CHAPTER 3

Covariant derivatives estimates

1. The maximum principle for functions

We first state and prove the maximum principle for subsolutions to reaction-diffusion equations on a closed manifold.

Lemma 3.1. *Let $(M, g(t))_{t\in[0,T)}$ be a smooth one-parameter family of metrics on a closed manifold M and let $u : M \times [0, T) \to \mathbb{R}$ be a smooth function that satisfies*

$$\frac{\partial u}{\partial t} - \Delta_{g(t)}u \leq g(t)(V(t), \nabla^{g(t)}u) + F(u),$$

where $V(t)$ is a vector field on M and F is a Lipschitz function. If $u(\cdot, 0) \leq c$ for some real constant c then $u(x, t) \leq U(t)$ where $U(t)$ is the solution to the ODE $\frac{dU}{dt} = F(U)$ satisfying $U(0) = c$.

Proof. Observe that by definition of the solution U,

$$\frac{\partial}{\partial t}(u - U) - \Delta_{g(t)}(u - U) \leq g(t)(V(t), \nabla^{g(t)}(u - U)) + F(u) - F(U),$$

since U depends on time only.

Since F is Lipschitz, there exists a constant C such that:

$$\frac{\partial}{\partial t}(u - U) - \Delta_{g(t)}(u - U) \leq g(t)(V(t), \nabla^{g(t)}(u - U)) + C|u - U|.$$

The previous differential inequality would let us conclude in case the absolute value on the righthand side was not there. To circumvent this issue, let us consider the auxiliary function $v := e^{-Ct}(u-U)$. Then v satisfies:

$$\frac{\partial}{\partial t}v - \Delta_{g(t)}v \leq g(t)(V(t), \nabla^{g(t)}v) + C\left(|v| - v\right).$$

Moreover, for $\varepsilon > 0$, we make use of the usual trick to force the desired contradiction, i.e. we define $v_\varepsilon(x, t) := v(x, t) - \varepsilon(1 + t)$ so that,

$$\frac{\partial}{\partial t}v_\varepsilon - \Delta_{g(t)}v_\varepsilon \leq g(t)(V(t), \nabla^{g(t)}v_\varepsilon) + C\left(|v| - v\right) - \varepsilon. \tag{1.1}$$

Now, we claim that $v_\varepsilon \leq 0$ for all time. Assume it is not the case and let us derive a contradiction.

Since $v_\varepsilon(x, 0) \leq -\varepsilon < 0$, $t_0 := \sup\{t \in [0, T), \quad \sup_M v_\varepsilon(\cdot, \tau) < 0,\ \tau \in [0, t]\}$ is positive and there must be some point $x_0 \in M$ such that $v_\varepsilon(x_0, t_0) = 0$. Here we have made use of the compactness of M and the continuity of v_ε. Therefore, at that point (x_0, t_0), the lefthand side of (1.1) is nonnegative and the gradient term vanishes so that $C\left(|v| - v\right)\big|_{(x_0,t_0)} - \varepsilon < 0$. Now, $v_\varepsilon(x_0, t_0) = 0$ so that $v(x_0, t_0) = \varepsilon > 0$. We have reached a contradiction.

We have proved that $v(x, t) \leq \varepsilon(1 + t)$ for all $t \in [0, T)$ and all $\varepsilon > 0$. Letting ε tend to 0, we get $u(x, t) \leq U(t)$ for all $t \in [0, T)$. □

Remark 3.2. The proof and the conclusion of Lemma 3.1 use the qualitative regularity of the function F only, i.e. its Lipschitz constant does not come into play here.

Remark 3.3. Lemma 3.1 admits a corresponding statement for supersolutions of the same differential inequality under the same assumptions on the data by reversing the signs.

