CHAPTER 2

Solitons and special solutions

1. Definitions and identities

Let us start with a static (i.e. time-independent) definition.

Definition 2.1. A *Ricci soliton* is a triple $(M,\, g,\, X)$, where M is a Riemannian manifold with a (complete) Riemannian metric g and a (complete) vector field X satisfying the equation

$$\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \frac{\lambda}{2} g \tag{1.1}$$

for some $\lambda \in \{-1,\, 0,\, 1\}$.

If $X = \nabla^g f$ for some real-valued smooth function f on M, then we say that $(M,\, g,\, X)$ is *gradient*. In this case, the soliton equation (1.1) reduces to

$$\mathrm{Ric}(g) + \nabla^{g,2} f = \frac{\lambda}{2} g. \tag{1.2}$$

For Ricci solitons $(M,\, g,\, X)$, the vector field X is called the *soliton vector field*. If the soliton is gradient, then the smooth real-valued function f satisfying $X = \nabla^g f$ is called the *soliton potential*. It is unique up to a constant unless the manifold splits a line:

Lemma 2.2. *Let (M^n, g) be a complete Riemannian manifold such that there exists a smooth function f on M satisfying $\nabla^{g,2} f = 0$. Then either f is constant or (M^n, g) is isometric to $(N^{n-1}\times\mathbb{R}, h+dt^2)$ where (N^{n-1}, h) is a complete Riemannian manifold.*

Proof. The condition $\nabla^{g,2} f = 0$ implies in particular that $\nabla^{g,2} f(\nabla^g f, X) = 0$ for all vector fields X on M, i.e. $X \cdot |\nabla^g f|^2_g = 0$ for all vector fields X on M. Therefore, $|\nabla^g f|^2_g = c$ on M (M is connected implicitly here) for some nonnegative constant c. If $c = 0$ then $\nabla^g f = 0$ identically, i.e. f is constant on M. If $c > 0$, we can assume $c = 1$ by rescaling f appropriately. In particular, f has no critical points on M which implies that level sets of f are hypersurfaces and the flow $(\Psi_t)_{t\in\mathbb{R}}$ generated by $\nabla^g f/|\nabla^g f|^2_g = \nabla^g f$ (also known as the Morse flow associated to f) defines the desired isometry: $\Psi : (x, t) \in f^{-1}\{0\} \times \mathbb{R} \to \Psi_t(x) \in M$. Indeed, one can check that $\Psi^* g(\partial_t, \partial_t) = 1$, $\Psi^* g(\partial_t, Y) = 0$ for all Y tangent to $N := f^{-1}\{0\}$ and if h denotes the metric on N induced by g via the embedding $\Psi(\cdot, 0)$ then (N^{n-1}, h) is totally geodesic since its second fundamental form vanishes identically, a fact that implies the completeness of the metric h. □

Finally, a Ricci soliton is called *steady* if $\lambda = 0$, *expanding* if $\lambda = -1$, and *shrinking* if $\lambda = 1$ in equations (1.1) and (2.1) respectively.

Definition 2.1 is consistent with that of being a fixed point of the Ricci flow (interpreted as a infinite-dimensional dynamical system on the space of metrics modulo the action of diffeomorphisms and scalings):

Definition 2.3. A solution to the Ricci flow $(M, g(t))_{t\in(a,b)}$ is called *self-similar* if there exist a smooth positive function $\sigma : (a, b) \to \mathbb{R}_+$ and a family of diffeomorphisms $(\psi_t)_{t\in(a,b)}$ such that:

$$g(t) = \sigma(t)\psi_t^* g, \quad t \in (a, b), \tag{1.3}$$

where g is a background Riemannian metric.

The following result shows that Definitions 2.1 and 2.3 are equivalent:

Lemma 2.4. *If $(M, g(t))_{t\in(a,b)}$ is a self-similar solution to the Ricci flow then there exist $\lambda \in \mathbb{R}$ and a vector field X on M such that (M, g, X) is a Ricci soliton with constant λ. Conversely, a Ricci soliton gives rise to a self-similar solution.*

Proof. If $(M, g(t)) = (M, \sigma(t)\psi_t^* g)$ is a solution to the Ricci flow then the Ricci flow equation evaluated at some time $t_0 \in (a, b)$:

$$-2\psi_{t_0}^* \operatorname{Ric}(g) = -2\operatorname{Ric}(\sigma(t_0)\psi_{t_0}^* g) = \partial_t\big|_{t=t_0} g(t) = \sigma'(t_0)\psi_{t_0}^* g + \sigma(t_0)\psi_{t_0}^* \mathcal{L}_X g,$$

where $X := \partial_t\big|_{t=t_0}\psi_t$. This makes this solution into a soliton metric (M, g, X) with soliton constant $-\sigma'(t_0)/\sigma(t_0)$.

