CHAPTER 8

Existence, uniqueness and convergence of the Ricci flow

1. Equivalence between the Ricci flow and the DeTurck Ricci flow

1.1. From the DeTurck Ricci flow to the Ricci flow. This short section explains formally how one can start the Ricci flow from a smooth Riemannian metric on a closed manifold. The main issue is that the Ricci flow equation is invariant under the group of diffeomorphisms of the underlying manifold which makes this parabolic equation degenerate. To circumvent this issue, Hamilton [**Ham82**] used a Nash-Moser iteration (roughly speaking, a fixed point theorem between suitable Fréchet spaces). Shortly after, DeTurck [**DeT83**] managed to prove the existence of a Ricci flow by coupling this equation with the so called Harmonic map heat flow that we explain below. This in turn let him to apply standard results from quasilinear parabolic equations for systems: see [**LSUc68**] for instance.

It is easier to understand how to define a solution of the Ricci flow from a solution to the Ricci-DeTurck flow that we now define: the Ricci-DeTurck equation on a smooth manifold M^n endowed with a smooth background metric g_b is:

$$\begin{aligned}\frac{\partial}{\partial t}g(t) &= -2\operatorname{Ric}(g(t)) + \mathcal{L}_{V(t)}(g(t)),\\ g(0) &:= g_0,\end{aligned}$$

where $V(t) = V(g(t), g_b)$ is a vector field defined by duality by

$$g(t)(V(t), X) := \operatorname{tr}_{g(t)}\left((u,v) \to \nabla^{g_b}_u (g(t)-g_b)_{v,X} - \frac{1}{2}\nabla^{g_b}_X (g(t)-g_b)_{u,v}\right), \quad X \in C^\infty(T^*M).$$

Equivalently, in coordinates, $V^i(t) := g(t)^{ij}g(t)^{kl}(\nabla^{g_b}_k (g(t)-g_b)_{lj} - \frac{1}{2}\nabla^{g_b}_j (g(t)-g_b)_{kl})$. By definition of the Christoffel symbols, we see that

$$V^i(t) = g(t)^{kl}(\Gamma(g(t))^i_{kl} - \Gamma(g_b)^i_{kl}), \quad i = 1, ..., n. \tag{1.1}$$

The following lemma links solutions to the DeTurck Ricci flow to solutions to the Ricci flow:

Lemma 8.1. *Let $(M^n, \tilde{g}(t))_{t\in(0,T)}$ be a solution to the DeTurck-Ricci flow with background metric g_b and let $(\psi_t)_{t\in(0,T)}$ be a one-parameter family of diffeomorphisms governed by the ODE $\partial_t\psi_t = -V(t)\circ\psi_t$ then the metrics $g(t) := \psi_t^*\tilde{g}(t)$ define a solution to the Ricci flow.*

Proof. Left as an exercise. □

In order to convince ourselves that the Ricci-DeTurck flow is a non-degenerate parabolic equation, we need to rewrite this equation in the following way:

Lemma 8.2.

$$\begin{aligned}\frac{\partial}{\partial t}g(t)_{ij} &= g(t)^{pq}\nabla^{g_b}_p\nabla^{g_b}_q g(t)_{ij} - g(t)^{pq}g(t)_{ik}g_b^{kl}\operatorname{Rm}(g_b)_{jplq} - g(t)^{pq}g(t)_{jk}g_b^{kl}\operatorname{Rm}(g_b)_{iplq}\\ &\quad + g(t)^{pq}g(t)^{kl}\left(\frac{1}{2}\nabla^{g_b}_i g(t)_{kp}\nabla^{g_b}_j g(t)_{lq} + \nabla^{g_b}_p g(t)_{jk}\nabla^{g_b}_l g(t)_{iq} - \nabla^{g_b}_p g(t)_{jk}\nabla^{g_b}_q g(t)_{il}\right)\\ &\quad - g(t)^{pq}g(t)^{kl}\left(\nabla^{g_b}_j g(t)_{kp}\nabla^{g_b}_q g(t)_{il} + \nabla^{g_b}_i g(t)_{kp}\nabla^{g_b}_q g(t)_{jl}\right).\end{aligned}$$

Equivalently,

$$\begin{aligned}\frac{\partial}{\partial t}(g(t)-g_b) &= \Delta_{L,g_b}(g(t)-g_b) + R_0(t) + R_1(t) + \nabla^{g_b} R_2(t) - 2\operatorname{Ric}(g_b),\\
R_0(t)_{ij} &:= \left(g(t)^{pq} - g_b^{pq}\right)(g(t)-g_b)_{ik} g_b^{kl}\operatorname{Rm}(g_b)_{jplq}\\
&\quad - \left(\left(g(t)^{-1}-g_b^{-1}\right)\circ(g(t)-g_b)\circ g_b^{-1}\right)_{pq}\operatorname{Rm}(g_b)_{jplq}\\
&= (g(t)^{-1}-g_b^{-1}) * (g(t)-g_b) * g_b^{-1} * \operatorname{Rm}(g_b),\\
R_1(t) &:= g(t)^{pq} g(t)^{kl}\left(\frac{1}{2}\nabla_i^{g_b} g(t)_{kp}\nabla_j^{g_b} g(t)_{lq} + \nabla_p^{g_b} g(t)_{jk}\nabla_l^{g_b} g(t)_{iq} - \nabla_p^{g_b} g(t)_{jk}\nabla_q^{g_b} g(t)_{il}\right)\\
&\quad - g(t)^{pq} g(t)^{kl}\left(\nabla_j^{g_b} g(t)_{kp}\nabla_q^{g_b} g(t)_{il} + \nabla_i^{g_b} g(t)_{kp}\nabla_q^{g_b} g(t)_{jl}\right)\\
&= g(t)^{-1} * g(t)^{-1} * \nabla^{g_b}(g(t)-g_b) * \nabla^{g_b}(g(t)-g_b),\\
R_2(t)_{ij}^p &:= \left(g(t)^{pq} - g_b^{pq}\right)\nabla_q^{g_b}(g(t)-g_b)_{ij}\\
&= \left(g(t)^{-1}-g_b^{-1}\right) * \nabla^{g_b}(g(t)-g_b).\end{aligned}$$

Proof. We refer the reader to the proof in coordinates given in [**CLN06**, Lemma 7.50] for instance.

