stochastic
processes
and their
applications

an official journal of the Bernoulli Society
for Mathematical Statistics and Probability

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Change of measure in the lookdown particle system

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Abstract

We perform various changes of measure in the lookdown particle system of Donnelly and Kurtz. The first example is a product type h-transform related to conditioning a Generalized Fleming–Viot process without mutation on coexistence of some genetic types in remote time. We give a pathwise construction of this h-transform by just "forgetting" some reproduction events in the lookdown particle system. We also provide an intertwining relationship for the Wright–Fisher diffusion and explicit the associated pathwise decomposition. The second example, called the linear or additive h-transform, concerns a wider class of measure-valued processes with spatial motion. Applications include a simple description of the additive h-transform of the Generalized Fleming–Viot process and an immortal particle representation for the additive h-transform of the Dawson–Watanabe process.

1. Introduction

Constant size population models evolving through resampling first appeared in the works of Wright (1930) and Fisher (1931) in discrete time, and Moran (1958) in continuous time. Building on the latter work, Fleming and Viot (1979) introduced a measure-valued process modelling a large population with constant size in which the individuals are also subject to a spatial motion. Bertoin and Le Gall [6–8] (2003) proposed a further generalization by allowing discontinuities

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in the Fleming–Viot process. In the so called Generalized Fleming–Viot (GFV) process, one single (infinitesimal) individual may beget a descendance comparable to the total population size. By comparison, in the usual Fleming–Viot processes, one (infinitesimal) individual is allowed to have at most two children. The backward genealogy of a Fleming–Viot process is therefore given by a Kingman coalescent (1982), whereas the genealogy of a GFV process is described by a Λ-coalescent. This class of coalescent processes was introduced independently by Pitman [35] (1999) and Sagitov [38] (1999). A common feature to all GFV processes is the following property, simplistic when considered from the genetic viewpoint, and sometimes called the Eve property, see Labbé [26]: one type eventually fixates the whole population (in finite or infinite time). Our first aim is therefore to define a model allowing for the coexistence of a fixed number of types in large time. This will be realized by a singular conditioning expressed in terms of Doob h-transforms.

Branching population models were introduced by Galton and Watson [44] (1874) in discrete time and space. Jirina [23] (1958) later defined continuous state branching (CB) processes evolving in continuous time, and Lamperti [29] (1967) proved that CB processes are the scaling limits of Galton–Watson processes. Adding a spatial motion to the individuals, Dawson [10] (1977) and Watanabe [43] (1968) later defined a measure-valued process with total mass given by a continuous CB process. The Dawson–Watanabe superprocess was further generalized to take into account the discontinuities arising in the sample paths of general CB processes, see Dynkin [15,16] (1991). We shall still call these processes Dawson–Watanabe superprocesses. Their genealogy is given by continuum Lévy trees, see Le Gall and Le Jan [30] (1998), or, alternatively, by flow of bridges, see Bertoin and Le Gall [5] (2000).

In both the constant and the branching population settings, the measure-valued processes arise as scaling limits of finite population models. These finite populations may be represented by particle systems, with the particles playing the role of the individuals. At the limit, the genealogy brought by the particles is lost. It is nevertheless useful to keep track of the particles in the limiting process, for instance to give a precise meaning to the infinitesimal individuals that have been mentioned in this introduction, or to give them a genealogy. This was achieved by Donnelly and Kurtz in [11,12] (1999). The idea is to order the particles by persistence, and give these particles a label called the "level" accordingly. The ranked particle system associated with a finite population model is then proved to weakly converge under suitable assumptions, and many measure-valued processes of interest (including GFV processes and Dawson–Watanabe superprocesses) may be simply recovered as the de Finetti measure of the limiting particle system, called the lookdown particle system. This particle system therefore unifies the constant and the branching population settings. A non-realistic property of the measure-valued processes defined from (this particular) lookdown particle system (see also the recent extension [25] by Kurtz and Rodriguez), at least when considered as populations models, is the absence of interaction between space and branching: each particle, independently of its spatial position, follows a similar spatial motion. Our second aim is to modify this feature in the direction of more interaction: a smooth way to do it is, again, by using Doob h-transforms. We now introduce this procedure, which associates to any Markov process a new Markov process with law absolutely continuous with respect to the former.

1.1. Doob h-transforms

We recall that Doob h-transform refers to the following operation: given a transition kernel $p_t(x, dy)$ of a Markov process $(X_t, t \geq 0)$ on a Polish space E and a non-negative space–time

harmonic function $(H(t, y), t \geq 0, y \in E)$ for this kernel, meaning that:

$$\int H(t, y) \, p_t(x, dy) = H(0, x)$$

for each $x \in E$ and $t \geq 0$, a new transition kernel is defined by the expression:

$$p_t(x, dy) \, \frac{H(t, y)}{H(0, x)},$$

as soon as $H(0, x) \neq 0$, and the associated Markov process is called an h-transform. The probabilistic counterpart of this definition is the following. Let $\mathcal{F}_t$ denote the sigma-algebra generated by the process $(X_s, 0 \leq s \leq t)$ up to time t. The h-transformed process is absolutely continuous with respect to the original process $(X_s, s \geq 0)$ with Radon–Nikodym derivative on $\mathcal{F}_t$ given by the martingale:

$$\frac{H(t, X_t)}{\mathbb{E}(H(0, X_0))}.$$

The consistency of this definition as t varies is ensured by the martingale property of $H(t, X_t)$, which is the probabilistic counterpart of the space–time harmonicity of the function H. Informally, the h-transform consists in reweighting the paths of the Markov process on $\mathcal{F}_t$ according to the value of the function H at point (t, X_t). The h-transformed processes *locally* look like the original process, but may have drastically different global behaviour. The two h-transform processes in which we are interested in this paper provide a good example of this point.

In the context of measure-valued processes, h-transform processes have been intensively studied. The long collection of papers on spine and backbone decompositions, which are instances of h-transforms, reflects this activity. Rather than providing an extensive bibliography, we refer to the discussion in the introduction of Engländer and Kyprianou [17], where a careful review of the literature on such decompositions is offered. Let us nevertheless mention the recent paper by Salisbury and Sezer [39], posterior to [17], which provides a nice example of investigation of the relation between conditioning and h-transform, in the same vein as the present paper, although the settings, the tools and the aim differ.

We now introduce the measure-valued processes of interest and, before, define the exchangeable lookdown particle system. This requires some preliminary definitions.

1.2. Exchangeable random partitions

We will need the following two measures on the set $\mathcal{P}_\infty$ of the partitions of $\mathbb{N}$. First, we define the probability measure ρ_x on $\mathcal{P}_\infty$ as the law of the random partition π with a unique non-trivial block with asymptotic frequency $x, 0 < x < 1$, constructed as follows. Let $(U_i, i \geq 1)$ be a sequence of independent and identically distributed Bernoulli random variables with parameter x, such that

$$\mathbb{P}(U_1 = 1) = 1 - \mathbb{P}(U_1 = 0) = x.$$

Then we declare that i and j are in the same block of π if and only if $U_i = U_j = 1$ and that i is a singleton of π if $U_i = 0$.

Second, we define the Kingman measure μ^k on $\mathcal{P}_\infty$ as the infinite measure on $\mathcal{P}_\infty$ which gives:

- mass 1 to the partitions with a unique non-trivial block formed by a doubleton $\{i, j\}$ with $i \neq j$, the other integers being isolated in singletons.
- mass 0 to the other partitions.

We are now ready to introduce the lookdown particle system.

1.3. The lookdown particle system

Let E be a Polish space. We consider a triple (R_0, Y, U) constructed as follows. R_0 stands for a random probability measure on E, $Y = (Y_t, t \geq 0)$ and $U = (U_t, t \geq 0)$ for two non-negative real valued processes. We invite the reader to have a look at the two following subsections for concrete examples of such processes Y and U. We assume that $U_0 = 0$ and U is non decreasing, so that U admits a unique decomposition $U_t = U_t^k + \sum_{s \leq t} \Delta U_s$ where U^k is continuous (with Stieltjes measure denoted by dU^k) and $\Delta U_s = U_s - U_{s-}$. We assume that 0 is an absorbing point for Y, and set $\tau(Y) = \inf\{t > 0, Y_t = 0\}$ the extinction time of Y. We also assume that for each $t \geq 0$, $\Delta U_t \leq Y_t^2$. Conditionally on U and Y, we define two point measures N^ρ and N^k on $\mathbb{R}_+ \times \mathcal{P}_\infty$.

- $N^\rho = \sum_{0 \leq t < \tau(Y), \Delta U_t \neq 0} \delta_{(t,\pi)}(dt, d\pi)$ and the exchangeable partitions π of $\mathbb{N}$ are independent with law $\rho_{\sqrt{\Delta U_t}/Y_t}$.
- $N^k = \sum_{0 \leq t < \tau(Y)} \delta_{(t,\pi)}(dt, d\pi)$ is an independent Poisson point measure with intensity $(dU_t^k/(Y_t)^2) \times \mu^k$.

Conditionally on (R_0, Y, U), we then define a particle system $X = (X_t(n), 0 \leq t < \tau(Y), n \in \mathbb{N})$ as follows:

- The initial state $(X_0(n), n \in \mathbb{N})$ is an exchangeable E-valued sequence with de Finetti's measure R_0.
- At each atom (t, π) of $N := N^k + N^\rho$, we associate a *reproduction event* as follows: let $j_1 < j_2 < \cdots$ be the elements of the unique block of the partition π which is not a singleton (either it is a doubleton if (t, π) is an atom of N^k or an infinite set if (t, π) is an atom of N^ρ). The individuals $j_1 < j_2 < \cdots$ at time t are declared to be the children of the individual j_1 at time $t-$, and receive at time t the type of the parent j_1 at time $t-$, whereas the types of the other individuals are shifted upwards accordingly, keeping the order they had before the birth event: for each integer ℓ, $X_t(j_\ell) = X_{t-}(j_1)$ and for each $k \notin \{j_\ell, \ell \in \mathbb{N}\}$, $X_t(k) = X_{t-}(k - \#J_k)$ with $\#J_k$ the cardinality of the set $J_k := \{\ell > 1, j_\ell \leq k\}$.
- Between the reproduction events, the type $X_t(n)$ of the particle at level n mutates according to a Markov process with càdlàg paths in E, independently for each n. The law of this Markov process will be denoted by $(P_x, x \in E)$ when started at $x \in E$. We shall say there is no mutation when the law P_x reduces to the Dirac mass at the constant path equal to x for each $x \in E$.