Lemma 3.1 is applied to a solution to the Ricci flow in order to bound the scalar curvature from below:

Proposition 3.4. *Let $(M^n, g(t))_{t\in[0,T)}$ be a solution to the Ricci flow on a closed manifold M. Then,*

- *if $\mathrm{R}_{g(0)} \geq c$ for some constant $c \in \mathbb{R}$ then $\mathrm{R}_{g(t)} \geq c$ for $t \in [0, T)$. In particular, nonnegativity of the scalar curvature is preserved along the Ricci flow.*
- *More generally,*

$$\mathrm{R}_{g(t)} \geq \frac{n \min_M \mathrm{R}_{g(0)}}{n - 2t \min_M \mathrm{R}_{g(0)}}.$$

Proof. Observe by Proposition 1.23 that the scalar curvature $\mathrm{R}_{g(t)}$ along a solution to the Ricci flow is a supersolution of the heat equation: $\frac{\partial}{\partial t}\mathrm{R}_{g(t)} \geq \Delta_{g(t)}\mathrm{R}_{g(t)}$ on $M\times(0,T)$. Let us apply Lemma 3.1 to $F \equiv 0$ combined with Remark 3.3 to get the expected lower bound, i.e. $\mathrm{R}_{g(t)} \geq U(t)$ where $\partial_t U(t) = F(U(t)) = 0$ with $U(0) = c$ such that $\mathrm{R}_{g(0)} \geq c$. This implies the desired lower bound.

Dropping the squared norm of the Ricci tensor in the evolution equation from Proposition 1.23 satisfied by the scalar curvature is a loss of (geometric) information. Observe that $|\mathrm{Ric}(g(t))|^2_{g(t)} \geq \frac{1}{n}\mathrm{R}^2_{g(t)}$ pointwise on M for each time $t \in (0, T)$ since $\mathrm{Ric}(g(t))$ is a symmetric 2-tensor. Therefore, $\mathrm{R}_{g(t)}$ satisfies:

$$\frac{\partial}{\partial t}\mathrm{R}_{g(t)} \geq \Delta_{g(t)}\mathrm{R}_{g(t)} + \frac{2}{n}\mathrm{R}^2_{g(t)}, \quad \text{on } M \times (0, T).$$

Applying Lemma 3.1 to $F(x) := \frac{2}{n}x^2$ for $x \in \mathbb{R}$ combined with Remark 3.3 gives: $\mathrm{R}_{g(t)} \geq U(t)$ where $\partial_t U(t) = \frac{2}{n}U(t)^2$, $U(0) = c$ if $\mathrm{R}_{g(0)} \geq c$. This again implies the desired lower bound. □

The following result is a straightforward consequence of Proposition 3.4.

Corollary 3.5. *(Finite time singularity) Let $(M^n, g(t))_{t\in[0,T)}$ be a solution to the Ricci flow on a closed manifold M such that $\min_M \mathrm{R}_{g(0)} > 0$. Then*

$$T \leq \frac{n}{2\min_M \mathrm{R}_{g(0)}}.$$

2. A toy example

Let us start with a toy example of great interest: let $(M, g(t))$ be a solution the Ricci flow and let $u : M \times [0, T) \to \mathbb{R}$ be a solution to the heat equation

$$\frac{\partial u}{\partial t} = \Delta_{g(t)}u.$$

Then, on the one hand,

$$\frac{\partial u^2}{\partial t} = 2u\Delta_{g(t)}u = \Delta_{g(t)}u^2 - 2|\nabla^{g(t)}u|^2_{g(t)}.$$

On the other hand, the norm of the gradient $|\nabla^{g(t)}u|^2_{g(t)}$ satisfies

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^{g(t)}u|^2_{g(t)} &= 2\,\mathrm{Ric}(g(t))(\nabla^{g(t)}u, \nabla^{g(t)}u) + 2g(t)(\nabla^{g(t)}\left(\Delta_{g(t)}u\right), \nabla^{g(t)}u)\\ &= 2g(t)\left(\Delta_{g(t)}\left(\nabla^{g(t)}u\right), \nabla^{g(t)}u\right)\\ &= \Delta_{g(t)}|\nabla^{g(t)}u|^2_{g(t)} - 2|\nabla^{g(t),2}u|^2_{g(t)}.\end{aligned} \tag{2.1}$$

Here we have used the Bochner formula for functions: $\Delta_g(\nabla^g u) = \nabla^g(\Delta_g u) + \mathrm{Ric}(g)(\nabla^g u)$ for a C^3_{loc} function $u : M \to \mathbb{R}$.

