Recurrence on affine grassmannians

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joint with C. Bruère

Plan:
1. Recurrence and stationary measures
2. Linear groups.
3. Semisimple groups.
4. Affine groups.
5. Stabilizer of a vector subspace.
6. Strategy of proof.

1/6 Recurrence in law and stationary measures

G : a Lie group acting on a manifold X,
$\mu \in \mathcal{P}(G)$: a probability measure on G,
Γ : the semigroup generated by suppμ.
Markov chain on X: $x \mapsto P_x = \mu * \delta_x = \int_G \delta_{gx} d\mu(g)$.

μ is recurrent (in law) at x if

$$\forall \varepsilon > 0 \ \exists K \Subset X, \forall n \ \ \mu^{*n} * \delta_x(K) \geq 1 - \varepsilon.$$

μ is recurrent (in law) if it is recurrent at all $x \in X$.

$\mu^{*n} * \delta_x$ is the law at time n of the walk starting from x.
A step of the Markov chain:

$$\mathcal{P}(X) \ni \nu \mapsto \mu * \nu := \int_G g_* \nu \, d\mu(g).$$

$\nu \in \mathcal{P}(X)$ is μ-stationary if $\mu * \nu = \nu$
ν is ergodic : ν is extremal among μ-stationary.

Fact μ is recurrent at some $x \iff$ there exists a μ-stationary probability measure ν on X.

Proof $\Rightarrow$ Take a weak limit ν_∞ of $\nu_n := \frac{1}{n} \sum_{k \leq n} \mu^{*k} * \delta_x$.
$\Leftarrow$ By Birkhoff theorem, μ is recurrent at ν-almost all x.

A closed subset $F \subset X$ is Γ-invariant if $\forall g \in \Gamma \ \ gF \subset F$.
F minimal means F minimal among Γ-invariant closed sets.

Two deterministic examples:
Example 1: $t \mapsto 2t$ in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.
Example 2: $t \mapsto t/2$ in $\mathbb{R}$.

2/6 Linear groups

Here $V = \mathbb{R}^d$, $\Gamma \subset GL(\mathbb{R}^d)$, $X = \mathbb{P}(V)$ and
$G =$ Zariski closure of Γ.
We assume that Γ acts irreducibly on V.

Theorem 1 with Quint There are bijections

$$\left\{\begin{array}{l}\nu \in \mathcal{P}(X) \text{ ergodic} \\ \mu\text{-stationary} \\ \text{probability measure}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}F \subset X \text{ minimal} \\ \Gamma\text{-invariant} \\ \text{closed subset}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\Omega \subset X \\ \text{compact} \\ G\text{-orbit}\end{array}\right\}$$

given by $\nu \mapsto F := supp\,\nu$; $F = \overline{\Gamma x} \mapsto \Omega := Gx$.

$\forall x \in X$, the limit $\nu_x := \lim_{n\to\infty} \mu^{*n} * \delta_x$ exists and is μ-stationary.

$\star$ When Γ is proximal in V,
this is due to Furstenberg and ν is unique.

When Γ is not proximal in V,
there might be uncountably many compact G-orbits in X.

Sketch of proof of the first bijection: Let $P : \mathcal{C}(X) \to \mathcal{C}(X)$
be the Markov operator $P\varphi(x) = \int_G \varphi(gx) d\mu(g)$.
$\mathcal{C}(X)^P := \{\varphi \in \mathcal{C}(X) \mid P\varphi = \varphi\}$ and
$\mathcal{M}(X)^P := \{\text{finite } \mu\text{-stationary measures}\}$.

Key Lemma
a) For all $\varphi \in \mathcal{C}(X)$, the family $(P^n\varphi)_{n\geq 1}$ is equicontinuous.
b) $\mathcal{M}(X)^P$ is naturally the dual of the Banach space $\mathcal{C}(X)^P$.

The implication $a) \Rightarrow b)$ is due to Raugi.

3/6 Semisimple Lie groups

Here G is a semisimple real algebraic Lie group,
H is an algebraic subgroup of G and $X = G/H$.
Assume that Γ is Zariski dense in G.

Theorem 2 with Quint The following are equivalent

- $\star$ μ is recurrent on X,
- $\star$ There exists a μ-stationary probability measure ν on X,
- $\star$ X is compact.

In this case ν is unique and, for all x in X, $\lim_{n\to\infty} \mu^{*n} * \delta_x = \nu$.

This is mainly a nice reformulation of Theorem 2 using
Chevalley embedding of algebraic homogeneous spaces.
Uniqueness is due to Guivarc'h-Raugi.

4/6 Affine groups

Here $G = Aff(\mathbb{R}^d) := \{g : v \mapsto A_g v + v_g\}$ and
$G_1 = SAff(\mathbb{R}^d) := \{g \in G \mid \det A_g = 1\}$,
$X = X_{k,d} := \{\text{affine } k\text{-dimensional subspaces of } \mathbb{R}^d\}$
We assume that Γ is Zariski dense in G or in G_1,
and that suppμ is compact.