This defines the particle system X on $[0, \tau(Y))$. For each $j \in \mathbb{N}$, the process $X_s(j)$ admits a limit as s goes to $\tau(Y)$, and we set $X_t(j) = \lim_{s \to \tau(Y)} X_s(j)$ for $t > \tau(Y)$. The sequence $(X_t(j), j \in \mathbb{N})$ is still exchangeable for $t \geq \tau_Y$ according to Proposition 3.1 of [12]. Conditionally on (R_0, Y, U), the sequence $(X_t(n), n \in \mathbb{N})$ is therefore exchangeable for each $t \geq 0$, and we
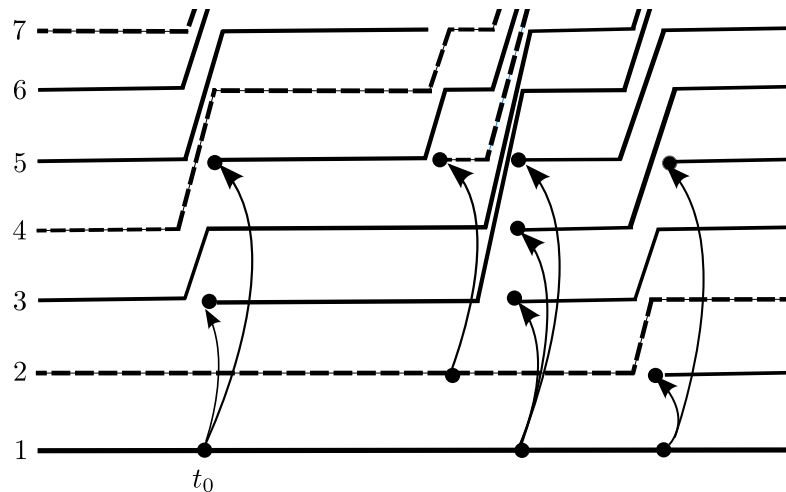
Fig. 1. A lookdown particle system restricted to its first 7 levels. The two types are represented by solid lines and dotted lines. At time t_0, there is a reproduction event, and the father at level $j_1 = 1$ at time t_0 gives its type to its children at levels $j_1 = 1, j_2 = 3, j_3 = 5, \ldots$ at time t_0. Notice the way the other types are transmitted.

denote by R_t its de Finetti[1] measure:

$$R_t(dx) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{X_t(n)}(dx). \tag{1}$$

Endowing the state space $\mathcal{M}_f(E)$ with the topology of weak convergence, we know that the probability measure-valued process $R = (R_t, t \geq 0)$ has a càdlàg version according to Theorem 3.2 of [12]. We shall work with this version from now on. See Fig. 1 for a schematic view of the look-down particle system $(X_t(n), n \in \mathbb{N})$.

We stress that, conditionally given R_t, the random variables $(X_t(n), n \in \mathbb{N})$ on E are independent and identically distributed according to the probability measure R_t thanks to de Finetti's theorem. This key fact will be used several times in the following.

We finally define the càdlàg $\mathcal{M}_f(E)$-valued process of interest Z by:

$$(Z_t, t \geq 0) = (Y_t \, R_t, t \geq 0). \tag{2}$$

Intuitively, the finite measure Z represents the distribution of a population distributed in the (type) space E, the process Y corresponds to the total population size, and U tracks the resampling inside the population.

We assume that (Y, U, X) have the prescribed law on an abstract probability space $(\Omega, \mathbb{P})$, which is then equipped with three different filtrations:

- $(\mathcal{F}_t = \sigma((Y_s, s \leq t), (X_s, s \leq t)), t \geq 0)$ corresponds to the filtration of the particle system and the total population size.
- $(\mathcal{G}_t = \sigma(Z_s, s \leq t), t \geq 0)$ corresponds to the filtration of the resulting measure-valued process.
- $(\mathcal{D}_t = \sigma(X_s(1), s \leq t), t \geq 0)$ is the filtration induced by the motion of the first level particle.

We have $Y_t = Z_t(\mathbf{1}) := \int_E Z_t(dx)$, and thus Y is $\mathcal{G}$-measurable. Notice also that X is a Markov process with respect to the filtration $\mathcal{F}$, and that Z is a Markov process with respect to the filtration $\mathcal{G}$.

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[1] Beware, that, in many papers, *the law* of R_t is rather called the de Finetti measure.

1.4. Generalized Fleming–Viot processes

The Generalized Fleming–Viot (GFV) process with mutation is the probability measure-valued process Z constructed in Section 1.3 with:

- $Y = 1$,
- U a subordinator with jumps no greater than 1.

We refer to the article [6] by Bertoin and Le Gall for more information on GFV processes, and to Bertoin [4] and Berestycki [2] for background on the associated Λ-coalescents. The GFV process has càdlàg sample paths in the space of probability measures on E endowed with the topology of weak convergence. We denote by $\phi(\lambda)$ the Laplace exponent of the subordinator U:

$$\phi(\lambda) = c\lambda + \int_{(0,1]} (1 - e^{-\lambda x}) \nu^U(dx)$$

where $c \geq 0$ and the Lévy measure ν^U satisfies $\int_{(0,1]} x \, \nu^U(dx) < \infty$. The genealogy of the GFV process is described by the Λ-coalescent of Pitman [35], where the finite measure Λ is derived from c and ν^U as follows:

$$\int_{[0,1]} g(x)\Lambda(dx) := cg(0) + \int_{(0,1]} g(\sqrt{x}) x \nu^U(dx).$$

The construction explained in Section 1.3 greatly simplifies in this setting: N is a Poisson point measure on $\mathbb{R}_+ \times \mathcal{P}_\infty$ with intensity:

$$dt \times \mu(d\pi) := dt \times \left(c\mu^k(d\pi) + \int_{(0,1]} \nu(dx)\rho_x(d\pi) \right),$$

where we have set $\nu(dx) = \Lambda_{|(0,1]}(dx)x^{-2}$, see Sections 3.1.4 and 5.1 of [12]. Notice that ν and ν^U are linked as follows:

$$\int_{(0,1]} g(x)x^2\nu(dx) = \int_{(0,1]} g(\sqrt{x})x\nu^U(dx),$$

for g bounded and measurable. This means that ν is the push-forward measure of ν^U by the map $x \to \sqrt{x}$.

Intuitively, the GFV process $(R_t, t \geq 0)$ describes a constant size population evolving through (time homogeneous) resampling. The evolution of this process is a superposition of a continuous evolution and a discontinuous one. In the continuous evolution, each pair of individuals is sampled at constant rate c, and the individual with lower level gives its type to the individual with upper level. For describing the discontinuous evolution, we need an independent Poisson point measure N' on $\mathbb{R}_+ \times (0, 1]$ with intensity $dt \, \nu(dx)$. If (t, x) is an atom of N' then t is a jump time of the process $(R_t, t \geq 0)$ and the conditional law of R_t given R_{t-} is $(1 - x)R_{t-} + x\delta_U$ where U is distributed according to R_{t-}. This translates as follows in the lookdown particle system: at time t, we sample independently each level with the same probability x. Then, the sampled individual with lower level is declared to be the father and it gives its type at time $t-$ to the other sampled individuals, which correspond to its children. It may be checked from the exchangeability of the particle system that conditionally given R_{t-}, the type of the father has distribution R_{t-}, like U. The other types are redistributed in order to preserve the ordering by persistence, following the rule specified in Section 1.3. After this redistribution, the de Finetti

measure of the particle system at time t has distribution $(1 - x)R_{t-} + x\delta_U$, conditionally given R_{t-}.

1.5. The Dawson–Watanabe superprocess

A continuous state branching process is a real-valued càdlàg strong Markov process characterized by a branching mechanism ψ taking the form

$$\psi(\lambda) = \frac{1}{2} \sigma^2\lambda^2 + \beta\lambda + \int_{(0,\infty)} (e^{-\lambda u} - 1 + \lambda u \mathbf{1}_{u\leq 1})\nu^Y(du), \tag{3}$$

for ν^Y a Lévy measure such that $\int_{(0,\infty)}(1 \wedge u^2)\nu^Y(du) < \infty$, $\beta \in \mathbb{R}$, and $\sigma^2 \in \mathbb{R}^+$. We refer to Dynkin [16] and Etheridge [18] for background on these processes. We will denote the continuous state branching process with branching mechanism ψ by CB(ψ). The branching mechanism yields the Laplace transform of the marginals of the CB(ψ) process through the following formula:

$$\mathbb{E}(e^{-\lambda Y_t}|Y_0 = x) = e^{-xu(\lambda,t)}, \tag{4}$$

where u is the unique non-negative solution of the integral equation:

$$u(\lambda, t) + \int_0^t ds\, \psi\,(u(\lambda, s)) = \lambda \tag{5}$$

holding for all $t \geq 0$, $\lambda \geq 0$. Let $(Y_t(x), t \geq 0)$ be a CB(ψ) process started at $x \geq 0$. From (4), we have the following primitive form of the branching property:

$$Y_t(x + x') = Y_t(x) + Y_t(x'),$$

where the equality is in distribution, and the two random variables on the right-hand side are chosen independent. In fact, the branching property also holds for the whole processes, and we have the following equality in distribution:

$$(Y_t(x + x'), t \geq 0) = (Y_t(x) + Y_t(x'), t \geq 0), \tag{6}$$

where the two processes on the right-hand side are independent. Conversely, the CB(ψ) processes are the only càdlàg Markov processes satisfying (6). Last, a CB(ψ) process $(Y_t, t \geq 0)$ may be constructed from a spectrally positive Lévy process $(L_t, t \geq 0)$ with Laplace exponent $\psi(\lambda)$ by a random time change, with the time running at speed given by L_t: this transformation is known as the Lamperti transform, see [29].

The Dawson–Watanabe process with branching mechanism $\psi(\lambda)$ is the measure-valued process $(Z_t, t \geq 0)$ constructed in Section 1.3 when:

- $(Y_t, t \geq 0)$ is a CB.
- $(U_t, t \geq 0)$ is the quadratic variation process of Y: $U_t = [Y](t)$. In particular, the condition $\Delta U_t = (\Delta Y_t)^2 \leq Y_t^2$ appearing in the construction of the lookdown particle system is satisfied.

This process has càdlàg sample paths in $\mathcal{M}_f(E)$ when the space $\mathcal{M}_f(E)$ of finite measures on E is endowed with the topology of weak convergence.

Intuitively, a Dawson–Watanabe superprocess $(Z_t, t \geq 0)$ is recovered from a CB(ψ) process by adding an independent spatial motion to the infinitesimal individuals. Like the CB(ψ) process,

the evolution of a Dawson–Watanabe process is the superposition of a continuous evolution and a discontinuous one. In the continuous evolution, each individual begets two individuals at constant rate. On the lookdown graph, this translates in a non-trivial way, due to the constant size of the ratio R: each pair of individuals is sampled at rate σ^2/Y_t at time t, and the individual with lower level gives its type to the individual with upper level. Notice the similarity with the continuous evolution in the GFV process, for which the total population size is constant equal to 1. For describing the discontinuous evolution, we need N'' a Poisson point measure on $(0,\infty)^3$ with intensity $dt\ du\ v^Y(dx)$. Given that (t,u,x) is an atom of N'' such that $0 \le u \le Y_{t-}$, the conditional law of Z_t given Z_{t-} is that of $Z_{t-} + x\delta_U$ where U is distributed according to Z_{t-}. This translates as follows in the lookdown particle system: at time t, we sample independently each level with the same probability $x/(Y_{t-}+x)$. Then, the sampled individual with lower level is declared to be the father and it gives its type U at time $t-$ to the other sampled individuals, which correspond to its children. The other types are redistributed in order to conserve the ordering by persistence.

1.6. The main results

Our interest is two-fold. First, we want to introduce an analogue of the GFV process, but with ultimate coexistence of types, and without appealing to some kind of mutation process. Second, we are interested in defining measure-valued processes with (soft) interaction between the spatial motion and the branching structure.