In particular, going back to the computation (2.1), one gets:

$$\frac{\partial}{\partial t}\left(t|\nabla^{g(t)}u|^2_{g(t)} + \frac{1}{2}u^2\right) = \Delta_{g(t)}\left(t|\nabla^{g(t)}u|^2_{g(t)} + \frac{1}{2}u^2\right) - 2|\nabla^{g(t),2}u|^2_{g(t)}.$$

This implies in particular that $t|\nabla^{g(t)}u|^2_{g(t)} + \frac{1}{2}u^2$ is a subsolution to the heat equation along the Ricci flow. The maximum principle as stated in Lemma 3.1 implies that $\sqrt{t}\sup_M|\nabla^{g(t)}u|_{g(t)} \leq C\sup_M|u_0|$ for some universal positive constant C.

Remark 3.6. Compare with the usual heat equation

$$\frac{\partial u}{\partial t} = \Delta_g u,$$

where (M, g) is a static Riemannian manifold, for which one can prove that:

$$\frac{\partial}{\partial t}|\nabla^g u|_g^2 = \Delta_g |\nabla^g u|_g^2 - 2|\nabla^{g,2} u|_g^2 - 2\operatorname{Ric}(g)(\nabla^g u, \nabla^g u).$$

3. Covariant derivatives of the curvature tensor

Lemma 3.7. *Let $(M^n, g(t))_{t\in[0,T]}$ be a solution to the Ricci flow on a closed manifold. Then,*

$$\frac{\partial}{\partial t}\nabla^{g(t),k}\operatorname{Rm}(g(t)) = \Delta_{g(t)}\nabla^{g(t),k}\operatorname{Rm}(g(t)) + \sum_{i=0}^{k}\nabla^{g(t),i}\operatorname{Rm}(g(t)) * \nabla^{g(t),k-i}\operatorname{Rm}(g(t)), \quad t \in [0,T].$$

Proof. The proof is by induction on $k \geq 0$. The case $k = 0$ is true by Proposition 1.24. Assume the result is true for indices $0 \leq i \leq k-1$. Then, on the one hand,

$$\begin{aligned}
&\nabla^{g(t)}\left(\frac{\partial}{\partial t}\nabla^{g(t),k-1}\operatorname{Rm}(g(t))\right) = \\
&\nabla^{g(t)}\left(\Delta_{g(t)}\nabla^{g(t),k-1}\operatorname{Rm}(g(t)) + \sum_{i=0}^{k-1}\nabla^{g(t),i}\operatorname{Rm}(g(t)) * \nabla^{g(t),k-1-i}\operatorname{Rm}(g(t))\right) \\
&= \Delta_{g(t)}\nabla^{g(t),k}\operatorname{Rm}(g(t)) + [\nabla^{g(t)}, \Delta_{g(t)}]\nabla^{g(t),k-1}\operatorname{Rm}(g(t)) \\
&\quad + \sum_{i=0}^{k-1}\nabla^{g(t),i+1}\operatorname{Rm}(g(t)) * \nabla^{g(t),k-1-i}\operatorname{Rm}(g(t)) + \nabla^{g(t),i}\operatorname{Rm}(g(t)) * \nabla^{g(t),k-i}\operatorname{Rm}(g(t)).
\end{aligned} \tag{3.1}$$

Now, by Lemma 1.1 applied to $T := \nabla^{g(t),k-1}\operatorname{Rm}(g(t))$, one gets:

$$[\nabla^{g(t)}, \Delta_{g(t)}]\nabla^{g(t),k-1}\operatorname{Rm}(g(t)) = \operatorname{Rm}(g(t)) * \nabla^{g(t),k}\operatorname{Rm}(g(t)) + \nabla^{g(t)}\operatorname{Rm}(g(t)) * \nabla^{g(t),k-1}\operatorname{Rm}(g(t)). \tag{3.2}$$

On the other hand, by Proposition 1.14,

$$\frac{\partial}{\partial t}\nabla^{g(t),k}\operatorname{Rm}(g(t)) = \nabla^{g(t)}\left(\frac{\partial}{\partial t}\nabla^{g(t),k-1}\operatorname{Rm}(g(t))\right) + \nabla^{g(t)}\operatorname{Rm}(g(t)) * \nabla^{g(t),k-1}\operatorname{Rm}(g(t)). \tag{3.3}$$

The result follows by inserting (3.2) and (3.3) back to (3.1).

