Y)$. Note that F is a complete metric space since a uniform limit of harmonic maps is harmonic.

We want to prove that the map $P : C(\partial X, Y) \to \mathcal{H}_b(X, Y)$ given by Proposition 1.16 has a unique continuous extension to $L^\infty(\partial X, Y)$. By Lemmas 4.4 and 4.2, it suffices to prove that this map P is uniformly continuous. We fix a compact $K \subset X$ and a point $o \in K$, and we set

$$C_K = \sup_{\xi \in \partial X,\, x \in K} P_\xi(x) \;<\; \infty\,,$$

where $P_\xi(x)$ is the Poisson kernel.

Let $0 < \varepsilon \leq 1$ and φ, φ' be two continuous maps from ∂X to Y such that $d(\varphi, \varphi') \leq \varepsilon$. This means that the function ψ on ∂X defined, for ξ in ∂X, by $\psi(\xi) := d(\varphi(\xi), \varphi'(\xi))$ satisfies

$$\sigma_o(\{\xi \in \partial X \mid \psi(\xi) \geq \varepsilon\}) \leq \varepsilon, \text{ where } \sigma_o = \sigma_o^X. \tag{4.7}$$

Note that this function ψ is bounded by the diameter δ_Y of Y.

Let $h = P\varphi$ and $h' = P\varphi'$ be their harmonic extensions to X.

By Lemma 3.2, the continuous function u on X given, for x in X, by $u(x) := d(h(x), h'(x))$ is subharmonic on X. This function is a continuous extension of ψ. Therefore it satisfies, for all x in X,

$$u(x) \;\leq\; \int_{\partial X} \psi(\xi)\, P_\xi(x)\, \mathrm{d}\sigma_o(\xi)\,.$$

Plugging (4.7) in this inequality, one gets for every x in K :

$$\begin{aligned} u(x) \;&\leq\; \varepsilon \int_{\partial X} P_\xi(x)\, \mathrm{d}\sigma_o(\xi) + \delta_Y \int_{\{\psi(\xi) \geq \varepsilon\}} P_\xi(x)\, \mathrm{d}\sigma_o(\xi) \\ &\leq\; \varepsilon + C_K \delta_Y \varepsilon. \end{aligned}$$

This proves, for any φ, φ' in $C(\partial X, Y)$ and any compact subset $K \subset X$, the inequality :

$$\sup_{x \in K} d(h(x), h'(x)) \leq (1 + C_K \delta_Y) d(\varphi, \varphi')\,.$$

This is the uniform continuity of the map P. $\square$

5 The boundary and the Poisson transform

In this chapter, we prove that the Poisson transform P is the inverse of the boundary transform β.

5.1 The Lebesgue density theorem

We recall here the generalized Lebesgue density theorem. See [7, Section 4.6] for a short and complete proof.

Definition 5.1. A quasi-distance on a space S is a map $d_0 : S \times S \to [0, \infty[$ for which there exists $b > 0$
such that $d_0(\xi_1, \xi_3) \leq b(d_0(\xi_1, \xi_2) + d_0(\xi_2, \xi_3))$, $\forall \xi_1, \xi_2, \xi_3 \in S$,
such that $d_0(\xi_1, \xi_2) = d_0(\xi_2, \xi_1)$ and
such that $d_0(\xi_1, \xi_2) = 0 \Leftrightarrow \xi_1 = \xi_2$.

- In this case, one says that S is a quasi-metric space. Then, there exists a topology on S for which the balls $B(\xi, \varepsilon) := \{\eta \in S \mid d_0(\xi, \eta) \leq \varepsilon\}$, with $\xi \in S$ and $\varepsilon > 0$, form a basis of neighborhood of the points ξ.
- One then has the inclusion $\overline{B(\xi, \varepsilon)} \subset B(\xi, b\varepsilon)$.

Let (S, d_0) be a compact quasi-metric space and σ be a finite Borel measure on S. One says that σ is doubling if there exists $C > 0$ such that, for all $\xi \in S$ and $r > 0$, one has $\sigma(B(\xi, 2r)) \leq C\,\sigma(B(\xi, r))$.

Let $F \subset S$ be a measurable subset. A point $\xi \in S$ is called a density point if

$$\lim_{\varepsilon \to 0} \frac{\sigma(B(\xi, \varepsilon) \cap F)}{\sigma(B(\xi, \varepsilon))} = 1.$$

Fact 5.2. (Lebesgue) *Let (S, d_0) be a compact quasi-metric space, σ be a doubling finite Borel measure on S, and let F be a measurable subset of S. Then σ-almost every point of F is a density point.*

In the next section, we will apply Fact 5.2 with $S = \partial X$ and with $\sigma = \sigma_o$.