- Assume $(Z_t, t \ge 0) = (R_t, t \ge 0)$ is the GFV process constructed in Section 1.4 without mutation and with finite state space E equal to $\{1,\ldots,K'\}$. Then, for $1 \le K \le K'$, the process

$$\frac{H(t,R_t)}{\mathbb{E}(H(0,Z_0))} = e^{r_K t}\frac{\prod_{i=1}^{K} R_t(\{i\})}{\mathbb{E}\left(\prod_{i=1}^{K} R_0(\{i\})\right)}$$

with

$$r_K = \frac{K(K-1)}{2}c + \int_{(0,1]} v(dx)\left(1 - (1-x)^K - Kx(1-x)^{K-1}\right),$$

defines a non-negative $\mathcal{G}$-martingale with expectation 1 as proved in Lemma 2.3. The associated h-transformed process $(R_t^h, t \ge 0)$ ponderates the paths for which the first K types are present in equal proportion. This new process admits a simple representation as the de Finetti measure of a new particle system $(X_t^h, t \ge 0)$, see Theorem 2.4, which is constructed as $(X_t, t \ge 0)$, except for the two following additional rules:
  - we impose $\{X_0^h(1),\ldots,X_0^h(K)\} = \{1,\ldots,K\}$.
  - All the reproductions events involving 2 of the first K levels, corresponding to $j_2 \le K$, are discarded, so that the equality $\{X_t^h(1),\ldots,X_t^h(K)\} = \{1,\ldots,K\}$ still prevails for each $t \ge 0$.

The process $(R_t, t \ge 0)$ is said to satisfy the absorption property when R_t is almost surely reduced to a Dirac mass in finite time. Under the assumption (15), which implies the almost sure absorption of $(R_t, t \ge 0)$, we prove that $(R_t^h, t \ge 0)$ has the law of the process $(R_t, t \ge 0)$ conditioned on non-absorption of its first K types in remote time, see Theorem 2.5. The analytical counterpart of this result was known for Wright–Fisher diffusions ($v = 0$) since

the work of Kimura [24], quoted by Lambert in [28]. Our method relies on the analysis of the lookdown particle system. As a consequence, our result has a more probabilistic flavour; in particular, it allows to interpret the GFV process conditioned on non-absorption as a GFV process where some reproduction events are *erased*. Interestingly, this is the opposite behaviour to that of branching populations conditioned on non-extinction, for which it is known since Kesten (see also Lyons, Pemantle and Peres [32]) that *additional* immigration is needed for the process not to extinct at 0. Another upshot of the construction of the particle system X^h is the possibility to compute the generator of the conditioned process $(R_t^h\{1\}, t \geq 0)$ in a rather straightforward way in the case of a two-type GFV process at Proposition 2.12. We also take the opportunity to present an intertwining relationship for the Wright–Fisher diffusion and explicit the associated pathwise decomposition, see Proposition 2.16: we prove that a Wright–Fisher diffusion may be constructed as a Wright–Fisher diffusion with a stochastic non-decreasing immigration driven by an independent Markov chain. This adds another decomposition to the striking one of Swart, see [42].

- Assume now that $(Z_t, t \geq 0)$ is the general measure-valued process on a Polish space E constructed in Section 1.3, which includes the GFV process and the Dawson–Watanabe superprocess. Notice that Z allows for mutation and non-constant population size. If $(Y_t, t \geq 0)$ and $(h(t, X_1(t)), t \geq 0)$ are two non-negative martingales (we do slightly better in the text), the process

$$\frac{H(t, Z_t)}{\mathbb{E}(H(0, Z_0))} = \frac{\int_E Z_t(du)h(t, u)}{\mathbb{E}\left(\int_E Z_0(du)h(0, u)\right)}$$

is again a non-negative martingale. The associated h-transform favours the paths for which Z is large where the function h is large. Once again, we construct the corresponding h-transform as the de Finetti measure of a new particle system X^h in Theorem 3.3. This particle system X^h is constructed as X, except for the two following rules:

  – the first level particle $(X_t^h(1), t \geq 0)$ follows the path of an h-transform of the underlying spatial motion (or mutation process) with $h = h(t, \cdot)$.
  – The total mass process Y_t^h is size biased.

  This result was suggested by Overbeck in the case of a Fleming–Viot process, see [34, p. 183]. It also relates in the branching setting to decompositions of the additive h-transforms of superprocesses found by the same author in [33] using Palm measures.

Our two examples, although similar, are independent: the first one may not be reduced to the second one, and vice versa. We stress on the change of filtration technique, learnt in Hardy and Harris [22], which allows us to give simple proofs of the main results. Our main contribution relies in understanding the interplay between the Doob h-transform and the exchangeable lookdown particle system. We take a probabilistic approach in the study of GFV processes conditioned on non-extinction of types. This approach notably yields pathwise results, which in turn allow to interpret and to effectively compute the generators of the conditioned processes arising in the analysis. This method using the lookdown particle system, is, to the best of our knowledge, new in this setting. The work on additive h-transforms has two antecedents, namely the papers [33,34] by Overbeck, the first paper being specific to the superbrownian setting, and the second paper dealing with diffusive measure-valued processes. Our contribution on additive h-transforms consists in a generalization of the previous decompositions known for this class of h-transforms to càdlàg measure-valued processes derived from the lookdown. This class is large enough to incorporate the two processes intensively studied since the parution of [33], namely

general Dawson–Watanabe superprocesses considered from the genealogical viewpoint, see the monograph [13], and GFV processes.

1.7. Outline of the paper

Section 2 is concerned with a product-type h-transform of a GFV process without mutation. We prove in Section 2.2 that the h-transform may be interpreted as the process conditioned on coexistence of some genetic types. In Section 2.3, we compute the generator of the conditioned process when the finite state space is composed of only two types, and recognize it as the generator of a GFV process with immigration. Section 2.3 also contains the statement and the interpretation of the intertwining relationship. Section 3 is concerned with the additive h-transform of a more general measure-valued process. Section 3.2 collects the two main applications, to GFV processes and Dawson–Watanabe superprocesses.

2. A product type h-transform

2.1. A pathwise construction of an h-transform

Along this subsection, we shall denote by $(R_t, t \geq 0)$ a GFV process without spatial motion (that means, P_x is the Dirac measure at the constant path equal to x). For the sake of simplicity, we shall assume that the type space is a finite state space: $E = \{1, 2, \ldots, K'\}$ for an integer $K' \geq 2$.

2.1.1. Results

This subsection is devoted to the construction of a multiplicative h-transform of the process $(R_t, t \geq 0)$ via a modified particle system. The interpretation of the resulting measure-valued process as a conditioned GFV process is postponed to Section 2.2. The proofs of the results contained in this subsection may be found in the next Section 2.1.2. Fix $1 \leq K \leq K'$. We assume from now on and until the end of Section 2 that:

$$\mathbb{E}\left(\prod_{i=1}^{K} R_0\{i\}\right) > 0, \tag{7}$$

to avoid empty definitions in the following. Recall the definition of the particle system X associated with R. We define from X a new particle system X^h, still defined on the suitably enriched probability space $(\Omega, \mathbb{P})$, as follows:

(i) The finite sequence $\left(X_0^h(j), 1 \leq j \leq K\right)$ is a uniform permutation of $\{1, \ldots, K\}$, and, independently, the sequence $\left(X_0^h(j), j \geq K + 1\right)$ is exchangeable with asymptotic frequencies R_0^H, where R_0^H is the random probability measure with law:

$$\mathbb{P}(R_0^H \in A) = \mathbb{E}\left(\mathbf{1}_A(R_0) \frac{\prod\limits_{i=1}^{K} R_0\{i\}}{\mathbb{E}\left(\prod\limits_{i=1}^{K} R_0\{i\}\right)}\right).$$

(ii) The reproduction events are given by the *restriction* of the Poisson point measure N to $V := \left\{(s, \pi), \pi_{|[K]} = \{\{1\}, \{2\}, \ldots, \{K\}\}\right\}$, where $\pi_{|[K]}$ is the restriction of the partition π of $\mathbb{N}$ to $\{1, \ldots, K\}$. This means that we keep only the atoms of N for which the reproductions events do not involve more than one of the first K levels.

Remark 2.1. Note that the particle system $\left(X_0^h(j), j \geq 1\right)$ is no more exchangeable due to the constraint on the K first levels. Nevertheless, the particle system $\left(X_0^h(j), j > K\right)$ is still exchangeable.

We also need the definition of the first level $L(t)$ at which the first K types appear:

$$L(t) = \inf\{i \geq K, \{1, \ldots, K\} \subset \{X_t(1), \ldots, X_t(i)\}\}, \tag{8}$$

with the convention that $\inf\{\varnothing\} = \infty$. The random variable $L(0)$ is finite if and only if $\prod_{i=1}^{K} R_0\{i\} > 0$, $\mathbb{P}$-a.s., thanks to de Finetti's theorem. The random variable $L(t)$ is $\mathcal{F}_t$ measurable, but not $\mathcal{G}_t$ measurable. This random variable counts the number of levels that we need for collecting the first K types: in that sense, it may be interpreted as an instance of the coupon collector problem. Viewed as a process, the collection of random variables $(L(t), t \geq 0)$ forms a Markov chain in continuous time with respect to its natural filtration. Its transition rates from i to j are a bit involved, except in the case $\nu = 0$, see the Section 2.3. The total jump rate from state i admits nevertheless a simple expression. It is equal to:

$$r_i = \frac{i(i-1)}{2} c + \int_{(0,1]} \nu(dx) \left(1 - (1-x)^i - ix(1-x)^{i-1}\right), \quad i \geq 1$$

and we shall call r_i the pushing rate at level i. Notice that $r_1 = 0$ and that r_i is finite for every $i \geq 1$ since $\int_{(0,1]} x^2 \, \nu(dx) < \infty$. From the construction of the lookdown particle system, the pushing rate r_i may be understood as the rate at which a type at level i is pushed up to higher levels (not necessarily $i+1$ if $\nu \neq 0$) by reproduction events at lower levels. Let us define a process $Q = (Q_t, t \geq 0)$ as follows:

$$Q_t = \frac{\mathbf{1}_{\{L(t)=K\}}}{\mathbb{P}(L(0) = K)} e^{r_K t}.$$

Lemma 2.2. *The process $Q = (Q_t, t \geq 0)$ is a non-negative $\mathcal{F}$-martingale, and*

$$\forall A \in \mathcal{F}_t, \quad \mathbb{P}(X^h \in A) = \mathbb{E}\left(\mathbf{1}_A(X) \, Q_t\right). \tag{9}$$

We need the following definition of the process:

$$M_t = \frac{\prod_{i=1}^{K} R_t\{i\}}{\mathbb{E}\left(\prod_{i=1}^{K} R_0\{i\}\right)} e^{r_K t}.$$

By projection on the smaller filtration $\mathcal{G}_t$, we deduce Lemma 2.3.

Lemma 2.3. *The process $M = (M_t, t \geq 0)$ is a non-negative $\mathcal{G}$-martingale.*

This fact allows to define the process $R^H = (R_t^H, t \geq 0)$ absolutely continuous with respect to $R = (R_t, t \geq 0)$ on each $\mathcal{G}_t, t \geq 0$, by the relation

$$\forall A \in \mathcal{G}_t, \quad \mathbb{P}(R^H \in A) = \mathbb{E}\left(\mathbf{1}_A(R) \, M_t\right). \tag{10}$$

The process R^H is the product type h-transform of interest. Intuitively, the ponderation by M favours the paths in which the first K types are present in equal proportion. Also notice that

Eq. (10) agrees with the definition of R_0^H. We shall deduce from Lemmas 2.2 and 2.3 the following theorem, which gives the pathwise construction of the h-transform R^H of R.