□

Proposition 3.8. *Let $(M^n, g(t))_{t\in[0,T]}$ be a solution to the Ricci flow on a closed manifold. If*

$$\sup_M |\operatorname{Rm}(g(t))|_{g(t)} \leq CT^{-1}, \quad \text{on } [0,T],$$

then for all $k \geq 0$, there exists a positive constant $C(n,k)$ such that:

$$|\nabla^{g(t),k}\operatorname{Rm}(g(t))|_{g(t)} \leq C(n,k)T^{-1}t^{-\frac{k}{2}}, \quad t \in (0,T].$$

Proof. The proof is again by induction on k. For $k = 0$, there is nothing to be proved since it is true by assumption. Assume the result is true for all indices $l \leq k-1$, i.e. for each such index l, there exists a positive constant $C = C(n,l)$ such that:

$$|\nabla^{g(t),l}\operatorname{Rm}(g(t))|_{g(t)} \leq CKt^{-\frac{l}{2}}, \quad t \in (0,T].$$

According to Lemma 3.7 applied to the index $k-1$,

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)|\nabla^{g(t),k-1}\operatorname{Rm}(g(t))|^2_{g(t)}\\
&\leq -2|\nabla^{g(t),k}\operatorname{Rm}(g(t))|^2_{g(t)}+C\sum_{l=0}^{k-1}|\nabla^{g(t),l}\operatorname{Rm}(g(t))|_{g(t)}|\nabla^{g(t),k-1-l}\operatorname{Rm}(g(t))|_{g(t)}|\nabla^{g(t),k-1}\operatorname{Rm}(g(t))|_{g(t)}\\
&\leq -2|\nabla^{g(t),k}\operatorname{Rm}(g(t))|^2_{g(t)}+CT^{-3}t^{-k+1}\\
&\leq -2|\nabla^{g(t),k}\operatorname{Rm}(g(t))|^2_{g(t)}+CT^{-2}t^{-k},
\end{aligned} \tag{3.4}$$

where we have used the induction assumption in the second inequality together with the fact that $t\leq T$ in the last line. Here C denotes a positive constant that depends on k and n only and that might vary from line to line.

Similarly,

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)|\nabla^{g(t),k}\operatorname{Rm}(g(t))|^2_{g(t)}\\
&\leq -2|\nabla^{g(t),k+1}\operatorname{Rm}(g(t))|^2_{g(t)}+C\sum_{l=0}^{k}|\nabla^{g(t),l}\operatorname{Rm}(g(t))|_{g(t)}|\nabla^{g(t),k-l}\operatorname{Rm}(g(t))|_{g(t)}|\nabla^{g(t),k}\operatorname{Rm}(g(t))|_{g(t)}\\
&\leq -2|\nabla^{g(t),k+1}\operatorname{Rm}(g(t))|^2_{g(t)}+CT^{-1}|\nabla^{g(t),k}\operatorname{Rm}(g(t))|^2_{g(t)}+CT^{-2}t^{-\frac{k}{2}}|\nabla^{g(t),k}\operatorname{Rm}(g(t))|_{g(t)}\\
&\leq -2|\nabla^{g(t),k+1}\operatorname{Rm}(g(t))|^2_{g(t)}+Ct^{-1}|\nabla^{g(t),k}\operatorname{Rm}(g(t))|^2_{g(t)}+CT^{-1}t^{-\frac{k}{2}-1}|\nabla^{g(t),k}\operatorname{Rm}(g(t))|_{g(t)},
\end{aligned} \tag{3.5}$$

where we have used the induction hypothesis in the third line.