Taken this expression for granted, let us prove the equivalent linearized quantitative version: the only term that needs to be rewritten concerns $R_0(t)$ since

$$g(t)^{pq}\nabla_p^{g_b}\nabla_q^{g_b} g(t)_{ij} = \Delta_{g_b} g(t)_{ij} + \nabla_p^{g_b}\left(\left(g(t)^{pq} - g_b^{pq}\right)\nabla_q^{g_b} g(t)_{ij}\right).$$

Now, by linearizing around g_b,

$$\begin{aligned}g(t)^{pq} g(t)_{ik} g_b^{kl}\operatorname{Rm}(g_b)_{jplq} &= \left(g(t)^{pq} - g_b^{pq}\right) g(t)_{ik} g_b^{kl}\operatorname{Rm}(g_b)_{jplq} + g_b^{pq} g(t)_{ik} g_b^{kl}\operatorname{Rm}(g_b)_{jplq}\\
&= \left(g(t)^{pq} - g_b^{pq}\right) g(t)_{ik} g_b^{kl}\operatorname{Rm}(g_b)_{jplq} - ((g(t)-g_b)\circ\operatorname{Ric}(g(t)))_{ij} - \operatorname{Ric}(g_b)_{ij}\\
&= \left(g(t)^{pq} - g_b^{pq}\right)(g(t)-g_b)_{ik} g_b^{kl}\operatorname{Rm}(g_b)_{jplq} + \left(g(t)^{pq} - g_b^{pq}\right)\operatorname{Rm}(g_b)_{jpiq}\\
&\quad - ((g(t)-g_b)\circ\operatorname{Ric}(g(t)))_{ij} - \operatorname{Ric}(g_b)_{ij}\\
&= \left(g(t)^{pq} - g_b^{pq}\right)(g(t)-g_b)_{ik} g_b^{kl}\operatorname{Rm}(g_b)_{jplq}\\
&\quad - \left(\left(g(t)^{-1}-g_b^{-1}\right)\circ h\circ g_b^{-1}\right)_{pq}\operatorname{Rm}(g_b)_{jplq} - (g_b^{-1}\circ h\circ g_b^{-1})_{pq}\operatorname{Rm}(g_b)_{jpiq}\\
&\quad - ((g(t)-g_b)\circ\operatorname{Ric}(g(t)))_{ij} - \operatorname{Ric}(g_b)_{ij}\\
&= \left(g(t)^{pq} - g_b^{pq}\right)(g(t)-g_b)_{ik} g_b^{kl}\operatorname{Rm}(g_b)_{jplq}\\
&\quad - \left(\left(g(t)^{-1}-g_b^{-1}\right)\circ(g(t)-g_b)\circ g_b^{-1}\right)_{pq}\operatorname{Rm}(g_b)_{jplq}\\
&\quad + \overset{\circ}{\operatorname{Rm}}(g_b)(g(t)-g_b)_{ij} - ((g(t)-g_b)\circ\operatorname{Ric}(g(t)))_{ij} - \operatorname{Ric}(g_b)_{ij}.\end{aligned}$$

Inverting the roles of indices i and j gives the expected result. □

1.2. From the Ricci flow to the DeTurck Ricci flow. Now, we finally answer the converse question, i.e. how one defines a solution to the DeTurck Ricci flow from a solution to the Ricci flow. This aspect is crucial to tackle uniqueness questions that we will explain in the next section.

The idea is to reverse the construction from above. Unfortunately, it does not lead to an ODE but rather leads to a PDE.

We first define the Laplacian of a C^2_{loc} map $\varphi : (M^n, g) \to (N^m, h)$.

Recall that $d\varphi$ is a section of $T^*M \otimes \varphi^*TN$ on which we consider the tensor product connection induced by those of g and φ^*h. Here the symbol φ^*h does not make sense if understood as the usual pull-back of the Riemannian metric h by an immersion φ. In general, we can still make sense of the pull-back connection on φ^*TN as follows: if s is a section of TN, then denote by φ^*s as the section

$s \circ \varphi$ of φ^*TN. Then there is a unique connection denoted by ∇^{φ^*TN} on φ^*TN induced by h such that

$$\nabla_X^{\varphi^*TN}(\varphi^* s)\Big|_p = \varphi^*\left(\nabla^h_{d\varphi(X)}s\right)\Big|_p, \quad \forall p \in M.$$

Here we emphasize on the fact that the previous definition is pointwise since $d\varphi(X)$ is not a section of TN. Now, since ∇^h is a connection, $\nabla^h_{d\varphi(X)}s$ only depends on the value of $d\varphi(X)$ at a point.

