Harmonic quasi-isometric maps II: negatively curved manifolds

Yves Benoist & Dominique Hulin

Abstract

We prove that a quasi-isometric map between pinched Hadamard manifolds is within bounded distance from a unique harmonic map.

1 Introduction

The aim of this article, which is a sequel to [4], is the following theorem.

Theorem 1.1. *Let $f : X \to Y$ be a quasi-isometric map between two pinched Hadamard manifolds. Then there exists a unique harmonic map $h : X \to Y$ which stays within bounded distance from f i.e. such that*

$$\sup_{x \in X} d(h(x), f(x)) < \infty \, .$$

We first recall that a few definitions. A pinched Hadamard manifold X is a complete simply-connected Riemannian manifold of dimension at least 2 whose sectional curvature is pinched between two negative constants: $-b^2 \leq K_X \leq -a^2 < 0$. A map $f : X \to Y$ between two metric spaces X and Y is said to be *quasi-isometric* if there exist constants $c \geq 1$ and $C \geq 0$ such that f is (c, C)-*quasi-isometric*. This means that one has

$$c^{-1}\, d(x, x') - C \;\leq\; d(f(x), f(x')) \;\leq\; c\, d(x, x') + C \tag{1.1}$$

for all x, x' in X. A $\mathcal{C}^2$ map $h : X \to Y$ between two Riemannian manifolds X and Y is said to be *harmonic* if it satisfies the elliptic nonlinear partial differential equation $\mathrm{tr}(D^2 h) = 0$ where $D^2 h$ is the second covariant derivative of h.

Partial results towards the existence statement were obtained in [27], [36], [15], [24], [5]. A major breakthrough was achieved by Markovic who solved the Schoen conjecture, i.e. the case where $X = Y$ is the hyperbolic

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plane $\mathbb{H}^2_{\mathbb{R}}$, and by Lemm–Markovic who proved the existence for the case $X = Y = \mathbb{H}^k_{\mathbb{R}}$ in [26], [25] and [20]. The existence when both X and Y are rank one symmetric spaces, which was conjectured by Li and Wang in [22, Introduction], was proved in our paper [4]. We refer to [4, Section 1.2] for more motivations and for a more precise historical perspective on this result.

Partial results towards the uniqueness statement were obtained by Li and Tam in [21], and by Li and Wang in [22]. All these papers were dealing with rank one symmetric spaces.

Note that Theorem 1.1 was conjectured by Markovic, at the end of the conference talk www.youtube.com/watch?v=A5Yt83I1FrY, during a 2016 Summer School in Grenoble. According to our knowledge, Theorem 1.1 is new even in the case where both X and Y are assumed to be surfaces.

The strategy of the proof of the existence follows the lines of the proof in [4]. As in [4], we replace the quasi-isometric map f by a $\mathcal{C}^\infty$-map whose first two covariant derivatives are bounded. But we need to modify the barycenter argument we used in [4] for this smoothing step. See Subsection 2.2.1 for more details on this step. As in [4], we then introduce the harmonic maps h_R which coincide with f on a sphere of X with large radius R and we need a uniform bound for the distance between h_R and f. The heart of our argument is in Chapter 3 which contains the boundary estimates and in Chapter 4 which contains the interior estimates. The proof of these interior estimates is based on a new simplification of an idea of Markovic in [25]. Indeed we will introduce a point x where $d(h_R(x), f(x))$ is maximal and focus on a subset U_{ℓ_0} of a sphere $S(x, \ell_0)$ whose definition (4.10) is much simpler than in [25] or [4]. This simplification is the key point which allows us to extend the arguments of [4] to pinched Hadamard manifolds. In this proof we use a uniform control on the harmonic measures on all the spheres of X, which is given in Proposition 4.9. We refer to Section 4.1 for more details on our strategy of proof of the existence.

In order to prove the uniqueness, we need to introduce Gromov-Hausdorff limits of the pointed metric spaces X and Y with respect to base points going to infinity and therefore to deal with $\mathcal{C}^2$-Riemannian manifolds with $\mathcal{C}^1$-metrics. This will be done in Chapter 5. We refer to Section 5.1 for more details on our strategy of proof of the uniqueness.

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2 Smoothing

In this chapter, we recall a few basic facts on Hadamard manifolds, and we explain how to replace our quasi-isometric map f by a $\mathcal{C}^\infty$-map whose first two covariant derivatives are bounded.

2.1 The geometry of Hadamard manifolds

We first recall basic estimates on Hadamard manifolds for triangles, for images of triangles under quasi-isometric maps, and for the Hessian of the distance function.

All the Riemannian manifolds will be assumed to be connected. We will denote by d their distance function.

A Hadamard manifold is a complete simply connected Riemannian manifold X of dimension $k \geq 2$ whose curvature is non positive $K_X \leq 0$. For instance, the Euclidean space $\mathbb{R}^k$ is a Hadamard manifold with zero curvature $K_X = 0$, and the real hyperbolic space $\mathbb{H}^k_{\mathbb{R}}$ is a Hadamard manifold with constant curvature $K_X = -1$. We will say that X is pinched if there exist constants $a, b > 0$ such that

$$-b^2 \leq K_X \leq -a^2 < 0.$$

For instance, the non-compact rank one symmetric spaces are pinched Hadamard manifolds.

Let x_0, x_1, x_2 be three points on a Hadamard manifold X. The *Gromov product* of the points x_1 and x_2 seen from x_0 is defined as

$$(x_1|x_2)_{x_0} := (d(x_0, x_1) + d(x_0, x_2) - d(x_1, x_2))/2. \tag{2.1}$$

We recall the basic comparison lemma which is one of the motivations for introducing the Gromov product.

Lemma 2.1. *Let X be a Hadamard manifold with $-b^2 \leq K_X \leq -a^2 < 0$. Let T be a geodesic triangle in X with vertices x_0, x_1, x_2, and let θ_0 be the angle of T at the vertex x_0.*
a) One has $(x_0|x_2)_{x_1} \geq d(x_0, x_1)\, \sin^2(\theta_0/2)$.
b) One has $\theta_0 \leq 4\, e^{-a\,(x_1|x_2)_{x_0}}$.
c) Moreover, if $\min((x_0|x_1)_{x_2}, (x_0|x_2)_{x_1}) \geq b^{-1}$*, one has* $\theta_0 \geq e^{-b(x_1|x_2)_{x_0}}$.

Proof. This is classical. See for instance [4, Lemma 2.1]. □

We now recall the effect of a quasi-isometric map on the Gromov product.

Lemma 2.2. *Let X, Y be Hadamard manifolds with $-b^2 \leq K_X \leq -a^2 < 0$ and $-b^2 \leq K_Y \leq -a^2 < 0$, and let $f : X \to Y$ be a (c, C)-quasi-isometric map. There exists $A = A(a, b, c, C) > 0$ such that, for all x_0, x_1, x_2 in X, one has*

$$c^{-1}(x_1|x_2)_{x_0} - A \;\leq\; (f(x_1)|f(x_2))_{f(x_0)} \;\leq\; c(x_1|x_2)_{x_0} + A\,. \tag{2.2}$$

Proof. This is a general property of quasi-isometric maps between Gromov δ-hyperbolic spaces which is due to M. Burger. See [12, Prop. 5.15]. $\square$

When x_0 is a point in a Riemannian manifold X, we denote by d_{x_0} the distance function defined by $d_{x_0}(x) = d(x_0, x)$ for x in X. We denote by $d_{x_0}^2$ the square of this function. When $F : X \to \mathbb{R}$ is a $\mathcal{C}^2$ function, we denote by DF its differential and by D^2F its second covariant derivative.

Lemma 2.3. *Let X be a Hadamard manifold and $x_0 \in X$.*
Assume that $-b^2 \leq K_X \leq -a^2 \leq 0$. The Hessian of the distance function d_{x_0} satisfies on $X \setminus \{x_0\}$

$$a \coth(a\, d_{x_0})\, g_0 \leq D^2 d_{x_0} \leq b \coth(b\, d_{x_0})\, g_0\,, \tag{2.3}$$

where $g_0 := g_X - Dd_{x_0} \otimes Dd_{x_0}$ and g_X is the Riemannian metric on X.

When $a = 0$ the left-hand side of (2.3) must be interpreted as $d_{x_0}^{-1} g_0$.

Proof. This is classical. See for instance [4, Lemma 2.3] $\square$

2.2 Smoothing rough Lipschitz maps

The following proposition will allow us to assume in Theorem 1.1 that the quasi-isometric map f we start with is $\mathcal{C}^\infty$ with bounded derivative and bounded second covariant derivative.

2.2.1 Rough Lipschitz maps

A map $f : X \to Y$ between two metric spaces X and Y is said to be *rough Lipschitz* if there exist constants $c \geq 1$ and $C \geq 0$ such that, for all x, x' in X, one has

$$d(f(x), f(x')) \;\leq\; c\, d(x, x') + C\,. \tag{2.4}$$

Proposition 2.4. *Let X, Y be two Hadamard manifolds with bounded curvatures $-b^2 \leq K_X \leq 0$ and $-b^2 \leq K_Y \leq 0$. Let $f : X \to Y$ be a rough Lipschitz map. Then there exists a $\mathcal{C}^\infty$ map $\widetilde{f} : X \to Y$ within bounded distance from f and whose first two covariant derivatives $D\widetilde{f}$ and $D^2\widetilde{f}$ are bounded on X.*

We denote $k = \dim X$ and $\ell = \dim Y$. We will first construct in 2.2.2 a regularized map $\widetilde{f} : X \to Y$ which is Lipschitz continuous. This first construction is the same as for rank one symmetric spaces in [4, Proposition 2.4]. This construction will not allow us to control the second covariant derivative, hence we will have to combine this first construction with an iterative smoothing process in local charts that we will explain in 2.2.3.

2.2.2 Lipschitz continuity

The first part of the proof of Proposition 2.4 relies on the following lemma.

Lemma 2.5. *Let Y be a Hadamard manifold.*
a) Let μ be a positive Borel measure on Y supported by a ball $B(y_0, R)$. The function Q_μ on Y defined by

$$Q_\mu(y) \;=\; \textstyle\int_Y d(y,w)^2 \,\mathrm{d}\mu(w)$$

has a unique minimum y_μ in Y called the center of mass of μ. This center of mass y_μ belongs to the ball $B(y_0, R)$.
b) Let μ_1, μ_2 be two positive Borel measures on Y. Assume that
(i) $\mu_1(Y) \geq m$ and $\mu_2(Y) \geq m$ for some $m > 0$,
(ii) both μ_1 and μ_2 are supported on the same ball $B(y_0, R)$,
(iii) The norm of $\|\mu_1 - \mu_2\| \leq \varepsilon$.
Then the distance between their center of mass y_{μ_1} and y_{μ_2} is bounded by

$$d(y_{\mu_1}, y_{\mu_2}) \leq 4\varepsilon R/m\,. \tag{2.5}$$

Proof of Lemma 2.5. a) Since the space Y is a proper space, i.e. its balls are compact, the function Q_μ is proper and admits a minimum y_μ. Since Y has non-positive curvature, one has the median inequality, for all y, y_1, y_2, y_3 in Y with y_3 middle of y_1 and y_2,

$$\tfrac{1}{2} d(y_1, y_2)^2 \leq d(y, y_1)^2 + d(y, y_2)^2 - 2d(y, y_3)^2. \tag{2.6}$$

Integrating (2.6) with respect to μ, one checks that the function Q_μ satisfies the following uniform convexity property : for y_3 middle of y_1 and y_2 in Y,

$$\tfrac{m}{2} d(y_1, y_2)^2 \leq Q_\mu(y_1) + Q_\mu(y_2) - 2Q_\mu(y_3)\,.$$

Applying this inequality with $y_1 = y_\mu$ and $y_2 = y$, one gets, for all y in Y,

$$\tfrac{m}{2} d(y_\mu, y)^2 \leq Q_\mu(y) - Q_\mu(y_\mu)\,, \tag{2.7}$$

Hence y_μ is the unique minimum of Q_μ.

We now check that $y_\mu \in B(x_0, R)$. By the median inequality (2.6), the ball $B(y_0, R)$ is convex, every point y in Y admits a unique nearest point y' on $B(y_0, R)$, and this point y' satisfies also the inequality

$$d(y', w) \leq d(y, w) \;\text{ for all } w \text{ in } B(y_0, R).$$

Therefore, one has $Q_\mu(y') \leq Q_\mu(y)$. This proves that the center of mass y_μ belongs to the ball $B(y_0, R)$.

b) Applying twice Inequality (2.7), one gets the two inequalities

$$\begin{array}{rcl}\tfrac{m}{2} d(y_{\mu_1}, y_{\mu_2})^2 & \leq & Q_{\mu_1}(y_{\mu_2}) - Q_{\mu_1}(y_{\mu_1})\,,\\ \tfrac{m}{2} d(y_{\mu_1}, y_{\mu_2})^2 & \leq & Q_{\mu_2}(y_{\mu_1}) - Q_{\mu_2}(y_{\mu_2})\,.\end{array}$$

Summing these two inequalities, one gets,

$$\begin{aligned} m\,d(y_{\mu_1},y_{\mu_2})^2 &\leq (Q_{\mu_1}-Q_{\mu_2})(y_{\mu_2})-(Q_{\mu_1}-Q_{\mu_2})(y_{\mu_1}) \\ &\leq \varepsilon \sup_{w\in B(y_0,R)} |d(y_{\mu_1},w)^2-d(y_{\mu_2},w)^2| \\ &\leq 4\varepsilon R\, d(y_{\mu_1},y_{\mu_2})\,, \end{aligned}$$

which proves (2.5). □

We choose a non-negative $\mathcal{C}^\infty$ function $\chi:\mathbb{R}\to\mathbb{R}$ with support included in $]-1,1[$, which is equal to 1 on a neighborhood of $[-\frac{1}{2},\frac{1}{2}]$ and whose first derivative is bounded by $|\chi'|\leq 4$.

Proof of Proposition 2.4. First step: Lipschitz continuity. We explain now this first construction. We can assume $b=1$. For x in X, we introduce the positive finite measure μ_x on Y such that

$$\mu_x(\varphi)=\int_X \varphi(f(z))\,\chi(d(x,z))\,\mathrm{dvol}_X(z)\,.$$

for any positive function φ on Y. This measure μ_x is the image by f of a measure supported in the ball $B(x,1)$. We will define $\widetilde{f}(x)\in Y$ to be the center of mass of the measure μ_x. Lemma 2.5.a tells us that the map $x\to\widetilde{f}(x)$ is well-defined. The Lipschitz continuity of this map, will follow from Lemma 2.5.b applied to two measures $\mu_1:=\mu_{x_1}$ and $\mu_2:=\mu_{x_2}$ with x_1, x_2 in X. Let us check the three assumptions in Lemma 2.5.b.

