# An introduction to Ricci flow

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# Introduction

The main sources for this series of lectures are the books [CLN06] and [Bre10]. When research articles are invoked, appropriate references are given. Other books on an introduction of the Ricci flow are: [MT07], [CK04] and [Top06] to mention a few. For a more in-depth study of this topic, see [CCG$^+$07], [CCG$^+$08], [CCG$^+$10] and [CCG$^+$15].

Standard books on Riemannian geometry are [Pet06], [GHL04] among many others.

A very nice introduction to the heat equation and the analysis of self-similar solutions for non-linear diffusion equations can be found in [GGS10].

This course is an introduction to the Ricci flow introduced by Hamilton in the early eighties in his seminal article [Ham82]. The Ricci flow can be interpreted as an intrinsic heat equation on the space of Riemannian metrics up to scalings and diffeomorphisms on a given (closed) smooth manifold. Heuristically, we expect (or hope) that the flow either converges to a canonical metric or decomposes the geometry into canonical ones.

The aim of this series of lectures is to give a proof of the uniformization theorem via the Ricci flow that was originally proved by Hamilton [Ham88] and Chow [Cho91]. The first chapter derives the evolution equations satisfied by the Ricci curvature and the scalar curvature. Chapter 2 is devoted to the introduction of fixed points of the Ricci flow also called Ricci solitons: the classification of shrinking gradient Ricci solitons on the 2-sphere lies at the heart of the uniformization theorem. Chapter 3 establishes crucial interior bounds on the higher covariant derivatives of the curvature tensor once a bound on the curvature holds: these are called Bernstein-Shi type estimates: the weak maximum principle for functions is the main tool here. Some curvature conditions such as nonnegative scalar curvature in all dimensions and nonnegative Ricci curvature in dimension 3 are shown to be preserved along the Ricci flow in Chapter 4. Rigidity statements follow from the strong maximum principle whose proof relies on local Harnack differential inequality. Chapter 5 explains some existence and uniqueness statements on the Ricci flow. This is the most technical part of these lectures. Chapter 6 specializes to 2-dimensional Ricci flows on a closed surface of higher genus, the torus case is left as a series of claims. Chapter 7 deals with the much more delicate case of the 2-sphere: Hamilton's entropy and sharp Harnack differential inequalities will be key tools to prove the convergence of the Ricci flow to a constant curvature metric. If time allows, the last chapter will introduce Perelman's entropy [Per02] whose monotonicity implies the non-existence of periodic (also known as breathers) solutions on a closed manifold.