Locally speaking, let $y \in N$ and let $(f_\alpha)_{1\leq\alpha\leq m}$ be a system of sections of TN providing a base of each fibre over a neighborhood V_y of y in N. Let $x \in M$ and a neighborhood U_x such that $\varphi(U_x) \subset V_y$ and consider a section ρ of φ^*TN so that $\rho = \rho^\alpha \varphi^* f_\alpha$ together with a section X of TM. If such a connection exists then necessarily:

$$\begin{aligned}\nabla_X^{\varphi^*TN}\rho &= (X\cdot\rho^\alpha)\,\varphi^* f_\alpha + \rho^\alpha \nabla_X^{\varphi^*TN}(\varphi^* f_\alpha) = (X\cdot\rho^\alpha)\,\varphi^* f_\alpha + \rho^\alpha d\varphi^\beta(X)\varphi^*\left(\nabla^h_{f_\beta} f_\alpha\right)\\ &= (X\cdot\rho^\alpha)\,\varphi^* f_\alpha + \rho^\alpha d\varphi^\beta(X)\varphi^*\left(\Gamma^\gamma_{\beta\alpha}(h) f_\gamma\right)\\ &= (X\cdot\rho^\alpha)\,\varphi^* f_\alpha + \rho^\alpha d\varphi^\beta(X)\varphi^*\Gamma^\gamma_{\beta\alpha}(h)\varphi^* f_\gamma.\end{aligned}$$

If $\nabla^{g,h}$ denotes the induced connection on $T^*M \otimes \varphi^*TN$ by the Riemannian metrics g and h, then the Hessian of φ is defined as:

$$(X,Y) \in TM \times TM \to \nabla^{g,h}(d\varphi)(X,Y) := \nabla_X^{\varphi^*TN}(d\varphi(Y)) - d\varphi(\nabla^g_X Y) \in \varphi^*TN.$$

It is a section of $T^*M \otimes T^*M \otimes \varphi^*TN$ and by definition, in coordinates:

$$\nabla^{g,h}(d\varphi)^\alpha_{ij} = \frac{\partial^2\varphi^\alpha}{\partial x_i \partial x_j} - \Gamma^k_{ij}(g)\frac{\partial\varphi^\alpha}{\partial x_k} + \Gamma^\alpha_{\beta\gamma}(h)\circ\varphi\frac{\partial\varphi^\beta}{\partial x_i}\frac{\partial\varphi^\gamma}{\partial x_j} = \nabla^{g,2}\varphi^\alpha_{ij} + \Gamma^\alpha_{\beta\gamma}(h)\circ\varphi\frac{\partial\varphi^\beta}{\partial x_i}\frac{\partial\varphi^\gamma}{\partial x_j}.$$

From the above expression, it is immediate that $\nabla^{g,h}(d\varphi)$ is symmetric, i.e. $\nabla^{g,h}(d\varphi)(X,Y) = \nabla^{g,h}(d\varphi)(Y,X)$ for all sections X and Y of TM.

Then the Laplacian of φ, denoted by $\Delta_{g,h}\varphi$ is defined by:

$$\Delta_{g,h}\varphi := \operatorname{tr}_g\left(\nabla^{g,h}(d\varphi)\right). \tag{1.2}$$

It is a section of φ^*TN. In coordinates, the Laplacian of such a map is:

$$\Delta_{g,h}\varphi^\alpha := g^{ij}\frac{\partial^2\varphi^\alpha}{\partial x_i\partial x_j} - g^{ij}\Gamma^k_{ij}(g)\frac{\partial\varphi^\alpha}{\partial x_k} + g^{ij}\Gamma^\alpha_{\beta\gamma}(h)\circ\varphi\frac{\partial\varphi^\beta}{\partial x_i}\frac{\partial\varphi^\gamma}{\partial x_j}, \quad \alpha = 1,...,m. \tag{1.3}$$

Let us summarize a few properties of such an operator:

Lemma 8.3. *Let $\varphi : (M^n, g) \to (N^m, h)$ be a C^2_{loc} map.*

(i) *If $(N,h) = (\mathbb{R}, \mathrm{eucl})$ then $\Delta_{g,h}\varphi$ coincides with the Laplacian for functions, i.e. $\Delta_{g,h}\varphi = \Delta_g\varphi$.*
(ii) *If φ is a local diffeomorphism then:*

$$\Delta_{g,h}\varphi = \left(\Delta_{(\varphi^{-1})^*g,h}\mathrm{Id}\right)(\varphi),$$

where $\mathrm{Id} : (N, (\varphi^{-1})^*g) \to (N,h)$.

Proof. Left as an exercise. □

Let $(M^n, g(t))_{t\in(0,T)}$ be a solution to the Ricci flow and let g_b be a background Riemannian metric on M.

A one-parameter family of self-maps $\varphi_t : (M, g(t)) \to (M, g_b)$ is a solution to the harmonic map heat flow coming out of the identity map if:

$$\frac{\partial}{\partial t}\varphi_t = \Delta_{g(t),g_b}\varphi_t, \quad \text{on } M\times(0,T), \quad \varphi_t\big|_{t=0} = \mathrm{Id}_M.$$

We summarize the main result of the end of this section on the lemma below:

Lemma 8.4. *Let $(M^n, g(t))_{t\in(0,T)}$ be a solution to the Ricci flow and let $(\varphi_t)_{t\in(0,T)}$ be a solution to the harmonic map heat flow coming out of the identity map. Then the one-parameter family of metrics $(\tilde{g}(t))_{t\in(0,T)}$ defined by $\tilde{g}(t) := (\varphi_t^{-1})^*g(t)$ is well-defined and solves the DeTurck-Ricci flow with initial condition $\tilde{g}(0) = g(0)$.*

Proof. It is a straightforward computation using the definitions of the various flows we introduced previously provided we assume the invertibility of the maps φ_t^{-1}:

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{g}(t) &= (\varphi_t^{-1})^*\frac{\partial}{\partial t}g(t) + \mathcal{L}_{W(t)}(\tilde{g}(t)), \quad W(t) := (\varphi_t)_*\left(\frac{\partial}{\partial t}\varphi_t^{-1}\circ\varphi_t\right)\\ &= -2(\varphi_t^{-1})^*\operatorname{Ric}(g(t)) + \mathcal{L}_{W(t)}(\tilde{g}(t)), \quad W(t) := (d\varphi_t)\left(\frac{\partial}{\partial t}\varphi_t^{-1}\right)\\ &= -2\operatorname{Ric}(\tilde{g}(t)) + \mathcal{L}_{W(t)}(\tilde{g}(t)), \quad W(t) = -\left(\frac{\partial}{\partial t}\varphi_t\right)(\varphi_t^{-1}).\end{aligned}$$