(i) Because of the pinching of the curvature on X, the Bishop volume estimates tell us that there exist positive constants $0<m_0<M_0$ such that for all x, one has

$$m_0\leq \mathrm{vol}(B(x,\tfrac{1}{2}))\leq \mu_x(Y)\leq \mathrm{vol}(B(x,1))\leq M_0\,.$$

(ii) When x_1, x_2 are two points of X with $d(x_1,x_2)\leq 1$, the bound (2.4) ensures that both μ_{x_1} and μ_{x_2} are supported by the ball $B(f(x_1),2c+C)$.

(iii) The norm of the difference of these measures satisfies,

$$\begin{aligned} \|\mu_{x_1}-\mu_{x_2}\| &\leq M_0\sup_{z\in X}|\chi(d(x_1,z))-\chi(d(x_2,z))| \\ &\leq 4M_0\,d(x_1,x_2)\,. \end{aligned}$$

Thus Lemma 2.5 applies and yields a bound on the Lipschitz constant of $\widetilde{f}$, namely

$$\mathrm{Lip}(\widetilde{f}):=\sup_{x_1\neq x_2}\frac{d(\widetilde{f}(x_1),\widetilde{f}(x_2))}{d(x_1,x_2)}\leq\frac{16(2c+C)M_0}{m_0}.$$ □

2.2.3 Bound on the second derivative

The second step of the proof of Proposition 2.4 relies on three lemmas. The first lemma gives a nice system of charts on Y.

Lemma 2.6. *Let Y be a Hadamard manifold with $-b^2 \leq K_Y \leq 0$ and $\ell = \dim Y$. There exist constants $r_0 = r_0(\ell, b) > 0$ and $c_0 = c_0(\ell, b) > 1$ such that, for each y in Y, there exists a $\mathcal{C}^\infty$ chart Φ_y for the open ball*

$$\Phi_y : \mathring{B}(y, r_0) \xrightarrow{\sim} U_y \subset \mathbb{R}^\ell \text{ with } \Phi_y(y) = 0 \tag{2.8}$$

and such that

$$\|D\Phi_y\| \leq c_0 \ , \ \|D\Phi_y^{-1}\| \leq c_0 \ , \ \|D^2\Phi_y\| \leq c_0 \ , \ \|D^2\Phi_y^{-1}\| \leq c_0 \ . \tag{2.9}$$

In particular, one has for all $r < r_0$

$$\Phi_y(B(y, c_0^{-1}r)) \subset B(0, r) \text{ and } B(0, c_0^{-1}r) \subset \Phi_y(B(y, r)). \tag{2.10}$$

We have endowed $\mathbb{R}^\ell$ with the standard Euclidean structure.

Proof of Lemma 2.6. This is classical. One can for instance choose the so-called almost linear coordinates, as in [17, Section 2] or [28, Section 3]. They are defined in the following way We fix an orthonormal basis $(e_i)_{1\leq i\leq \ell}$ of the tangent space T_yY and introduce the points $y_i := exp_y(-e_i)$ in Y. The map Φ_y is defined by the formula

$$\Phi_y(z) = (d(z, y_1){-}1, \ldots, d(z, y_\ell){-}1),$$

where z belongs to a sufficiently small ball $\mathring{B}(y, r_0)$. See [17, p. 43 and 58] for a detailed proof. □

There exist better systems of coordinates, the so-called harmonic coordinates. We will not need them in this chapter, but we will need them in Chapter 5 to prove uniqueness (see Lemma 5.2).

The second lemma gives a way to modify a Lipschitz map g inside a tiny ball $B(x, r)$ of X so that this new map $g_{x,r}$ is constant on the ball $B(x, \frac{r}{2})$ and the first two derivatives of $g_{x,r}$ are controlled by those of g. We recall that $\chi : \mathbb{R} \to \mathbb{R}$ is a non-negative $\mathcal{C}^\infty$ function with support included in $]-1, 1[$, which is equal to 1 on a neighborhood of $[-\frac{1}{2}, \frac{1}{2}]$ and which is 4-Lipschitz, i.e. $|\chi'| \leq 4$.

Lemma 2.7. *Let X and Y be two Hadamard manifolds with bounded curvatures $-b^2 \leq K_X \leq 0$, $-b^2 \leq K_Y \leq 0$. Let $r_0 > 0$ and $c_0 \geq 1$ be as in Lemma 2.6. Let $g : X \to Y$ be a Lipschitz map, x be a point in X, $y = g(x)$ and let $0 < r < r_0$. Assume that*

$$\mathrm{Lip}(g) < \frac{r_0}{c_0^2 r}. \tag{2.11}$$

Then the following formulas define a Lipschitz map $g_{r,x} : X \to Y$

$$
\begin{array}{rcll}
g_{r,x}(z) & = & g(x) & \text{when } d(z,x) \leq \frac{r}{2}\,, \\
& = & \Phi_y^{-1}\Big((1-\chi(\frac{d(z,x)}{r}))\,\Phi_y(g(z))\Big) & \text{when } \frac{r}{2} \leq d(z,x) \leq r\,, \\
& = & g(z) & \text{when } d(z,x) \geq r\,.
\end{array}
$$

One has the inequality between the Lipschitz constant on the balls $B(x,r)$,

$$\mathrm{Lip}_{B(x,r)}(g_{r,x}) \leq 5{c_0}^2\,\mathrm{Lip}_{B(x,r)}(g). \tag{2.12}$$

In particular, one has

$$\mathrm{Lip}(g_{r,x}) \leq 5{c_0}^2\,\mathrm{Lip}(g). \tag{2.13}$$

Moreover if g is $\mathcal{C}^2$ in a neighborhood of a point z in X, then $g_{r,x}$ is also $\mathcal{C}^2$ in this neighborhood and one has

$$\|D^2 g_{r,x}(z)\| \leq \Big(\|D^2 g(z)\| + \mathrm{Lip}_{B(x,r)}(g)^2 + 1\Big)\,M_r \tag{2.14}$$

where the constant $M_r \geq 1$ depends only on r, b, k, ℓ and χ.

Proof of Lemma 2.7. Note that, by Condition (2.11), for any point z in the ball $B(x,r)$, the image $g(z)$ belongs to the ball $\mathring{B}(y, c_0^{-2} r_0)$. Therefore, by (2.10), the vector $\Phi_y(g(z))$ belongs to the ball $\mathring{B}(0, c_0^{-1} r_0) \subset \mathbb{R}^\ell$. When we multiply this vector by the scalar $1-\chi(.)$, the new vector is still in the same ball. This is why, using again (2.10), the element $g_{r,x}(z)$ is well-defined and belongs to $B(y, r_0)$.

The upper bound (2.12) follows from the chain rule. Indeed, when z is a point in $B(x,r)$ where g is differentiable, one has, using the bound (2.9),

$$
\begin{array}{rcl}
\|Dg_{r,x}(z)\| & \leq & c_0\left(\dfrac{4}{r}\|\Phi_y(g(z))\| + \|D\Phi_y \circ g(z)\|\right) \\
& \leq & 5c_0\,\mathrm{Lip}_{B(x,r)}(\Phi_y \circ g) \;\leq\; 5{c_0}^2\,\mathrm{Lip}_{B(x,r)}(g).
\end{array}
$$

The upper bound (2.14) follows from similar and longer computations left to the reader, which also use the bounds (2.3) for $D^2 d_x$. □

We will also need a third lemma. We recall that a subset X_0 of a metric space X is said to be r-separated if the distance between two distinct points of X_0 is at least r.

Lemma 2.8. *Let X be a Hadamard manifold with $-b^2 \leq K_X \leq 0$. Let $k = \dim X$ and $N_0 := 100^k$. There exists a radius $r_0 = r_0(k,b) > 0$ such that, for all $r < r_0$, every $\frac{r}{2}$-separated subset X_0 of X can be decomposed as a union of at most N_0 subsets which are $2r$-separated.*

Proof of Lemma 2.8. The bound on the curvature of X and the Bishop volume estimates ensure that we can choose $r_0 > 0$ such that, one has

$$\mathrm{vol}B(x, 4r) \leq N_0 \,\mathrm{vol}B(x, \tfrac{r}{4}) \quad \text{for all } r < r_0 \text{ and } x \text{ in } X. \tag{2.15}$$

This r_0 works. Indeed, let $X_1, X_2, \ldots$ be a sequence of disjoint $2r$-separated subsets of X_0 with X_1 maximal in X_0, X_2 maximal in $X_0 \smallsetminus X_1$, and so on. Every point x of X_0 must be in one of the X_i's with $i \leq N_0$, because, if it is not the case, each X_i contains a point in $B(x, 2r)$, contradicting (2.15). □

Proof of Proposition 2.4. Second step: bound on $D^2\widetilde{f}$. According to the first step of this proof, we can now assume that the map $f : X \to Y$ is c-Lipschitz with $c \geq 1$.

We can choose a new radius $r_0 = r_0(k, \ell, b)$ which satisfies both the conclusion of Lemma 2.8 for X and of Lemma 2.6 for Y. We will use freely the notation of these two lemmas. Now let

$$r_1 = \frac{r_0}{5^{N_0}\, c_0^{2N_0+2}\, c}$$

and pick a maximal $\frac{r_1}{4}$-separated subset X_0 of X. Thanks to Lemma 2.8, we write this set X_0 as a union

$$X_0 = X_1 \cup \cdots \cup X_{N_0}$$

of N_0 subsets X_i which are $2r_1$-separated.

In order to construct $\widetilde{f}$ from f, we will use a finite iterative process based on Lemma 2.7. We start with $f_0 = f$, we construct by induction a finite sequence of maps f_i for $i \leq N_0$ and we set $\widetilde{f} := f_{N_0}$. Given f_{i-1}, the map f_i is defined, using the notations of Lemma 2.7, by

$$\begin{aligned} f_i(z) &= (f_{i-1})_{r_1,x}(z) \qquad \text{when } d(z,x) \leq r_1 \text{ for some } x \text{ in } X_{i+1}, \\ &= f_{i-1}(z) \qquad \text{otherwise,} \end{aligned}$$

and the Lipschitz constant of these maps satisfy

$$\mathrm{Lip}(f_i) \leq 5c_0^2 \,\mathrm{Lip}(f_{i-1}) \leq 5^i {c_0}^{2i} c. \tag{2.16}$$

Indeed, once f_i is known to be well defined and to satisfy (2.16), it also satisfies the bound (2.13): $\mathrm{Lip}(f_i) < \frac{r_0}{c_0^2 r_1}$. Therefore, by Lemma 2.7, the map f_{i+1} is well defined and, using the bound (2.12), satisfies also (2.16):

$$\mathrm{Lip}(f_{i+1}) \leq 5c_0^2 \,\mathrm{Lip}(f_i) \leq 5^{i+1} {c_0}^{2(i+1)} c.$$

Let $\Lambda := M_{r_1} + 25c_0^4 + 1$. By (2.14) and (2.16), for $i \leq N_0$, and z in X, one has

$$\|D^2 f_i(z)\| + \mathrm{Lip}(f_i)^2 + 1 \leq \Lambda \left(\|D^2 f_{i-1}(z)\| + \mathrm{Lip}(f_{i-1})^2 + 1\right) \tag{2.17}$$

Since X_0 is a maximal $\frac{r_1}{4}$-separated subset of X, every z in X belongs to at least one ball $\mathring{B}(x, \frac{r}{2})$ where x is in one of the sets X_{i_0}. But then the function f_{i_0} is constant in a neighborhood of z. Therefore, using (2.16) and applying $(N_0 - i_0)$ times the bound (2.17) one deduces that $\widetilde{f}$ is a $\mathcal{C}^2$-map which satisfies the uniform upper bound

$$\|D^2\widetilde{f}(z)\| \leq ((5^{i_0}c_0^{2i_0}c)^2 + 1)\Lambda^{N_0 - i_0} \leq \Lambda^{N_0}c^2. \qquad \square$$

3 Harmonic maps

In this chapter we begin the proof of the existence part in Theorem 1.1. We first recall basic facts satisfied by harmonic maps. We explain why a standard compactness argument reduces this existence part to proving a uniform upper bound on the distance between f and the harmonic map h_R which is equal to f on the sphere $S(O, R)$. Then we provide this upper bound near this sphere $S(O, R)$.

3.1 Harmonic functions and the distance function

We recall basic facts on the Laplace operator on Hadamard manifolds.

The Laplace-Beltrami operator Δ on a Riemannian manifold X is defined as the trace of the Hessian. In local coordinates, the Laplacian of a function φ is

$$\Delta\varphi = \mathrm{tr}(D^2\varphi) = \tfrac{1}{v}\textstyle\sum_{i,j} \frac{\partial}{\partial x_i}(v\, g_X^{ij} \frac{\partial}{\partial x_j}\varphi) \tag{3.1}$$

where $v = \sqrt{\det(g_X^{ij})}$ is the volume density. The function φ is said to be harmonic if $\Delta\varphi = 0$ and subharmonic if $\Delta\varphi \geq 0$.

We will need the following basic lemma.

Lemma 3.1. *Let X be a Hadamard manifold with $K_X \leq -a^2 \leq 0$ and x_0 be a point of X. Then, the function d_{x_0} is subharmonic. More precisely, the distribution $\Delta\, d_{x_0} - a$ is non-negative.*

Proof. This is [4, Lemma 2.5]. $\square$

3.2 Harmonic maps and the distance function

In this section, we recall two useful facts satisfied by a harmonic map h : the subharmonicity of the functions $d_{y_0} \circ h$, and Cheng's estimate for the differential Dh.

Definition 3.2. *Let $h : X \to Y$ be a $\mathcal{C}^2$ map between two Riemannian manifolds. The tension field of h is the trace of the second covariant derivative $\tau(h) := \mathrm{tr}D^2h$. The map h is said to be harmonic if $\tau(h) = 0$.*

Note that the tension field $\tau(h)$ is a Y*-valued vector field on* X, i.e. it is a section of the pulled-back of the tangent bundle $TY \to Y$ under the map $h : X \to Y$.

For instance, an isometric immersion with minimal image is always harmonic. The problem of the existence, regularity and uniqueness of harmonic maps under various boundary conditions is a very classical topic (see [10], [31], [17], [9], [35], [34] or [23]). In particular, when Y is simply connected and has non positive curvature, a harmonic map is always $\mathcal{C}^\infty$ i.e. it is indefinitely differentiable, and is a minimum of the energy functional among maps that agree with h outside a compact subset of X.

Lemma 3.3. *Let $h : X \to Y$ be a harmonic $\mathcal{C}^\infty$ map between Riemannian manifolds. Let $y_0 \in Y$ and let $\rho_h : X \to \mathbb{R}$ be the function $\rho_h := d_{y_0} \circ h$. If Y is Hadamard, the continuous function ρ_h is subharmonic on X.*

Proof. See [4, Lemma 3.2]. □

Another crucial property of harmonic maps is the following bound for their differential due to Cheng.