Conversely, if (M, g, X) satisfies (1.1) with $\lambda = 1$ for instance, then $g(t) := (-t)\varphi_t^* g$, $t < 0$, defines a self-similar solution if (φ_t) is generated by $X/(-t)$. □

Let us derive identities holding on a Ricci soliton:

Lemma 2.5 (Ricci soliton identities)**.** *Let (M^n, g, X) be a gradient Ricci soliton satisfying (1.2) with $\lambda \in \mathbb{R}$ with soliton vector field $X = \nabla^g f$ for a smooth real-valued function $f : M \to \mathbb{R}$. Then the trace and first order soliton identities are:*

$$\begin{aligned} \Delta_g f + \mathrm{R}_g &= \frac{\lambda}{2} n, && (1.4)\\ \nabla^g \mathrm{R}_g - 2\operatorname{Ric}(g)(X) &= 0, && (1.5)\\ |\nabla^g f|_g^2 + \mathrm{R}_g - \lambda f &= \text{const}. && (1.6) \end{aligned}$$

Identity (1.6) is a first order identity with respect to the potential function f and is a Hamilton-Jacobi equation. We will see important applications from it to derive bounds on the potential function under curvature bounds.

Proof. Identity (1.4) is easily obtained by tracing the soliton equation (1.2). Identity (1.6) is equivalent to showing that the differential of its lefthand side is 0 since M is assumed to be connected (always implicitly in these notes). Now, observe that:

$$\begin{aligned} d\left(|\nabla^g f|_g^2 + \mathrm{R}_g - \lambda f\right)(Y) &= 2g(\nabla^g_Y(\nabla^g f), \nabla^g f) + g(\nabla^g \mathrm{R}_g, Y) - \lambda g(\nabla^g f, Y)\\ &= 2\nabla^{g,2}(f)(\nabla^g f, Y) + 2\operatorname{Ric}(g)(\nabla^g f, Y) - \lambda g(\nabla^g f, Y)\\ &= 0, \end{aligned}$$

where we have used identity (1.5) in the second equality together with the soliton identity (1.2). We are then left with proving identity (1.5).

Consider the divergence (with respect to the metric g) of the lefthand side of the soliton equation (1.2) multiplied by a factor 2:

$$\begin{aligned} \operatorname{div}_g\left(2\operatorname{Ric}(g) + 2\nabla^{g,2}(f)\right)(Y) &= g(\nabla^g \mathrm{R}_g, Y) + \operatorname{div}_g(\mathcal{L}_{\nabla^g f}(g))(Y)\\ &= g(\nabla^g \mathrm{R}_g, Y) + 2g\left(\nabla^g \Delta_g f, Y\right) + 2\operatorname{Ric}(g)(\nabla^g f, Y)\\ &= -g(\nabla^g \mathrm{R}_g, Y) + 2\operatorname{Ric}(g)(\nabla^g f, Y). \end{aligned}$$

Here we have used the commutation formula $\operatorname{div}_g \nabla^{g,2}(f) = \nabla^g \Delta_g f + \operatorname{Ric}(g)(\nabla^g f)$ together with identity (1.4) in the last equality.

Since the divergence of the righthand side of the soliton equation (1.2) is 0, the result follows. □

A first interesting consequence from Lemma 2.5 is that the completeness of the soliton vector field is guaranteed by the completeness of the soliton metric [Zha09]:

Proposition 2.6. *Let $(M, g, \nabla^g f)$ be a gradient Ricci soliton such that (M^n, g) is a complete Riemannian manifold with scalar curvature bounded from below, i.e. $\mathrm{R}_g \geq -C$ for some nonnegative constant C. Then the vector field $\nabla^g f$ is complete, i.e. its integral curves are defined on $\mathbb{R}$.*

PROOF. We recall the following basic fact. A vector field X on a complete Riemannian manifold (M^n, g) is complete if it satisfies the following condition: there exists a constant C and a point $p \in M$ such that $|X(x)|_g \leq C(1 + d_g(p, x))$ for all $x \in M$. We say that the vector field X has at most linear growth at infinity.

In case $\lambda = 0$, (1.6) implies that the soliton vector field is bounded on M if $\mathrm{R}_g \geq -C$.

In case $\lambda = 1$ (the case where $\lambda = -1$ is similar), the assumption on the scalar curvature together with (1.6) imply:

$$|\nabla^g f|_g^2 \leq f + C, \tag{1.7}$$

for some constant C. In particular, the function $2\sqrt{f + C}$ is Lipschitz with constant 1 which means that $2\sqrt{f + C}$ has at most linear growth at infinity: $2\sqrt{f(x) + C} \leq 2\sqrt{f(p) + C} + d_g(p, x)$ for some arbitrary point $p \in M$ and all $x \in M$. Inserting this growth back to inequality (1.7) leads to the linear growth of $\nabla^g f$ at infinity. $\square$

2. A short digression on Kähler Ricci solitons

If g is complete and Kähler with Kähler form ω, then we say that $(M,\, g,\, X)$ (or $(M,\, \omega,\, X)$) is a *Kähler-Ricci soliton* if the vector field X is complete and real holomorphic, i.e. $\mathcal{L}_X J = 0$ and the pair $(g,\, X)$ satisfies the equation

$$\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{2.1}$$

for $\lambda \in \mathbb{R}$. If g is a Kähler-Ricci soliton and if $X = \nabla^g f$ for some real-valued smooth function f on M, then we say that $(M,\, g,\, X)$ is *gradient*.