Define now the auxiliary function for some positive constant K to be specified later:

$$R_k:=t^{k+1}|\nabla^{g(t),k}\operatorname{Rm}(g(t))|^2_{g(t)}+Kt^k|\nabla^{g(t),k-1}\operatorname{Rm}(g(t))|^2_{g(t)}.$$

Based on estimates (3.4) and (3.5), one arrives at:

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta_{g(t)}\right)R_k&\leq Ct^k|\nabla^{g(t),k}\operatorname{Rm}(g(t))|^2_{g(t)}+CT^{-1}t^{\frac{k}{2}}|\nabla^{g(t),k}\operatorname{Rm}(g(t))|_{g(t)}\\
&+Kt^k\left(-2|\nabla^{g(t),k}\operatorname{Rm}(g(t))|^2_{g(t)}+CT^{-2}t^{-k}+\frac{k}{t}|\nabla^{g(t),k-1}\operatorname{Rm}(g(t))|^2_{g(t)}\right)\\
&\leq\left(\frac{3C}{2}-2K\right)t^k|\nabla^{g(t),k}\operatorname{Rm}(g(t))|^2_{g(t)}+\left((k+C)K+\frac{C}{2}\right)T^{-2},
\end{aligned} \tag{3.6}$$

where we have used Young's inequality $2ab\leq a^2+b^2$ together with the induction hypothesis in the last line.

In particular, if K is large enough so that $4K\geq 3C$, then the function $R_k-\left((k+C)K+\frac{C}{2}\right)tT^{-2}$ is a subsolution of the heat equation along the underlying solution to the Ricci flow $(g(t))_{t\in[0,T]}$. The maximum principle applied to $M\times[0,t]\subset M\times[0,T]$ implies then the expected result:

$$t^{k+1}\sup_M|\nabla^{g(t),k}\operatorname{Rm}(g(t))|^2_{g(t)}\leq\sup_{M\times[0,t]}R_k\leq\left((k+C)K+\frac{C}{2}\right)tT^{-2}.$$

□

As a corollary, one obtains interior in time estimates for the covariant derivatives of the curvature tensor along a solution to the Ricci flow:

Corollary 3.9. *Let $(M^n, g(t))_{t\in[0,T]}$ be a solution to the Ricci flow on a closed manifold. If*

$$\sup_M |\operatorname{Rm}(g(t))|_{g(t)} \leq CT^{-1}, \qquad \text{on } [0,T],$$

then for all $k \geq 0$, there exists a positive constant $C(n,k)$ such that:

$$|\nabla^{g(t),k} \operatorname{Rm}(g(t))|_{g(t)} \leq C(n,k) T^{-\frac{k}{2}-1}, \quad t \in \left[\frac{T}{2}, T\right].$$

4. A criteria for convergence of the Ricci flow

The main result of this section is the following (necessary and) sufficient condition to get convergence of the flow as time approaches some time eventually equal to $+\infty$.

Proposition 3.10. *Let $(M^n, g(t))_{t\in[0,T)}$, $T \leq +\infty$, be a smooth one-parameter family of metrics on a closed manifold. If the following conditions,*

$$\int_0^T \sup_M \left|\nabla^{g(t),k}\left(\frac{\partial}{\partial t} g(t)\right)\right|_{g(t)} dt < +\infty,$$

hold for all $k \geq 0$ then the metrics $g(t)$ converge to a smooth metric g_T in the C^∞ topology as t tends to T.

Before giving the proof of Proposition 3.10, we state and prove the following lemma that rephrases the conditions from that in terms of a time-independent Levi-Civita connection.