Lemma 3.4. *Let X, Y be two Hadamard manifolds with $-b^2 \leq K_X \leq 0$. Let $k = \dim X$, z be a point of X, $r > 0$ and let $h : B(z,r) \to Y$ be a harmonic $\mathcal{C}^\infty$ map such that the image $h(B(z,r))$ lies in a ball of radius R_0. Then one has the bound*

$$\|Dh(z)\| \leq 2^5\, k\, \tfrac{1+br}{r}\, R_0 \,.$$

In the applications, we will use this inequality with $r = b^{-1}$.

Proof. This is a simplified version of [8, Formula 2.9]. □

3.3 Existence of harmonic maps

In this section we prove Theorem 1.1, taking for granted Proposition 3.5 below.

Let X and Y be two Hadamard manifolds whose curvatures are pinched $-b^2 \leq K_X \leq -a^2 < 0$ and $-b^2 \leq K_Y \leq -a^2 < 0$. Let $k = \dim X$ and $\ell = \dim Y$. Let $f : X \to Y$ be a (c, C)-quasi-isometric $\mathcal{C}^\infty$ map whose first two covariant derivatives are bounded.

We fix a point O in X. For $R > 0$, we denote by $B_R := B(O, R)$ the closed ball in X with center O and radius R and by ∂B_R the sphere that bounds B_R. Since the manifold Y is a Hadamard manifold, there exists a unique harmonic map $h_R : B_R \to Y$ satisfying the Dirichlet condition $h_R = f$ on the sphere ∂B_R. Thanks to Schoen and Uhlenbeck in [32] and

[33], the harmonic map $h_{_R}$ is known to be $\mathcal{C}^\infty$ on the closed ball $B_{_R}$. We denote by

$$d(h_{_R}, f) = \sup_{x \in B(O,R)} d(h_{_R}(x), f(x))$$

the distance between these two maps. The main step for proving existence in Theorem 1.1 is the following uniform estimate.

Proposition 3.5. *There exists a constant $\rho \geq 1$ such that, for any $R \geq 1$, one has $d(h_{_R}, f) \leq \rho$.*

The constant ρ is a function of a, b, c, C, k and ℓ. More precisely, when f satisfies (4.1), ρ needs only to satisfy Conditions (4.6), (4.7) and (4.8).

We briefly recall the classical argument used to deduce Theorem 1.1 from this Proposition

Proof of Theorem 1.1. As explained in Proposition 2.4, we may assume that the (c, C)-quasi-isometric map f is $\mathcal{C}^\infty$ and that its first two covariant derivatives are bounded. Pick an increasing sequence of radii R_n converging to ∞ and let $h_{R_n} : B_{R_n} \to Y$ be the harmonic $\mathcal{C}^\infty$ map which agrees with f on the sphere ∂B_{R_n}. Proposition 3.5 ensures that the sequence of maps h_{R_n} is locally uniformly bounded. Using the Cheng Lemma 3.4 it follows that the first derivatives are also uniformly bounded on the balls $B_{_S}$. The Ascoli-Arzela theorem implies that, after extracting a subsequence, the sequence h_{R_n} converges uniformly on every ball $B_{_S}$ towards a continuous map $h : X \to Y$. Using Schauder's estimates, one also gets a uniform bound for the $\mathcal{C}^{2,\alpha}$-norms of h_{R_n} on $B_{_S}$. If needed, note that these classical estimates will be recalled in Formulas (5.29), (5.32) and (5.33) in Section 5.6. Therefore, using again the Ascoli-Arzela theorem, the sequence h_{R_n} converges in $\mathcal{C}^2$-norm and the limit map h is $\mathcal{C}^2$ and harmonic. By construction this limit map h stays within bounded distance from the quasi-isometric map f. □

Remark 3.6. By the uniqueness part of our Theorem 1.1 that we will prove in Chapter 5, the harmonic map h which stays within bounded distance from f is unique. Hence the above argument also proves that the whole family of harmonic maps h_R converges to h uniformly on the compact subsets of X when R goes to infinity.

3.4 Boundary estimate

In this section we begin the proof of Proposition 3.5 : we bound the distance between $h_{_R}$ and f near the sphere $\partial B_{_R}$.

Proposition 3.7. *Let X, Y be Hadamard manifolds and $k = \dim X$. Assume moreover that $K_X \leq -a^2 < 0$ and $-b^2 \leq K_Y \leq 0$. Let $c \geq 1$ and $f : X \to Y$ be a $\mathcal{C}^\infty$ map with $\|Df(x)\| \leq c$ and $\|D^2 f(x)\| \leq bc^2$. Let $O \in X$, $R > 0$, $B_{_R} := B(O, R)$.*

Let $h_R : B_R \to Y$ be the harmonic $\mathcal{C}^\infty$ map whose restriction to the sphere ∂B_R is equal to f. Then, for all x in B_R, one has

$$d(h_R(x), f(x)) \leq \tfrac{3kbc^2}{a}\, d(x, \partial B_R)\ . \tag{3.2}$$

An important feature of this upper bound is that it does not depend on the radius R, provided the distance $d(x, \partial B_R)$ remains bounded. This is why we call (3.2) the *boundary estimate*. The proof relies on an idea of Jost in [17, Section 4].

Proof. This proposition is already in [4, Proposition 3.8]. We give here a slightly shorter proof. Let x be a point in B_R and y be a point of Y chosen such that $d(y, f(B_R)) \geq b^{-1}$ and

$$d_y(h_R(x)) - d_y(f(x)) = d(f(x), h_R(x))\ . \tag{3.3}$$

This point y is far away on the geodesic ray starting at $h_R(x)$ and containing $f(x)$. Let φ be the $\mathcal{C}^\infty$ function on the ball B_R given by

$$\varphi(z) := d_y(h_R(z)) - d_y(f(z)) - \tfrac{3kbc^2}{a}(R - d_O(z)) \ \text{ for all } z \text{ in } B_R. \tag{3.4}$$

This function is the sum of three functions $\varphi = \varphi_1 + \varphi_2 + \varphi_3$.

The first function $\varphi_1 : z \mapsto d_y(h_R(z))$ is subharmonic on B_R i.e. one has $\Delta\varphi_1 \geq 0$. This follows from Lemma 3.3 and the harmonicity of the map h_R.

The second function $\varphi_2 : z \mapsto -d_y(f(z))$ has a Laplacian which is bounded $|\Delta\varphi_2| \leq 3kbc^2$. Indeed, since y is far away, using Formula (2.3), one has the bound $\|D^2 d_y\| \leq 2b$ on $f(B_R)$ and one computes

$$|\Delta\varphi_2| = |\Delta(d_y \circ f)| \leq k\|D^2 d_y\|\|Df\|^2 + k\|Dd_y\|\|D^2 f\| \leq 3kbc^2\ .$$

The third function $\varphi_3 : z \mapsto -\frac{3kbc^2}{a}(R - d_O(z))$ has a Laplacian bounded below $\Delta\varphi_3 \geq 3kbc^2$. This follows from Lemma 3.1 which says that $\Delta d_O \geq a$.

Hence the function φ is subharmonic: $\Delta\varphi \geq 0$. Since φ is null on ∂B_R, one gets $\varphi(x) \leq 0$ as required. $\square$

4 Interior estimate

In this chapter we complete the proof of Proposition 3.5.

4.1 Strategy

We first explain more precisely the notations and the assumptions that we will use in the whole chapter.

Let X and Y be Hadamard manifolds whose curvatures are pinched $-b^2 \leq K_X \leq -a^2 < 0$ and $-b^2 \leq K_Y \leq -a^2 < 0$. Let $k = \dim X$ and $\ell = \dim Y$. We start with a $\mathcal{C}^\infty$ quasi-isometric map $f : X \to Y$ whose first and second covariant derivatives are bounded. We fix constants $c \geq 1$ and $C > 0$ such that, for all x, x' in X, one has

$$\|Df(x)\| \leq c \ , \quad \|D^2 f(x)\| \leq bc^2 \quad \text{and} \tag{4.1}$$

$$c^{-1}\, d(x,x') - C \ \leq \ d(f(x), f(x')) \ \leq \ c\, d(x,x') \, . \tag{4.2}$$

Note that the additive constant C on the right-hand side term of (1.1) has been removed since the derivative of f is now bounded by c.

4.1.1 Choosing the radius ℓ_0

We fix a point O in X. We introduce a fixed radius ℓ_0 depending only on a, b, k, ℓ, c and C. This radius ℓ_0 is only required to satisfy the following three inequalities (4.3), (4.4) and (4.5) that will be needed later on.

The first condition we impose on the radius ℓ_0 is

$$b\ell_0 > 1. \tag{4.3}$$

The second condition we impose on the radius ℓ_0 is

$$\ell_0 > \frac{(A + b^{-1})c}{\sin^2(\varepsilon_0/2)} \ \text{ where } \ \varepsilon_0 := (3c^2 M)^{-N}, \tag{4.4}$$

where A is the constant given by Lemma 2.2, and M, N the constants given by Proposition 4.9.

The third condition we impose on the radius ℓ_0 is

$$16\, e^{\frac{aC}{2}}\, e^{-\frac{a\ell_0}{4\,c}} < \theta_0 \ \text{ where } \ \theta_0 := e^{-bA}\, (\varepsilon_0/4)^{\frac{bc}{a}}. \tag{4.5}$$

4.1.2 Assuming ρ to be large

We want to prove Proposition 3.5. For $R > 0$, let $h_R : B(O,R) \to Y$ be the harmonic $\mathcal{C}^\infty$ map whose restriction to the sphere $\partial B(O,R)$ is equal to f. We let

$$\rho := \sup_{x \in B(O,R)} d(h_R(x), f(x)) \, .$$

We argue by contradiction. If this supremum ρ is not uniformly bounded, we can fix a radius R such that ρ satisfies the following three inequalities (4.6), (4.7) and (4.8) that we will use later on.

The first condition we impose on the radius ρ is

$$a\rho > 8kbc^2\ell_0 \, . \tag{4.6}$$

The second condition we impose on the radius ρ is

$$\frac{2^7(a\rho)^2}{\sinh(a\rho/2)} < \theta_0\,. \tag{4.7}$$

The third condition we impose on the radius ρ is

$$\rho > 4c\ell_0 M\,(2^{10}e^{b\ell_0}k)^N \tag{4.8}$$

where M, N are the constants given by Proposition 4.9.

We denote by x a point of $B(O,R)$ where the supremum (4.1.2) is achieved:

$$d(h_{_R}(x), f(x)) = \rho\,.$$

According to the boundary estimate in Proposition 3.7, one has, using Conditions (4.6),

$$d(x,\partial B(O,R)) \geq \frac{a\rho}{3kbc^2} \geq 2\ell_0\,.$$

Combined with Condition (4.3), this ensures that the ball $B(x,\ell_0)$ with center x and radius ℓ_0 satisfies the inclusion $B(x,\ell_0)\subset B(O,R{-}b^{-1})$. This inclusion will allow us to apply Cheng 's lemma 3.4 at each point z of the ball $B(x,\ell_0)$.

4.1.3 Getting a contradiction

We will focus on the restrictions of both maps f and $h_{_R}$ to this ball $B(x,\ell_0)$. We introduce the point $y := f(x)$. For y_1, y_2 in $Y \smallsetminus \{y\}$, we denote by $\theta_y(y_1,y_2)$ the angle at y of the geodesic triangle with vertices y, y_1, y_2. For z on the sphere $S(x,\ell_0)$, we will analyze the triangular inequality:

$$\theta_y(f(z),h_{_R}(x)) \leq \theta_y(f(z),h_{_R}(z)) + \theta_y(h_{_R}(z),h_{_R}(x)), \tag{4.9}$$

and prove that on a subset U_{ℓ_0} of the sphere, each term on the right-hand side is small (Lemmas 4.5 and 4.6) while the measure of U_{ℓ_0} is large enough (Lemma 4.4) to ensure that the left-hand side is not that small (Lemma 4.8), giving rise to the contradiction. These arguments rely on uniform lower and upper bounds for the harmonic measures on the spheres of X which will be given in Proposition 4.9.

We denote by ρ_h the function on $B(x,\ell_0)$ given by $\rho_h(z) = d(y,h_{_R}(z))$ where again $y = f(x)$. By Lemma 3.3, this function is subharmonic.

Definition 4.1. *The subset U_{ℓ_0} of the sphere $S(x,\ell_0)$ is given by*

$$U_{\ell_0} \quad = \quad \{z\in S(x,\ell_0) \mid \rho_h(z)\geq \rho - \frac{\ell_0}{2c}\}\,. \tag{4.10}$$

4.2 Measure estimate

We first notice that one can control the size of $\rho_h(z)$ and of $Dh_{_R}(z)$ on the ball $B(x, \ell_0)$. We will then derive a lower bound for the measure of U_{ℓ_0}.

Lemma 4.2. *For z in $B(x, \ell_0)$, one has*

$$\rho_h(z) \leq \rho + c\,\ell_0.$$

Proof. The triangle inequality and (4.2) give, for z in $B(x, \ell_0)$,

$$\rho_h(z) \leq d(h_{_R}(z), f(z)) + d(f(z), y) \leq \rho + c\,\ell_0\,. \qquad \square$$

Lemma 4.3. *For z in $B(x, \ell_0)$, one has*

$$\|Dh_{_R}(z)\| \leq 2^8 kb\rho.$$

Proof. For all z, z' in $B(O, R)$ with $d(z, z') \leq b^{-1}$, the triangle inequality and (4.2) yield

$$\begin{aligned} d(h_{_R}(z), h_{_R}(z')) &\leq d(h_{_R}(z), f(z)) + d(f(z), f(z')) + d(f(z'), h_{_R}(z')) \\ &\leq \rho + b^{-1}c + \rho \leq 2\,\rho + c\ell_0 \leq 3\,\rho\,. \end{aligned}$$

For these last two inequalities, we used Conditions (4.3) and (4.6). Applying Cheng's lemma 3.4 with $R_0 = 3\rho$ and $r = b^{-1}$, one then gets for all z in $B(O, R-b^{-1})$ the bound $\|Dh_{_R}(z)\| \leq 2^8 kb\rho$. $\square$

We now give a lower bound for the measure of U_{ℓ_0}.

Lemma 4.4. *Let $\sigma = \sigma_{x,\ell_0}$ be the harmonic measure on the sphere $S(x, \ell_0)$ for the center point x. Then one has*

$$\sigma(U_{\ell_0}) \;\; \geq \;\; \frac{1}{3\,c^2}\,. \tag{4.11}$$

Proof. By Lemma 3.3, the function ρ_h is subharmonic on the ball $B(x, \ell_0)$. Hence this function ρ_h is not larger than the harmonic function on the ball with same boundary values on the sphere $S(x, \ell_0)$. Comparing these functions at the center x, one gets

$$\int_{S(x,\ell_0)} (\rho_h(z) - \rho)\,\mathrm{d}\sigma(z) \geq 0\,. \tag{4.12}$$

By Lemma 4.2, the function ρ_h is bounded by $\rho + c\,\ell_0$. Hence Equation (4.12) and the definition of U_{ℓ_0} implies

$$c\,\ell_0\,\sigma(U_{\ell_0}) - \frac{\ell_0}{2c}\,(1 - \sigma(U_{\ell_0})) \geq 0$$

so that $\sigma(U_{\ell_0}) \geq \frac{1}{3c^2}$. $\square$

4.3 Upper bound for $\theta_y(f(z), h_R(z))$

For all z in U_{ℓ_0}, we give an upper bound for the the angle between $f(z)$ and $h_R(z)$ seen from the point $y = f(x)$.