Now, by assumption on the maps φ_t together with Lemma 8.3,

$$W(t) = -(\Delta_{g(t),g_b}\varphi_t)(\varphi_t^{-1}) = -\Delta_{\tilde{g}(t),g_b}\mathrm{Id} = V(t),$$

where $V(t)$ is defined in (1.1). Indeed, the last equality can be checked with the help of (1.3) for instance. □

2. Uniqueness

We start this section by a uniqueness statement for solutions to the DeTurck Ricci flow:

Proposition 8.5. *Let $(M^n, g_i(t))_{t\in[0,T]}$, $i = 1, 2$, be two solutions to the DeTurck Ricci flow with background metric g_b on a closed manifold. Assume that the solutions satisfy:*

$$C_i^{-1}g_b \leq g_i(t) \leq C_i g_b, \quad |\nabla^{g_b}g_i(t)|_{g_b} + \sqrt{t}|\nabla^{g_b,2}g_i(t)|_{g_b} \leq C_i, \quad \text{on } M,$$

for some uniform positive constant C_i. If $g_1(0) = g_2(0)$ then $g_1(t) = g_2(t)$ for all $t \in [0,T]$.

Remark 8.6. The conditions required in Proposition 8.5 are easily satisfied for solutions to the DeTurck Ricci flow coming out of smooth initial metrics. In this case, all the covariant derivatives of the solutions are bounded up to $t = 0$.

Proof. Denote $h(t) := g_2(t) - g_1(t)$ and let us derive the evolution equation satisfied by $h(t)$ thanks to (the proof of) Lemma 8.2: if $\operatorname{tr}_{g_1(t)}\left(\nabla^{g_b,2}h(t)\right) := g_1(t)^{ij}\nabla_i^{g_b}\nabla_j^{g_b}h(t)$,

$$\frac{\partial}{\partial t}h(t) - \operatorname{tr}_{g_1(t)}\left(\nabla^{g_b,2}h(t)\right) = R_0^2(t) - R_0^1(t) + R_1^2(t) - R_1^1(t) + (g_2(t)^{-1} - g_1(t)^{-1}) * \nabla^{g_b,2}g_2(t),$$

where $R_j^i(t)$ refers to $R_j(t)$ as defined in Lemma 8.2 with respect to the solution $g_i(t)$.

Observe that schematically:

$$\begin{aligned}&|R_0^2(t) - R_0^1(t)|_{g_b} =\\ &\left|(g_2(t)^{-1} - g_b^{-1}) * (g_2(t) - g_b) * g_b^{-1} * \operatorname{Rm}(g_b) - (g_1(t)^{-1} - g_b^{-1}) * (g_1(t) - g_b) * g_b^{-1} * \operatorname{Rm}(g_b)\right|_{g_b}\\ &\leq \left|(g_2(t)^{-1} - g_b^{-1}) * h(t) * g_b^{-1} * \operatorname{Rm}(g_b)\right|_{g_b} + \left|(g_2(t)^{-1} - g_1(t)^{-1}) * (g_1(t) - g_b) * g_b^{-1} * \operatorname{Rm}(g_b)\right|_{g_b}\\ &\leq C|\operatorname{Rm}(g_b)|_{g_b}|h(t)|_{g_b}\\ &\leq C|h(t)|_{g_b},\end{aligned}$$

where $C = C(g_b, C_1, C_2, n)$ is a time-independent positive constant that may vary from line to line. Here we have used the fact that $g_2(t)^{-1} - g_1(t)^{-1} = -g_2(t)^{-1}\circ h(t)\circ g_1(t)^{-1}$ together with the fact that $|g_i(t)|_{g_b} \leq C_i$ for $t \in [0,T]$ and $\sup_M|\operatorname{Rm}(g_b)|_{g_b} < +\infty$.

Similarly, since $|\nabla^{g_b} g_i(t)|_{g_b} \leq C_i$ for $i=1,2$,

$$|R_1^2(t) - R_1^1(t)|_{g_b} =$$
$$\left|g_2(t)^{-1} * g_2(t)^{-1} * \nabla^{g_b}(g_2(t)-g_b) * \nabla^{g_b}(g_2(t)-g_b) - g_1(t)^{-1} * g_1(t)^{-1} * \nabla^{g_b}(g_1(t)-g_b) * \nabla^{g_b}(g_1(t)-g_b)\right|_{g_b}$$
$$\leq C|h(t)|_{g_b} + C|\nabla^{g_b} h(t)|_{g_b}.$$

Finally, since $\sqrt{t}|\nabla^{g_b,2} g_i(t)|_{g_b} \leq C_i$ for $i=1,2$,

$$\left|(g_2(t)^{-1} - g_1(t)^{-1}) * \nabla^{g_b,2} g_2(t)\right|_{g_b} \leq \frac{C}{\sqrt{t}}|h(t)|_{g_b}.$$