Theorem 2.4. *Let $1 \leq K \leq K'$. We have that:*

(a) *The limit of the empirical measure:*

$$R_t^h(dx) := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{X_t^h(n)}(dx)$$

exists a.s.

(b) *The process $(R_t^h, t \geq 0)$ is distributed as $(R_t^H, t \geq 0)$.*

Let us comment on these results. The process X^h is constructed by changing the initial condition and forgetting (as soon as $K \geq 2$) specific reproduction events in the lookdown particle system of X. Lemma 2.2 tells us that this procedure selects the configurations of X in which the first K levels are filled with the first K types at initial time without any "interaction" between these first K levels at a further time. Theorem 2.4 claims that the process R^h constructed in this way is an h-transform of R and, from Lemma 2.3, we have the following probabilistic interpretation of the Radon–Nikodym derivative in Eq. (10): the numerator is proportional to the probability that the first K levels are occupied by the first K types at time t, whereas the denominator is proportional to the probability that the first K levels are occupied by the first K types at time 0.

2.1.2. Proofs

Proof of Lemma 2.2. From the de Finetti theorem, conditionally on R_t, the random variables $(X_t(i), i \in \mathbb{N})$ are independent and identically distributed according to R_t. This implies that:

$$\mathbb{P}(L(t) = K | \mathcal{G}_t) = K! \prod_{i=1}^{K} R_t\{i\}. \tag{11}$$

In particular, we have:

$$\mathbb{P}(L(0) = K) = K! \, \mathbb{E}\left(\prod_{i=1}^{K} R_0\{i\}\right),$$

which, together with (7), ensures that Q_t is well defined.

Then, we define $W = \{\pi, \pi_{|[K]} = \{\{1\}, \{2\}, \dots, \{K\}\}\}$, and $V_t = \{(s, \pi), 0 \leq s \leq t, \pi \in W\}$, and also the set differences $W^c = \mathcal{P}_\infty \setminus W$ and $V_t^c = \{(s, \pi), 0 \leq s \leq t, \pi \in W^c\}$. We observe that:

- From the de Finetti Theorem, X_0^h defined in (i) is distributed as X_0 conditioned on $\{L(0) = K\}$.
- The law of the restriction of a Poisson point measure on a given subset is that of a Poisson point measure conditioned on having no atoms outside this subset: thus N conditioned on having no atoms in V_t^c (this event has positive probability) is the restriction of N to V_t.

Since the two conditionings are independent, we have, for $A \in \mathcal{F}_t$:

$$\mathbb{P}(X^h \in A) = \mathbb{P}(X \in A | \{L(0) = K\} \cap \{N(V_t^c) = 0\})$$

$$= \mathbb{E}\left(\mathbf{1}_A(X) \frac{\mathbf{1}_{\{L(0)=K\} \cap \{N(V_t^c)=0\}}}{\mathbb{P}(L(0) = K)\mathbb{P}(N(V_t^c) = 0)}\right). \tag{12}$$

We compute:

$$
\mu(W^c) = c\mu^k(W^c) + \int_{(0,1]} \nu(dx)\rho_x(W^c)
$$

$$
= c\,\frac{K(K-1)}{2} + \int_{(0,1]} \nu(dx)\left(1 - (1-x)^K - Kx(1-x)^{K-1}\right)
$$

$$
= r_K.
$$

This implies from the construction of N that:

$$
\mathbb{P}(N(V_t^c) = 0) = e^{-\mu(W^c)t} = e^{-r_K t}. \tag{13}
$$

Notice that

$$
\{L(t) = K\} = \{L(0) = K\} \cap \{N(V_t^c) = 0\}. \tag{14}
$$

From (12)–(14), we deduce that:

$$
\mathbb{P}(X^h \in A) = \mathbb{E}\left(\mathbf{1}_A(X)\frac{\mathbf{1}_{\{L(t)=K\}}}{\mathbb{P}(L(0) = K)}e^{r_K t}\right) = \mathbb{E}\left(\mathbf{1}_A(X)Q_t\right).
$$

Observe now that A also belongs to $\mathcal{F}_s$ as soon as $s \geq t$, which yields:

$$
\mathbb{P}(X^h \in A) = \mathbb{E}(\mathbf{1}_A(X)Q_s).
$$

Comparing the two last equalities ensures that $(Q_t, t \geq 0)$ is a $\mathcal{F}$-martingale. $\square$

Proof of Lemma 2.3. We know from Lemma 2.2 that $(Q_t, t \geq 0)$ is a $\mathcal{F}$-martingale. Since $\mathcal{G}_t \subset \mathcal{F}_t$ for every $t \geq 0$, we deduce that $(\mathbb{E}(Q_t|\mathcal{G}_t), t \geq 0)$ is a $\mathcal{G}$-martingale. But

$$
\mathbb{E}(Q_t|\mathcal{G}_t) = \mathbb{E}\left(\frac{\mathbf{1}_{\{L(t)=K\}}}{\mathbb{P}(L(0) = K)}e^{r_K t}\Big|\mathcal{G}_t\right) = \frac{\prod_{i=1}^{K} R_t\{i\}}{\mathbb{E}\left(\prod_{i=1}^{K} R_0\{i\}\right)}e^{r_K t} = M_t,
$$

using (11) for the second equality, so that $(M_t, t \geq 0)$ is a $\mathcal{G}$-martingale. $\square$

Proof of Theorem 2.4. From Lemma 2.2, X^h is absolutely continuous with respect to X on $\mathcal{F}_t$. The existence of the almost sure limit of the empirical measure claimed in point (a) follows from (1). We now project on $\mathcal{G}_t$ the absolute continuity relationship on $\mathcal{F}_t$ given in Lemma 2.3. Let $A \in \mathcal{G}_t$:

$$
\mathbb{P}(R^h \in A) = \mathbb{E}\left(\mathbf{1}_A(R)Q_t\right) = \mathbb{E}\left(\mathbf{1}_A(R)\mathbb{E}(Q_t|\mathcal{G}_t)\right) = \mathbb{E}\left(\mathbf{1}_A(R)M_t\right) = \mathbb{P}(R^H \in A),
$$

where we use Lemma 2.2 for the first equality and the definition of R^H for the last equality. This proves point (b). $\square$

2.2. The h-transform as a conditioned process

We gave in the previous subsection a pathwise construction of the h-transform R^H. We now study the conditioning associated with this h-transform.

Let $1 \leq K \leq K'$. Assumption (7) allows us to define a family of processes $R^{(\geq t)}$ on $\mathcal{G}$ by:

$$\forall A \in \mathcal{G}_t, \quad \mathbb{P}(R^{(\geq t)} \in A) = \mathbb{P}\left(R \in A \mid \prod_{i=1}^{K} R_t\{i\} \neq 0 \right),$$

and the associated particle system $X^{(\geq t)}$ on $\mathcal{F}$ by:

$$\forall A \in \mathcal{F}_t, \quad \mathbb{P}(X^{(\geq t)} \in A) = \mathbb{P}\left(X \in A \mid \prod_{i=1}^{K} R_t\{i\} \neq 0 \right).$$

The process $R^{(\geq t)}$ thus corresponds to the process R conditioned on coexistence of the *first K* types at time t. It is not easy to derive the probabilistic structure of the particle system $X^{(\geq t)}$ on all $\mathcal{F}_t$. Nevertheless, for a fixed $s \geq 0$, the probabilistic structure of $X^{(\geq t)}$ on the sigma algebra $\mathcal{F}_s$ simplifies as t goes to infinity, as shown by the following theorem, which may be seen as a generalization of Theorem 3.7.1.1 of Lambert [28]. The latter Theorem builds on the work of Kimura [24] and corresponds to the case $\nu^U = 0$. We use a new method, based on the lookdown particle system, which offers new probabilistic insights, described in the next Section 2.3. We need some further notations: We write $\mathbb{P}_i$ for the law of L (defined in (8)) conditionally on $\{L(0) = i\}$. For I an interval of $\mathbb{R}^+$ and F a process indexed by $\mathbb{R}^+$, we denote by F_I the restriction of F on the interval I.

Theorem 2.5. *Let $s \geq 0$ be fixed. Assume that*

$$\lim_{t \to \infty} \frac{\mathbb{P}_{K+1}(L(t) < \infty)}{\mathbb{P}_K(L(t) < \infty)} = 0. \tag{15}$$

Then:

(i) *The family of processes $X^{(\geq t)}_{[0,s]}$ weakly converges as $t \to \infty$ towards the process $X^h_{[0,s]}$.*

(ii) *The family of processes $R^{(\geq t)}_{[0,s]}$ weakly converges as $t \to \infty$ towards the process $R^h_{[0,s]}$.*

We refer to Lemma 2.9 for a sufficient condition for (15) to be satisfied, and notice that the case $K = 1$ corresponds to a non degenerate conditioning since the event $\{R_t\{1\} \neq 0$ for every $t\}$ has positive probability under (7).

Remark 2.6. The following property

$$\text{(CDI)} \quad \mathbb{P}(\inf\{t > 0, L(t) = \infty\} < \infty) = 1,$$

is independent of K used to define L in (8), as soon as $K \geq 2$. This property corresponds to the Coming Down from Infinity property for the Λ-coalescent associated with the GFV process R, whence the acronym (CDI). The key points to see this connection are:

- the fact that $L(0)$ is an upper bound on the number of blocks in the standard coalescent started at any time greater than $\inf\{t > 0, L(t) = \infty\}$ (and run backward in time).
- The 0–1 law of Pitman, see Proposition 23 of [35], according to which the number of blocks in a standard Λ-coalescent either stays infinite at every time $t \geq 0$ with probability 1, or is finite at each positive time $t > 0$ with probability 1.

We refer to Schweinsberg [40] for more details about the Coming Down from Infinity property. We conjecture this property is in fact equivalent to our assumption (15), but we were unable to prove it. See Remark 2.8 nevertheless.

Remark 2.7. It should still be possible to interpret the processes X^h and R^h as conditioned processes, without assuming (15). Our guess in that more general case is that X^h corresponds to X conditioned by the event $\{\limsup_{t\to\infty} \prod_{i=1}^K R_t\{i\} > 0\}$ (which has null probability as soon as $K \geq 2$).

Proof. First observation is that, from the de Finetti Theorem on exchangeable random partitions, we have: $\prod_{i=1}^K R_t\{i\} \neq 0$ if and only if $L(t) < \infty$, $\mathbb{P}$ a.s. This gives, for any $A \in \mathcal{F}_s$:

$$\mathbb{P}\left(A \Big| \prod_{i=1}^K R_t\{i\} \neq 0\right) = \frac{\mathbb{P}\left(A \cap \left\{\prod_{i=1}^K R_t\{i\} \neq 0\right\}\right)}{\mathbb{P}\left(\prod_{i=1}^K R_t\{i\} \neq 0\right)}$$

$$= \frac{\mathbb{P}\left(A \cap \{L(t) < \infty\}\right)}{\mathbb{P}(L(t) < \infty)}.$$

Now, using the Markov property, we have:

$$\mathbb{P}(A \cap \{L(t) < \infty\}) = \mathbb{P}(A \cap \{L(s) = K\} \cap \{L(t) < \infty\})$$
$$+ \mathbb{P}(A \cap \{L(s) \geq K+1\} \cap \{L(t) < \infty\})$$
$$= \mathbb{P}(A \cap \{L(s) = K\})\mathbb{P}_K(L(t-s) < \infty)$$
$$+ \mathbb{E}(\mathbf{1}_{A\cap\{L(s)\geq K+1\}}\mathbb{P}_{L(s)}(\tilde{L}(t-s) < \infty)),$$

where $\tilde{L}$ is an independent copy of L.