Lemma 4.5. *For z in U_{ℓ_0}, one has*

$$\theta_y(f(z), h_R(z)) \quad \leq \quad 4\, e^{\frac{aC}{2}}\, e^{-\frac{a\ell_0}{4\,c}}. \tag{4.13}$$

Proof. For z in U_{ℓ_0}, we consider the triangle with vertices y, $f(z)$ and $h_R(z)$. Its side lengths satisfy

$$d(h_R(z), f(z)) \leq \rho \;\;,\;\; d(y, f(z)) \geq \frac{\ell_0}{c} - C \;\;,\;\; d(y, h_R(z)) \geq \rho - \frac{\ell_0}{2c}$$

using successively, the definition of ρ, the quasi-isometry lower bound (4.2), and the definition of U_{ℓ_0}. Hence, one gets the following lower bound for the Gromov product

$$(f(z)|h_R(z))_y \geq \frac{\ell_0}{4c} - \frac{C}{2}.$$

Since $K_Y \leq -a^2$, Lemma 2.1 yields

$$\theta_y(f(z), h_R(z)) \leq 4\, e^{\frac{aC}{2}}\, e^{-\frac{a\ell_0}{4\,c}}\,. \qquad \square$$

4.4 Upper bound for $\theta_y(h_R(z), h_R(x))$

For all z in $S(x, \ell_0)$, we give an upper bound for the angle between $h_R(z)$ and $h_R(x)$ seen from the point $y = f(x)$.

Lemma 4.6. *For all z in the sphere $S(x, \ell_0)$, one has*

$$\theta_y(h_R(z), h_R(x)) \quad \leq \quad \frac{2^5\,(a\rho)^2}{\sinh(a\rho/2)}. \tag{4.14}$$

The proof will rely on the following lemma which also ensures that this angle $\theta_y(h_R(z), h_R(x))$ is well defined.

Lemma 4.7. *For all z in the ball $B(x, \ell_0)$, one has $\rho_h(z) \geq \rho/2$.*

Proof of Lemma 4.7. Assume by contradiction, that there exists a point z_1 in the ball $B(x, \ell_0)$ such that $\rho_h(z_1) = \rho/2$. Set $r_1 := d(x, z_1)$. One has $0 < r_1 \leq \ell_0$. According to Lemma 4.3, one can bound the differential of h_R on the ball $B(x, \ell_0)$, namely

$$\sup_{B(x,\ell_0)} \|Dh_R\| \leq 2^8 kb\rho\,.$$

Hence one has

$$\rho_h(z) \leq \frac{3\rho}{4} \text{ for all } z \text{ in } S(x, r_1) \cap B(z_1, \tfrac{1}{2^{10}kb}).$$

By comparison with the hyperbolic plane with curvature $-b^2$, this intersection contains the trace on the sphere $S(x, r_1)$ of a cone C_α with vertex x and angle α as soon as $\sin\frac{\alpha}{2} \le \frac{\sinh(2^{-11}/k)}{\sinh(br_1)}$. For instance we will choose for α the angle $\alpha := e^{-b\ell_0}2^{-10}/k$.

Let $\sigma' = \sigma_{x,r_1}$ be the harmonic measure on the sphere $S(x, r_1)$ for the center point x. Using the subharmonicity of the function ρ_h as in the proof of Lemma 3.3, one gets the inequality

$$\int_{S(x,r_1)} (\rho_h(z) - \rho)\, \mathrm{d}\sigma'(z) \ge 0\,. \tag{4.15}$$

By Lemma 4.2, the function ρ_h is bounded by $\rho + c\,\ell_0$. Using the bound $\rho_h(z) \le \frac{3}{4}\rho$ when z is in the cone C_α, hence Equation (4.15) implies

$$c\,\ell_0 - \frac{\rho}{4}\,\sigma'(C_\alpha) \ge 0\,.$$

Using the uniform lower bounds for the harmonic measures on the spheres of X in Proposition 4.9, one gets

$$\rho \le 4c\ell_0 M\,\alpha^{-N} = 4c\ell_0 M\,(2^{10}e^{b\ell_0}k)^N,$$

which contradicts Condition (4.8). $\square$

Proof of Lemma 4.6. Let us first sketch the proof. Let z be a point on the sphere $S(x, \ell_0)$. We denote by $t \mapsto z_t$, for $0 \le t \le \ell_0$, the geodesic segment between x and z. By Lemma 4.7, the curve $t \mapsto h_R(z_t)$ lies outside of the ball $B(y, \rho/2)$ and by Cheng's bound on $\|Dh_R(z_t)\|$ one controls the length of this curve.

We now detail the argument. We denote by $(\rho(y'), v(y')) \in]0,\infty[\times T^1_yY$ the polar exponential coordinates centered at y. For a point y' in $Y \smallsetminus \{y\}$, they are defined by the equality $y' = \exp_y(\rho(y')v_\rho(y'))$. Since $K_Y \le -a^2$ the Alexandrov comparison theorem for infinitesimal triangles and the Gauss lemma ([11, 2.93]) yield,

$$\sinh(a\rho(y'))\,\|Dv(y')\| \le a\,.$$

Writing $v_h := v \circ h_R$, one gets, for z' in $B(x, \ell_0)$,

$$\sinh(a\rho_h(z'))\,\|Dv_h(z')\| \le a\|Dh_R(z')\|\,.$$

Hence, using Lemma 4.7, we have the inequality

$$\begin{aligned} \theta_y(h_R(z), h_R(x)) &\le \ell_0 \sup_{0\le t\le \ell_0} \|Dv_h(z_t)\| \\ &\le \frac{a\ell_0}{\sinh(a\rho/2)} \sup_{0\le t\le \ell_0} \|Dh_R(z_t)\|\,. \end{aligned}$$

Therefore, using Lemma 4.3 and Condition (4.6), one gets

$$\theta_y(h_R(z), h_R(x)) \le \frac{2^8 kb\rho\, a\ell_0}{\sinh(a\rho/2)} \le \frac{2^5(a\rho)^2}{\sinh(a\rho/2)}\,. \qquad \square$$

4.5 Lower bound for $\theta_y(f(z), h_R(x))$

We find a point z in U_{ℓ_0} for which the angle between $f(z)$ and $h(x)$ seen from $y = f(x)$ has an explicit lower bound.

Lemma 4.8. *There exist two points z_1, z_2 in U_{ℓ_0} such that the angle*

$$\theta_y(f(z_1), f(z_2)) \geq \theta_0 \,,$$

where θ_0 is the angle given by (4.5)*.*

Proof of Lemma 4.8. Let $\sigma_0 := \frac{1}{3c^2}$. According to Lemma 4.4, one has $\sigma(U_{\ell_0}) \geq \sigma_0 > 0$ Thus, using the uniform upper bounds for the harmonic measures on the spheres of X in Proposition 4.9, one can find z_1, z_2 in U_{ℓ_0} such that the angle $\theta_x(z_1, z_2)$ between z_1 and z_2 seen from x satisfies,

$$\sigma_0 \leq M\, \theta_x(z_1, z_2)^{\frac{1}{N}}$$

This can be rewritten as

$$\theta_x(z_1, z_2) \geq \varepsilon_0 \,, \tag{4.16}$$

where ε_0 is the angle introduced in (4.4) by the equality $\sigma_0 = M\, \varepsilon_0^{\frac{1}{N}}$. Therefore, using Lemma 2.1.a and Condition (4.4), we get the following lower bound on the Gromov products

$$\min((x|z_1)_{z_2}, (x|z_2)_{z_1}) \geq \ell_0 \sin^2(\varepsilon_0/2) \geq (A + b^{-1})c.$$

Using then Lemma 2.2, one gets

$$\min((y|f(z_1))_{f(z_2)}, (y|f(z_2))_{f(z_1)}) \geq b^{-1}. \tag{4.17}$$

This inequality (4.17) allows us to apply Lemma 2.1.c, which gives

$$\theta_y(f(z_1), f(z_2)) \;\; \geq \;\; e^{-b(f(z_1)|f(z_2))_y}.$$

Therefore, by Lemma 2.2, one has

$$\theta_y(f(z_1), f(z_2)) \;\; \geq \;\; e^{-bA}\, e^{-bc\,(z_1|z_2)_x}.$$

Using Lemma 2.1.b and Condition (4.16), one gets

$$\begin{aligned} \theta_y(f(z_1), f(z_2)) &\geq e^{-bA}\, (\theta_x(z_1, z_2)/4)^{\frac{bc}{a}} \\ &\geq e^{-bA}\, (\varepsilon_0/4)^{\frac{bc}{a}} = \theta_0, \end{aligned}$$

according to the definition (4.5) of θ_0. $\square$

End of the proof of Proposition 3.5. Using Lemmas 4.5 and 4.6 and the triangle inequality (4.9), one gets, for any two points $z_i = z_1$ or z_2 in U_{ℓ_0},

$$\begin{aligned} \theta_y(f(z_i), h_R(x)) &\leq 4\, e^{\frac{aC}{2}}\, e^{-\frac{a\ell_0}{4c}} + \frac{2^5(a\rho)^2}{\sinh(a\rho/2)} \\ &< \tfrac{1}{2}\theta_0 \qquad \text{by Conditions (4.5) and (4.7).} \end{aligned}$$

Therefore, using again a triangle inequality, one has

$$\theta_y(f(z_1), f(z_2)) < \theta_0,$$

which contradicts Lemma 4.8. □

4.6 Harmonic measures

The following proposition gives the uniform lower and upper bounds for the harmonic measure on a sphere for the center which were used in the proof of Lemmas 4.7 and 4.8.

Proposition 4.9. *Let $0 < a < b$ and $k \geq 2$ be an integer. There exist positive constants M, N depending solely on a, b, k such that for every Hadamard manifold X with pinched curvature $-b^2 \leq K_X \leq -a^2$, for every point x in X, every radius $r > 0$ and every angle $\theta \in [0, \pi]$ one has*

$$\tfrac{1}{M}\, \theta^N \leq \sigma_{x,r}(C_{x,\theta}) \leq M\, \theta^{\frac{1}{N}} \tag{4.18}$$

where $\sigma_{x,r}$ denotes the harmonic measure on the sphere $S(x,r)$ for the point x and where $C_{x,\theta}$ stands for any cone with vertex x and angle θ.

We recall that, by definition, $\sigma_{x,r}$ is the unique probability measure on the sphere $S(x,r)$ such that, for every continuous function h on the ball $B(x,r)$ which is harmonic in the interior $\mathring{B}(x,r)$, one has

$$h(x) = \int_{S(x,r)} h(z)\, \mathrm{d}\sigma_{x,r}(z).$$

A proof of Proposition 4.9 is given in [3]. It relies on various technical tools of the potential theory on pinched Hadamard manifolds: the Harnack inequality, the barrier functions constructed by Anderson and Schoen in [2] and upper and lower bounds for the Green functions due to Ancona in [1]. Related estimates are available like the one by Kifer–Ledrappier in [18, Theorem 3.1 and 4.1] where (4.18) is proven for the sphere at infinity or by Ledrappier–Lim in [19, Proposition 3.9] where the Hölder regularity of the Martin kernel is proven.

5 Uniqueness of harmonic maps

In this chapter we prove the uniqueness part in Theorem 1.1.

5.1 Strategy

In other words we will prove the following proposition.

Proposition 5.1. *Let X, Y be two pinched Hadamard manifolds and let $h_0, h_1 : X \to Y$ be two quasi-isometric harmonic maps that stay within bounded distance of one another:*

$$\sup_{x \in X} d(h_0(x), h_1(x)) < \infty\,.$$

Then one has $h_0 = h_1$.

When $X = Y = \mathbb{H}^2$, this proposition was first proven by Li and Tam in [21]. When both X and Y admit a cocompact group of isometries, this proposition was then proven by Li and Wang in [22, Theorem 2.3]. The aim of this chapter is to explain how to get rid of these extra assumptions.

Note that the assumption that the h_i are quasi-isometric is useful. Indeed there does exist non constant bounded harmonic functions on X. Note that there also exist bounded harmonic maps with open images. Here is a very simple example. Let $0 < \lambda < 1$. The map h_λ from the Poincaré unit disk $\mathbb{D}$ of $\mathbb{C}$ into itself given by $z \mapsto \lambda z$ is harmonic. More generally, for any harmonic map $h : \mathbb{D} \to \mathbb{D}$, the map $h_\lambda : \mathbb{D} \to \mathbb{D} : z \mapsto h(\lambda z)$ is a harmonic map with bounded image.

Before going into the technical details, we first explain the strategy of the proof of this uniqueness.

Strategy of proof of Proposition 5.1. We recall that the distance function $x \mapsto d(h_0(x), h_1(x))$ is a subharmonic function on X and that, by the maximum principle, a subharmonic function which achieves its maximum value is constant. Unfortunately since X is non-compact we can not a priori ensure that this bounded function achieves its maximum. This is why we will use a recentering argument.

We assume, by contradiction, that $h_0 \neq h_1$ and we choose a sequence of points p_n in X for which the distances

$$d(h_0(p_n), h_1(p_n)) \text{ converge to } \delta := \sup_{x \in X} d(h_0(x), h_1(x)) > 0 \tag{5.1}$$

and we set $q_n := h_0(p_n)$.

The pinching conditions on X and Y ensures that, after extracting a subsequence, the pointed metric spaces (X, p_n) and (Y, q_n) converge in the Gromov–Hausdorff topology to pointed metric spaces (X_∞, p_∞) and (Y_∞, q_∞) which are $\mathcal{C}^2$ Hadamard manifold with a $\mathcal{C}^1$ Riemannian metric satisfying the same pinching conditions (Proposition 5.14). Moreover, extracting again a subsequence, the harmonic map h_0 (resp. h_1) seen as a sequence of maps between the pointed Hadamard manifolds (X, p_n) and (Y, q_n) converges locally uniformly to a map $h_{0,\infty}$ (resp $h_{1,\infty}$) between the pointed C^2 Hadamard manifolds (X_∞, p_∞) and (Y_∞, q_∞). These harmonic maps $h_{0,\infty}$ and $h_{1,\infty}$ are still harmonic quasi-isometric maps (Lemma 5.15).