We will use the quasi-distance on ∂X defined, for two points η_1 and η_2, by the exponential inverse of the Gromov product :

$$d_0(\eta_1, \eta_2) = e^{-(\eta_1|\eta_2)_o}. \tag{5.1}$$

This formula defines indeed a quasi-distance on ∂X, because of (1.1). Note that one can modify this formula so that d_0 is actually a distance, see [18].

The balls for this quasi-distance in ∂X are the trace at infinity of the half-spaces of X. The doubling property for σ_o is proven in Lemma 2.17.

5.2 Limit of subharmonic functions

In this section we prove the technical lemma 5.3 that plays a crucial role in the proof of the surjectivity of the boundary transform.

We fix a point o in X. For all $\xi \in \partial X$ we define $N\xi$ as the union of the geodesic rays $o\xi$ from o to ξ. We define then *the tube NF over a compact set $F \subset \partial X$* as the union

$$NF := \bigcup_{\xi \in F} N\xi \subset X. \tag{5.2}$$

Lemma 5.3. *Let X be a* **GGG** *Riemannian manifold, and fix a point $o \in X$. Let $\psi_n : \partial X \to [0,1]$ be a sequence of continuous functions that converges σ_o-almost surely to 0. Let $u_n : X \to [0,1]$ be non-negative subharmonic functions on X that extend continuously ψ_n. Then, for all $\varepsilon > 0$, there exists a compact subset $F_\varepsilon \subset \partial X$ with $\sigma_o(F_\varepsilon^c) \leq \varepsilon$, such that the sequence (u_n) converges uniformly to 0 on the tube NF_ε.*

The arguments of Section 4.3 tell us that the sequence (u_n) converges to 0 uniformly on the compact subsets K of X. Lemma 5.3 tells us that this convergence is still uniform on “large radial subsets of X”.

Proof. **First step** *We control the Poisson kernel on tubes.*

For ξ in ∂X and $m \geq 0$ we denote by

$$B_m(\xi) := B(\xi, e^{-m}) = \{\eta \in \partial X \mid (\eta|\xi)_o \geq m\}$$

the balls for the quasidistance d_0 in (5.1), and we introduce the annuli

$$A_m(\xi) := B_m(\xi) \smallsetminus B_{m+1}(\xi),$$

so that one has $\cup_{m\geq 0} A_m(\xi) = \partial X \smallsetminus \{\xi\}$.

By Proposition 2.8, there exists $C_3 > 0$ such that, for $\xi \in \partial X$ and $m \geq 0$,

$$\frac{P_{\eta_1}(x)}{P_{\eta_2}(x)} \leq C_3 \text{ holds for all } x \in N\xi \text{ and } \eta_1, \eta_2 \text{ in } A_m(\xi). \tag{5.3}$$

Second step *We apply the Lebesgue density theorem.*

Let $\varepsilon > 0$. Since the sequence (ψ_n) converges σ_o-almost surely to 0, there exist an integer $n_\varepsilon \geq 1$ and a compact subset $K_\varepsilon \subset \partial X$ with $\sigma_o(K_\varepsilon^c) \leq \varepsilon/2$, and such that $\psi_n(\xi) \leq \varepsilon$ for all $n \geq n_\varepsilon$ and $\xi \in K_\varepsilon$.

By the Lebesgue density theorem (Fact 5.2), applied to the doubling measures $\sigma_o = \sigma_o^X$ and the family of balls $B_m(\xi)$, the sequence of functions

$$f_m(\xi) := \frac{\sigma_o(B_m(\xi) \cap K_\varepsilon^c)}{\sigma_o(B_m(\xi))}$$

converges to zero for σ_o-almost all $\xi \in K_\varepsilon$. Since the measure σ_o is doubling, the ratios $\dfrac{\sigma_o(B_m(\xi))}{\sigma_o(B_{m+1}(\xi))}$ are uniformly bounded, and this implies that the ratios $\dfrac{\sigma_o(B_m(\xi))}{\sigma_o(A_m(\xi))}$ are also uniformly bounded. Hence the sequence of functions $g_m(\xi) := \dfrac{\sigma_o(A_m(\xi)\cap K_\varepsilon^c)}{\sigma_o(A_m(\xi))}$ also converges to zero for σ_o-almost all $\xi \in K_\varepsilon$.