Let $\ell \in \mathbb{N}$. We can couple the processes L under $\mathbb{P}_\ell$ and L under $\mathbb{P}_{\ell+1}$ on the same lookdown graph by using the same reproduction events. Let us denote by $(L_\ell, L_{\ell+1})$ this coupling: L_ℓ is distributed as L under $\mathbb{P}_\ell$ and $L_{\ell+1}$ is distributed as L under $\mathbb{P}_{\ell+1}$. By the ordering by persistence property of the lookdown graph, we then have, for every $t \geq 0$:

$$L_\ell(t) \leq L_{\ell+1}(t),$$

whence:

$$\mathbb{P}_{\ell+1}(L(t) < \infty) \leq \mathbb{P}_\ell(L(t) < \infty) \tag{16}$$

for every integer ℓ. Therefore, we have:

$$\mathbb{E}(\mathbf{1}_{A\cap\{L(s)\geq K+1\}}\mathbb{P}_{L(s)}(\tilde{L}(t-s) < \infty)) \leq \mathbb{P}(A \cap \{L(s) \geq K+1\})$$
$$\times \mathbb{P}_{K+1}(L(t-s) < \infty).$$

Our assumption (15) now implies:

$$\frac{\mathbb{P}(A \cap \{L(t) < \infty\})}{\mathbb{P}_K(L(t-s) < \infty)} \underset{t\to\infty}{\to} \mathbb{P}(A \cap \{L(s) = K\}).$$

Setting $A = \Omega$, this also yields:

$$\frac{\mathbb{P}(L(t) < \infty)}{\mathbb{P}_K(L(t-s) < \infty)} \underset{t\to\infty}{\to} \mathbb{P}(L(s) = K).$$

Taking the ratio, we find that:

$$\frac{\mathbb{P}(A \cap \{L(t) < \infty\})}{\mathbb{P}(L(t) < \infty)} \underset{t\to\infty}{\to} \frac{\mathbb{P}(A \cap \{L(s) = K\})}{\mathbb{P}(L(s) = K)}.$$

We also have that $\mathbb{P}(L(s) = K) = \mathbb{P}(L(0) = K)\mathrm{e}^{-r_K s}$ since Q is a $\mathcal{G}$-martingale from Lemma 2.2. Altogether, we find that:

$$\lim_{t \to \infty} \mathbb{P}\left(A \mid \prod_{i=1}^{K} R_t\{i\} \neq 0\right) = \mathbb{E}\left(\mathbf{1}_A(X) \frac{\mathbf{1}_{\{L(s)=K\}}}{\mathbb{P}(L(0) = K)} \mathrm{e}^{r_K s}\right) = \mathbb{P}(X^h \in A)$$

where the last equality corresponds to Lemma 2.2. This implies the convergence in law of $X^{(\geq t)}$ towards X^h as $t \to \infty$, and proves (i). The proof of (ii) is similar to the one of (i). $\quad\square$

Remark 2.8. Having introduced in the previous proof the coupling (L_K, L_{K+1}), we may complete the Remark 2.6: It is possible to prove that, if (CDI) holds and for each $t \geq 0$,

$$(j \to \mathbb{P}(L_{K+1}(t) < \infty | L_K(t) \leq j)) \text{ is non increasing,}$$

then (15) holds. This monotonicity assumption is made plausible by the ordering by persistence property of the lookdown graph, see (16).

We now give a sufficient condition for (15) to be satisfied, which is then checked in the most interesting cases.

Lemma 2.9. *If* $\sum_{j \geq K} \frac{1}{r_j} < \infty$, *then* (15) *holds.*

Proof. A lower bound for $\mathbb{P}_K(L(t) < \infty)$ is easily found:

$$\mathrm{e}^{-r_K t} = \mathbb{P}_K(L(t) = K) \leq \mathbb{P}_K(L(t) < \infty). \tag{17}$$

We now look for an upper bound for $\mathbb{P}_{K+1}(L(t) < \infty)$. Recall the non decreasing pure jump process L jumps with intensity r_j when $L = j$. We may write, under $\mathbb{P}_{K+1}$:

$$\sup\{t, L(t) < \infty\} = \sum_{j \geq K+1} \tilde{T}_j$$

where, given the range $\{L(t), t \geq 0\} = \{L^{K+1}, L^{K+2}, \ldots\}$ (with $L^{K+1} < L^{K+2} < \cdots$) of the random function L, the sequence $(\tilde{T}_j, j \geq K + 1)$ is a sequence of independent exponential random variables with parameter r_{L^j}. Since $(r_j)_{j \geq K+1}$ forms an increasing sequence and the function L has jumps greater than or equal to 1, we have for each $j \geq K + 1$,

$$r_{L^j} \geq r_j. \tag{18}$$

Let $(T_j, j \geq K + 1)$ be a sequence of independent exponential random variables with parameter $(r_j, j \geq K)$. We have, for $0 < \lambda < r_{K+1}$:

$$\begin{aligned}
\mathbb{P}_{K+1}(L(t) < \infty) &= \mathbb{P}\left(\sum_{j \geq K+1} \tilde{T}_j > t\right) \\
&\leq \mathbb{P}\left(\sum_{j \geq K+1} T_j > t\right) \\
&= \mathbb{P}\left(\exp\left(\lambda \sum_{j \geq K+1} T_j\right) > \exp(\lambda t)\right) \\
&\leq \exp(-\lambda t) \, \mathbb{E}\left(\exp\left(\lambda \sum_{j \geq K+1} T_j\right)\right)
\end{aligned}$$

$$= \exp(-\lambda t) \prod_{j \geq K+1} \frac{r_j}{r_j - \lambda}$$

$$= \exp\left(-\lambda t + \sum_{j \geq K+1} \log\left(1 + \frac{\lambda}{r_j - \lambda}\right)\right)$$

$$\leq \exp\left(-\lambda t + \lambda \sum_{j \geq K+1} \frac{1}{r_j - \lambda}\right),$$

where we use (18) for the first inequality and the Markov inequality for the second inequality. By assumption, the sum $\sum_{j \geq K+1} 1/r_j$ is finite, which implies that $\sum_{j \geq K+1} 1/(r_j - \lambda)$ is finite. Taking $\lambda = (r_K + r_{K+1})/2$, we obtain that:

$$\mathbb{P}_{K+1}(L(t) < \infty) < C \exp\left(-\frac{r_K + r_{K+1}}{2} t\right) \tag{19}$$

for the finite constant $C = \exp \lambda \sum_{j \geq K} 1/(r_j - \lambda)$ associated with this choice of λ. Using (17) and (19), we have that:

$$0 \leq \frac{\mathbb{P}_{K+1}(L(t) < \infty)}{\mathbb{P}_K(L(t) < \infty)} \leq C \exp\left(-\frac{r_{K+1} - r_K}{2} t\right).$$

Letting t tend to ∞, we get the required limit.   □

The following corollary ensures that (15) is satisfied in the most interesting cases.

Corollary 2.10. *If either $c > 0$, either $c = 0$ and ν satisfies $\nu(dx) = f(x)dx$ with*

$$\liminf_{x \to 0} f(x)x^{\alpha+1} > 0 \quad \text{for some } 1 < \alpha < 2,$$

then (15) holds.

Remark 2.11. Notice that, for $1 < \alpha < 2$, the Beta$(2 - \alpha, \alpha)$ GFV process, for which $\nu(dx) = x^{-1-\alpha}(1-x)^{\alpha-1}\mathbf{1}_{(0,1)}(x)dx$, satisfies this assumption.

Proof. If $c > 0, r_j \geq cj(j-1)/2$, and thus $\sum_{j \geq K} 1/r_j < \infty$. Assume now $c = 0$ and $\liminf_{x \to 0} f(x)x^{\alpha+1} > 0$ for some $1 < \alpha < 2$. From Lemma 2 of Limic and Sturm [31], we have the equality:

$$r_{j+1} - r_j = \int_{(0,1]} j(1-x)^{j-1}x^2\nu(dx).$$

We deduce that there exists an integer n, and a positive constant C such that:

$$r_{j+1} - r_j > C \int_{(0,1/n]} j(1-x)^{j-1}x^{1-\alpha}dx \geq \frac{C}{n} \int_{(0,1]} j(1-x)^{j-1}x^{1-\alpha}dx$$

$$= \frac{C}{n} j \operatorname{Beta}(2 - \alpha, j + \alpha - 1),$$

using the definition of lim inf at the first inequality, and the fact that the map $x \mapsto (1-x)^{j-1}x^{1-\alpha}$ is non-increasing at the second inequality. Let us define a sequence $(s_j, j \geq K)$ by:

$$s_K = 0 \quad \text{and} \quad s_{j+1} - s_j = \frac{C}{n} j \operatorname{Beta}(2 - \alpha, j + \alpha - 1) \quad \text{for } j \geq K.$$

Since:

$$\text{Beta}(2 - \alpha, j + \alpha - 1) \underset{j \to \infty}{\sim} \Gamma(2 - \alpha) j^{\alpha - 2},$$

we deduce that:

$$s_j \underset{j \to \infty}{\sim} \frac{C}{n} \Gamma(2 - \alpha) j^\alpha / \alpha.$$

By definition of the sequence $(s_j)_{j \geq K}$, we have the inequality $r_j \geq s_j$ for $j \geq K$, and we deduce that

$$\sum_{j \geq K} 1/r_j \leq \sum_{j \geq K} 1/s_j < \infty.$$

Lemma 2.9 allows to conclude that (15) holds in both cases. $\square$

2.3. The immigration interpretation

We develop further the two following examples:

(i) $K = K' = 2$: this amounts (provided condition (15) is satisfied) on conditioning a two-type GFV process on coexistence of each type.
(ii) $1 = K < K' = 2$: this amounts (provided (15) is satisfied) on conditioning a two-type GFV process on absorption by the first type.

We regard the $K (=1 \text{ or } 2)$ first level particles in X^h as K external sources of immigration in a population assimilated to the particle system $(X^h(n), n \geq K + 1)$. We deduce from the construction of the particle system X^h a transparent derivation of the generator of the GFV process conditioned on non-absorption or absorption by some given type. We refer to Foucart [21] for a study of GFV processes with one source of immigration ($K = 1$ here).

Since $K' = 2$, the resulting probability measure-valued process $R = (R_t, t \geq 0)$ and $R^h = (R_t^h, t \geq 0)$ on $\{1, 2\}$ may be described by the $[0, 1]$-valued processes $R\{1\} = (R_t\{1\}, t \geq 0)$ and $R^h\{1\} = (R_t^h\{1\}, t \geq 0)$ respectively. For the sake of simplicity, we will just write R for $R\{1\}$ and R^h for $R^h\{1\}$ respectively. We recall that the infinitesimal generator of R is given by:

$$Gf(x) = \frac{1}{2} cx(1 - x) f''(x) + x \int_{(0,1]} \nu(dy)[f(x(1 - y) + y) - f(x)]$$
$$+ (1 - x) \int_{(0,1]} \nu(dy)[f(x(1 - y)) - f(x)]$$

for all $f \in \mathcal{C}^2([0, 1])$, the space of twice differentiable functions with continuous derivatives, and $x \in [0, 1]$, see Bertoin and Le Gall [6].