Lemma 2.7. *Let M be a Kähler manifold with Kähler metric g and complex structure J and let X be a real holomorphic vector field on M. Then the following are equivalent.*

(i) *JX is Killing.*
(ii) *$g(\nabla^g_Y X,\, Z) = g(Y,\, \nabla^g_Z X)$ for all real vector fields Y, Z on M.*
(iii) *The g-dual one-form of X is closed.*

PROOF. Since X is holomorphic, we have that

$$\nabla^g_{JY} X = J\nabla^g_Y X$$

for every real vector field Y on M. Hence, for every real vector field Y and Z on M, we see that

$$\begin{aligned}(\mathcal{L}_{JX} g)(Y,\, Z) = g(\nabla^g_Y (JX),\, Z) + g(Y,\, \nabla^g_Z (JX)) &= g(J\nabla^g_Y X,\, Z) + g(Y,\, J\nabla^g_Z X)\\ &= g(\nabla^g_{JY} X,\, Z) + g(Y,\, J\nabla^g_Z X)\\ &= g(\nabla^g_{JY} X,\, Z) - g(JY,\, \nabla^g_Z X).\end{aligned}$$

The equivalence of (i) and (ii) now follows.

The equivalence of (ii) and (iii) can be seen from the identity

$$\begin{aligned}d\eta_X(Y,\, Z) &= 2g(\nabla^g_Y X,\, Z) - (\mathcal{L}_X g)(Y,\, Z)\\ &= 2g(\nabla^g_Y X,\, Z) - (g(\nabla^g_Y X,\, Z) + g(Y,\, \nabla^g_Z X))\\ &= g(\nabla^g_Y X,\, Z) - g(Y,\, \nabla^g_Z X)\end{aligned}$$

for every real vector field Y and Z on M, where η_X denotes the g-dual one-form of X. $\square$

Using this, we can prove:

Corollary 2.8. *Let M be a Kähler manifold with Kähler metric g and complex structure J and let X be a real holomorphic vector field on M.*

(i) *If $H^1_{dR}(M) = 0$ and JX is Killing, then there exists a smooth real-valued function $f \in C^\infty(M)$ such that $X = \nabla^g f$.*
(ii) *Conversely, if $X = \nabla^g f$ for a smooth real-valued function $f \in C^\infty(M)$, then JX is Killing.*

3. First examples

3.1. Trivial examples. Riemannian metrics with constant sectional curvature or more generally, those which are Einstein, i.e. $\mathrm{Ric}(g) = \lambda g$ for some $\lambda \in \mathbb{R}$ are obvious Ricci solitons.

Einstein metrics are *trivial* Ricci solitons in the sense that the soliton vector field is $X \equiv 0$. Their study is however non trivial: see [**Bes08**] as a reference book on the subject.

3.2. Gaussian solitons. The first non-trivial examples of gradient Ricci solitons whose geometries are trivial are the so called Gaussian solitons that live on Euclidean space $\mathbb{R}^n$, $n \geq 2$.

For $\lambda \in \{-1, 0, 1\}$, define $\sigma(t) := 1 - \lambda t$ and consider the family of diffeomorphisms

$$\varphi_t(x) := \frac{x}{\sigma(t)^{\frac{1}{2}}}, \qquad \text{for } x \in \mathbb{R}^n.$$

Then the Gaussian soliton metric with parameter λ is defined as

$$g(t) := \sigma(t)\varphi_t^* \,\mathrm{eucl}\,.$$

Then one defines the potential function compatible with the previous family of diffeomorphisms in the following sense:

$$f(x,t) := \frac{\lambda}{4}\frac{|x|^2}{\sigma(t)} = (\varphi_t^* f(0))(x), \quad \frac{\partial}{\partial t}\varphi_t(x) = \frac{1}{\sigma(t)}\nabla^{g(0)} f(0)(\varphi_t(x)).$$

With these definitions and remarks in hand, we can check that $(\mathbb{R}^n, \mathrm{eucl}, \nabla^{g(0)} f(0))$ is a shrinking (respectively steady, respectively expanding) gradient Ricci soliton if $\lambda = 1$ (respectively $\lambda = 0$, respectively $\lambda = -1$). Let us check the case $\lambda = 1$, since $\mathrm{Ric}(g(t)) = 0$,

$$\mathrm{Ric}(g(t)) + \mathrm{Hess}_{g(t)} f(t) = \mathrm{Hess}_{g(t)} f(t) = \varphi_t^* \,\mathrm{Hess}_{g(0)} f(0) = \sigma(t)^{-1}\frac{\sigma(t)\varphi_t^* g(0)}{2} = \frac{g(t)}{2\sigma(t)}.$$

3.3. Metric products. If $(M_i^{n_i}, g_i, \nabla^{g_i} f_i)$, $i = 1, 2$ are two gradient Ricci solitons of the same kind then the metric product defined on $M_1^{n_1} \times M_2^{n_2}$ by $g := g_1 + g_2$ together with the potential function defined by $f := f_1 + f_2$ define a new gradient Ricci soliton of the same kind on $M_1^{n_1} \times M_2^{n_2}$.