The limit distance function $x \mapsto d(h_{0,\infty}(x), h_{1,\infty}(x))$ is a subharmonic function on X_∞ and achieves its maximum $\delta > 0$ at the point p_∞. Hence, by the maximum principle this distance function is constant and equal to δ (Lemma 5.16). Generalizing [22, Lemma 2.2], we will see in Corollary 5.19 that this equidistance property implies that both $h_{0,\infty}$ and $h_{1,\infty}$ take their values in a geodesic of Y_∞. This contradicts the fact that these maps $h_{0,\infty}$ and $h_{1,\infty}$ are quasi-isometric maps, and concludes this strategy of proof. $\square$

In the following sections of Chapter 5, we fill in the details of the proof.

5.2 Harmonic coordinates

We first introduce the so-called harmonic coordinates, which improve the quasilinear coordinates introduced in Lemma 2.6. We refer to [14] or [17] for more details.

These harmonic coordinates have been introduced by DeTurk and Kazdan and extensively used by Cheeger, Jost, Karcher, Petersen... to prove various compactness results for compact Riemannian manifolds. Beyond being harmonic, the main advantage of these coordinates is that, for every $\alpha \in]0,1[$, they are uniformly bounded in $\mathcal{C}^{2,\alpha}$-norm, i.e. they are uniformly bounded in $\mathcal{C}^2$-norm and one also has a uniform control of the α-Hölder norm of their second covariant derivatives. Moreover, one has a uniform control on the size of the balls on which these harmonic charts are defined. This is what the following lemma tells us.

We endow $\mathbb{R}^k$ with the standard Euclidean structure.

Lemma 5.2. *Let X be a k-dimensional Hadamard manifold with bounded curvature $-1 \leq K_X \leq 0$. Let $0 < \alpha < 1$. There exist two constants $r_0 = r_0(k) > 0$ and $c_0 = c_0(k, \alpha) > 0$ such that, for every x in X, there exists a $\mathcal{C}^\infty$-diffeomorphism*

$$\Psi_x : \mathring{B}(x, r_0) \xrightarrow{\sim} U_x \subset \mathbb{R}^k \text{ with } \Psi_x(x) = 0 \,, \tag{5.2}$$

$$\|D\Psi_x\| \leq c_0 \;\;,\;\; \|D\Psi_x^{-1}\| \leq c_0 \;\;,\;\; \|D^2\Psi_x\| \leq c_0 \;\;,\;\; \|D^2\Psi_x^{-1}\| \leq c_0 \tag{5.3}$$

and such that each component $z_1, \ldots, z_k$ of Ψ_x is a harmonic function.

In particular, one has for all $r < r_0$:

$$\Psi_x(B(x, c_0^{-1}r)) \subset B(0, r) \text{ and } B(0, c_0^{-1}r) \subset \Psi_x(B(x, r)). \tag{5.4}$$

(iii) The second covariant derivatives of Ψ_x are also uniformly α-Hölder :

$$\|D^2\Psi_x\|_{\mathcal{C}^\alpha} \leq c_0. \tag{5.5}$$

This α-Hölder semi-norm $\|D^2\Psi_x\|_{\mathcal{C}^\alpha}$ is defined as follows. Using the vector fields $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_k}$ on $\mathring{B}(x, r_0)$ associated to our coordinate system $\Psi_x = (z_1, \ldots, z_k)$, we reinterpret the tensor $D^2\Psi_x$ as a family of vector valued functions on $\mathring{B}(x, r_0)$. Indeed, we set

$$T_x^{ij}(z) = D^2\Psi_x(z)(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial z_j}) \in \mathbb{R}^k \ , \ \text{ for } i, j \text{ in } \{1, \ldots, k\},$$

and the bound (5.5) means that

$$\|D^2\Psi_x\|_{\mathcal{C}^\alpha} := \max_{i,j} \sup_{z,z'} \frac{\|T_x^{ij}(z) - T_x^{ij}(z')\|}{d(z, z')^\alpha} \leq c_0. \tag{5.6}$$

These uniform bounds (5.3) and (5.5) have three consequences.

First, in the harmonic coordinate systems $\Psi_x = (z_1, \ldots, z_k)$, the Christoffel coefficients Γ_{ij}^ℓ are uniformly bounded in $\mathcal{C}^\alpha$-norm. Indeed, these coefficients $(\Gamma_{ij}^\ell)_{1 \leq \ell \leq k}$ are the components of the vector $-D^2\Psi_x(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}) \in \mathbb{R}^k$

Second, on their domain of definition, the transition functions

$$\Psi_{x'} \circ \Psi_x^{-1} \text{ are uniformly bounded in } \mathcal{C}^{2,\alpha}\text{-norm.} \tag{5.7}$$

Third, in coordinate systems $\Psi_x = (z_1, \ldots, z_k)$, the coefficients of the metric tensor

$$g_{ij} := g(\tfrac{\partial}{\partial z_i}, \tfrac{\partial}{\partial z_j}) \text{ are uniformly bounded in } \mathcal{C}^{1,\alpha}\text{-norm.} \tag{5.8}$$

Proof of Lemma 5.2. See [17, p. 62 and 65] or [28, Section 4]. □

5.3 Gromov-Hausdorff convergence

In this section, we recall the definition of Gromov–Hausdorff convergence for pointed metric spaces and some of its key properties. We refer to [7] for more details.

5.3.1 Definition

When X is a metric space, we will denote by d or d_X the distance on X. We denote by $B(x, R)$ the closed ball of center x and radius R and by $\mathring{B}(x, R)$ the open ball. We recall that a metric space X is *proper* if all its balls are compact or, equivalently, if X is complete and for all $R > 0$ and $\varepsilon > 0$ every ball of radius R can be covered by finitely many balls of radius ε.

We also recall the notion of Gromov–Hausdorff distance between two (isometry class of proper) pointed metric spaces.

Definition 5.3. *The Gromov–Hausdorff distance between two pointed metric spaces (X, p) and (Y, q) is the infimum of the $\varepsilon > 0$ for which there exists a subset $\mathcal{R}$ of $X \times Y$, called a correspondence, such that:*

(*i*) *the correspondence* $\mathcal{R}$ *contains the pair* (p, q),
(*ii*) *for all* x *in the ball* $B(p, \varepsilon^{-1})$, *there exists* y *in* Y *with* (x, y) *in* $\mathcal{R}$,
(*iii*) *for all* y *in the ball* $B(q, \varepsilon^{-1})$, *there exists* x *in* X *with* (x, y) *in* $\mathcal{R}$,
(*iv*) *for all* (x, y) *and* (x', y') *in* $\mathcal{R}$, *one has* $|d(x, x') - d(y, y')| \leq \varepsilon$.

Heuristically, this correspondence $\mathcal{R}$ must be thought as an ε-rough isometry between these two balls of radius ε^{-1}.

Based on this definition, a sequence (X_n, p_n) of pointed metric spaces converges to a pointed metric space (X_∞, p_∞) if, for all $\varepsilon > 0$, there exists n_0 such that for $n \geq n_0$, there exists a map $f_n : B(p_n, \varepsilon^{-1}) \to X_\infty$ such that
(α) $d(f_n(p_n), p_\infty) \leq \varepsilon$,
(β) $|d(f_n(x), f_n(x')) - d(x, x')| \leq \varepsilon$, for all x, x' in $B(p_n, \varepsilon^{-1})$,
(γ) the ε-neighborhood of $f_n(B(p_n, \varepsilon^{-1}))$ contains the ball $B(p_\infty, \varepsilon^{-1} - \varepsilon)$.

This definition 5.3 is only useful for complete metric spaces. Indeed, the Gromov–Hausdorff topology does not distinguish between a metric space and its completion. It does not either distinguish between two pointed metric spaces that are isometric : the Gromov–Hausdorff distance is a distance on the set of isometry classes of proper pointed metric spaces. See [7, Theorem 8.1.7]

The following equivalent definition of Gromov–Hausdorff convergence is useful when one wants to get rid of the ambiguity coming from the group of isometries of (X_∞, p_∞).

Fact 5.4. *Let* (X_n, p_n), *for* $n \geq 1$, *and* (X_∞, p_∞) *be pointed proper metric spaces. The sequence* (X_n, p_n) *converges to* (X_∞, p_∞) *if and only if there exists a complete metric space* Z *containing isometrically all the metric spaces* X_n *and* X_∞ *as disjoint closed subsets, and such that*
(*a*) *the sequence of points* p_n *converges to* p_∞ *in* Z,
(*b*) *the sequence of closed subsets* X_n *converges to* X_∞ *for the Hausdorff topology.*

The statement (*b*) means that
- every point z of X_∞ is the limit of a sequence $(x_n)_{n\geq 1}$ with $x_n \in X_n$,
- every cluster point $z \in Z$ of a sequence $(x_n)_{n\geq 1}$ with $x_n \in X_n$ belongs to X_∞.

Sketch of proof of Fact 5.4. Assume that the sequence (X_n, p_n) converges to (X_∞, p_∞). We want to construct the metric space Z. We can choose a sequence $\varepsilon_n \searrow 0$, and correspondences $\mathcal{R}_n$ on $X_n \times X_\infty$ as in Definition 5.3 with $p = p_n$, $q = p_\infty$ and $\varepsilon = \varepsilon_n$. This allows us to construct, for every $n \geq 1$, a metric space Y_n which is the disjoint union of X_n and X_∞, which contains isometrically both X_n and X_∞ and such that the distance between two points x in X_n and y in X_∞ is given by

$$d_{Y_n}(x, y) = \inf\{d_{X_n}(x, x') + \varepsilon + d_{X_\infty}(y', y)\}, \tag{5.9}$$

where the infimum is over all the pairs (x', y') which belong to $\mathcal{R}_n$.

The space Z is defined as the disjoint union of all the X_n and of X_∞. The distance on Z is given on each union $Y_n := X_n \cup X_\infty$ by (5.9) and the distance between points x in X_m and z in X_n with $m \neq n$ is given by

$$d_Z(x, z) = \inf\{d_{Y_m}(x, y) + d_{Y_n}(y, z)\}, \tag{5.10}$$

where the infimum is over all the points y in X_∞.

Then (a) follows from (i) and (b) follows from (ii), (iii) and (iv). $\square$

The choice of such isometric embeddings of all X_n and X_∞ in a fixed metric space Z will be called a realization of the Gromov-Hausdorff convergence. Such a realization is not unique. It is useful since it allows us to define the notion of a converging sequence of points x_n in X_n to a limit x_∞ in X_∞ by the condition $d_Z(x_n, x_\infty) \xrightarrow[n\to\infty]{} 0$.

5.3.2 Compactness criterion

A fundamental tool in this topic is the following compactness result for *uniformly proper* pointed metric spaces due to Cheeger–Gromov:

Fact 5.5. *Let $(X_n, p_n)_{n\geq 1}$ be a sequence of pointed proper metric spaces. Suppose that, for all $R > 0$ and $\varepsilon > 0$, there exists an integer $N = N(R, \varepsilon)$ such that, for all $n \geq 1$, the ball $B(p_n, R)$ of X_n can be covered by N balls of radius ε. Then there exists a subsequence of (X_n, p_n) which converges to a proper pointed metric space (X_∞, p_∞).*

For a proof see [7, Theorem 8.1.10].

The following lemma gives us a compactness property for sequences of Lipschitz functions between pointed metric spaces.

Lemma 5.6. *Let $(X_n, p_n)_{n\geq 1}$ and $(Y_n, q_n)_{n\geq 1}$ be sequences of pointed proper metric spaces which converge respectively to proper pointed metric spaces (X_∞, p_∞) and (Y_∞, q_∞). As in Fact 5.4, we choose metric spaces Z_X and Z_Y which realize these Gromov–Hausdorff convergences as Hausdorff convergences.*

Let $c > 1$ and let $(f_n : X_n \to Y_n)_{n\geq 1}$ be a sequence of c-Lipschitz maps such that $f_n(p_n) = q_n$. Then there exists a c-Lipschitz map $f_\infty : X_\infty \to Y_\infty$ such that, after extracting a subsequence, the sequence of maps f_n converges to f_∞. This means that for each sequence $x_n \in X_n$ which converges to $x_\infty \in X_\infty$, the sequence $f_n(x_n) \in Y_n$ converges to $f_\infty(x_\infty) \in Y_\infty$.

Proof. This follows from basic topology arguments.

First step. We first choose a point x_∞ in X_∞ and a sequence x_n in X_n converging to x_∞. Since the metric space Z_Y is proper and the sequence $f_n(x_n)$ is bounded in Z_Y we can assume after extracting a subsequence that the sequence $f_n(x_n)$ converges to a point $y_\infty \in Y_\infty$. Since the f_n are c-Lipschitz, this limit y_∞ does not depend on the choice of the sequence x_n converging to x_∞. We define $f_\infty(x_\infty) := y_\infty$.

Second step. We choose a countable dense subset $S_\infty \subset X_\infty$ and use Cantor's diagonal argument to ensure that the first step is valid simultaneously for all points x_∞ in S_∞.

Last step. One checks that the limit map $f_\infty : S_\infty \to Y_\infty$ is c-Lipschitz. Hence it extends uniquely as a c-Lipschitz map $f_\infty : X_\infty \to Y_\infty$ and the sequence f_n converges locally uniformly to f_∞. □

5.3.3 Length spaces and Alexandrov spaces

We recall a few well known definitions (see [7]).

A *length space* is a complete metric space for which the distance δ between two points is the infimum of the length of the curves joining them. When X is proper, any two points at distance δ can be joined by a curve of length δ. Such a curve is called a *geodesic segment*.

Let $K \leq 0$. A CAT*(K)-space* or CAT-*space with curvature at most* K is a length space in which any geodesic triangle (P, Q, R) is thinner than a comparison triangle $(\overline{P}, \overline{Q}, \overline{R})$ in the plane $\overline{X}$ of constant curvature K. Let us explain what this means. A *comparison triangle* is a triangle in $\overline{X}$ with same sidelength. For every point P' on the geodesic segment $[P, Q]$ we denote by $\overline{P}'$ the corresponding point on the geodesic segment $[\overline{P}, \overline{Q}]$ i.e. the point such that $d(P, P') = d(\overline{P}, \overline{P}')$. *Thinner* means that one always has $d(P', R) \leq d(\overline{P}', \overline{R})$. Note that a CAT(0)-space is always simply connected (See [6, Corollary II.1.5]). We also recall that in a proper CAT(0)-space, any two points can be joined by a geodesic and that this geodesic is unique.

Similarly, a *metric space with curvature at least* K is a length space in which any geodesic triangle (P, Q, R) is thicker than a comparison triangle $(\overline{P}, \overline{Q}, \overline{R})$ in the plane $\overline{X}$ of constant curvature K. *Thicker* means that one always has $d(P', R) \geq d(\overline{P}', \overline{R})$.

The following proposition tells us that these properties are closed for the Gromov–Hausdorff topology.

Fact 5.7. *Let $(X_n, p_n)_{n\geq 1}$ and (X_∞, p_∞) be pointed proper metric spaces. Let $K \leq 0$. Assume that the sequence (X_n, p_n) converges to (X_∞, p_∞).*
(i) If the X_n's are length spaces then X_∞ is also a length space.
(ii) If the X_n's are CAT(K) *spaces then X_∞ is also a* CAT(K) *space.*
(iii) If moreover the X_n's have curvature at least K then X_∞ too.