2.3.1. We assume $K = K' = 2$

Results of Section 2 allow us to compute the generator of the conditioned process in two different ways. We define, for $f \in \mathcal{C}^2([0, 1])$, and $x \in [0, 1]$:

$$G^0 f(x) = c(1 - 2x) f'(x) + \int_{(0,1]} y(1 - y)\nu(dy)[f(x(1 - y) + y) - f(x)]$$
$$+ \int_{(0,1]} y(1 - y)\nu(dy)[f(x(1 - y)) - f(x)],$$

and

$$G^1 f(x) = \frac{1}{2}cx(1-x)f''(x) + x \int_{(0,1]} (1-y)^2 \nu(dy)[f(x(1-y)+y) - f(x)]$$
$$+ (1-x) \int_{(0,1]} (1-y)^2 \nu(dy)[f(x(1-y)) - f(x)].$$

Proposition 2.12. *Assume $K = K' = 2$. The operator $G^0 + G^1$ is a generator for R^h.*

Remark 2.13. When the measure ν is null, the process R is called a Wright–Fisher (WF) diffusion. In that case, the process R^h may be seen as a WF diffusion with immigration, where the two first level particles induce continuous immigration (according to G^0) of both types 1 and 2 in the original population (evolving according to $G^1 = G$).

When the measure ν is not null, the process R^h is a GFV process with immigration, but the generator G^1 is *no more* that of the initial GFV process G. The two first level particles induce both continuous and discontinuous immigration (according to G^0) of types 1 and 2 in a population with a *reduced* reproduction (the measure $\nu(dy)$ is ponderated by a factor $(1-y)^2 \leq 1$ in G^1).

Proof. Let us denote by G^h the generator of R^h. The process R^h is the Doob h-transform of R for the space–time harmonic function $h(x)e^{r_2 t}$, where:

$$h(x) = x(1-x).$$

From the definition of the generator, for $f \in \mathcal{C}^2([0, 1])$, and $x \in [0, 1]$:

$$(hf)(R_t)e^{r_2 t} - (hf)(R_0) - \int_0^t ds \; G(hf)(R_s)e^{r_2 s} - \int_0^t ds \; r_2 \; (hf)(R_s)e^{r_2 s}$$

is a $\mathcal{G}$-martingale. Therefore, on the event $\{h(R_0) \neq 0\}$, the process

$$\frac{h(R_t)e^{r_2 t}}{h(R_0)} f(R_t) - f(R_0) - \int_0^t ds \; \frac{h(R_s)e^{r_2 s}}{h(R_0)} \frac{G(hf)(R_s)}{h(R_s)} - \int_0^t ds \; \frac{h(R_s)e^{r_2 s}}{h(R_0)} r_2 \; f(R_s)$$

is again a $\mathcal{G}$-martingale under $\mathbb{P}$. This implies that:

$$f(R_t^h) - f(R_0^h) - \int_0^t ds \; \frac{G(hf)(R_s^h)}{h(R_s^h)} - \int_0^t ds \; r_2 \; f(R_s^h)$$

is a $\mathcal{G}$-martingale under $\mathbb{P}$. We thus have:

$$G^h f(x) = \frac{G(hf)}{h}(x) + r_2 \; f(x). \tag{20}$$

In the case $\nu = 0$, we have

$$Gf(x) = \frac{1}{2}c \; x(1-x)f''(x) = \frac{1}{2}c \; h(x)f''(x),$$

and the Eq. (20) reduces to:

$$G^h f(x) = Gf(x) + c(1-2x)f'(x).$$

The general case $\nu \neq 0$ consists in a longer but straightforward calculus, left to the reader. $\quad\square$

Using the particle system X^h, we also have the following intuitive interpretation of the generator G^h in the case of a pure jump GFV process ($c = 0$). Let us decompose the measure ν as follows:

$$\nu(dy) = 2y(1 - y)\nu(dy) + (1 - y)^2\nu(dy) + y^2\nu(dy).$$

(1) The first term is the sum of the two measures $y(1 - y)\nu(dy)$ appearing in each integrand of the generator G^0 and each of these measures corresponds to the intensity of the reproduction events involving level 1 and not level 2, or level 2 and not level 1 (these events have probability $y(1 - y)$ when the reproduction involves a fraction y of the population). We interpret them as immigration events.
(2) The second term is the measure $(1 - y)^2\nu(dy)$ appearing in the generator G^1 and corresponds to the intensity of the reproduction events involving neither level 1 nor level 2 (this event has probability $(1 - y)^2$ when the reproduction involves a fraction y of the population). We interpret them as reproduction events.
(3) The third term does not appear in the generators G^0 and G^1: it corresponds to the intensity of the reproduction events involving both level 1 and 2, and these events have been discarded in the construction of X^h.

2.3.2. We assume $K = 1$, $K' = 2$

Note that the case $K = 1$ differs from the case $K = 2$, since the event $\{R_t \neq 0$ for every $t\}$ has positive probability under (7). Let us define, for $f \in C^2([0, 1])$, and $x \in [0, 1]$:

$$I^0 f(x) = c(1 - x)f'(x) + \int_{(0,1]} y\nu(dy)[f(x(1 - y) + y) - f(x)]$$

and

$$I^1 f(x) = \frac{1}{2}cx(1 - x)f''(x) + x\int_{(0,1]} (1 - y)\nu(dy)[f(x(1 - y) + y) - f(x)]$$
$$+ (1 - x)\int_{(0,1]} (1 - y)\nu(dy)[f(x(1 - y)) - f(x)].$$

We can then prove the analogue of Proposition 2.12 in that setting.

Proposition 2.14. *Assume $K = 1$, $K' = 2$. The operator $I^0 + I^1$ is a generator for the Markov process R^h.*

In particular, we recover the well-known fact that a WF diffusion conditioned on fixation at 1 (that is, $R_t = 1$ for t large enough) may be viewed as a WF process with immigration, see [14] for instance.

Proof. The proof is similar to that of Proposition 2.12. Here we use an h-transform with the function $h(x) = x$. This function is harmonic according to Lemma 2.3 (recall $r_1 = 0$). □

Here again, we have the following intuitive interpretation of the generator $I^0 + I^1$ in the case $c = 0$. We now decompose the measure ν as follows:

$$\nu(dy) = y\nu(dy) + (1 - y)\nu(dy).$$

(1) The first term is the measure $y\nu(dy)$ appearing in the generator I^0. This is the intensity of the reproduction events involving level 1 particle. We interpret them as immigration events.

(2) The second term is the measure $(1 - y)\nu(dy)$ appearing in the generator I^1. This is the intensity of the reproduction events not involving level 1 particle. We interpret them as reproduction events.

Summing the two measures $y\nu(dy)$ and $(1 - y)\nu(dy)$, we recover this time the full measure $\nu(dy)$ since no reproduction events are discarded in the case $K = 1$.

2.3.3. Intertwining

This subsection starts with a little digression on intertwining theory following the original idea Rogers and Pitman [37]. The link with the paper is then carefully explained.

Let us recall the following piece of intertwining theory. Given a Markov process $((A_t, B_t), t \geq 0)$, or more precisely its generator, we ask whether $(A_t, t \geq 0)$ is a Markov process in its own filtration and, in that case, what is his generator. The following theorem answers by the affirmative under the algebraic relationship (21), that we shall call the intertwining relationship. This Theorem is an adaptation (formulated in terms of the infinitesimal generator) of the original one of Rogers and Pitman [37], see also Fill [20] and Athreya and Swart [1]. In view of the application we have in mind, we present it in the case where B is a process with values in a discrete state space T.

Theorem 2.15. *Let* $((A_t, B_t), t \geq 0)$ *be a Markov process with state space* $S \times T$, *and with generator* $\hat{G}$, *let* K *be a probability kernel from* S *to* T. *Define the operator* $\hat{K}$ *by*

$$\hat{K} f(x) = \sum_{y \in T} K(x, y) f(x, y).$$

Let G *be the generator of a Markov process in* S *and assume that, for each* $f : S \times T \to \mathbb{R}$ *in the domain of* $\hat{G}$,

$$\hat{K}\hat{G}(f)(x) = G\hat{K}(f)(x), \quad \text{for each } x \in S. \tag{21}$$

Then:

$$\mathbb{P}(B_0 = y | A_0) = K(A_0, y) \quad a.s.$$

implies that for each $t \geq 0$

$$\mathbb{P}(B_t = y | (A_s, 0 \leq s \leq t)) = K(A_t, y) \quad a.s.$$

and $(A_t, t \geq 0)$ *is a Markov process on* S *in its own filtration, with generator* G.

We assume $K' = 2$ and $\nu = 0$ (for the sake of simplicity). We denote by $\hat{G}$ the generator defined for $\ell \in \mathbb{N}$ and $x \in [0, 1]$ by:

$$\hat{G} f(x, \ell) = \frac{1}{2} cx(1 - x)\partial_{xx} f(x, \ell) + c \left[(1 - x) - (\ell - 1)x\right] \partial_x f(x, \ell)$$
$$+ c \frac{\ell(\ell - 1)}{2} \left[f(x, \ell + 1) - f(x, \ell)\right].$$

This generator acts on functions f such that f, as a function of x, belongs to $\mathcal{C}^2([0, 1])$. The kernel K is defined by:

$$K(x, \ell) = (1 - x)^{\ell - 1} x, \quad x \in (0, 1), \ \ell \in \mathbb{N}.$$

Last, we slightly abuse of notation by still denoting G the generator of the Wright–Fisher diffusion:

$$Gf(x) = \frac{1}{2}cx(1-x)f''(x)$$

acting on $f \in \mathcal{C}^2([0,1])$. Our intertwining relationship reads as follows.

Proposition 2.16. *Let f be in the domain of $\hat{G}$ and $x \in (0,1)$. The kernel K intertwins the generators G and $\hat{G}$ in the sense that:*

$$\hat{K}\hat{G}(f)(x) = G\hat{K}(f)(x).$$

The proof consists in a long but simple calculation and is eluded. A similar intertwining relation also holds for $\nu \neq 0$, but the generator $\hat{G}$ is then more complicated. We deduce from the Rogers–Pitman Theorem 2.15 and Proposition 2.16 that the first coordinate of the Markov process with generator $\hat{G}$ is a Markov process on its own, with generator G. This gives a pathwise decomposition of the Wright–Fisher diffusion.

This decomposition may also be read directly from the lookdown particle system, as we now explain. Nevertheless, it seemed worth to us to recast this decomposition, which may be seen as another instance of the so-called "backbone" decompositions, see Kyprianou et al. [3] in the branching setting, in the framework of the intertwining theory.

We introduce

$$L^1(t) = \inf\{i \geq 1, 1 \in \{X_t(1), \ldots, X_t(i)\}\}$$

the first level occupied by a type 1 particle in the lookdown particle system associated to a WF diffusion R. Notice that the process R^h studied in Section 2.3.2 is the WF diffusion R conditioned on $\{L^1 = 1\}$.