Of particular importance are the *generalized shrinking cylinders* $\mathbb{R}^p \times \mathbb{S}^n$, $n \geq 2$, endowed with the metric

$$g_{cyl}(t) := \mathrm{eucl} + 2(n-1)(1-t)g_{\mathbb{S}^n}, \quad t < 1,$$

where $g_{\mathbb{S}^n}$ has constant curvature 1 and where the potential function is

$$f_{cyl}(x,y,t) := f_{\mathrm{eucl}}(x,t) = \frac{|x|^2}{1-t}, \quad x \in \mathbb{R}^p,\ y \in \mathbb{S}^n,\ t < 1.$$

4. The cigar soliton

The cigar soliton $(\Sigma^2, g_{\mathrm{cigar}}, \nabla^{g_{\mathrm{cigar}}} f)$ is a steady gradient Ricci soliton living on $\mathbb{R}^2$ defined as follows:

$$g_{\mathrm{cigar}} := \frac{1}{1+|x|^2}dx^2, \quad \nabla^{g_{\mathrm{cigar}}} f(x) := 2x, \quad x \in \mathbb{R}^2.$$

The scalar curvature can be computed without too much pain since g_{cigar} is conformal to a Euclidean metric: $\mathrm{R}_{g_{\mathrm{cigar}}}(x) = -(1+|x|^2)\Delta \log(1+|x|^2)^{-1}$. Therefore,

$$\mathrm{R}_{g_{\mathrm{cigar}}}(x) = \frac{4}{1+|x|^2}, \quad x \in \mathbb{R}.$$

In particular, the family $(\varphi_t)_{t\in\mathbb{R}}$ of diffeomorphisms of $\mathbb{R}^2$ associated to the flow of $-\nabla^{g_{\mathrm{cigar}}} f$ satisfies: $\varphi_t(x) = e^{-2t}x$ for $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$. By pulling back g_{cigar} by the aforementioned family of diffeomorphisms, one gets the associated solution to the Ricci flow:

$$g_{\mathrm{cigar}}(t) = \varphi_t^* g_{\mathrm{cigar}}(0) = \frac{1}{e^{4t}+|x|^2}dx^2, \quad t \in \mathbb{R}.$$

We check immediately that $\partial_t g_{\text{cigar}}(t) = -\,\mathrm{R}_{g_{\text{cigar}}(t)}\, g_{\text{cigar}}(t), \quad t\in\mathbb{R}$.

The asymptotics of the cigar soliton can be read in suitable coordinates making the metric rotationally symmetric. We define the following new system of coordinates on $\mathbb{R}^2$, $s := \operatorname{arcsinh}(r)$, $r := |x|$, $x\in\mathbb{R}^2$, so that $(1+r^2)ds^2 = dr^2$ and $g_{\text{cigar}} = ds^2 + \tanh(s)^2 d\theta^2$. In particular,

$$\mathrm{R}_{g_{\text{cigar}}} = \frac{4}{\cosh s^2}, \quad s>0.$$

Let us sum up the qualitative properties of the cigar soliton we just derived in the next proposition:

Proposition 2.9. *The cigar soliton is the unique non-flat rotationally symmetric steady gradient Ricci soliton on $\mathbb{R}^2$ up to scalings. Its scalar curvature decays exponentially fast at infinity and it is asymptotic to a flat cylinder.*

Proof. The only thing left to prove is to show that a non-flat rotationally symmetric steady gradient Ricci soliton on $\mathbb{R}^2$ must be the cigar soliton up to scalings. Let $(\mathbb{R}^2, g, \nabla^g f)$ be a rotationally symmetric non-flat steady gradient Ricci soliton, i.e. $g = ds^2+\phi(s)^2d\theta^2$ and $f = f(s)$. Then $K_g = -\frac{\phi''(s)}{\phi(s)}$. Now since $\mathrm{Ric}(g) = K_g g$ in dimension 2, the soliton equation $\mathrm{Ric}(g) = \nabla^{g,2}f$ imposes:

$$\begin{aligned}
-\frac{\phi''(s)}{\phi(s)} &= K_g g\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right) = \nabla^{g,2}f\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right) = f''(s)\\
-\frac{\phi''(s)}{\phi(s)} &= K_g g\left(\phi(s)^{-1}\frac{\partial}{\partial\theta},\phi(s)^{-1}\frac{\partial}{\partial\theta}\right) = \nabla^{g,2}f\left(\phi(s)^{-1}\frac{\partial}{\partial\theta},\phi(s)^{-1}\frac{\partial}{\partial\theta}\right)\\
&= -\nabla^g_{\phi(s)^{-1}\frac{\partial}{\partial\theta}}\left(\phi(s)^{-1}\frac{\partial}{\partial\theta}\right)\cdot f = \frac{\phi'(s)}{\phi(s)}f'(s).
\end{aligned}$$

We end up by a separation of variables with $f'(s) = c\phi(s)$ for some constant c. The constant c is not 0 since the soliton is assumed to be non-flat. According to Proposition 3.4 that we admit for the moment in its non-compact version, the curvature is positive since it is an ancient solution to the Ricci flow. Therefore, the second equation from the previous system of ODEs forces c to be positive. Injecting this relation back to the first equation of the previous system of ODEs leads to:

$$\phi''(s)+c\phi(s)\phi'(s) = 0,$$

which gives after integration:

$$\phi'(s)+\frac{c}{2}\phi^2(s) = c',$$

for some constant c'. Since the metric is smooth up to the origin, $\phi(0) = 0$ and $\phi'(0) = 1$ which in turn imposes $c' = 1$. Therefore the we need to solve the ODE $\phi'(s)+\frac{c}{2}\phi^2(s) = 1$ with initial condition $\phi(0) = 1$. This gives $\phi(s) = \sqrt{\frac{2}{c}}\tanh\left(\sqrt{\frac{c}{2}}s\right)$. $\square$

The ideas from Section 6.2 will show that the cigar soliton is the unique non-flat steady gradient Ricci soliton on $\mathbb{R}^2$ up to scalings without assuming rotational symmetry: see exercises.