Lemma 3.11. *Let $(M^n, g(t))_{t\in[0,T)}$, $T \leq +\infty$, be a smooth one-parameter family of metrics on a closed manifold. The following conditions are equivalent:*

(i)
$$\int_0^T \sup_M \left|\nabla^{g(t),k}\left(\frac{\partial}{\partial t} g(t)\right)\right|_{g(t)} < +\infty, \quad \forall\, k \geq 0,$$

(ii)
$$\int_0^T \sup_M \left|\nabla^{g(0),k}\left(\frac{\partial}{\partial t} g(t)\right)\right|_{g(0)} < +\infty, \quad \forall\, k \geq 0.$$

Proof. We only show the implication $(i) \Rightarrow (ii)$. Observe first that for $t \in [0,T)$ and $k \geq 1$,

$$\nabla^{g(0),k} g(t) = \int_0^t \nabla^{g(0),k}\left(\frac{\partial}{\partial s} g(s)\right) ds$$

since the metric $g(0)$ is parallel with respect to the Levi-Civita connection with respect to $g(0)$. Therefore, for $t \in [0,T)$ and $i \geq 1$,

$$\sup_M |\nabla^{g(0),i} g(t)|_{g(0)} \leq \int_0^t \sup_M \left|\nabla^{g(0),i}\left(\frac{\partial}{\partial s} g(s)\right)\right|_{g(0)} ds, \quad t \in [0,T). \tag{4.1}$$

If $k = 0$, the assumption implies that the metrics $g(t)$ are uniformly equivalent: there exists $C > 0$ such that $C^{-1} g(0) \leq g(t) \leq C g(t)$ for $t \in [0,T)$.

Let us assume that the implication $(i) \Rightarrow (ii)$ holds true for all indices less than or equal to $k-1$ for some $k \geq 1$ and let us prove the assertion for index k. Thanks to Lemma 3.13 applied to the one-parameter family of metrics $g(t)$ and the tensor $T := \partial_t g(t)$ for each $t \in [0,T)$:

$$\nabla^{g(t),k}\left(\frac{\partial}{\partial t} g(t)\right) - \nabla^{g(0),k}\left(\frac{\partial}{\partial t} g(t)\right) = \sum_{l=0}^{k-1} \sum_{i_1+\ldots+i_q=k-l} \nabla^{g(0),i_1} g(t) * \ldots * \nabla^{g(0),i_q} g(t) * \nabla^{g(0),l}\left(\frac{\partial}{\partial t} g(t)\right).$$

In particular, there exists a time-independent constant C such that:

$$\left|\nabla^{g(0),k}\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} \leq \left|\nabla^{g(t),k}\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} + C\sum_{l=0}^{k-1}\sum_{i_1+\ldots+i_q=k-l}\left|\nabla^{g(0),i_1}g(t)\right|_{g(0)}\cdots\left|\nabla^{g(0),i_q}g(t)\right|_{g(0)}\left|\nabla^{g(0),l}\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)}.$$

By induction together with (4.1),

$$\left|\nabla^{g(0),k}\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} \leq \left|\nabla^{g(t),k}\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} + C\left(1+\left|\nabla^{g(0),k}g(t)\right|_{g(0)}\right)\left|\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} + C\sum_{i=0}^{k-1}\left|\nabla^{g(0),i}\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)}.$$

By taking the supremum on M, if $y_k(t) := \int_0^t \sup_M \left|\nabla^{g(0),k}\left(\frac{\partial}{\partial s}g(s)\right)\right|_{g(0)} ds$,

$$\begin{aligned} y_k'(t) &\leq \sup_M \left|\nabla^{g(t),k}\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} + C\left(1+\sup_M\left|\nabla^{g(0),k}g(t)\right|_{g(0)}\right)\sup_M\left|\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} \\ &\quad + C\sum_{i=0}^{k-1}\sup_M\left|\nabla^{g(0),i}\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} \\ &\leq \sup_M \left|\nabla^{g(t),k}\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} + C\left(1+y_k(t)\right)\sup_M\left|\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} + C\sum_{i=0}^{k-1}\sup_M\left|\nabla^{g(0),i}\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} \end{aligned}$$

for some large constant C that may vary from line to line. Here we have invoked (4.1) in the second line. By Grönwall's lemma and the induction assumption again:

$$\log(1+y_k(t)) \leq \log(1+y_k(0)) + \int_0^T \sup_M\left|\nabla^{g(t),k}\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} dt + C\int_0^T\left|\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} dt + C\sum_{i=0}^{k-1}\int_0^T\sup_M\left|\nabla^{g(0),i}\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} dt < +\infty.$$