We now claim that the generator $\hat{G}$ is the generator of (R, L^1) up to the hitting time of 0 by R. Let us explain why. The process L^1 is from the construction of the lookdown particle system a Markov process in its own filtration and jumps from ℓ to $\ell + 1$ at rate $c\ell(\ell-1)/2$. Then, conditionally on the value of $L^1 = \ell$, we view the ℓ first particles as ℓ sources of immigration: $\ell - 1$ sources of type 2 and 1 source of type 1, whence the drift term $c[(1-x) - (\ell-1)x]$ (this may be checked thanks to similar calculations as in 2.3.1). We may sum up as follows the resulting pathwise decomposition of the WF diffusion:

- Conditionally on $\{R_0 = x\}$, $x > 0$, the initial value $L^1(0)$ has law:
  $$\mathbb{P}(L^1(0) = \ell) = (1-x)^{\ell-1}x = K(x, \ell), \quad \ell \geq 1.$$

- Conditionally on $(R_0, L^1(0))$, the process L^1 is a pure jump Markov process, which jumps from ℓ to $\ell + 1$ at rate $c\ell(\ell-1)/2$ if $\ell < \infty$, and has $+\infty$ as an absorbing point.
- Conditionally on (R_0, L^1), the process R is a Wright–Fisher diffusion with immigration, with generator given by:
  $$\frac{1}{2}cx(1-x)f''(x) + \mathbf{1}_{\{L^1 < \infty\}}c\left[(1-x) - (L^1-1)x\right]f'(x).$$

  These two last points are encoded in the definition of the generator $\hat{G}$.
- Among the conclusions of the Rogers–Pitman Theorem is the fact that the conditional law of L^1 given R is constant as time evolves, given by the kernel K: let us notice that, in our specific framework, this is a simple consequence of the exchangeability of the particles in the lookdown model; de Finetti theorem in fact ensures that the types of the particles are i.i.d., equal to 1 with probability R_t at time t.

3. The additive *h*-transform

In this section, we derive another example of *h*-transform of measure-valued processes, which admits a simple construction in terms of the lookdown particle system. Unlike the first example of *h*-transform, the additive *h*-transform applies to measure-valued processes with spatial motion.

3.1. The additive h-transform

Let $(Z_t, t \geq 0)$ be the measure-valued process constructed in Section 1.3. We call a non-negative function H on $[0, \infty) \times \mathcal{M}_f$ a space–time harmonic function for $\mathbb{P}$ when the process $(H(t, Z_t), t \geq 0)$ is a martingale under $\mathbb{P}$. The *h*-transform Z^H of Z associated with H is then defined by:

$$\forall A \in \mathcal{G}_t, \quad \mathbb{P}(Z^H \in A) = \mathbb{E}\left(\frac{H(t, Z_t)}{\mathbb{E}(H(0, Z_0))} \mathbf{1}_A(Z) \right) \tag{22}$$

for every $t \geq 0$. Furthermore, an *h*-transform is called additive if there exists a non-negative function $(h_t(x), t \geq 0, x \in E)$ such that $H(t, Z_t) = Z_t(h_t) := \int_E h_t(x) Z_t(dx)$. An additive *h*-transform intuitively favours the paths for which the population (represented by the measure-valued process Z) is large where h is large.

3.1.1. Statement of the results

Let ξ be the canonical process under P_x. We assume there exists a deterministic positive function m such that $(Y_t/m(t), t \geq 0)$ and $(m(t)h_t(\xi_t), t \geq 0)$ are martingales in their own filtrations. We also assume from now on that

$$\mathbb{E}(Y_0 R_0(h_0)) > 0.$$

Under this assumption, we define (the law of) a new process

$$(Y^h, U^h, X^h)$$

by the following requirements:

(i) The initial condition satisfies:

$$\forall A \in \mathcal{G}_t, \quad \mathbb{P}((Y_0^h, R_0^h) \in A) = \mathbb{E}\left(\frac{Y_0 R_0(h_0)}{\mathbb{E}(Y_0 R_0(h_0))} \mathbf{1}_A(Y_0, R_0) \right).$$

(ii) Conditionally on (Y_0^h, R_0^h), and provided $R_0^h(h_0) > 0$, $X_0^h(1)$ is distributed according to:

$$\forall A \in \mathcal{D}_0, \quad \mathbb{P}(X_0^h(1) \in A | R_0^h = \mu) = \mathbb{E}\left(\frac{h_0(X_0(1))}{\mu(h_0)} \mathbf{1}_A(X_0(1)) | R_0 = \mu \right),$$

and $(X_0^h(n), n \geq 2)$ is a random sequence with de Finetti's measure R_0^h.

(iii) Conditionally on $(Y_0^h, R_0^h, X_0^h(1))$, the process (Y^h, U^h) is distributed according to:

$$\forall A \in \mathcal{G}_t, \quad \mathbb{P}((Y^h, U^h) \in A | Y_0^h = x) = \mathbb{E}\left(\frac{Y_t}{x} \frac{m(0)}{m(t)} \mathbf{1}_A(Y, U) | Y_0 = x \right). \tag{23}$$

(iv) Conditionally on $(Y^h, U^h, R_0^h, X_0^h(1))$, $X^h(1)$ is distributed according to:

$$\forall A \in \mathcal{D}_t, \quad \mathbb{P}(X^h(1) \in A | X_0^h(1) = x)$$

$$= \mathbb{E}\left(\frac{h_t(X_t(1))}{h_0(x)} \frac{m(t)}{m(0)} \mathbf{1}_A(X(1)) g | X_0(1) = x \right). \tag{24}$$

(v) The rest of the definition of X^h is the same as the one given for X, namely:
  - for $n \geq 2$, between the reproduction events, the type $X_t^h(n)$ of the particle at level n mutates according to a Markov process in E with law $(P_x, x \in E)$ when started at $x \in E$, independently for each n.
  - at each atom (t, π) of $N = N^k + N^\rho$, with N^k and N^ρ derived from U^h and Y^h, a reproduction event is associated as previously.

Notice that the law of the initial condition Z_0^h specified by (i) differs from that of Z_0 only for random Z_0. Also, items (iii) and (iv) are meaningful since both $(Y_t/m(t), t \geq 0)$ and $(m(t)h_t (X_t(1)), t \geq 0)$ are assumed to be martingales. Last, observe from (23) that

$$\mathbb{P}(Y_t^h = 0) = 0$$

for each $t \geq 0$, which implies $\mathbb{P}(\tau(Y^h) = \infty) = 1$ since 0 is assumed to be absorbing. We will assume that (Y, U, X) and (Y^h, U^h, X^h) are defined on a common probability space with probability measure $\mathbb{P}$, and denote the expectation by $\mathbb{E}$.

Let us define a process $S = (S_t, t \geq 0)$ by:

$$S_t = \frac{h_t(X_t(1)) \, Y_t}{\mathbb{E}(Z_0(h_0))}.$$

Lemma 3.1. *The process $(S = S_t, t \geq 0)$ is a non-negative $\mathcal{F}$-martingale, and*

$$\forall A \in \mathcal{F}_t, \quad \mathbb{P}(X^h \in A) = \mathbb{E}(\mathbf{1}_A(X) \, S_t). \tag{25}$$

We then define a process T by setting:

$$T_t = \frac{Z_t(h_t)}{\mathbb{E}(Z_0(h_0))}.$$

Using Lemma 3.1, and projecting on the filtration $\mathcal{G}_t$, we deduce Lemma 3.2.

Lemma 3.2. *The process $T = (T_t, t \geq 0)$ is a non-negative $\mathcal{G}$-martingale.*

This fact allows to define the process $Z^H := (Z_t^H, t \geq 0)$ absolutely continuous with respect to $Z := (Z_t, t \geq 0)$ on each $\mathcal{G}_t, t \geq 0$:

$$\forall A \in \mathcal{G}_t, \quad \mathbb{P}(Z^H \in A) = \mathbb{E}(\mathbf{1}_A(Z) \, T_t).$$

We deduce from Lemmas 3.1 and 3.2 the following theorem, which gives a pathwise construction of the additive h-transform.

Theorem 3.3. *We have that:*

(a) *The limit of the empirical measure:*

$$R_t^h(dx) := \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \delta_{X_t^h(n)}(dx)$$

  exists a.s.

(b) *The process $(Z_t^h := Y_t^h R_t^h, t \geq 0)$ is distributed as $(Z_t^H, t \geq 0)$.*

We may interpret Theorem 3.3 as follows. The effect of the additive h-transform factorizes in two parts, according to the decomposition of the Radon–Nikodym derivative:

$$Z_t(h_t) = Y_t \, R_t(h_t).$$

The first term Y_t induces a size bias of the total population size $Z^h(\mathbf{1}) = Y^h$, see formula (23), whereas the second term $R_t(h_t)$ forces the first level particle to follow an h-transform of P, see formula (24). The sequence $(X_t^h(n), n \in \mathbb{N})$ is no more exchangeable in general contrary to the initial sequence $(X_t(n), n \in \mathbb{N})$. The following proposition shows that, loosely speaking, the first level particle is precursory.

Proposition 3.4. *Conditionally on* $\{R_t^h = \mu\}$, $X_t^h(1)$ *is distributed according to:*

$$\mathbb{P}(X_t^h(1) \in dx) = \frac{h_t(x)}{\mu(h_t)} \mu(dx),$$

and $(X_t^h(n))_{n \geq 2}$ *is an independent exchangeable random sequence with de Finetti's measure* μ.

3.1.2. Proofs

Proof of Lemma 3.1. It is enough to observe that, by construction, the law of (Y^h, U^h, X^h) is absolutely continuous with respect to the law of (Y, U, X) on $\mathcal{F}_t$, and:

$$\forall A \in \mathcal{F}_t, \quad \mathbb{P}((Y^h, U^h, X^h) \in A)$$
$$= \mathbb{E}\left(\frac{Y_0 R_0(h_0)}{\mathbb{E}(Y_0 R_0(h_0))} \frac{h_0(X_0^h(1))}{R_0(h_0)} \frac{Y_t}{Y_0} \frac{m(0)}{m(t)} \frac{h_t(X_t(1))}{h_0(X_0^h(1))} \frac{m(t)}{m(0)} \mathbf{1}_A(Y, U, X) \right)$$
$$= \mathbb{E}\left(\frac{Y_t \, h_t(X_t(1))}{\mathbb{E}(Z_0(h_0))} \mathbf{1}_A(Y, U, X) \right).$$

This also yields (the obvious fact) that $(S_t, t \geq 0)$ is a $\mathcal{F}$-martingale, arguing as in the proof of Lemma 2.2. □

Proof of Lemma 3.2. Since $\mathcal{G}_t \subset \mathcal{F}_t$ and S is a $\mathcal{F}$-martingale, the projection $\mathbb{E}(S_t | \mathcal{G}_t)$ is a $\mathcal{G}$-martingale. We also have that:

$$\mathbb{E}(S_t | \mathcal{G}_t) = \mathbb{E}\left(\frac{Y_t \, h_t(X_t(1))}{\mathbb{E}(Z_0(h_0))} | \mathcal{G}_t \right) = \frac{Z_t(h_t)}{\mathbb{E}(Z_0(h_0))} = T_t,$$

where we used that $X_t(1)$ has law R_t conditionally on $\mathcal{G}_t$ at the third equality. Thus $(T_t, t \geq 0)$ is a $\mathcal{G}$-martingale. □

