On morphisms onto quadrics

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Let X, Y be smooth projective varieties and $f : X \to Y$ a surjective morphism. It is natural to ask under which conditions the family of possible f is bounded, that is, whether the parameterizing space for such morphisms has only finitely many irreducible components. Clearly, if $Y = \mathbb{P}^n$, then our family is not bounded (provided that $n > 0$); whereas if Y is of general type, then we know from a theorem by Kobayashi and Ochiai ([KO]) that there is only a finite number of morphisms (and even of dominant rational maps) from X to Y.

Consider the following weaker version of our question: let $dim(X) = dim(Y) = n$, and let is impose some condition on the cohomology of X and Y, for example, suppose that $Pic(X) = Pic(Y) = \mathbb{Z}$. For which Y is it true that the degree of a morphism $f : X \to Y$ is bounded from above by some function of the discrete invariants of X and Y?

Obviously, this is not the case when Y is the projective space, a torus or an unramified quotient of a torus, that is, a flat variety (notice that if Y has a ramified covering which is a torus and $Pic(Y) = \mathbb{Z}$, then $Y = \mathbb{P}^n$ by [HM2]; and that a smooth image of the projective space is the projective space itself (if not a point) by Lazarsfeld's theorem). Are there other examples, or can one prove that the degree of f is bounded for any other Y?

It is especially interesting to check the boundedness when Y is an n-dimensional quadric ($n > 2$), because among the varieties with Picard number one the quadric is in some sense the "closest" to $\mathbb{P}^n$. Carmen Schuhmann ([Sch]), using Miyaoka's bound for the number of double points on a surface, proved the boundedness in dimension three. This proof cannot be generalized to arbitrary n.

In [A], theorem 4.1.1, I have proposed another proof of this fact, based on the so-called Infinitesimal Noether-Lefschetz theorem, and indicated (remark 4.1.2) that this argument does generalize. The purpose of this note is to state and prove such a generalization.

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So we shall prove the following result:

Theorem *Let X be a smooth projective variety of dimension n with cyclic Néron-Severi group, and let Q denote an n-dimensional quadric ($n \geq 3$). The degree of a* *morphism $f : X \to Q$ is bounded in terms of the discrete invariants of X.*

More precisely, let L denote an ample generator of $NS(X)$, and let k, r be such that $K_X \equiv kL$ and rL is very ample. If $f^*\mathcal{O}_Q(1) \equiv mL$, then $m < 2k + (2n+5)r$ when $n \geq 4$, and $m < 3k + 16r$ for $n = 3$.

Clearly, the degree of f is $m^n L^n/2$.

Let us remind that Hwang and Mok ([HM]) proved the boundedness of the degree of a morphism between Fano manifolds with Picard number one which have rational curves with trivial normal bundle. Obviously the quadric does not contain such curves - its canonical class is "too positive". The method of Hwang and Mok probably cannot work for a homogeneous Y. Unfortunately, our method does not seem to generalize easily to other homogeneous varieties. Of course one would like to see this result as an evidence for the following conjecture: if X, Y are n-dimensional with Picard number one and Y is Fano different from $\mathbb{P}^n$, then the degree of a morphism $f : X \to Y$ is bounded in terms of the discrete invariants of X, Y (cf. [ARV] where the case $K_Y = 0$ is treated).

Proof of the theorem:

Let $f : X \to Q$, $NS(X) \cong \mathbb{Z}$, be a morphism; f is automatically finite (if non-constant). Take l a sufficiently general line on Q, and let $S \subset Q$ be a general 2-dimensional linear section of $Q \subset \mathbb{P}^{n+1}$ containing l (so S is a two-dimensional quadric). Write $S = Q \cap H_1 \cap H_2 \cap \ldots \cap H_n$, with H_i sufficiently general hyperplanes.

One has: $N_{l,S} = \mathcal{O}_{\mathbb{P}^1}$ and $N_{S,Q}|_l = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-2}$ That is, the exact sequence of normal bundles

$$0 \to N_{l,S} \to N_{l,Q} \to N_{S,Q}|_l \to 0$$

splits. Let $C = f^{-1}l$ and $M = f^{-1}S$. As l and S are chosen to be sufficiently general, C and M are smooth by Kleiman's version of Bertini theorem ([H], chapter III.10). The exact sequence of normal bundles

$$0 \to N_{C,M} \to N_{C,X} \to N_{M,X}|_C \to 0$$

is just the pull-back of the corresponding sequence on Q, that is, it splits. So the induced map $\alpha : H^0(C, N_{C,X}) \to H^0(C, N_{M,X}|_C)$ is surjective.

In particular, $Im(\alpha)$ contains the image of the restriction map $\beta : H^0(M, N_{M,X}) \to H^0(C, N_{M,X}|_C)$. This is interpreted as follows: any infinitesimal deformation of M in X contains an infinitesimal deformation of C. That is, the class of C in $H^{1,1}(M)$ is "infinitesimally fixed" according to the terminology of [CGGH].

Now there are various results of "infinitesimal Noether-Lefschetz" type, which state that if M is sufficiently ample on X, then all infinitesimally fixed classes in $H^{1,1}(M)$ are restrictions of (1,1)-classes on X. Our class $[C]$ obviously cannot be such a restriction: indeed, $N_{C,M} = \mathcal{O}_C$, that is, $[C]^2 = 0$; but, thanks to the hypothesis that $NS(X) = \mathbb{Z}$, any (1,1)-class which has zero square on M and is a restriction of a (1,1)-class on X, must be numerically trivial, which is obviously not the case for $[C]$. So such results give the non-existence of f as soon as M is sufficiently ample, that is, $deg(f)$ is sufficiently high.