□

Proof of Proposition 3.10. Thanks to Lemma 3.11, we know that

$$\int_0^T \sup_M\left|\nabla^{g(0),k}\left(\frac{\partial}{\partial t}g(t)\right)\right|_{g(0)} < +\infty, \quad \forall\, k\geq 0.$$

Therefore, $\nabla^{g(0),k}g(t)$ is a Cauchy sequence for all $k \geq 0$ as t tends to T. A diagonal argument shows then that $g(t)$ converges to a smooth symmetric 2-tensor denoted by g_T in the C^∞ topology. The tensor g_T is a Riemannian metric since the bound $g(t) \geq C^{-1}g(0)$ holds for all $t \geq 0$ for some uniform positive constant C. □

5. Curvature blow-up at finite-time singularities

Before proving the main result of this section, we need several preliminary results:

Lemma 3.12. *Let g_0 and g be two Riemannian metrics on M. Then the difference of the Levi-Civita connections $D_{g,g_0} : (X, Y) \to \nabla^g_X Y - \nabla^{g_0}_X Y$ is a tensor. More precisely,*

$$2g(D_{g,g_0}(X,Y), Z) = \nabla^{g_0}_X g(Y,Z) + \nabla^{g_0}_Y g(X,Z) - \nabla^{g_0}_Z g(X,Y).$$

Proof. Left as an exercise: use the formula that gives the Levi-Civita connection in terms of the Lie bracket. □

We continue with a lemma independent of the Ricci flow which estimates the difference of the covariant derivatives of a tensor in terms of two Riemannian metrics:

Lemma 3.13. *Let g_0 and g be two Riemannian metrics on M and let T a tensor on M. Then for all $k \geq 1$,*

$$\nabla^{g,k}T - \nabla^{g_0,k}T = \sum_{l=0}^{k-1} \sum_{i_1+\ldots+i_q=k-l} \nabla^{g_0,i_1} g * \ldots * \nabla^{g_0,i_q} g * \nabla^{g_0,l}T.$$

PROOF. As usual, the proof is by induction on $k \geq 1$. If $k = 1$, Lemma 3.12 implies that: $\nabla^g T - \nabla^{g_0}T = \nabla^{g_0} g * T$. Now, assume the result is true for indices less than or equal to $k-1$. Then by applying Lemma 3.12 repeatedly :

$$\begin{aligned}
\nabla^{g,k}T - \nabla^{g_0,k}T &= \nabla^g \left(\nabla^{g,k-1}T - \nabla^{g_0,k-1}T\right) + (\nabla^g - \nabla^{g_0})\nabla^{g_0,k-1}T \\
&= \nabla^g \left(\sum_{l=0}^{k-2} \sum_{i_1+\ldots+i_q=k-1-l} \nabla^{g_0,i_1} g * \ldots * \nabla^{g_0,i_q} g * \nabla^{g_0,l}T \right) \\
&\quad + \nabla^{g_0} g * \nabla^{g_0,k-1}T \\
&= \sum_{l=0}^{k-2} \sum_{i_1+\ldots+i_q=k-1-l} \nabla^{g_0,i_1} g * \ldots * \underbrace{\nabla^g \nabla^{g_0,i_q} g}_{=\nabla^{g_0,i_q+1}g + \nabla^{g_0}g*\nabla^{g_0,i_q}g} * \nabla^{g_0,i}T \\
&\quad + \sum_{l=0}^{k-2} \sum_{i_1+\ldots+i_q=k-1-l} \nabla^{g_0,i_1} g * \ldots * \nabla^{g_0,i_q} g * \underbrace{\nabla^g \nabla^{g_0,i}T}_{=\nabla^{g_0,i+1}T + \nabla^{g_0}g*\nabla^{g_0,i}T} \\
&\quad + \nabla^{g_0} g * \nabla^{g_0,k-1}T \\
&= \sum_{l=0}^{k-1} \sum_{i_1+\ldots+i_q=k-l} \nabla^{g_0,i_1} g * \ldots * \nabla^{g_0,i_q} g * \nabla^{g_0,i}T,
\end{aligned}$$

as expected.