5. Expanders coming out of two dimensional cones

In this section, we look for expanding gradient Ricci solitons with positive scalar curvature on $\mathbb{R}^2$. We look for rotational symmetric metrics of the form $ds^2+\varphi(s)^2d\theta^2$. It turns out that it will be more convenient to look for metrics of the form $\phi(r)^2dr^2+r^2d\theta^2$ on $\mathbb{R}^2\setminus\{0\}$. Here we consider the variables (r,θ) to belong to $(0,\infty)\times\mathbb{S}^1(2\pi\phi(0))$ in order for the metric to extend smoothly at the origin.

Exercise 2.10. *Show that if $g := ds^2+\varphi(s)^2d\theta^2$ is a complete positively curved rotationally symmetric metric on $\mathbb{R}^2$ then g can be written as $\phi(r)^2dr^2+r^2d\theta^2$ for some coordinates (r,θ) and some smooth positive function ϕ.*

Let us compute the curvature of such a metric: if we recall that $K_g = -\varphi''(s)/\varphi(s)$ then the change of variable $r := \varphi(s)$ together with $\phi(r) := (\varphi'(s(r)))^{-1}$ gives

$$K_g = \frac{\phi'(r)}{r\phi(r)^3}.$$

Let us compute the gradient and the Hessian of a smooth radial function f:

$$\nabla^g f = \frac{1}{\phi(r)^2} f'(r)\partial_r, \quad \nabla^{g,2} f(\partial_r, \partial_r) = f''(r) - \frac{\phi'(r)}{\phi(r)} f'(r), \quad \nabla^{g,2} f(\partial_\theta, \partial_\theta) = \frac{r}{\phi(r)^2} f'(r).$$

Inserting these expressions back to the soliton equation (1.2) with $\lambda = -1$ gives the system of ODEs:

$$\frac{\phi'(r)}{r\phi(r)} + f''(r) - \frac{\phi'(r)}{\phi(r)} f'(r) = -\frac{\phi(r)^2}{2}, \quad \frac{r\phi'(r)}{\phi(r)^3} + \frac{r}{\phi(r)^2} f'(r) = -\frac{r^2}{2}.$$

Multiplying the second equation by ϕ^2/r^2 and combining this new equation together with the first one lead to:

$$f''(r) = \left(\frac{\phi'(r)}{\phi(r)} + \frac{1}{r}\right) f'(r),$$

This gives after integration, $f'(r) = -cr\phi(r)$ for some constant c which must be positive since $\nabla^{g,2} f$ is negative definite. Therefore, ϕ solves:

$$\phi'(r) = r\left(c - \frac{\phi(r)}{2}\right)\phi(r)^2.$$

Let us solve this ODE by separating variables by observing that $\phi(r) \in (0, 2c)$ since $K_g = c\phi(r)^{-1} - 2^{-1} > 0$:

$$h(r) + \log h(r) = -(cr)^2 + h(0) + \log h(0), \quad h(r) := \frac{2c}{\phi(r)} - 1.$$

Let us consider W (also known as the Lambert-W function) to be the inverse of the function xe^x and let us reinterpret the previous result as $h(r) = W(h(0)e^{h(0)-(cr)^2})$. Finally, we have obtained:

Theorem 2.11. *There exists a one-parameter family $(M^2, g_c, \nabla^{g_c} f_c)_{c>0}$ of expanding gradient Ricci solitons with positive curvature on $\mathbb{R}^2$ such that:*

(i) *(M^2, g_c) is isometric to $(\mathbb{R}^2, \phi_c(r)^2 dr^2 + r^2 d\theta^2)$ with*

$$\phi_c(r) = \frac{2c}{W\left(\left(\frac{2c}{\phi_c(0)} - 1\right)\exp\left(\frac{2c}{\phi_c(0)} - 1 - (cr)^2\right)\right) + 1},$$

where $W : \mathbb{R}_+ \to \mathbb{R}_+$ is the inverse of xe^x.