We can combine the previous estimates in order to derive a differential inequality satisfied by the norm of h with respect to g_b: if $g(t)^{-1}(\nabla^{g_b} h(t), \nabla^{g_b} h(t)) := g(t)^{ij} g_b^{kl} g_b^{pq} \nabla_i^{g_b} h(t)_{kp} \nabla_j^{g_b} h(t)_{lq}$ and $\operatorname{tr}_{g(t)} \nabla^{g_b,2} u := g(t)^{ij} \nabla_i^{g_b} \nabla_j^{g_b} u$ for a C^2_{loc} function,

$$\begin{aligned}
\left(\partial_t - \operatorname{tr}_{g(t)} \nabla^{g_b,2}\right)|h|^2_{g_b} &\leq -2g(t)^{-1}(\nabla^{g_b} h(t), \nabla^{g_b} h(t)) + C\left(\frac{1}{\sqrt{t}}+1\right)|h(t)|^2_{g_b} + C|\nabla^{g_b} h(t)|_{g_b}|h(t)|_{g_b} \\
&\leq -C^{-1}|\nabla^{g_b} h(t)|^2_{g_b} + \left(\frac{1}{\sqrt{t}}+1\right)|h(t)|^2_{g_b} + C|\nabla^{g_b} h(t)|_{g_b}|h(t)|_{g_b} \\
&\quad - \frac{1}{2}C^{-1}|\nabla^{g_b} h(t)|^2_{g_b} + C\left(\frac{1}{\sqrt{t}}+1\right)|h(t)|^2_{g_b} \\
&\leq C\left(\frac{1}{\sqrt{t}}+1\right)|h(t)|^2_{g_b},
\end{aligned}$$

where C is a positive constant that may vary from line to line. Here we have used Young's inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$ for $\varepsilon > 0$ in the last line.

In particular, the function $H(t) := e^{-C(\sqrt{t}+t)}|h(t)|^2_{g_b}$ satisfies for some sufficiently large positive constant C:

$$\left(\partial_t - \operatorname{tr}_{g(t)} \nabla^{g_b,2}\right) H(t) \leq 0.$$

The maximum principle as illustrated in Chapter 3 applied to the function H gives: $\sup_M H(t) \leq \sup_M H(0) = 0$ for $t \geq 0$ since $g_2(0) = g_1(0)$ by assumption. □

As a consequence of the uniqueness statement from Proposition 8.5, we obtain the following uniqueness statement for solutions to the Ricci flow:

Proposition 8.7. *Let $(M^n, g_i(t))_{t\in(0,T)}$, $i=1,2$, be two solutions to the Ricci flow on a closed Riemannian manifold coming out smoothly of the same initial metric g_0. Then $g_1(t) = g_2(t)$ for all $t \in (0,T)$.*

Proof. For $i=1,2$, let $\varphi_t^i : (M, g_i(t)) \to (M, g_0)$ be the solution to the harmonic map heat flow with respect to initial condition Id: $\partial_t \varphi_t^i = \Delta_{g_i(t), g_0} \varphi_t^i$ for $t \in (0,T)$ and $\varphi_t^i\big|_{t=0} = \mathrm{Id}$. Then the one-parameter families of metrics $\tilde{g}_i(t) := ((\varphi_t^i)^{-1})^* g_i(t)$, $i=1,2$, solve the DeTurck Ricci flow with initial condition (and background metric) g_0 by Lemma 8.4. According to Proposition 8.5, $\tilde{g}_1(t) = \tilde{g}_2(t)$ for all $t \in (0,T)$. Now, let $(\psi_t^i)_{t\in(0,T)}$ be the flow generated by the vector field $V^i(t)$ defined as in (1.1) starting from the identity at $t=0$. By definition, $V^1(t) = V^2(t)$ for all $t \in (0,T)$. Since these two flows coincide at $t=0$, they must coincide for all $t \in (0,T)$. Now, by Lemma 8.1, each one-parameter family of metrics $\bar{g}_i(t) := (\psi_t^i)^* \tilde{g}_i(t)$ solves the Ricci flow and starts from the same metric g_0. Moreover, $\bar{g}_1(t) = \bar{g}_2(t)$ for all $t \in (0,T)$. Since by construction, $\bar{g}_i(t) = g_i(t)$ for $i=1,2$, we are able to conclude the proof of the desired uniqueness statement. □

A direct corollary of the previous result is that isometries of the initial condition remain isometries of the flow:

Corollary 8.8. *Let $(M^n, g(t))_{t\in(0,T)}$ be a solution to the Ricci flow on a closed manifold coming out of a smooth metric g_0. Assume ϕ is an isometry of g_0 then ϕ is an isometry of $g(t)$ for all $t \in (0,T)$.*

In other words, the isometry group of the initial metric $\mathrm{Isom}(g_0)$ embeds in the isometry group of the flow $\mathrm{Isom}(g(t))$ as long as the flow is smooth.

A more subtle result due to Kotschwarr [**Kot10**] is that the isometry group of the flow is constant in time, i.e. $\mathrm{Isom}(g(t)) = \mathrm{Isom}(g_0)$ for all $t \in (0, T)$. This result is connected to backward uniqueness of the heat equation, i.e. if two solutions to a parabolic equation agree at some positive time, they must coincide at previous times. As such, it is an ill-posed problem because the heat equation is not reversible, a fact that explains the analytic difficulty of this problem.

3. Existence

We state the following general existence theorem due to Shi that extends Hamilton's seminal work [**Ham82**] on the existence of a solution to the Ricci flow on a closed manifold endowed with a C^2 metric.

Theorem 8.9 (Shi's solution)**.** *Let (M^n, g_0) be a complete non-compact Riemannian manifold with bounded curvature, i.e. $\sup_M |\mathrm{Rm}(g_0)|_{g_0} < +\infty$. Then there exists a complete solution to the Ricci flow $(g(t))_{t\in[0,T)}$ starting from g_0 with bounded curvature on each compact time intervals such that is maximal in the sense that either $T = +\infty$ or $\sup_{M\times[0,T)} |\mathrm{Rm}(g(t))|_{g(t)} = +\infty$.*

The proof of this theorem is a long technical one. It consists in solving the corresponding DeTurck Ricci flow on a sequence of domains that exhausts the manifold M with Dirichlet boundary conditions. Standard results from the theory of parabolic equations for systems ensure the existence for each domain. The main difficulty is that the maximal existence time for each domain may depend on its geometry. Shi's tour de force is to show that it does not by establishing so called a priori estimates on these solutions that do not depend on the sequence of exhausting domains.