Proof of Theorem 3.3. From Lemma 3.1, the law of X^h is absolutely continuous with respect to the law of X. The existence of the a.s. limit of the empirical measure of X^h follows from that of X (but not the exchangeability of the sequence) and yields point (a). We prove point (b) now. Take $A \in \mathcal{G}_t$.

$$\mathbb{P}(Z^h \in A) = \mathbb{E}\left(S_t \mathbf{1}_A(Z)\right)$$
$$= \mathbb{E}\left(\mathbb{E}\left(S_t | \mathcal{G}_t\right) \mathbf{1}_A(Z)\right)$$
$$= \mathbb{P}(T_t \, \mathbf{1}_A(Z))$$
$$= \mathbb{P}(Z^H \in A),$$

where we use Lemma 3.2 at the third equality and the definition of Z^H at the fourth. □

Proof of Proposition 3.4. Let $n \in \mathbb{N}$ be fixed, and let $(\phi_i, 1 \le i \le n)$ be a collection of bounded and measurable functions on E.

$$
\mathbb{E}\left(\prod_{1 \le i \le n} \phi_i(X_i^h(t))\right) = \mathbb{E}\left(\frac{Y_t \, h_t(X_t(1))}{E(Z_0(h_0))} \prod_{1 \le i \le n} \phi_i(X_t(i))\right)
$$

$$
= \frac{1}{\mathbb{E}(Z_0(h_0))}\mathbb{E}\left(Y_t \, \mathbb{E}\left(h_t(X_t(1))\phi_1(X_t(1)) \prod_{2 \le i \le n} \phi_i(X_t(i))|\mathcal{G}_t\right)\right)
$$

$$
= \frac{1}{\mathbb{E}(Z_0(h_0))}\mathbb{E}\left(Y_t \, R_t(h_t \, \phi_1) \prod_{2 \le i \le n} R_t(\phi_i)\right)
$$

$$
= \mathbb{E}\left(\frac{Z_t(h_t)}{\mathbb{E}(Z_0(h_0))} \, R_t\left(\frac{h_t \, \phi_1}{R_t(h_t)}\right) \prod_{2 \le i \le n} R_t(\phi_i)\right)
$$

$$
= \mathbb{E}\left(R_t^h\left(\frac{h_t \, \phi_1}{R_t^h(h_t)}\right) \prod_{2 \le i \le n} R_t^h(\phi_i)\right),
$$

where we use Lemma 3.1 at the first equality, the de Finetti Theorem at the third equality, and Theorem 3.3 at the last equality. Since functions of the type $\prod_{1 \le i \le n} \phi_i$ characterize the law of n-uple, this proves the proposition. $\square$

3.2. Applications

Overbeck investigated in [34] h-transform of measure-valued diffusions, among which the Dawson–Watanabe process (with quadratic branching mechanism) and the Fleming–Viot (FV) process (which is the GFV process for $\nu = \nu^U = 0$) using a martingale problem approach. He also provided a pathwise construction in the first case, see [33]. We shall see in this last section how Theorem 3.3 applies in both cases and sheds new light on Overbeck's results.

3.2.1. Generalized Fleming–Viot processes

Since $Y_t = 1$, Y is a martingale and we may apply results of Section 3.1.1 for any non-negative space–time harmonic function $(h_t(x), t \ge 0, x \in E)$ for the spatial motion P, that is any function such that $(h_t(\xi_t), t \ge 0)$ is a non-negative martingale where ξ stands for the canonical process under P.

First, we notice that we may use the same lookdown graph (that is the same reproduction events) for both the GFV process and its additive h-transform according to (23) and the fact that:

$$
(U^h, Y^h) \overset{\text{(law)}}{=} (U, Y) = (U, \mathbf{1}).
$$

Secondly, regarding the spatial motion of the particles, Overbeck suggested in [34] that in the particular case of the FV process, an additive h-transform looks like a FV process where "*the gene type of at least one family mutates as an h-transform of the one particle motion*". This suggestion was made "*plausible*" by similar results known for superprocesses, see [33] and the next subsection, and a well-known connection between quadratic Dawson–Watanabe superprocesses and FV processes which goes back to Shiga [41]. We did not attempt to follow this line of reasoning since the connection between superprocesses and GFV processes is restricted to stable superprocesses and Beta-GFV processes, see Birkner et al. [9]. Theorem 3.3 confirms (and generalizes to GFV processes) the suggestion and ensures that the family which "*mutates as*

an h-transform" is the family generated by the first level particle in the lookdown process, which is actually the only lineage with a perturbed spatial motion. Proposition 3.4 also gives the law of the position $X_t^h(1)$ of the first particle conditionally on the (*h*-transformed) GFV process R_t^h: we stress that, contrary to the following particles $X_t^h(2), X_t^h(3), \ldots$, this particle is not distributed according to R_t^h. To sum up, the spatial motion is added on the lookdown graph independently for each level, but the law now differs according to the level: the first level particle follows the path of an *h*-transformed spatial motion, whereas the particles at the subsequent levels follow the path of the original spatial motion.

Third, we may also interpret the *h*-transform as a conditioned process. For fixed $s \geq 0$, the additive *h*-transform of the GFV process on $[0, s]$ may be obtained by conditioning a random particle chosen at time t, t large, to move as an *h*-transform.

Remark 3.5. In the case of the FV process, the truncated processes obtained by considering the first N particles:

$$Z_t^N(dx) := \frac{1}{N} \sum_{1 \leq n \leq N} \delta_{X_t(n)}(dx) \quad \text{and} \quad Z_t^{N,h}(dx) := \frac{1}{N} \sum_{1 \leq n \leq N} \delta_{X_t^h(n)}(dx)$$

correspond respectively to the Moran model with N particles (see [12]) and its additive *h*-transform. Therefore, our approach is robust, in the sense that we can also consider discrete population.

3.2.2. The Dawson–Watanabe superprocess

Let $(Y_t, t \geq 0)$ be a CB(ψ) process. We assume that $\psi'(0+) > -\infty$, so that the (necessarily conservative) CB(ψ) has integrable marginals. Under this assumption, the process $(Y_t e^{\psi'(0+)t}, t \geq 0)$ is a martingale. Notice also that $\psi'(0+) < \infty$ since ψ a convex function. If $(h_t(\xi_t) e^{-\psi'(0+)t}, t \geq 0)$ is a martingale, Theorem 3.3 applies and yields a description of the additive *h*-transform associated to the non-negative space–time harmonic function $(h_t(x), t \geq 0, x \in E)$, as given by (22). According to this theorem, performing the additive *h*-transform has two effects: the first level particle follows an *h*-transform of P, as in the GFV setting, but also the total population Y^h is size-biased with respect to the original one Y, according to (23). We shall now concentrate on this second effect, and explain how a "spinal" decomposition may be partly recovered.

Let $\phi(\lambda)$ be the Laplace exponent of a subordinator. Recall a continuous state branching process with immigration with branching mechanism $\psi(\lambda)$ and immigration mechanism $\phi(\lambda)$, CBI(ψ, ϕ) for short, is a strong Markov process $(Y_t^i, t \geq 0)$ characterized by the Laplace transform:

$$\mathbb{E}(e^{-\lambda Y_t^i} | Y_0^i = x) = e^{-xu(\lambda,t) - \int_0^t ds\, \phi(u(\lambda,s))}.$$

We recall for the ease of reference the following well-known lemma. The proof is classical and relies on computation of the Laplace transforms.

Lemma 3.6. *The process* Y^h *defined by* (23) *with* $m(t) = e^{-\psi'(0+)t}$ *is a* CBI(ψ, ϕ) *with immigration mechanism given by* $\phi(\lambda) = \psi'(\lambda) - \psi'(0+)$.

Notice that in the case where the CB process Y extincts almost surely, the CBI process Y^h may also be interpreted as the CB process Y conditioned on non-extinction in remote time, see

Lambert [27]. The total mass process $Y^h = Z^h(\mathbf{1})$ is thus a CBI process. Next question is to identify the source of the immigration in the population represented by the particle system X^h. The following lemma identifies the offsprings of the first level particle as the immigrants when $c = 0$. The general case $c \neq 0$ is treated in the following Remark 3.8. Recall j_1 refers to the first level sampled in the lookdown construction. Let us denote $j_1(s)$ instead of j_1 for indicating the dependence in s.

Lemma 3.7. *The process* $\left(\sum_{0 \le s \le t} \Delta Y_s^h \, \mathbf{1}_{\{j_1(s)=1\}}, t \ge 0 \right)$ *is a pure jump subordinator with Lévy measure* $u \nu^Y(du)$.

Proof. By assumption, the process Y is a CB(ψ) and from Lemma 3.6, Y^h is a CBI(ψ, ϕ). From the Poissonian construction of CBI, we have that the point measure

$$\sum_{0 \le s \le t} \delta_{(s, \Delta Y_s^h)}(ds, du)$$

has for predictable compensator

$$ds(Y_{s-}^h \nu^Y(du) + u \nu^Y(du)).$$

The expression of the compensator may be explained as follows. The term $ds \, Y_{s-}^h \nu^Y(du)$ comes from the time change of the underlying spectrally positive Lévy process, called the Lamperti time change (for CBs). The term $ds \, u \nu^Y(du)$ is independent of the current state of the population and corresponds to the immigration term. Then, conditionally on the value of the jump $\Delta Y_s^h = u$, the event $\{j_1(s) = 1\}$ has probability

$$\frac{u}{Y_s^h} = \frac{u}{Y_{s-}^h + u}$$

independently for each jump. Therefore, the predictable compensator of the point measure

$$\sum_{0 \le s \le t} \delta_{(s, \Delta Y_s^h)}(ds, du) \mathbf{1}_{\{j_1(s)=1\}}$$

is

$$ds \left(\frac{u}{Y_{s-}^h + u} \right) (Y_{s-}^h \nu^Y(du) + u \nu^Y(du)) = ds \, u \nu^Y(du).$$

But the measure $u \nu^Y(du)$ is the Lévy measure associated with the immigration mechanism $\phi(\lambda)$, which has no drift under the assumption that $\sigma^2 = 0$. This ends up the proof. $\quad\square$

Remark 3.8. Understanding the action of the continuous part of the subordinator requires to work with the particle system generated by the first N particles. Namely, it is possible to prove that the family of processes

$$\left(\sum_{0 \le s \le t} Y_s^h \, \frac{\#\{1 \le i \le N, j_i(s) \le N\}}{N} \mathbf{1}_{\{j_1(s)=1, j_2(s) \le N\}}, t \ge 0 \right)$$

converges almost surely as $N \to \infty$ in the Skorohod topology towards a subordinator with Laplace exponent $\psi'(\lambda) - \psi'(0+)$.

The link with literature is the following:

(1) When Y is a subcritical CB process, meaning that $\psi'(0+) \geq 0$, setting $m(t) = e^{-\psi'(0+)t}$ and choosing $h_t(x)$ independent of x equal to $e^{\psi'(0+)t}$, Theorem 3.3 yields (part of) the Roelly and Rouault [36] and Evans [19] decomposition. Lambert [27] proved that this h-transform may be interpreted as the process conditioned on non-extinction in remote time.

(2) When Y is a critical Feller diffusion, P is the law of a Brownian motion and $h_t(x)$ is a space–time harmonic function for P, then, setting $m(t) = 1$, Theorem 3.3 reduces to the decomposition of the h-transform of the Dawson–Watanabe process provided by Overbeck in [33].

Acknowledgements

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