(ii) *the potential function f_c is radial and the soliton vector field satisfies:*

$$\nabla^{g_c} f_c = -\frac{cr}{\phi(r)}\partial_r,$$

(iii) *each metric g_c is asymptotic to a cone of angle $\pi\phi_c(0)/c \in (0, 2\pi)$.*

6. Classification results

6.1. Soliton equations and rigidity results. In this section, we prove that expanding and steady gradient Ricci solitons are Einstein, i.e. the soliton vector field vanishes identically. Before we do so, we establish an elliptic equation satisfied by the scalar curvature on a gradient Ricci soliton:

Lemma 2.12. *Let $(M^n, g, \nabla^g f)$ be a gradient Ricci soliton with constant λ. Then the scalar curvature satisfies:*

$$\Delta_g \mathrm{R}_g - g(\nabla^g f, \nabla^g \mathrm{R}_g) - \lambda \mathrm{R}_g + 2|\mathrm{Ric}(g)|_g^2 = 0. \tag{6.1}$$

Lemma 2.12 suggests the following definition:

Definition 2.13. The weighted laplacian associated to a gradient Ricci soliton $(M^n, g, \nabla^g f)$ is defined for a tensor T by:

$$\Delta_f T := \Delta_g T - \nabla^g_{\nabla^g f} T.$$

Observe the following formal integration by parts:

$$\int_M \langle \Delta_f S, T\rangle_g \, e^{-f} d\mu_g = \int_M \langle S, \Delta_f T\rangle_g \, e^{-f} d\mu_g,$$

that holds for instance for any compactly supported tensors S and T of the same type. The weighted laplacian is therefore symmetric with respect to the weighted measure $e^{-f} d\mu_g$ and as such, a gradient Ricci soliton can be interpreted as a metric measure space endowed with the weighted measure $e^{-f} d\mu_g$.

Corollary 2.14. *Steady and expanding gradient Ricci solitons on a closed manifold are trivial, i.e. they are Einstein.*

Proof. If $(M^n, g, \nabla^g f)$ is a gradient Ricci soliton with constant λ then Lemma 2.12 implies:

$$\begin{aligned} \Delta_f \left(\mathrm{R}_g - \frac{n}{2}\lambda\right) &= -2|\operatorname{Ric}(g)|^2_g + \lambda \left(\mathrm{R}_g - \frac{n}{2}\lambda\right) + \frac{n}{2}\lambda^2 \\ &= -2\left|\operatorname{Ric}(g) - \frac{\lambda}{2} g\right|^2_g - \lambda \left(\mathrm{R}_g - \frac{n}{2}\lambda\right). \end{aligned} \tag{6.2}$$

On the one hand, if $\lambda \leq 0$, the minimum principle implies that at a point $x_0 \in M$ such that $\min_M \left(\mathrm{R}_g - \frac{n}{2}\lambda\right) = \left(\mathrm{R}_g(x_0) - \frac{n}{2}\lambda\right)$,

$$0 \leq 2\left|\operatorname{Ric}(g) - \frac{\lambda}{2} g\right|^2_g \leq -\lambda \left(\mathrm{R}_g - \frac{n}{2}\lambda\right). \tag{6.3}$$

On the other hand, by the soliton identity $\mathrm{R}_g + \Delta_g f = \lambda \frac{n}{2}$, and by integrating on M, one gets:

$$\int_M \left(\mathrm{R}_g - \lambda \frac{n}{2}\right) d\mu_g = 0.$$

The combination of the minimum principle and the previous remark implies that $\mathrm{R}_g = \frac{n}{2}\lambda$ identically which in turn gives by (6.2) that $\operatorname{Ric}(g) = \frac{\lambda}{2} g$ identically on M. $\square$

6.2. 2-dimensional shrinkers.

The goal of this section is to prove the following rigidity statement:

Theorem 2.15. *A shrinking gradient Ricci soliton on a closed surface has constant (positive) scalar curvature.*

The proof of this theorem is due to Chen-Lu-Tian [CLT06] and is a fundamental remark in order to circumvent the use of the uniformization theorem in Chapter 6. We start by proving that a non-trivial gradient Ricci soliton on a surface is a warped product (locally):

Lemma 2.16. *A Riemannian manifold (M^n, g) is a local warped product, i.e. there exists a C^1_{loc} function h such that g is isometric to $dr^2 + h(r)^2 g_0$ if and only if there exists a non-trivial C^2_{loc} function f and a C^0_{loc} function c such that $\nabla^{g,2} f = c\, g$.*

Proof. If $g = dr^2 + h(r)^2 g_0$ then define $f(r)$ to be a primitive of h. Then $\nabla^g f = f'(r)\nabla^g r = h(r)\nabla^g r$ and

$$\nabla^{g,2} f = f''(r) dr^2 + f'(r) h'(r) h(r) g_0 = h'(r)\left(dr^2 + h(r)^2 g_0\right) = h'(r) g.$$