Therefore, by Egorov theorem, there exist a compact subset $L_\varepsilon \subset K_\varepsilon$ and an integer $m_\varepsilon \geq 1$ such that $\sigma_o(L_\varepsilon^c) \leq \varepsilon$ and with

$$\sigma_o(A_m(\xi)\cap K_\varepsilon^c) \leq \varepsilon\, \sigma_o(A_m(\xi)) \ \text{ for all } m \geq m_\varepsilon \text{ and } \xi \in L_\varepsilon. \tag{5.4}$$

Third step *We bound the functions u_n by using the Poisson kernel.*

Since each function u_n is subharmonic with boundary value ψ_n, one has

$$u_n(x) \ \leq \ \int_{\partial X} \psi_n(\eta)\, P_\eta(x)\, \mathrm{d}\sigma_o(\eta) \ \text{ for all } x \in X.$$

We now assume that x belongs to a tube $N\xi$ with $\xi \in L_\varepsilon$. We write

$$u_n(x) \ \leq \ \sum_{m=0}^{\infty} I_{m,n}(x,\xi) \ \text{ where } I_{m,n}(x,\xi) := \int_{A_m(\xi)} \psi_n(\eta)\, P_\eta(x)\, \mathrm{d}\sigma_o(\eta).$$

We split this sum into two parts, according to whether $m < m_\varepsilon$ or $m \geq m_\varepsilon$.

First assume that $m < m_\varepsilon$. The function $(x,\eta) \mapsto P_\eta(\xi)$ being continuous on $(\overline{X} \times \partial X) \smallsetminus \Delta_{\partial X}$, there exists a constant $C_8 = C_8(m_\varepsilon) > 0$ such that one has, for all $\xi \in \partial X$:

$$P_\eta(x) \leq C_8 \quad \text{for all } x \in N\xi \text{ and } \eta \in \partial X \smallsetminus B_{m_\varepsilon}(\xi).$$

This gives the bound

$$\sum_{m<m_\varepsilon} I_{m,n}(x,\xi) \ \leq \ C_8 \int_{\partial X} \psi_n(\eta)\, \mathrm{d}\sigma_o(\eta)\,, \tag{5.5}$$

where the integral converges to 0 when $n \to \infty$ by the Lebesgue dominated convergence theorem.

Now assume that $m \geq m_\varepsilon$. One splits the integral $I_{m,n}(x,\xi)$ as a sum

$$\begin{aligned} I_{m,n}(x,\xi) &= I'_{m,n}(x,\xi) + I''_{m,n}(x,\xi) \ \text{ where} \\ I'_{m,n}(x,\xi) &:= \int_{A_m(\xi)\cap K_\varepsilon^c} \psi_n(\eta)\, P_\eta(x)\, \mathrm{d}\sigma_o(\eta), \\ I''_{m,n}(x,\xi) &:= \int_{A_m(\xi)\cap K_\varepsilon} \psi_n(\eta)\, P_\eta(x)\, \mathrm{d}\sigma_o(\eta). \end{aligned}$$

Since $m \geq m_\varepsilon$ and $\xi \in L_\varepsilon$, we obtain by using (5.3) and (5.4) and the bound $\|\psi_n\|_\infty \leq 1$:

$$\begin{aligned} I'_{m,n}(x,\xi) &\leq \max_{\eta \in A_m(\xi)} P_\eta(x)\ \ \sigma_o(A_m(\xi) \cap K^c_\varepsilon) \\ &\leq C_3 \min_{\eta \in A_m(\xi)} P_\eta(x)\ \ \varepsilon\, \sigma_o(A_m(\xi)) \\ &\leq \varepsilon\, C_3 \int_{A_m(\xi)} P_\eta(x)\, \mathrm{d}\sigma_o(\eta). \end{aligned} \tag{5.6}$$

Assume moreover that $n \geq n_\varepsilon$. Using the definition of K_ε, we obtain :

$$I''_{m,n}(x,\xi) \ \ \leq\ \ \varepsilon \int_{A_m(\xi)} P_\eta(x)\, \mathrm{d}\sigma_o(\eta). \tag{5.7}$$