Instead, we decide to illustrate the existence of the DeTurck Ricci flow for metrics that are close to Euclidean space in the L^∞ sense only to give a flavor about how one can prove such an existence result. This result is due to Koch and Lamm [**KL12**] and the techniques are different from those used in the proof of Theorem 8.9. The advantage of this approach lies in the weak regularity of the initial condition whose curvature is a priori not defined since it is assumed to be essentially bounded only!

Mimicking the proof of the Cauchy-Lipschitz theorem for ODEs, we recall the following integral formula called the Duhamel principle. A one-parameter family of metrics $(g(t))_{t>0}$ is a solution to the DeTurck Ricci flow with a background euclidean metric starting from a metric g_0 if (and only if) the tensor $h(t) := g(t) - g_{\mathrm{eucl}}$ satisfies for $t > 0$ and $x \in \mathbb{R}^n$,

$$h(x,t) := \int_{\mathbb{R}^n} K(x,t,y,0)h_0(y)\,dy + \int_0^t \int_{\mathbb{R}^n} K(x,t,y,s)\left(R_1[h](y,s) + \nabla R_2[h](y,s)\right) dyds, \quad (3.1)$$

where $K(x,t,y,s)$ denotes the Euclidean heat kernel:

$$K(x,t,y,s) := \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \exp\left\{-\frac{|x-y|^2}{4(t-s)}\right\}, \quad 0 \leq s < t, \quad x, y \in \mathbb{R}^n. \quad (3.2)$$

According to (3.1), the solution can be decomposed into two parts, the first one being the solution to the linear heat equation starting from $h_0 := g_0 - g_{\mathrm{eucl}}$ and the second one being the solution to the heat equation with a source term $S(h) := R_1[h] + \nabla R_2[h]$ starting from 0 which depends in h implicitly. The second solution will be eventually neglectable compared to the first one if we impose our solutions to stay close to g_{eucl}.

Before stating their main result, we need some preliminaries about the function spaces we will use as the all proof boils down to the application of the contraction principle applied to a suitable map defined between suitable function spaces! Define the source space X to be:

$$X := \Big\{ (h(t) \in S^2T^*\mathbb{R}^n)_{t>0} \mid \|h\|_X := \sup_{t>0} \|h(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{(x,R)\times\mathbb{R}^n\times\mathbb{R}_+} \left(R^{-\frac{n}{2}} \|\nabla h\|_{L^2(B(x,R)\times(0,R^2))} + R^{\frac{2}{n+4}} \|\nabla h\|_{L^{n+4}(B(x,R)\times(\frac{R^2}{2},R^2))} \right) < +\infty \Big\}.$$

The target space Y is decomposed as $Y_0 + \nabla Y_1$ is:

$$Y_1 := \Big\{ (H(t) \in S^2T^*\mathbb{R}^n)_{t>0} \mid \|H\|_{Y_0} := \sup_{(x,R)\times\mathbb{R}^n\times\mathbb{R}_+} \left(R^{-n} \|H\|_{L^1(B(x,R)\times(0,R^2))} + R^{\frac{4}{n+4}} \|H\|_{L^{\frac{n+4}{2}}(B(x,R)\times(\frac{R^2}{2},R^2))} \right) < +\infty \Big\}.$$

$$Y_2 := \Big\{ (H(t) \in S^2T^*\mathbb{R}^n)_{t>0} \mid \|H\|_{Y_1} := \sup_{(x,R)\times\mathbb{R}^n\times\mathbb{R}_+} \left(R^{-\frac{n}{2}} \|H\|_{L^2(B(x,R)\times(0,R^2))} + R^{\frac{2}{n+4}} \|H\|_{L^{n+4}(B(x,R)\times(\frac{R^2}{2},R^2))} \right) < +\infty \Big\}.$$

The introduction of such function spaces is legitimated by their invariance under parabolic rescalings, i.e. if $h \in X$ and $\lambda > 0$, then $h_\lambda(x,t) := h(\lambda x, \lambda^2 t)$ lies in X and $\|h_\lambda\|_X = \|h\|_X$. The same holds for the function space Y. The exponent $n+4$ (respectively $\frac{n+4}{2}$) in the definition of X (respectively Y) can be replaced by any exponent $p > n+2$ (respectively $q := \frac{p}{2} > \frac{n+2}{2}$).

Lemma 8.10. *For every $\gamma \in (0,1)$, the operator $R_1 + \nabla R_2 : B_X(0,\gamma) \to Y$ is well-defined and moreover:*

$$\|R_1[h] + \nabla R_2[h]\|_Y \leq C(n,\gamma)\|h\|_X^2, \quad h \in B_X(0,\gamma),$$

and

$$\|R_1[h'] + \nabla R_2[h'] - R_1[h] - \nabla R_2[h]\|_Y \leq C(n,\gamma) \left(\|h\|_X + \|h'\|_X\right) \|h' - h\|_X, \quad h, h' \in B_X(0,\gamma).$$

This lemma is left to the reader, it is a simple consequence of the very definitions of the function spaces and those of R_1 and ∇R_2 introduced in the previous section.