In order to make this more precise, we shall prove the following proposition, which is a very direct generalization of Proposition 3.4 from [EL]:

Proposition *Let X be a smooth projective n-fold and Y a smooth complete intersection of ample divisors $D_1, D_2, \ldots, D_{n-2}$ on X (so that dim(Y)=2). Suppose that for some very ample A, one has $D_i \in |K_X + (n+1)A + B_i|$ and $D = D_1 + D_2 + \ldots + D_{n-2} \in |nK_X + (n+1)^2 A + E|$ with B_i and E nef. Then the infinitesimal Noether-Lefschetz theorem holds for $Y \subset X$, that is, any (1,1)-class on Y which remains of type (1,1) under all infinitesimal deformations of Y in X, is a restriction of a class on X.*

Proof: We follow the exposition of [EL]. Let $N = \oplus_i \mathcal{O}_Y(D_i)$ be the normal bundle of Y in X, so that $H^0(Y,N)$ is the space of infinitesimal deformations of Y in X. bundle According to the theory of [CGGH], the space of infinitesimally fixed classes is the kernel of the map

$$\alpha : H^1(Y, \Omega^1_Y) \to H^2(Y, N^*) \to Hom(H^0(Y,N), H^2(Y, \mathcal{O}_Y)),$$

γ being the usual coboundary map, and β coming from the multiplication map $\mu : H^0(Y,N) \otimes H^0(Y,K_Y) \to H^0(Y, N \otimes K_Y)$ by Serre duality. We shall be done if we check that

1) β is injective, that is, μ is surjective;

2) the restriction map $H^1(X, \Omega^1_X) \to H^1(Y, \Omega^1_X|_Y)$ is surjective.

To check the surjectivity of μ, it is enough to check the surjectivity of each "component"

$$\mu_i : H^0(Y, \mathcal{O}_Y(D_i)) \otimes H^0(Y, K_Y) \to H^0(Y, \mathcal{O}_Y(D_i) \otimes K_Y)$$

.

From the resolution

$$(*)\ 0 \to \mathcal{O}_X(-D) \to \ldots \to \oplus_{i<j} \mathcal{O}_X(-D_i - D_j) \to \oplus_i \mathcal{O}_X(-D_i) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

one sees that $H^0(X, \mathcal{O}_X(D_i) \otimes K_X \otimes \mathcal{O}_X(D))$ surjects onto $H^0(Y, \mathcal{O}(D_i) \otimes K_Y)$ (indeed, the D_i are ample, so this is implied by the Kodaira vanishing). That is, the surjectivity of μ_i would, by restriction, follow from the surjectivity of the multiplication map on X:

$$H^0(X, \mathcal{O}_X(D_i)) \otimes H^0(X, \mathcal{O}_X(D) \otimes K_X) \to H^0(X, \mathcal{O}_X(D) \otimes K_X \otimes \mathcal{O}_X(D_i)).$$

This, in turn, follows from the case $p = 0$ of [EL], Theorem 2, by taking (in the notations of [EL]) $L_d = \mathcal{O}_X(D_i)$, $W = H^0(X, L_d)$, $N_f = K_X + D$. Indeed, $N_f = K_X + (n+1)A + n(K_X + (n+1)A) + E$, and $n(K_X + (n+1)A)$ is nef; and D_i is globally generated because it's of the form $K_X + (n+1)A + B_i$ with A very ample.

As for the surjectivity of the restriction map, one sees from the resolution $(*)$ and the Kodaira vanishing that it is enough to check that $H^{n-1}(X, \Omega^1_X(-D)) = 0$, or, dually, $H^1(X, \Omega^{n-1}_X(D)) = 0$. Now the bundle $\Omega^{n-1}_X(nA)$ is globally generated, so one can apply Griffiths' vanishing: $H^i(X, E \otimes F) = 0$ for $i > 0$, where E is a globally generated vector bundle and F is a line bundle such that $F \otimes (det(E) \otimes K_X)^{-1}$ is ample. As $det(\Omega^{n-1}_X(nA)) = n^2A + (n-1)K_X$, the required vanishing follows.

Let us return to our $M \subset X$, $M = D_1 \cap D_2 \cap \ldots \cap D_{n-2}$ with $D_i \in |f^*\mathcal{O}_Q(1)|$, violating the infinitesimal Noether-Lefschetz condition. It is clear that if $f^*\mathcal{O}_Q(1)$ is sufficiently ample, then the D_i shall satisfy the two conditions of our proposition, leading to a contradiction. More precisely, suppose that D_i is of the form $2K_X+(2n+5)A$ for some very ample A; an easy calculation (using the fact that $K_X+(n+1)A$ is nef) shows that the conditions of the proposition are satisfied provided that $n \geq 4$. (For $n = 3$, these hold when $D = D_1 = 3K_X + 16A$; this is already in [EL]). So D_i cannot get that ample; in other words, if m is such that $f^*\mathcal{O}_Q(1) \equiv mL$, then $m < 2k + (2n+5)r$ for $n \geq 4$ and $m < 3k + 16r$ for $n = 3$.

Remark: This can be slightly improved by remarking that, as X maps onto an $n-$quadric, X is not $\mathbb{P}^n$, and if X is a quadric, then f is an isomorphism; so one may suppose that X is neither $\mathbb{P}^n$ nor a quadric, in which case already $K_X + (n-1)A$ (rather than $K_X + (n+1)A$) is nef.

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