Conversely, observe that $|\nabla^g f|_g$ is constant on each level set of f as if Y is a vector field orthogonal to $\nabla^g f$: $Y \cdot |\nabla^g f|^2_g = 2\nabla^{g,2} f(\nabla^g f, Y) = c g(\nabla^g f, Y) = 0$, by assumption. Therefore, if f is not constant, let $N := f^{-1}\{0\}$ be a level set of f defining a hypersurface of M and consider the flow (Ψ_t) generated by $\nabla^g f / |\nabla^g f|$ to define a local diffeomorphism $\Phi : (x, t) \in N \times I \to M$, I being an interval containing 0. Since $|\nabla^g f|_g$ is constant on each level set of f, $\Phi^* f = \tilde{f}(t)$ for some function $\tilde{f}$ defined on I and if $\tilde{g} := \Phi^* g$ then as in the computation we did in the first step,

$$\nabla^{\tilde{g},2}\tilde{f}(\partial_t, \partial_t) = \tilde{f}''(t),$$

which implies that $\tilde{c} = \tilde{f}''(t)$ and $\nabla^{\tilde{g},2}\tilde{f}(\partial_t, Y) = 0$ for all Y tangent to N. If g_0 denotes the induced metric on N via the embedding $\Phi(\cdot, 0)$ then $\partial_t \tilde{g}(Y,Y) = 2\tilde{f}''(t)/\tilde{f}'(t)\tilde{g}(Y,Y)$ so that after integration, $\tilde{g} = dt^2 + (\tilde{f}'(t)/\tilde{f}'(0))^2 g_0$. □

Lemma 2.17. *A non-trivial gradient Ricci soliton on a closed surface M^2 is globally rotationally symmetric.*

Proof. We define for $p \in \Sigma$ and $v \in T_p\Sigma$ the vector Jv as the unique vector in $T_p\Sigma$ such that the following conditions hold:

$$g(p)(v, Jv) = 0, \quad g(p)(Jv, Jv) = g(p)(v, v), \quad (v, Jv) \text{ is a positively oriented basis of } T_p\Sigma.$$

It is an exercise to show that the only almost complex structures compatible with the metric g are of the form $\pm J$. It is a theorem that J defines a complex structure on $T\Sigma$ and (Σ, g, J) is Kähler, i.e. J is parallel with respect to g.

Since Σ has complex dimension 1, the vector field $\nabla^g f$ is a conformal vector field: $\mathcal{L}_{\nabla^g f}(g) = \left(\frac{1}{2}\mathrm{R}_g - 1\right) g$. Therefore, $\nabla^g f$ is a real holomorphic vector field. According to Lemma 2.7, $J(\nabla^g f)$ is a Killing vector field. By the Hopf-Poincaré index theorem applied to the Killing vector field $J(\nabla^g f)$, then since its zeroes (necessarily isolated) have index 1, it must have 2 zeroes. These correspond to the maximum and the minimum of the potential function f.

Lemma 2.16 ensures that g is a rotationally symmetric outside these two points.

To see that there is an $\mathbb{S}^1$-action, if $(\Phi_t)_{t\in\mathbb{R}}$ denotes the flow of isometries generated by the soliton vector field $\nabla^g f$ then $\Phi_t(p) = p$ for all $t \in \mathbb{R}$ and Φ_t induces through its differential a nontrivial homomorphism $t \in \mathbb{R} \to \mathrm{Isom}^+(T_pM, g(p))$. Since $\mathrm{Isom}^+(T_pM, g(p))$ is $\mathbb{S}^1$, there must be some time $t_0 \neq 0$ such that $d_p\Phi_{t_0} = d_p\Phi_0$. But since $\Phi_0(p) = \Phi_{t_0}(p) = p$, this identifies Φ_{t_0} with Φ_0. In particular, there is a non-trivial action of $\mathbb{S}^1$ on M. This implies the expected result. □

Proof of Theorem 6.2 . Assume by contradiction that f is non constant. Then Lemma 2.17 shows that g is rotationally symmetric: $g = dr^2 + \varphi(r)^2 d\theta^2$ for $r \in [0, r_0]$ and $\theta \in [0, 2\pi]$. Recall that the Gauss curvature of such a metric is equal to $-\varphi''(r)/\varphi(r)$. The soliton equation imposes:

$$-\frac{\varphi''(r)}{\varphi(r)} = 1 + f''(r), \quad -\frac{\varphi''(r)}{\varphi(r)} = 1 + \frac{\varphi'(r)}{\varphi(r)} f'(r), \quad r \in (0, r_0).$$

Therefore, we get a first order ODE for the derivative of f: $f''(r) = \frac{\varphi'(r)}{\varphi(r)} f'(r)$ that integrate to get $f'(r) = c\varphi(r)$ for $r \in (0, r_0)$ for some constant c. Inserting this information back to the second ODE, one gets that $-\frac{\varphi''}{\varphi} = 1 + c\varphi'$. Multiplying across by $\varphi\varphi'$, and integrating from $r = 0$ to $r = r_0$:

$$-\frac{1}{2}\left[\varphi'(r)^2\right]_0^{r_0} = \frac{1}{2}\left[\varphi(r)^2\right]_0^{r_0} + c\int_0^{r_0} \varphi(r)\varphi'(r)^2\, dr.$$

Now, completeness of the metric imposes that $\varphi(0) = \varphi(r_0) = 0$ and $\varphi'(0) = -\varphi'(r_0) = 1$ so that the integral on the righthand side must vanish. In particular, either $c = 0$ or the integrand must vanish identically since it is nonnegative. The second case cannot happen so that $c = 0$ which leads to a contradiction. This ends the proof. □

7. Exercises

Exercise 2.18. *Show that the cigar soliton is the unique complete steady gradient Ricci soliton living on* $\mathbb{R}^2$.