The crucial result is the following lemma:

Lemma 8.11. *Let $R := R_1 + \nabla R_2 \in Y$. Then the solution to $\frac{\partial}{\partial t} h = \Delta h + R$, $h(0) = h_0 \in L^\infty(\mathbb{R}^n)$ given by Duhamel's principle lies in X and moreover:*

$$\|h\|_X \leq C(n) \left(\|h_0\|_{L^\infty(\mathbb{R}^n)} + \|R\|_Y \right).$$

Let us show first that the combination of these two lemmata leads to the proof of the main theorem:

Theorem 8.12 (Koch-Lamm)**.** *There exists a neighborhood $B_{L^\infty(\mathbb{R}^n)}(0,\varepsilon)$ and a constant C such that there exists an immortal solution to the DeTurck-Ricci flow $g(t) - g_{\text{eucl}} \in X$ starting from $g_0 \in B_{L^\infty(\mathbb{R}^n)}(0,\varepsilon)$ and which satisfies $\|g(t) - g_{\text{eucl}}\|_X \leq C\|g_0 - g_{\text{eucl}}\|_{L^\infty(\mathbb{R}^n)}$. Moreover, such a solution is unique in the ball $B_X(0, C\varepsilon)$.*

It can be further proved that the solution obtained in Theorem 8.12 is analytic in space and time for positive time. However, if g_0 is not assumed to be smooth, the relation with the Ricci flow needs to be clarified since the flow generated by the (dual of) the Bianchi one-form is not well-defined up to $t = 0$!

Proof of Theorem 8.12. To an element $h \in X$, consider $R[h] := R_1[h] + \nabla R_2[h]$ that lies in Y by Lemma 8.10 and consider the solution to the Cauchy problem: $\frac{\partial}{\partial t} h' = \Delta h' + R[h]$, $h'(0) = h_0 \in L^\infty(\mathbb{R}^n)$. The solution h' lies in X thanks to Lemma 8.11. Therefore, define the self-map $\Phi_0 : h \in X \to h' \in X$ for $h_0 := g_0 - g_{\text{eucl}}$ fixed in a ball $B_{L^\infty(\mathbb{R}^n)}(0,\varepsilon)$ and observe that Φ_0 preserves a ball $B_X(0,C'\varepsilon)$ for some uniform positive constant C' and ε sufficiently small:

$$\begin{aligned} d_X(\Phi_0(h),0) = \|\Phi_0(h)\|_X &\leq C\left(\|h_0\|_{L^\infty(\mathbb{R}^n)} + \|R[h]\|_Y\right) \\ &\leq C\left(\varepsilon + C(n)\|h\|_X^2\right) \leq C(\varepsilon + C(n)C'^2\varepsilon^2) \\ &< C'\varepsilon, \end{aligned}$$

if C' is chosen sufficiently large and if $\varepsilon = \varepsilon(C')$ is chosen sufficiently small. Here we have made use of Lemma 8.10 in the antepenultimate estimate and we have invoked Lemma 8.11 in the penultimate estimate. Moreover, Φ_0 is a contraction in a neighborhood $B_X(0,C'\varepsilon)$ of $0 \in X$:

$$\begin{aligned} \|\Phi_0(h_2) - \Phi_0(h_1)\|_X &\leq C(n)\|R[h_2] - R[h_1]\|_Y \leq C'(n)\left(\|h_1\|_X + \|h_2\|_X\right)\|h_2 - h_1\|_X \\ &\leq C'(n)\varepsilon\|h_2 - h_1\|_X, \end{aligned}$$

where $C'(n)$ is a positive constant that may vary from line to line. Here we have used Lemma 8.10 in the first inequality and then Lemma 8.11 was used in the second estimate. Therefore, there must exists a unique fixed point h of Φ_0 in $B_X(0,C'\varepsilon)$ thanks to Picard's fixed point theorem. □

Let us sketch the proof of Lemma 8.11, we refer the reader to the original article [**KL12**] for details:

We first record quantitative estimates on the Euclidean heat kernel and its (covariant) derivatives in time and space:

Lemma 8.13. *For every non-zero integers k and l, there exists a constant $C_{k,l}$ such that:*

$$\left|\frac{\partial^l}{\partial t^l}\nabla^k K(x,t,y,s)\right| \leq C_{k,l}\left(|x-y| + \sqrt{t-s}\right)^{-n-k-2l}, \quad x,y \in \mathbb{R}^n, \quad s < t. \tag{3.3}$$

$$\left\|\frac{\partial^l}{\partial t^l}\nabla^k K(\cdot,t,y,s)\right\|_{L^1(\mathbb{R}^n)} \leq C_{k,l}(t-s)^{-l-\frac{k}{2}}, \quad s < t. \tag{3.4}$$

Moreover, for all $k \geq 0$, $\nabla^k K(x,t,0,0) \in L^p(B(0,1)\times(0,1))$ if and only if $p < \frac{n+2}{n+k}$.

Finally, if $(x,t) \in \mathbb{R}^n \times (0,1) \setminus B(0,1)\times(0,1/4)$,

$$K(x,t,0,0) + |\nabla K(x,t,0,0)| + |\nabla^2 K(x,t,0,0)| \leq Ce^{-\alpha|x|}, \tag{3.5}$$

for some uniform positive constants C and α.

Proof of Lemma 8.11. We divide the proof into two claims.

Claim 8.14. If $h(t)$ is the unique *bounded* solution to the *linear* heat equation coming out of h_0 then

$$\|h\|_X \leq C(n)\|h_0\|_{L^\infty(\mathbb{R}^n)}.$$

Proof of Claim 8.14. The L^∞ bound on the solution $h(t)$ follows from a uniform L^1 bound on the heat kernel together with the (essential) boundedness of the initial condition h_0. The bound on the L^{n+4} norm of the gradient of h over $B(x,R)\times(R^2/2,R^2)$ follows from the following L^∞ stronger bound: $\sup_{t>0}\sqrt{t}\|\nabla h(t)\|_{L^\infty(\mathbb{R}^n)} < +\infty$. This bound in turn can be obtained by showing that $|\nabla h(0,1)| \leq C(n)\|h_0\|_{L^\infty(\mathbb{R}^n)}$ since this bound is invariant by translations in space and parabolic rescalings. This pointwise gradient bound is then obtained thanks to Lemma 8.13 by differentiating under the integral sign.