Combining (5.6) and (5.7) and summing over $m \geq m_\varepsilon$, one gets for $n \geq n_\varepsilon$:

$$\sum_{m \geq m_\varepsilon} I_{m,n}(x,\xi) \ \ \leq\ \ (1+C_3)\,\varepsilon \int_{\partial X} P_\eta(x)\, \mathrm{d}\sigma_o(\eta) \ \ =\ \ (1+C_3)\,\varepsilon\,. \tag{5.8}$$

We now define the compact F_ε as the intersection $F_\varepsilon := \cap_{\ell \geq 1} L_{\varepsilon_\ell}$ with $\varepsilon_\ell := 2^{-\ell}\varepsilon$, so that $\sigma_o(F^c_\varepsilon) \leq \varepsilon$. Combining (5.5) and (5.8) we observe that one has, for all x in NF_ε and every integers $\ell \geq 1$ and $n \geq n_{\varepsilon_\ell}$:

$$u_n(x) \ \ \leq\ \ C_8 \int_{\partial X} \psi_n(\eta)\, \mathrm{d}\sigma_o(\eta) \ + (1+C_3)\,\varepsilon_\ell\,.$$

If ℓ is large enough the second term is small. And, as we have already seen by using the Lebesgue dominated convergence theorem, the first term is small if n is large enough.

This proves that the sequence (u_n) converges uniformly to 0 on NF_ε. □

5.3 Surjectivity of the boundary transform

Proof of Proposition 1.18. Let $\varphi \in L^\infty(\partial X, Y)$. We want to prove the equality $\varphi = \beta P \varphi$. The metric space Y has been assumed to be proper so that we can use the existence of the boundary map β from Proposition 1.13. Let δ_Y be the diameter of Y.

Let (φ_n) be a sequence in $C^0(\partial X, Y)$ that converges almost surely to φ. Such a sequence also converges to φ in probability i.e. for the distance (4.1). Let $h_n = P\varphi_n$ and $h = P\varphi$. By construction, one has $\varphi_n = \beta h_n$ and the harmonic map h is the limit of the harmonic maps h_n, where the convergence is uniform on compact sets of X. This is not enough to conclude. But we will prove below that, for all $\varepsilon > 0$, this convergence is also uniform on the tube NF_ε over a compact set $F_\varepsilon \subset \partial X$ such that $\sigma_o(F_\varepsilon^c) \leq \varepsilon$.

Since the sequence (φ_n) converges almost surely to φ, the continuous functions $\psi_{m,n} : \partial X \to [0, \delta_Y]$ defined, for ξ in ∂X, by

$$\psi_{m,n}(\xi) = d(\varphi_m(\xi), \varphi_n(\xi))$$

converge almost surely to 0 when m, n go to ∞.

The functions $u_{m,n} : X \to [0, \delta_Y]$ defined by

$$u_{m,n}(x) = d(h_m(x), h_n(x))$$

for x in X extend continuously the functions $\psi_{m,n}$. By Lemma 3.2, these functions $u_{m,n}$ are subharmonic on X.

Let $\varepsilon > 0$. Lemma 5.3 ensures that there exists a compact subset $F_\varepsilon \subset \partial X$ such that $\sigma_o(F_\varepsilon^c) \leq \varepsilon$, and such that the sequence $(u_{m,n})$ converges uniformly to 0 on the tube NF_ε. This tells us that the convergence of the sequence (h_n) to h is uniform on the tube NF_ε. By Egorov theorem, we may also assume that the sequence of continuous functions (φ_n) converges uniformly to φ on F_ε. Therefore the function $\overline{h} : \overline{X} \to Y$ equal to h on X and equal to φ on ∂X is continuous on $NF_\varepsilon \cup F_\varepsilon$.

This proves that, when ξ is in F_ε, the non-tangential limit $\operatorname{NTlim}_{x \to \xi} h(x)$ given by Proposition 1.13 is equal to $\varphi(\xi)$. Since the measure of F_ε^c is arbitrarily small, the map φ is the boundary map of h. $\square$

5.4 A concrete example

We give an example of Theorem 1.3 in a situation where $X = Y$ is the hyperbolic plane $\mathbb{H}^2$ and the boundary map $\varphi : \partial\mathbb{H}^2 \to \mathbb{H}^2$ has finite image.