Proof. (i) See [7, Theorem 8.1.9].
(ii) See [6, Corollary II.3.10].
(iii) See [7, Theorem 10.7.1]. □

5.4 Hadamard manifolds with $\mathcal{C}^1$ metrics

In this section we focus on $\mathcal{C}^2$ Hadamard manifolds when the Riemannian metric is only assumed to be $\mathcal{C}^1$. These Hadamard manifolds will occur in Section 5.5 as Gromov-Hausdorff limits of pinched $\mathcal{C}^\infty$ Hadamard manifolds.

5.4.1 Definition

We need first to clarify the definitions. We will deal with $\mathcal{C}^2$-manifolds X. This means that X has a system of charts $x \mapsto (x_1, \ldots, x_k)$ into $\mathbb{R}^k$ for which the transition functions are of class $\mathcal{C}^2$. These manifolds will be endowed with a $\mathcal{C}^1$ Riemannian metric g. This means that in any $\mathcal{C}^2$ chart, the functions $g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ are continuously differentiable.

In general, on such a Riemannian manifold, there might exist two different geodesics which are tangent at the same point (See [16] for an example with a $\mathcal{C}^{1,\alpha}$-Riemannian metric). The following lemma tells us that this kind of examples will not occur here since we are dealing only with CAT(0)-spaces whose curvature is bounded below. Note that, since the metric tensor is not assumed to be twice differentiable, the expression "curvature bounded below" refers to the definitions in Section 5.3,

Definition 5.8. *By a $\mathcal{C}^2$ Hadamard manifold with $\mathcal{C}^1$ metric, we mean a $\mathcal{C}^2$ manifold endowed with a $\mathcal{C}^1$ Riemannian metric which is* CAT(0) *and complete.*

5.4.2 Exponential map

Lemma 5.9. *Let X be a $\mathcal{C}^2$ Hadamard manifold with a $\mathcal{C}^1$ metric of bounded curvature.*
a) For all x in X and v in T_xX there is a unique geodesic $t \mapsto \exp_x(tv)$ starting from x at speed v. This geodesic is of class $\mathcal{C}^2$.
b) This exponential map induces an homeomorphism $\Psi : TX \xrightarrow{\sim} X \times X$ given by, $\Psi(x, v) = (x, \exp_x(v)$ for x in X and v in T_xX.

Proof. This lemma looks very familiar. But, since the Christoffel coefficients might not be Lipschitz continuous, we cannot apply Cauchy–Lipschitz theorem on Existence and Uniqueness of solutions of differential equations.

a) Since the Christoffel coefficients are continous, we can apply Peano–Arzela theorem. It tells us that there exists at least one geodesic of class $\mathcal{C}^2$ starting from x at speed v. Uniqueness follows from the lower bound on the curvature.

b) Since X is CAT(0), the map Ψ is a bijection. Since a uniform limit of geodesic on X is also a geodesic, the map Ψ is continuous. This map Ψ is also proper therefore it is an homeomorphism. $\square$

5.4.3 Geodesic interpolation of h_0 and h_1

In the sequel of this section we prove a few technical properties of the interpolation h_t of two equidistant Lipschitz maps h_0 and h_1 with values in a Hadamard manifold (lemmas 5.10). In Section 5.8, we will apply this lemma to two equidistant harmonic maps h_0 and h_1 obtained by a limit process. This lemma 5.10 will be used to compare the energy of h_0 and h_1 with the energy of some small perturbations of h_0 and h_1. However in this section 5.4, we do not need to assume h_0 and h_1 to be harmonic. Here are the precise assumptions and notations for Lemma 5.10.

Let X be a $\mathcal{C}^2$ Riemannian manifold with $\mathcal{C}^1$ metric and Y be a $\mathcal{C}^2$ Hadamard manifold with $\mathcal{C}^1$ metric. Let $h_0, h_1 : X \to Y$ be two $\mathcal{C}^1$ maps such that, one has

$$d(h_0(x), h_1(x)) = 1 \text{ for all } x \text{ in } X. \tag{5.11}$$

Since Y is a Hadamard manifold there exists a unique map,

$$\begin{array}{rcl} h : [0,1] \times X & \to & Y \\ (t,x) & \mapsto & h(t,x) = h_t(x) \end{array} \tag{5.12}$$

such that, for all x in X the path $t \mapsto h_t(x)$ is the unit speed geodesic joining $h_0(x)$ and $h_1(x)$. This map h is called the *geodesic interpolation* of h_0 and h_1. By convexity of the distance function, h is Lipschitz continuous. Therefore, by Rademacher's theorem, the map h is differentiable on a subset of full measure (with respect to the Riemannian measure on X). In particular, there exists a subset $X' \subset X$ of full measure such that, for all x in X', the map h is differentiable at (x,t) for almost all t in $[0,1]$. In particular, for all tangent vector $V \in T_xX$ at a point $x \in X'$, the following derivative

$$t \mapsto J_V(t) := D_x h_t(V) \in T_{h_t(x)}Y \tag{5.13}$$

is well-defined for almost all t in $[0,1]$. Such a measurable vector field J_V on the geodesic $t \mapsto h_t(x)$ will be called a *Jacobi field*. We denote by

$$t \mapsto \tau_x(t) := \partial_t h_t(x) \in T_{h_t(x)}Y \tag{5.14}$$

the unit tangent vector to the geodesic $t \mapsto h_t(x)$.

Lemma 5.10. *We keep these assumptions and notations. Let x be a point in X' and $V \in T_xX$.*
a) There exists a constant $\alpha_V \in \mathbb{R}$ such that

$$\langle J_V(t), \tau_x(t)\rangle = \alpha_V \ , \ \textit{for all } t \textit{ in } [0,1] \textit{ where } J_V(t) \textit{ is defined.} \tag{5.15}$$

b) There exists a convex function $t \mapsto \varphi_V(t)$ on $[0,1]$ such that

$$\varphi_V(t) = \|J_V(t)\| \ , \ \textit{for all } t \textit{ in } [0,1] \textit{ where } J_V(t) \textit{ is defined.} \tag{5.16}$$

c) The function $\psi_V := (\varphi_V^2 - \alpha_V^2)^{1/2}$ is also a convex function on $[0,1]$.

Proof. When Y is a $\mathcal{C}^\infty$ Hadamard manifold, the vector field $J_V(t)$ is a classical Jacobi field and this lemma is well known. Indeed, the function ψ_V is the norm of the orthogonal component K_V of the Jacobi field J_V, and Inequality (5.12) follows from the Jacobi equation satisfied by this Jacobi field K_V. We now explain how to adapt the classical proof when Y is only assumed to be a $\mathcal{C}^2$ Hadamard manifold with $\mathcal{C}^1$ metric.

a) Since the path $t \mapsto h_t(x)$ is a unit speed geodesic, one has the equality $d(h_s(x), h_t(x)) = |t-s|$, for all s, t in $[0,1]$. Differentiating this equality gives, when $J_V(s)$ and $J_V(t)$ are defined,

$$\langle J_V(s), \tau_x(s)\rangle = \langle J_V(t), \tau_x(t)\rangle \, .$$

Hence this scalar product is almost surely constant.

b) Let c be a $\mathcal{C}^1$ curve $c : [-\varepsilon_0, \varepsilon_0] \to X$ with $c(0) = x$ and $\partial_s c(0) = V$. Since the space Y is CAT(0), when $s > 0$, the functions

$$t \mapsto \varphi_s(t) := \frac{1}{s} d(h_t(c(0)), h_t(c(s))$$

are convex on $[0,1]$. Let $S_V := \{t \in [0,1] \mid J_V(t) \text{ is defined}\}$. This set S_V has full measure and contains the endpoints 0 and 1. For all t in this set S_V, one can compute the limit of these functions $\lim_{s\to 0} \varphi_s(t) = \|J_V(t)\|$. Since these functions φ_s are convex, the limit $\varphi_V(t) := \lim_{s\to 0} \varphi_s(t)$ exists for all t in $[0,1]$ and is a convex function.

c) We slightly change the parametrization of the geodesic interpolation: the function $k : (t,s) \mapsto k_t(s) := h_{t-s\alpha_V}(c(s))$ is well defined when $t - s\alpha_V$ is in $[0,1]$, and the paths $t \mapsto k_t(s)$ are also unit speed geodesics. Hence, for almost all t in $[0,1]$, the vector field

$$t \mapsto K_V(t) := \partial_s k_t(0) \in T_{k_t(0)} Y \tag{5.17}$$

is well-defined and one has the orthogonal decomposition

$$J_V(t) = K_V(t) + \alpha_V \tau_x(t).$$

In particular, one has the equality,

$$\psi_V(t) = \|K_V(t)\| \, . \tag{5.18}$$

The same argument as in *b*) with the Jacobi fields K_V proves that the function ψ_V is also convex. □

5.4.4 Geodesic interpolation in negative curvature

The following Lemma 5.11 improves Lemma 5.10 when the curvature of Y is uniformly negative. Indeed it tells us that the norm $t \mapsto \psi_V(t)$ of the Jacobi field K_V is uniformly convex.

Lemma 5.11. *We keep the assumptions and notations of Lemma 5.10. Moreover we assume that Y is a* $\mathrm{CAT}(-a^2)$*-space with* $a>0$*. Then the function* ψ_V *satisfies the following uniform convexity property,*

$$\psi_V(t) \leq \tfrac{\sinh(a(1-t))}{\sinh(a)}\psi_V(0) + \tfrac{\sinh(at)}{\sinh(a)}\psi_V(1) \quad \textit{for all } t \textit{ in } [0,1]. \tag{5.19}$$

Remark 5.12. *One can reformulate* (5.19) *as the following inequality between positive measures*

$$\tfrac{d^2}{dt^2}\psi_V \geq a^2\psi_V\,.$$

Proof. This inequality (5.19) will follow from an upper bound for the norm of the Jacobi field $t \mapsto K_V(t)$ by the norm of a well chosen Jacobi field $t \mapsto \overline{K}(t)$ along a geodesic segment in the hyperbolic plane of curvature $-a^2$. Here are the details of the construction of this Jacobi field $t \mapsto \overline{K}(t)$.

Using a slight rescaling, we can assume without loss of generality that the geodesics $t \mapsto k_t(s)$ are defined for t in $[0,1]$ and that the Jacobi field $K_V(t)$ are well defined for $t=0$ and for $t=1$. We choose $s>0$. Later on we will let s goes to 0. We set $P_t := k_t(0)$ and $Q_{s,t} := k_t(s)$, and we apply Reshetnyak Lemma 5.13 to the four points P_0, P_1, $Q_{s,1}$, $Q_{s,0}$. According to this lemma, there exists a convex quadrilateral $\overline{C}_s$ in the hyperbolic plane $\overline{Y}$ of curvature $-a^2$ with vertices $\overline{P}_0$, $\overline{P}_1$, $\overline{Q}_{s,1}$, $\overline{Q}_{s,0}$, and a 1-Lipschitz map $j : \overline{C}_s \to Y$ whose restriction to each of the four geodesic sides $\overline{P}_0\overline{P}_1$, $\overline{P}_1\overline{Q}_{s,1}$, $\overline{Q}_{s,1}\overline{Q}_{s,0}$, $\overline{Q}_{s,0}\overline{P}_0$ is an isometry onto each of the four geodesic segments P_0P_1, $P_1Q_{s,1}$, $Q_{s,1}Q_{s,0}$, $Q_{s,0}P_0$. Indeed, since $d(\overline{P}_0,\overline{P}_1)=1$, we can assume that the two vertices $\overline{P}_0$ and $\overline{P}_1$ do not depend on s and that the quadrilateral $\overline{C}_s$ is positively oriented.

Since the vectors $K_V(0)$ and $K_V(1)$ are orthogonal to the geodesic segment $t \mapsto k_t(0)$, by Lemma 5.9, each of the four successive angles θ_i, $i=1,\ldots,4$ between the four successive geodesic segments P_0P_1, $P_1Q_{s,1}$, $Q_{s,1}Q_{s,0}$, $Q_{s,0}P_0$ in Y are equal to $\frac{\pi}{2}+o(1)$ where $o(1)$ denotes a quantity that goes to 0 when s goes to 0. Since j is 1-Lipschitz, each of the corresponding four successive angles $\overline{\theta}_i$ between the four successive geodesic sides $\overline{P}_0\overline{P}_1$, $\overline{P}_1\overline{Q}_{s,1}$, $\overline{Q}_{s,1}\overline{Q}_{s,0}$, $\overline{Q}_{s,0}\overline{P}_0$ in the hyperbolic plane $\overline{Y}$ is not smaller than θ_i. Since the sum of these four angles $\overline{\theta}_i$ is bounded by 2π, each of these four angles $\overline{\theta}_i$ satisfies also the equality when s goes to 0,

$$\overline{\theta}_i = \frac{\pi}{2} + o(1)\,. \tag{5.20}$$

Denote by $t \mapsto \overline{P}_t$ and $t \mapsto \overline{Q}_{s,t}$ the unit speed parametrization of the sides $\overline{P}_0\overline{P}_1$ and $\overline{Q}_0\overline{Q}_1$. For t in $[0,1]$, one has $j(\overline{P}_t)=P_t$ and $j(\overline{Q}_{s,t})=Q_{s,t}$, and also

$$d(P_t,Q_{s,t}) \leq d(\overline{P}_t,\overline{Q}_{s,t})\,, \tag{5.21}$$

with equality when $t=0$ or 1 :

$$d(P_0,Q_{s,0}) = d(\overline{P}_0,\overline{Q}_{s,0}) \ \text{ and } \ d(P_1,Q_{s,1}) = d(\overline{P}_1,\overline{Q}_{s,1})\,. \tag{5.22}$$

We now focus on these convex quadrilateral $\overline{C}_s$ in the hyperbolic plane $\overline{Y}$ of curvature $-a^2$. We write $\overline{Q}_{s,t} = \exp_{\overline{P}_t}(s\overline{K}_{s,t})$ where $\overline{K}_{s,t}$ belongs to $T_{\overline{P}_t}\overline{Y}$. Since $K_V(0)$ and $K_V(1)$ are well defined, by (5.17), (5.18), (5.20) and (5.22), the limits

$$\overline{K}(0) = \lim_{s\to 0} \overline{K}_{s,0} \text{ and } \overline{K}(0) = \lim_{s\to 0} \overline{K}_{s,1}$$

exist and satisfy

$$\|\overline{K}(0)\| = \psi_V(0) \text{ and } \|\overline{K}(1)\| = \psi_V(1) \tag{5.23}$$

Therefore for all t in $[0,1]$ the limits

$$\overline{K}(t) = \lim_{s\to 0} \overline{K}_{s,t}$$

exist. Moreover, by (5.17), (5.18) and (5.21), they satisfy the inequalities

$$\psi_V(t) \leq \|\overline{K}(t)\| \tag{5.24}$$

Since the vector field $t \mapsto \overline{K}(t)$ is a Jacobi field on the geodesic segment $t \mapsto \overline{P}_t$ which is orthogonal to the tangent vector, its norm

$$\overline{\psi}(t) := \|\overline{K}(t)\|$$

satisfies the Jacobi differential equation

$$\frac{d^2}{dt^2}\overline{\psi} = a^2\overline{\psi}\,.$$

Hence, one has the equality

$$\overline{\psi}(t) = \tfrac{\sinh(a(1-t))}{\sinh(a)}\overline{\psi}(0) + \tfrac{\sinh(at)}{\sinh(a)}\overline{\psi}(1) \quad \text{for all } t \text{ in } [0,1]. \tag{5.25}$$

We deduce Inequality (5.19) directly from (5.23), (5.24) and (5.25). □

We have used the following existence result of a majorizing quadrilateral due to Reshetnyak in [30]. More precisely we have used the boundary of this majorizing quadrilateral $\overline{C}$.