Theorem 3 with Bruère The following are equivalent
$\star$ μ is recurrent on $X_{k,d}$,
$\star$ There exists a μ-stationary probability measure ν on $X_{k,d}$,
$\star$ $\lambda_{k+1} < 0$.
In this case ν is unique and, for all x in X, $\lim_{n\to\infty} \mu^{*n} * \delta_x = \nu$.

Here λ_j is the j^{th} Lyapunov exponent defined by

$$\lambda_1 + \cdots + \lambda_j = \lim_{n\to\infty} \frac{1}{n} \int_G log \|\Lambda^j A_g\| d\mu^{*n}(g)$$

Corollary 1 For $k = d - 1$ and $\Gamma \subset G_1$,
μ is always recurrent on $X_{k,d}$.

Use Guivarc'h simplicity of Lyapunov spectrum: $\lambda_1 > \cdots > \lambda_d$.

This corollary is surprising since $X_{k,d}$ is not compact.

Corollary 2 When μ is symmetric,
μ is recurrent on $X_{k,d} \iff 2k \geq d$.

Note also that, for μ symmetric, one has $\lambda_j = -\lambda_{d+1-j}$.

5/6 Stabilizer of a vector subspace (proving Theorem 3)

Here $W \subset V$ are vector spaces,
$W' := V/W$, $X := \mathbb{P}(V) \smallsetminus \mathbb{P}(W)$ and
$G = \{g \in GL(V) \mid gW = W\}$. Assume that

- $\star$ $\Gamma \subset G$ acts strongly irreducibly and proximally on W and W'.
- $\star$ W has no Γ-invariant complementary subspace in V.

Proposition The following are equivalent

- $\star$ μ is recurrent on X,
- $\star$ There exists a μ-stationary probability measure ν on X,
- $\star$ $\lambda_{W,1} < \lambda_{W',1}$.

In this case ν is unique and, for all x in X, $\lim_{n\to\infty} \mu^{*n} * \delta_x = \nu$.

See also : Aoun & Guivarc'h JEMS (2020).

Why Proposition $\Longrightarrow$ Theorem 3

Embed the affine space $E = \mathbb{R}^d$ in a vector space:

$$E \simeq \{u = (v, 1) \mid v \in \mathbb{R}^d\} \subset \mathbb{R}^d \oplus \mathbb{R} = \mathbb{R}^{d+1}$$

$W := \Lambda^{k+1}\mathbb{R}^d \subset V := \Lambda^{k+1}\mathbb{R}^{d+1}$ so that $W' \simeq \Lambda^k \mathbb{R}^d$.

$$\begin{array}{ccccc} \mathrm{Aff}(u_1, \dots, u_{k+1}) & & X_{k,d} & \hookrightarrow & \mathbb{P}(V) \smallsetminus \mathbb{P}(W) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Vec}(u_1, \dots, u_{k+1}) & & \mathbb{G}_{k+1,d+1} & \hookrightarrow & \mathbb{P}(V) \end{array}$$

so that $\lambda_{W,1} = \lambda_1 + \cdots + \lambda_{k+1}$ and $\lambda_{W',1} = \lambda_1 + \cdots + \lambda_k$

Hence : $\lambda_{W,1} < \lambda_{W',1} \iff \lambda_{k+1} < 0$.

6/6 Strategy of proof

Why is the Proposition true?

Write in a non canonical way $V \simeq W \oplus W'$.
Choose $g_1, \dots g_n$ independent with law μ.
Write $g_1 \cdots g_n = \begin{pmatrix} a_n & b_n \\ 0 & d_n \end{pmatrix}$.

$\star$ If there exists a μ-stationary probability ν on X, then the limit $\nu_b := \lim_{n\to\infty} (g_1 \dots g_n)_* \nu \in \mathcal{P}(X)$ exists almost surely.
Therefore, one has $\limsup_{n\to\infty} \frac{\|a_n\|}{\|d_n\|} < \infty$.
By the law of large numbers,
this implies $\lambda_{W,1} < \lambda_{W',1}$.

$\star$ Conversely, if $\lambda_{W,1} < \lambda_{W',1}$.
Then one constructs a proper function
$u : X \to [0, \infty)$ such that $Pu \leq au + b$ with $a < 1$.
Choose $u([w, w']) = \left(\frac{\|w\|}{\|w'\|}\right)^\delta$ with $\delta > 0$ small.
This forces μ to be recurrent.

For more go to www.math.u-psud.fr/~benoist or read
Benoist & Quint in Compositio Mathematica 150 (2014) p.1579-1606
Benoist & Bruère in Ergodic Theory and Dynamical Systems 39 (2019) p.3207-3223