Corollary 5.4. *Let $X := \mathbb{H}^2$ and $x_1, \ldots, x_n$ be n points on $\partial\mathbb{H}^2$ cutting $\partial\mathbb{H}^2$ into n open arcs $I_1, \ldots, I_n$. Let $y_1, \ldots, y_n$ be n points on $Y := \mathbb{H}^2$. Then there exists a unique bounded harmonic map $h : \mathbb{H}^2 \to \mathbb{H}^2$ that extends continuously to the arcs I_j with $h(I_j) = \{y_j\}$, for all j.*

Proof. By Theorem 1.3 the map h has to be the Poisson transform of the map $\varphi : \partial\mathbb{H}^2 \to \mathbb{H}^2$ that sends the sides I_j to the point y_j, for all j. We only have to check that this map h extends continuously φ outside the points x_j. This is the content of Proposition 5.5. $\square$

Proposition 5.5. *Let X be a* **GGG** *Riemannian manifold, and Y be a proper* CAT(0)*-space. Let $h : X \to Y$ be a bounded harmonic map and $\varphi : \partial X \to Y$ be its boundary map. Let $I \subset \partial X$ be an open set on which φ is continuous. Then, for all $\xi \in I$, one has $\varphi(\xi) := \lim_{x \to \xi} h(x)$.*

Proof. The argument is as in the proof of Proposition 1.16 in Section 4.2. □

5.5 Bounded Lipschitz domain

A "bounded Lipschitz Riemannian domain" Ω means a connected bounded open subset of a smooth Riemannian manifold (M, g_0) such that Ω is the interior of a connected compact submanifold $\overline{\Omega}$ of M whose boundary $\partial\Omega$ is a non-empty Lipschitz continuous codimension one submanifold.

The same argument as for Theorem 1.3 will give the following corollary

Corollary 5.6. *Let Ω be a bounded Lipschitz Riemannian domain, and Y be a proper* $\mathrm{CAT}(0)$*-space.*
a) Let $h:\Omega\to Y$ be a bounded harmonic map. Then, for σ-almost all $\xi\in\partial\Omega$, the map h admits a non-tangential limit $\varphi(\xi) := \operatorname*{NTlim}_{x\to\xi} h(x)$ at the point ξ.
b) The map $h \mapsto \beta(h) := \varphi$ gives a bijection $\beta : \mathcal{H}_b(\Omega, Y) \xrightarrow{\sim} L^\infty(\partial\Omega, Y)$.

In this case, a non-tangential limit means a limit along all sequences x_n such that $\sup\limits_{n\geq 1} \dfrac{d(x_n, \xi)}{d(x_n, \partial\Omega)} < \infty$.

The measure σ is any finite Borel measure on $\partial\Omega$ which is equivalent to any of the harmonic measures of Ω. Since Ω is a bounded Lipschitz Riemannian domain, by Dahlberg's theorem in [14], one can choose σ to be the Riemannian measure on $\partial\Omega$.

As in Section 1.4, when Y is a Riemannian manifold, Corollary 5.6.a is due to Aviles, Choi, Micallef in [6, Thm 5.1], and Theorem 5.6.b is expected to be true as a final observation in [6, Section 1]. The first cases of Corollary 5.6.b that seem to be new is when X is the Euclidean unit ball in $\mathbb{R}^k$ and Y is the hyperbolic space $\mathbb{H}^\ell$.

Proof. Corollary 5.6 is a corollary of the proof of Theorem 1.3. The strategy is the same, relying on variations of Propositions 2.8 and 2.18 for bounded Lipschitz Riemannian domains. The proofs of these variations are very similar. The only difference is that they rely on [2] instead of [3]. □

Remark 5.7. The fact that Corollary 5.6 is a special case of Theorem 1.3 can also be explained thanks to a trick due to Bonk, Heinonen and Koskela in [10, Chapter 8]. This trick consists in replacing the Riemannian metric g_0 on Ω by $g = \overline{d}(x)^{-2} g_0$ where $\overline{d}$ is a suitable C^∞ function roughly equal to the distance to the boundary, obtaining this way a **GGG** Riemannian manifold (Ω, g). One then sees the harmonic and subharmonic functions on (Ω, g_0) as

$\mathcal{L}$-harmonic and $\mathcal{L}$-subharmonic functions on X where $\mathcal{L} := \overline{d}(x)^2\Delta_{g_0}$ is an elliptic differential operator of order 2 which is equal to the Laplacian Δ_g up to terms of order 1 and which also has spectral gap. All the arguments we developed in this paper for the Laplace operator Δ_g also apply to the operator $\mathcal{L}$. This trick could be applied to a much wider class of bounded Riemannian domains called "inner uniform domains". See also Aikawa in [1, Theorem 1.2] for more on these domains.

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