Lemma 5.13. *Let Y be a* $\mathrm{CAT}(-a^2)$ *metric space and $\overline{Y}$ be the hyperbolic plane of curvature $-a^2$. Then for every four points P_0, P_1, Q_1 Q_0 in Y there exists a convex quadrilateral $\overline{C}$ in $\overline{Y}$ with vertices $\overline{P}_0$,$\overline{P}_1$, $\overline{Q}_1$,$\overline{Q}_0$ and a* 1*-Lipschitz map $j : \overline{C} \to Y$ which is an isometry on each of the four geodesic sides of $\overline{C}$, and which sends each of these four vertices $\overline{R}_i$ on the corresponding given point R_i in Y.*

5.5 Limits of Hadamard manifolds

In this section we describe the Gromov–Hausdorff limits of pinched Hadamard manifolds.

The following proposition is a variation on the Cheeger compactness theorem.

Proposition 5.14. *Let $(X_n, p_n)_{n\geq 1}$ be a sequence of k-dimensional pointed Hadamard manifolds with pinched curvature $-1 \leq K_{X_n} \leq -a^2 \leq 0$.*
a) There exists a subsequence of (X_n, p_n) which converge to a pointed proper CAT-*space (X_∞, p_∞) with curvature between -1 and $-a^2$.*
b) This space X_∞ has a structure of $\mathcal{C}^2$ Hadamard manifold such that the distance on X_∞ comes from a $\mathcal{C}^1$ Riemannian metric.

The same proof shows that X_∞ has a structure of $\mathcal{C}^{2,\alpha}$ Hadamard manifold with a $\mathcal{C}^{1,\alpha}$ Riemannian metric, for all $0 < \alpha < 1$. We will not use this improvement.

Even though this proposition follows from [29, Theorem 72 p. 311], we will give a sketch of proof below.

Proof. *a*) The assumption on the curvature of X_n ensures that for all $R > 0$, one has uniform estimates on the volumes of balls in X_n : for all $n \geq 1$ and x in X_n, one has

$$\mathrm{vol} B_{\mathbb{R}^k}(O, R) \leq \mathrm{vol} B_{X_n}(x, R) \leq \mathrm{vol} B_{\mathbb{H}^k}(O, R)\,.$$

Therefore, for all $0 < \varepsilon < R$, there exists an integer $N = N(R, \varepsilon)$ such that every ball $B_{X_n}(p_n, R)$ of X_n can be covered by N balls of radius ε. Hence, according to Fact 5.5, there exists a subsequence of (X_n, p_n) which converges to a proper pointed metric space (X_∞, p_∞). According to Fact 5.7, X_∞ is a CAT-space with curvature between -1 and $-a^2$.

b) It remains to check that X_∞ is a $\mathcal{C}^2$-manifold with a $\mathcal{C}^1$ Riemannian metric. We isometrically imbed the converging sequence (X_n, p_n) in a proper metric space Z as in Fact 5.4. We fix $r_0 > 0$, $c_0 > 0$ as in Lemma 5.2 which introduces the harmonic coordinates and we choose a maximal $\frac{r_0}{2c_0}$-separated subset S_∞ of X_∞. For all x_∞ in S_∞ we choose a sequence x_n of points in X_n that converges to x_∞. By (5.3), the harmonic charts

$$\Psi_{x_n} : \mathring{B}(x_n, \tfrac{r_0}{c_0}) \to \mathbb{R}^k \tag{5.26}$$

are uniformly bilipschitz. More precisely, for all z, z' in $\mathring{B}(x_n, \frac{r_0}{c_0})$, one has

$$c_0^{-1} d(z, z') \leq \|\Psi_{x_n}(z) - \Psi_{x_n}(z')\| \leq c_0\, d(z, z').$$

Hence after extracting a subsequence, this sequence of charts Ψ_{x_n} converge toward a bilipschitz map

$$\Psi_{x_\infty} : \mathring{B}(x_\infty, \tfrac{r_0}{c_0}) \to \mathbb{R}^k. \tag{5.27}$$

The extraction can be chosen simultaneously for all the points x_∞ in the countable set S_∞. These maps Ψ_{x_∞} endow X_∞ with a structure of a Lipschitz manifold.

We now want to prove that this manifold X_∞ is a $\mathcal{C}^2$-manifold. Indeed we will check that, for x_∞ and x'_∞ in S_∞, the transition functions $\Phi_{x'_\infty} \circ \Phi^{-1}_{x_\infty}$ are of class $\mathcal{C}^2$. This just follows from the fact that these transition functions are uniform limit on compact sets of the transition functions $\Phi_{x'_n} \circ \Phi^{-1}_{x_n}$ which are, by (5.7), uniformly bounded in $\mathcal{C}^{2,\alpha}$-norm.

Finally, we check that the distance d on X_∞ comes from a $\mathcal{C}^1$ Riemannian metric on X_∞. By (5.8), the Riemannian metrics $(g_n)_{ij}$ on X_n, seen as functions in the charts Ψ_{x_n} of X_n, are uniformly bounded in $\mathcal{C}^{1,\alpha}$-norm. Extracting again a subsequence, there exists a $\mathcal{C}^1$ Riemannian metric $(g_\infty)_{ij}$ in the charts Ψ_{x_∞} of X_∞ such that the metric

$$(g_n)_{ij} \text{ converge to } (g_\infty)_{ij} \text{ in } \mathcal{C}^1 \text{ topology.} \tag{5.28}$$

Let d_∞ be the associated distance on X_∞. We check that $d_\infty = d$ on X_∞. Let x'_∞ and x''_∞ be points in X_∞. They are limit of points x'_n and x''_n in X_n. Let c_n be the geodesic joining them. Extracting once more a subsequence, the curves c_n converge uniformly to a curve joining x'_∞ and x''_∞. This curve must be a geodesic for g_∞. This proves that $d_\infty(x'_\infty, x''_\infty) = d(x'_\infty, x''_\infty)$. □

5.6 Convergence of harmonic maps

We now explain how to obtain the limit harmonic maps.

First we notice that we can extend Definition 3.2 : A $\mathcal{C}^2$ map $h : X \to Y$ between two $\mathcal{C}^2$ Riemannian manifolds with $\mathcal{C}^1$ metrics X and Y is said to be *harmonic* if its tension field is zero $\tau(h) := \mathrm{tr} D^2 h = 0$. Indeed the tension field of a $\mathcal{C}^2$ map h at a point x depends only the 2-jet of h and 1-jet of the metrics of X and Y at the points x and $h(x)$. More precisely, writing h in a coordinate system $h : (x_1, \ldots, x_k) \mapsto (h_1, \ldots, h_\ell)$, the equation $\mathrm{tr} D^2 h = 0$ reads as for all $\lambda \le \ell$

$$\Delta h_\lambda = -\sum_{ij\mu\nu} g^{ij} \Gamma^\lambda_{\mu\nu} \frac{\partial h_\mu}{\partial x_i} \frac{\partial h_\nu}{\partial x_j}, \tag{5.29}$$

where $\Gamma^\lambda_{\mu\nu}$ are the Christoffel coefficients on Y and where Δ is the Laplace operator on X defined as in (3.1)

$$\Delta : \varphi \mapsto \tfrac{1}{v} \tfrac{\partial}{\partial x_i} (v\, g^{ij} \tfrac{\partial \varphi}{\partial x_j}) \tag{5.30}$$

with $v = \sqrt{\det(g_{ij})}$ being the volume density on X. See [17, Section 1.3] for more details.

Lemma 5.15. *Let $(X_n, p_n)_{n\geq 1}$ and $(Y_n, q_n)_{n\geq 1}$ be two sequences of equi-dimensional pointed Hadamard manifolds with curvature between -1 and 0. Let $c, C > 0$ and let $h_n : X_n \to Y_n$ be a sequence of (c, C)-quasi-isometric harmonic maps such that $\sup_n d(h_n(p_n), q_n) < \infty$. After extracting a subsequence, the sequences of pointed metric spaces (X_n, p_n) and (Y_n, q_n) converge respectively to pointed $\mathcal{C}^2$ manifolds with $\mathcal{C}^1$ Riemannian metrics (X_∞, p_∞) and (Y_∞, q_∞), and the sequence of maps h_n converges to a c-quasi-isometric map $h_\infty : X_\infty \to Y_\infty$. This map h_∞ is of class $\mathcal{C}^2$ and is harmonic.*

Proof. Since they are harmonic, the maps h_n are $\mathcal{C}^\infty$. Since these maps are also (c, C)-quasi-isometric, according to Cheng's Lemma 3.4, there exists some constant $C' > 0$ such that the maps h_n are C'-Lipschitz. The first two statements then follow from Proposition 5.14 and Lemma 5.6.

It remains to show that the limit map h_∞ is of class $\mathcal{C}^2$ and harmonic. The key point will be a uniform bound on the $\mathcal{C}^{2,\alpha}$-norm of h_n in suitable harmonic coordinates. Let $k := \dim X_n$ and $\ell := \dim Y_n$. Let x_∞ be a point in X_∞ and $y_\infty := h_\infty(x_\infty)$. Let x_n be a sequence of X_n converging to x_∞ and let $y_n := h_n(x_n)$.

We look at the maps h_n through the harmonic charts Ψ_{x_n} of X_n and Ψ_{y_n} of Y_n as in (5.26). By (5.27), these charts converge respectively to charts Ψ_{x_∞} of X_∞ and Ψ_{y_∞} of Y_∞. By (5.28), in these charts the Riemannian metric of X_n and Y_n converge to the Riemannian metric of X_∞ and Y_∞ in $\mathcal{C}^{1,\alpha}$-norm.

Let $0 < \alpha < 1$. When one writes in these harmonic coordinates Equation (5.29) for $h = h_n$ on a small open ball $\Omega := \mathring{B}(0, \frac{r_0}{c_0 C'})$ of $\mathbb{R}^k$ which does not depend on n, one gets

$$\sum_{ij} g^{ij} \frac{\partial^2 h_\lambda}{\partial z_i \partial z_j} = - \sum_{ij\mu\nu} g^{ij} \Gamma^\lambda_{\mu\nu} \frac{\partial h_\mu}{\partial z_i} \frac{\partial h_\nu}{\partial z_j}. \tag{5.31}$$

The coefficients of this equation depend on n but by Lemma 5.2, they are uniformly bounded in $\mathcal{C}^\alpha$-norm. The Schauder estimates for functions u on Ω and compact sets K of Ω as in [29, Theorem 70 p. 303] thus tell us that

$$\|u\|_{\mathcal{C}^{1,\alpha},K} \leq M\left(\|\Delta u\|_{\mathcal{C}^0,\Omega} + \|u\|_{\mathcal{C}^\alpha,\Omega}\right) \tag{5.32}$$

$$\|u\|_{\mathcal{C}^{2,\alpha},K} \leq M\left(\|\Delta u\|_{\mathcal{C}^\alpha,\Omega} + \|u\|_{\mathcal{C}^\alpha,\Omega}\right) \tag{5.33}$$

for a constant $M = M(k, \Omega, K)$. Therefore, since the maps h_n are C'-Lipschitz, combining (5.29), (5.32) and (5.33), one gets a uniform bound for the $\mathcal{C}^{2,\alpha}$-norm of the maps h_n. Hence, by the Ascoli Lemma, after extracting a subsequence, the sequence of maps h_n converges towards a $\mathcal{C}^2$ maps in $\mathcal{C}^2$ topology. Hence the limit map h_∞ is $\mathcal{C}^2$ and harmonic. $\square$

5.7 Construction of the limit equidistant harmonic maps

We now explain why the limit harmonic maps $h_{0,\infty}$ and $h_{1,\infty}$ constructed in the strategy of Proposition 5.1 are equidistant.

We first sum up the construction of these limit maps.

We start with two Hadamard manifolds X, Y of bounded curvature and with two distinct quasi-isometric harmonic maps $h_0, h_1 : X \to Y$ such that $\delta := d(h_0, h_1)$ is finite and non-zero. We choose a sequence of points p_n in X such that $d(h_0(p_n), h_1(p_n))$ converge to δ and we set $q_{0,n} := h_0(p_n)$ and $q_{1,n} := h_1(p_n)$. We will frequently replace this sequence by subsequences without mentioning it. By Proposition 5.14, there exist two $\mathcal{C}^2$ Hadamard manifolds with $\mathcal{C}^1$ metric (X_∞, p_∞) and $(Y_\infty, q_{0,\infty})$ which are Gromov–Hausdorff limit of the pointed metric spaces (X, p_n) and $(Y, q_{0,n})$. These limit Hadamard manifolds also have bounded curvature. We denote by $q_{1,\infty}$ the limit in Y_∞ of the sequence $q_{1,n}$. By Cheng Lemma 3.4, the harmonic quasi-isometric maps h_0 and h_1 are Lipschitz continuous. By Lemma 5.6, there exists a limit map $h_{0,\infty} : (X_\infty, p_\infty) \to (Y_\infty, q_{0,\infty})$ of the sequence of Lipschitz continuous maps $h_0 : (X, p_n) \to (Y, q_{0,n})$. There exists also a limit map $h_{1,\infty} : (X_\infty, p_\infty) \to (Y_\infty, q_{1,\infty})$ of the sequence of Lipschitz continuous maps $h_1 : (X, p_n) \to (Y, q_{1,n})$. By Lemma 5.15, these limit maps $h_{0,\infty}$ and $h_{1,\infty}$ are still harmonic quasi-isometric maps.

Lemma 5.16. *With the above notation, the two limit harmonic quasi-isometric maps $h_{0,\infty}$, $h_{1,\infty}$ are equidistant. More precisely, for all x in X_∞, one has $d(h_{0,\infty}(x), h_{1,\infty}(x)) = \delta > 0$, where $\delta := d(h_0, h_1)$.*

We will apply this lemma to two pinched Hadamard manifolds X, Y. In this case, the limit $\mathcal{C}^2$ Hadamard manifolds X_∞, Y_∞ will also be pinched.