Exercise 2.19. (i) *Show that the only expanding solitons with positive curvature living on* $\mathbb{R}^2$ *are rotationally symmetric.*

(ii) *Show that there exists expanding gradient Ricci solitons with negative curvature living on* $\mathbb{R}^2$ *which are asymptotic to cones with angle in* $(2\pi, \infty)$.

Exercise 2.20. *Prove the following commutation formula for an arbitrary vector field* X *on a Riemannian manifold* (M, g)*:*

$$\mathrm{div}_g\, \mathcal{L}_X(g) - \frac{1}{2} d\mathrm{tr}_g\, \mathcal{L}_X(g) = g(\Delta_g X, \cdot) + \mathrm{Ric}(g)(X, \cdot),$$

as 1*-forms on* M.

Exercise 2.21. *Deduce from this fact that a Killing field, i.e. a vector field that generates isometries, i.e.* $\mathcal{L}_X(g) = 0$*, satisfies* $\Delta_g X + \mathrm{Ric}(g)(X) = 0$. *In particular, show that*

- *a closed Riemannian manifold with negative Ricci curvature does not support any Killing field,*
- *a Killing field on a closed Riemannian manifold with Ricci flat curvature is parallel, i.e.* $\nabla^g X = 0$.

Exercise 2.22. *If* (M^n, g, X) *is a Ricci soliton then show that* $\Delta_g X + \mathrm{Ric}(g)(X) = 0$.

Exercise 2.23. *Reprove Corollary 2.14 as follows: consider the potential function* f *of a steady or expanding gradient Ricci soliton.*

- *Prove that,*
$$\Delta_f f = \frac{n\lambda}{2} - \lambda f - c,$$
for some constant c.
- *If* $\lambda = 0$*, show that* $c \geq 0$ *and conclude with the help of the strong minimum principle.*
- *If* $\lambda < 0$*, show that the drift laplacian satisfies* $\Delta_f v = -\lambda v$ *for some function* v *to be determined and conclude by integration.*

Exercise 2.24. *Prove similar elliptic equations for the curvature operator and the Ricci tensor of a gradient expanding Ricci soliton:*

$$\Delta_f \mathrm{Rm}(g) + \mathrm{Rm}(g) + \mathrm{Rm}(g) * \mathrm{Rm}(g) = 0, \tag{7.1}$$
$$\Delta_f \mathrm{Ric}(g) + \mathrm{Ric}(g) + 2\,\mathrm{Rm}(g) * \mathrm{Ric}(g) = 0, \tag{7.2}$$
$$\tag{7.3}$$

where, if A *and* B *are two tensors,* $A * B$ *denotes any linear combination of contractions of the tensorial product of* A *and* B.

Exercise 2.25. *On a closed Riemannian manifold* (M^n, g) *endowed with a function* $f : M \to \mathbb{R}$*, if a function* u *satisfies* $\Delta_f u \geq \lambda u$ *for some* $\lambda \geq 0$*, show that* u *is constant. (Hint: consider first the case where* $u \geq 0$ *and use integration by parts once the differential inequality is multiplied accross by a suitable function).*

Exercise 2.26. *Prove the following Bochner identity for functions:*

$$\Delta_f |\nabla^g u|_g^2 = 2|\nabla^{g,2} u|_g^2 + 2(\mathrm{Ric}(g) + \nabla^{g,2} f)(\nabla^g u, \nabla^g u) + 2g(\nabla^g(\Delta_f u), \nabla^g u).$$

Exercise 2.27. *Show that if $\tilde{g} := e^{2u}g$ for some smooth function u on M then:*

(i)

$$\nabla^{\tilde{g}}_X Y = \nabla^g_X Y + (X \cdot u)Y + (Y \cdot u)X - g(X,Y)\nabla^g u.$$

(ii) *If $\tilde{X} := e^{-u}X$ and $\tilde{Y} := e^{-u}Y$,*

$$e^{2u}\,\mathrm{Ric}(\tilde{g})(\tilde{X},\tilde{Y}) = \left(\mathrm{Ric}(g) - (n-2)\nabla^{g,2}u - (\Delta_g u)g - (n-2)|\nabla^g u|^2_g g + (n-2)\nabla^g u \otimes \nabla^g u\right)(X,Y).$$

(iii)

$$\mathrm{R}_{\tilde{g}} = e^{-2u}\left(\mathrm{R}_g - 2(n-1)\Delta_g u - (n-2)(n-1)|\nabla^g u|^2_g\right).$$

Exercise 2.28. *Let $(M^n, g, \nabla^g f)$ an expanding gradient Ricci soliton with non-negative Ricci curvature.*

- *Show that the potential function f is equivalent to $\frac{d_g(p,\cdot)^2}{4}$ for some (hence any) $p \in M$. (Hint: use the soliton equation and the soliton identities)*
- *Show that M is diffeomorphic to Euclidean space. (Hint: use the potential function as a Morse function)*
- *Show that an expanding gradient Ricci soliton with non-negative Ricci curvature is volume non-collapsed, i.e. the Asymptotic Volume Ratio defined by $\mathrm{AVR}(g) := \lim_{r\to+\infty}\frac{\mathrm{vol}_g B_g(p,r)}{r^n}$ is positive. (Hint: use the trace of the soliton equation and mimic the proof of Bishop-Gromov theorem)*