□

Theorem 3.14. *Let $(M^n, g(t))_{t\in[0,T)}$ be a maximal solution to the Ricci flow on a closed manifold. Then either $T = +\infty$ or $T < \infty$ and $\limsup_{t\to T} \sup_M |\mathrm{Rm}(g(t))|_{g(t)} = +\infty$.*

PROOF. Assume that $T < +\infty$ and assume by contradiction that $\limsup_{t\to T} \sup_M |\mathrm{Rm}(g(t))|_{g(t)} < +\infty$. Then Corollary 3.9 ensures that for all $k \geq 1$, there exists $C(n, k, T) > 0$ such that:

$$|\nabla^{g(t),k} \mathrm{Rm}(g(t))|_{g(t)} \leq C(n, k, T), \quad t \in \left[\frac{T}{2}, T\right).$$

In particular, this implies by the very definition of the Ricci flow, that for all $k \geq 0$:

$$\left|\nabla^{g(t),k} \left(\frac{\partial}{\partial t} g(t)\right)\right|_{g(t)} \leq C(n, k, T). \tag{5.1}$$

We claim that the metrics $g(t)$ converge to a limiting smooth Riemannian metric g_T as t tends to T.

For this purpose, let us define the following time integrals:

$$R_k(t) := \int_0^t \sup_M \left|\nabla^{g(s),k} \left(\frac{\partial}{\partial s} g(s)\right)\right|_{g(s)} ds, \quad t \in [0, T), \quad k \geq 0.$$

Conditions (5.1) imply that for all $k \geq 0$, $R_k(T)$ are finite.

This ends the proof by applying the existence result from Proposition 3.10.

□

6. Exercices

Exercise 3.15. *Show that if a tensor S satisfies $\frac{\partial}{\partial t}S = \Delta_{g(t)}S + \mathrm{Rm}(g(t)) * S$ along a solution $(g(t))_{t\in[0,T]}$ to the Ricci flow on a closed manifold M then interior estimates also holds for the covariant derivatives of S. More precisely, if $\sup_{M\times[0,T]} |\mathrm{Rm}(g(t))|_{g(t)} \leq T^{-1}$ and $\sup_{M\times[0,T]} |S(t)|_{g(t)} \leq S$, show that for all $k \geq 0$, there exists a positive constant $C = C(n,k)$ such that:*

$$|\nabla^{g(t),k} S(t)|_{g(t)} \leq CSt^{-\frac{k}{2}}, \quad t \in (0,T].$$

Exercise 3.16. *([Ban87]) Let $(\mathbb{T}^n, g)$ be a flat torus and let u be a solution to the heat equation with initial condition a continuous function u_0.*

(i) *Show that for $k \geq 0$,*

$$\frac{\partial}{\partial t}|\nabla^{g,k}u|^2_g = \Delta_g|\nabla^{g,k}u|^2_g - 2|\nabla^{g,k+1}u|^2_g.$$

(ii) *Show that for $m \geq 1$,*

$$\frac{\partial}{\partial t}\left(\sum_{k=0}^{m}\frac{(2t)^k}{k!}|\nabla^{g,k}u|^2_g\right) = \Delta_g\left(\sum_{k=0}^{m}\frac{(2t)^k}{k!}|\nabla^{g,k}u|^2_g\right) - \frac{2(2t)^m}{m!}|\nabla^{g,m+1}u|^2_g.$$

(iii) *Show that the series $\sum_{k\geq 0}\frac{(2t)^k}{k!}|\nabla^{g,k}u|^2_g$ converges uniformly in space and time over a time interval to be defined with the help of the maximum principle.*

(iv) *Conclude about the analyticity in space of the solution u.*

(v) *Does it imply the analyticity in time of the solution u?*