Proof. Let Δ_∞ be the Laplace operator on X_∞ defined as in (5.30). We first check that the function $\varphi_\infty : x \mapsto d(h_{0,\infty}(x), h_{1,\infty}(x))$ is subharmonic on X_∞. This means that $\Delta_\infty \varphi_\infty$ is a positive measure on X_∞. Assume first that the Riemannian metric on Y_∞ is $\mathcal{C}^\infty$. In this case φ_∞ is the composition of a harmonic map $h = (h_0, h_1) : X_\infty \to Y_\infty \times Y_\infty$ and a convex $\mathcal{C}^\infty$-function $F = d : Y_\infty \times Y_\infty \to \mathbb{R}$, and the function φ_∞ is subharmonic on X_∞ because of the formula

$$\Delta_\infty(F \circ h) = \sum_{1 \le i \le k} D^2 F(D_{e_i} h, D_{e_i} h) + \langle DF, \tau(h) \rangle \,,$$

where $(e_i)_{1 \le i \le k}$ is an orthonormal basis of the tangent space to X,

Since the Riemannian metric on Y might not be of class $\mathcal{C}^\infty$, we will use instead a limit argument. We fix a point x_∞ in X_∞. In a chart $(x_1, \ldots, x_k)$, the Laplace operator Δ_∞ of the Riemannian metric $(g_\infty)_{ij}$ of X_∞ reads as

$$\psi \mapsto \Delta_\infty \psi = \frac{1}{v_\infty} \frac{\partial}{\partial x_i} (v_\infty g_\infty^{ij} \frac{\partial \psi}{\partial x_j}) \,, \tag{5.34}$$

where v_∞ is the volume density. We want to prove that for every $\mathcal{C}^2$ function $\psi \geq 0$ with compact support in a small neighborhood of x_∞, one has

$$\int_{\mathbb{R}^k} \varphi_\infty \, \Delta_\infty \psi \, v_\infty \mathrm{d}x \geq 0 \,. \tag{5.35}$$

This function φ_∞ on the pointed metric space (X_∞, p_∞) is equal to the limit of the sequence of functions $\varphi_n : x \mapsto d(h_0(x), h_1(x))$ on the pointed metric spaces (X, p_n), as defined in Lemma 5.6. Note that the dependence on n comes from the base point p_n which moves with n. We choose a sequence x_n in X_n converging to x_∞. As in the proof of Lemma 5.15, we look at the functions φ_n through the harmonic charts Ψ_{x_n} of X_n. By (5.27), these charts converge to a chart Ψ_{x_∞} of X_∞. By (5.28), in these charts $(x_1, \ldots, x_k)$ the Riemannian metric $(g_n)_{ij}$ of X_n converge to the Riemannian metric $(g_\infty)_{ij}$ of X_∞ in $\mathcal{C}^1$ topology.

Since, by the above argument, the functions φ_n are subharmonic for the metric $(g_n)_{ij}$, for every $\mathcal{C}^2$ function $\psi \geq 0$ with compact support in these charts, one has at each step n

$$\int \varphi_n \Delta_n \psi \, v_n \mathrm{d}x \geq 0 \tag{5.36}$$

where Δ_n and v_n are the Laplace operator and the volume density of the metric $(g_n)_{ij}$. Letting n go to ∞ in (5.36) gives (5.35). This proves that the function φ_∞ is subharmonic.

By construction, this subharmonic function φ_∞ on X_∞ achieves its maximum $\delta > 0$ at the point p_∞. By (5.34), the Laplace operator is an elliptic linear differential operator with continuous coefficients. Hence, by the strong maximum principle in [13, Theorem 8.19 p.198], this function φ_∞ is constant and equal to δ. □

The aim of Sections 5.8 and 5.9 is to prove that such equidistant harmonic quasi-isometric maps $h_{0,\infty}$ and $h_{1,\infty}$ can not exist (Corollary 5.19) when Y_∞ is pinched. This will conclude the proof of Proposition 5.1.

5.8 Equidistant harmonic maps

We first study equidistant harmonic maps without pinching assumption.

The following lemma 5.17 extends [22, Lemma 2.2] to the case where the source space X is only assumed to be a C^2-Hadamard manifold.

Lemma 5.17. *Let X, Y be two $\mathcal{C}^2$-Hadamard manifolds with a $\mathcal{C}^1$ Riemannian metric of bounded curvature. Let $h_0, h_1 : X \to Y$ be two harmonic maps such that the distance function $x \mapsto d(h_0(x), h_1(x))$ is constant. For t*

in $[0,1]$*, let* h_t *be the geodesic interpolation of* h_0 *and* h_1 *as in* (5.12)*. Then for almost all* x *in* X*,* t *in* $[0,1]$ *and* V *in* T_xX*, one has*

$$\|Dh_0(V)\| = \|Dh_t(V)\| = \|Dh_1(V)\| . \tag{5.37}$$

Note that we can not conclude that Equality (5.37) is valid for all x and t since the interpolation h_t might not be of class $\mathcal{C}^1$.

We will use the following straightforward inequality for convex functions.

Lemma 5.18. *Let* $t \mapsto \Phi_t$ *be a non-negative convex function on* $[0,1]$*. Then, for all* t *in* $[0, \frac{1}{2}]$*, one has*

$$\Phi_t + \Phi_{1-t} \leq \Phi_0 + \Phi_1 - 2t\,(\Phi_0 + \Phi_1 - 2\Phi_{1/2}) \tag{5.38}$$

Proof of Lemma 5.18. We just add the following two convexity inequalities $\Phi_t \leq (1-2t)\Phi_0 + 2t\Phi_{1/2}$ and $\Phi_{1-t} \leq (1-2t)\Phi_1 + 2t\Phi_{1/2}$. □

Proof of Lemma 5.17. The idea is to construct two small perturbations f_ε and g_ε of the harmonic maps h_0 and h_1 with support in a compact set K of X and to compare the sum of the energies of f_ε and g_ε with the sum of the energies of h_0 and h_1.

Let $0 \leq \varepsilon \leq 1$. Here is the definition of the two maps $f_\varepsilon : X \to Y$ and $g_\varepsilon : X \to Y$. We fix a $\mathcal{C}^1$ cutoff function $\eta : X \mapsto [0,1]; x \mapsto \eta_x$ with compact support K, and we define, for all x in X,

$$f_\varepsilon(x) := h_{\varepsilon\eta_x}(x) \;\text{ and }\; g_\varepsilon(x) := h_{1-\varepsilon\eta_x}(x) . \tag{5.39}$$

These functions are Lipschitz continuous, hence they are almost everywhere differentiable. In order to compute their differentials, we use the notations (5.13) and (5.14): for all x in a subset $X' \subset X$ of full measure, for all V in T_xX, for almost all t in $[0,1]$, we set

$$J_V(t) := D_x h_t(V) \quad \text{and} \quad \tau_x(t) := \partial_t h_t(x) .$$

For such a tangent vector V, according to Lemma 5.10.b there exists a convex function $t \mapsto \varphi_V(t)$ such that $\varphi_V(t) = \|J_V(t)\|$ for all t where the derivative $J_V(t)$ exists. By the chain rule, for almost all ε in $[0,1]$, the differentials of f_ε and g_ε are given by, for almost all x in X and all V in T_xX

$$\begin{aligned} Df_\varepsilon(V) &= \quad J_V(\varepsilon\eta_x) \;\;+\;\; \varepsilon\, V.\eta\, \tau_x(\varepsilon\eta_x), && (5.40)\\ Dg_\varepsilon(V) &= J_V(1-\varepsilon\eta_x) - \varepsilon\, V.\eta\, \tau_x(1-\varepsilon\eta_x), && (5.41)\end{aligned}$$

where $V.\eta = d\eta(V)$ is the derivative of the function η in the direction V.

According to Lemma 5.10.a, for almost all x in X and all V in T_xX, the scalar product $\langle J_V(t), \tau_x(t)\rangle$ is almost surely constant. Therefore, for almost all ε in $[0,1]$, x in X and V in the unit tangent bundle T^1_xX, one has the equality

$$\|Df_\varepsilon(V)\|^2 + \|Dg_\varepsilon(V)\|^2 = \varphi_V(\varepsilon\eta_x)^2 + \varphi_V(1-\varepsilon\eta_x)^2 + 2\varepsilon^2(V.\eta)^2 \tag{5.42}$$

We introduce the convex function $t \mapsto \Phi^V_t := \varphi_V(t)^2$. Using Inequality (5.38), one gets, for almost all ε in $[0,1]$, x in X and V in T^1_xX, the bound

$$\|Df_\varepsilon(V)\|^2 + \|Dg_\varepsilon(V)\|^2 \leq \Phi^V_0 + \Phi^V_1 - 2\varepsilon\eta_x(\Phi^V_0 + \Phi^V_1 - 2\Phi^V_{1/2}) + 2\varepsilon^2(V.\eta)^2.$$

We recall that the energy over K of a Lipschitz map $h : X \to Y$ is

$$E_K(h) := \int_K \|D_xh\|^2 \,\mathrm{d}x = \int_{T^1K} \|Dh(V)\|^2 \,\mathrm{d}V\,,$$

where $\mathrm{d}x$ is the Riemannian measure on X and $\mathrm{d}V$ the Riemannian measure on T^1X. Integrating the previous inequality on the unit tangent bundle of K one get the following inequality relating the energy over K of f_ε, g_ε, h_0 and h_1,

$$E_K(f_\varepsilon) + E_K(g_\varepsilon) - E_K(h_0) - E_K(h_1) \leq -\varepsilon \int_{T^1_K} A(V)\,\mathrm{d}V + O(\varepsilon^2)\,, \tag{5.43}$$

where A is the function on T^1X given, for almost all x in X and V in T^1_xX, by

$$A(V) := 2\eta_x\,(\Phi^V_0 + \Phi^V_1 - 2\Phi^V_{1/2}).$$

Since the harmonic maps h_0 and h_1 are critical points for the energy functional, Inequality (5.43) implies that

$$\int_{T^1K} A(V)\,\mathrm{d}V \leq 0\,. \tag{5.44}$$

Since the function Φ^V is convex, the function A is non-negative. Therefore Inequality (5.44) implies that the function A is almost surely zero. Since the function η was arbitrary, this tells us that, for almost all V in T^1X, one has

$$2\Phi^V_{1/2} = \Phi^V_0 + \Phi^V_1\,.$$

Since Φ^V is the square of the convex function φ_V, this tells us that for almost all V in TX, the function φ_V is constant. This proves (5.37). □

5.9 Equidistant harmonic maps in negative curvature

The following Corollary 5.19 improves the conclusion of Lemma 5.17 when the curvature of Y is uniformly negative.

Corollary 5.19. *Let $a > 0$. Let X, Y be two $\mathcal{C}^2$-Hadamard manifolds with a $\mathcal{C}^1$ Riemannian metric. Assume moreover that Y is $\mathrm{CAT}(-a^2)$. Let $h_0, h_1 : X \to Y$ be two harmonic maps such that the distance function $x \mapsto d(h_0(x), h_1(x))$ is constant. Then either $h_0 = h_1$ or*

$$h_0 \text{ and } h_1 \text{ takes their values in the same geodecic } \Gamma \text{ of } Y. \tag{5.45}$$

This means that, when $h_0 \neq h_1$, there exists a geodesic $t \to \gamma(t)$ in Y and two harmonic functions u_0, u_1 on X such that $h_0 = \gamma \circ u_0$, $h_1 = \gamma \circ u_1$ and the difference $u_1 - u_0$ is a bounded harmonic function on X.

Note that this case is not possible when h_0 and h_1 are within bounded distance of a quasi-isometric map $f : X \to Y$ since X has dimension $k \geq 2$.

Proof of Corollary 5.19. We can assume that the distance between h_0 and h_1 is equal to 1. We recall a few notations we have already used. For t in $[0, 1]$, let h_t be the geodesic interpolation of h_0 and h_1. For x in X, let $\tau_x(t) := \partial_t h_t(x)$. Since the map $(t, x) \mapsto h_t(x)$ is Lipschitz continuous, for almost all t in $[0, 1]$, x in X and V in T_xX, the vector $J_V(t) := Dh_t(V)$ is defined. For all such t, x, V, we set

$$\alpha_V(t) := \langle J_V(t), \tau_x(t)\rangle\,, \quad \varphi_V(t) := \|J_V(t)\|\,, \quad \psi_V(t) := (\varphi_V(t)^2 - \alpha_V(t)^2)^{1/2}\,.$$

By Lemmas 5.10.a and 5.17, for almost all t in $[0, 1]$ and almost all V in TX, one has the equality

$$\alpha_V(0) = \alpha_V(t) = \alpha_V(1) \quad \text{and} \quad \varphi_V(0) = \varphi_V(t) = \varphi_V(1)\,, \tag{5.46}$$

and therefore

$$\psi_V(0) = \psi_V(t) = \psi_V(1)\,.$$

Comparing these equalities with the uniform convexity of the function ψ_V in (5.19), one deduces $\psi_V(t) = 0$. Hence, when $J_V(t)$ is defined, one has

$$J_V(t) = \alpha_V(0)\, \tau_x(t)\,. \tag{5.47}$$

We now explain why (5.47) implies (5.45). It is enough to check that, for every $\mathcal{C}^1$ curve

$$c : [0, 1] \to X; s \mapsto c_s$$

with speed at most 1/3, the images

$$h_0(c_{[0,1]}) \text{ and } h_1(c_{[0,1]}) \text{ are both included in the geodesic } \Gamma \tag{5.48}$$

of Y containing both $h_0(c_0)$ and $h_1(c_0)$.

The idea is to construct an auxiliary curve C with zero derivative. Let $\beta : [0, 1] \to [-1/3, 1/3]$ be the function given by $s \mapsto \beta_s := \int_0^s \alpha_{c'_r}(0)\, \mathrm{d}r$. For t_0 in $[1/3, 2/3]$, let C be the curve

$$C : [0, 1] \to Y; s \mapsto C(s) := h_{t_0 - \beta_s}(c_s).$$

Since the speed of c is bounded by $1/3$, the curve C is well-defined. By construction C is a Lipschitz continuous path, and by (5.46) and (5.47), for almost all s, its derivative is,

$$C'(s) = \big(\alpha_{c'_s}(t_0-\beta_s) - \alpha_{c'_s}(0)\big)\,\tau_{c_s}(t_0-\beta_s) = 0$$

Therefore, for all s in $[0,1]$, one has $C(s) = C(0)$, that is

$$h_{t_0-\beta_s}(c_s) = h_{t_0}(c_0).$$

Using this equality for two distinct values of t_0, we deduce that the geodesic segments $h_{[0,1]}(c_0)$ and $h_{[0,1]}(c_s)$ meet in at least two points. This proves (5.48) and ends the proof of Corollary 5.19. $\square$

This also ends the proof of Proposition 5.1.

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Y. Benoist & D. Hulin, CNRS & Université Paris-Sud, Orsay 91405 France
yves.benoist@math.u-psud.fr & dominique.hulin@math.u-psud.fr