Finally, the L^2 bound on the gradient over $B(x,R)\times(0,R^2)$ follows from the corresponding one over $B(0,1)\times(0,1)$. Now, let φ be a smooth cut-off function with support in $B(0,2)$ such it equals

1 identically on $B(0,1)$:

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^n}\varphi^2(x)|h(x,t)|^2\,dx &= \int_{\mathbb{R}^n}\varphi^2(x)\langle\Delta h(x,t),h(x,t)\rangle_{g_{\mathrm{eucl}}}\,dx\\
&=\frac{1}{2}\int_{\mathbb{R}^n}\varphi^2(x)\Delta|h(x,t)|^2\,dx-\int_{\mathbb{R}^n}\varphi^2(x)|\nabla h(x,t)|^2\,dx\\
&=-\int_{\mathbb{R}^n}2\langle\nabla_{\nabla\varphi(x)}h(x,t),\varphi(x)h(x,t)\rangle+\varphi^2(x)|\nabla h(x,t)|^2\,dx\\
&\leq-\frac{1}{2}\int_{\mathbb{R}^n}\varphi^2(x)|\nabla h(x,t)|^2\,dx+2\int_{\mathbb{R}^n}|\nabla\varphi(x)|^2|h(x,t)|^2\,dx,
\end{aligned}
$$

where we have used an integration by parts together with the elementary inequality $2ab\leq\frac{1}{2}a^2+2b^2$. Integration in time shows the expected result since we have already proved $\sup_{t>0}\|h(t)\|_{L^\infty(\mathbb{R}^n)}\leq\|h_0\|_{L^\infty(\mathbb{R}^n)}$. This ends the proof of the claim.

□

Claim 8.15. If $h(t)$ is the unique *bounded* solution to the *linear* heat equation with a source term of the form $R:=R_1+\nabla R_2\in Y$ coming out of 0, i.e. if $\partial_t h-\Delta h=R\in Y$ on $\mathbb{R}^n\times\mathbb{R}_+$ with $h|_{t=0}=h_0=0$ then

$$\|h\|_X\leq C(n)\|R\|_Y.$$

Proof of Claim 8.15. The desired estimate is invariant under translations in space and parabolic scalings so that it is enough to show that:

$$|h(0,1)|+\|\nabla h\|_{L^2(B(0,1)\times(0,1))}+\|\nabla h\|_{L^{n+4}(B(0,1)\times(\frac{1}{2},1))}\leq c\|R\|_Y.$$

Let us start with the L^∞ bound: the idea consists in splitting the space-time integral into two complementary subsets depending on the singularity of the heat kernel. Let $P:=B(0,1)\times(1/2,1)$ and observe that by Hölder's inequality:

$$
\begin{aligned}
\left|\int_P K(0,1,x,t)R_1(x,t)\,dxdt\right| &\leq \|K(0,1,\cdot,\cdot)\|_{L^{\frac{n+4}{2}*}(P)}\|R_1\|_{L^{\frac{n+4}{2}}(P)}\\
&\leq\|K(0,0,\cdot,\cdot)\|_{L^{\frac{n+4}{n+2}}(B(0,1)\times(0,1/2))}\|R\|_Y\leq c\|R\|_Y.
\end{aligned}
$$

Here we have used [(3.3), Lemma 8.13] in the last line to ensure the finiteness of the $L^{\frac{n+4}{n+2}}(B(0,1)\times(0,1/2))$-norm of the heat kernel centered at $(0,0)$. Similarly, one can prove that:

$$\left|\int_P\nabla K(0,1,x,t)R_2(x,t)\,dxdt\right|\leq c\|R\|_Y.$$

Now, we invoke [(3.5), Lemma 8.13] to estimate the remaining terms integrated on $\mathbb{R}^n\times(0,1)\setminus P$ as follows:

$$
\begin{aligned}
\left|\int_{\mathbb{R}^n\times(0,1)\setminus P}K(0,1,x,t)R_1(x,t)\,dxdt\right| &\leq c\sum_{y\in\mathbb{Z}^n}\int_{B(y,1)\times(0,1)}e^{-c|x|}|R_1(x,t)|\,dxdt\\
&\leq c\|R\|_Y,
\end{aligned}
$$

where c is a positive constant that may vary from lie to line. Here we have used Hölder's inequality in the last estimate.

The $L^2(B(0,1)\times(0,1))$-norm can be estimated as in the previous claim, by integration by parts together with the L^∞ bound we just proved.

The same reasoning we used for proving the L^∞ bound shows:

$$\sup_{(x,t)\in P}\left|\int_{\mathbb{R}^n\times(0,1)\setminus B(0,2)\times(1/4,1)}\left(\nabla K(x,t,y,s)R_1(y,s)-\nabla^2K(x,t,y,s)R_2(y,s)\right)\,dyds\right|\leq c\|R\|_Y,$$

thanks to [(3.5), Lemma 8.13] again. This bounds in particular the L^{n+4} norm of ∇h on P as desired. We can therefore assume that the data R_0 and R_1 has support in $B(0,2)\times(1/4,1)$. However, a similar approach fails in that case since $\nabla^k K(\cdot,\cdot,0,0)$ is in $L^p(B(0,1)\times(0,1))$ if and only if $p<\frac{n+2}{n+k}$.

Instead, we invoke Young's inequality on $\mathbb{R}^n\times\mathbb{R}_+$:

$$\|f*g\|_{L^r(\mathbb{R}^n\times\mathbb{R}_+)}\leq c\|f\|_{L^p(\mathbb{R}^n\times\mathbb{R}_+)}\|g\|_{L^q(\mathbb{R}^n\times\mathbb{R}_+)},$$

where $p\geq 1$, $q\geq 1$ and $r\geq 1$ are related so that $p^{-1}+q^{-1}=1+r^{-1}$.

We apply this inequality to $\nabla K(0,1,\cdot,\cdot)$ and R_1 with $r:=n+4$, $q:=\frac{n+4}{2}$ and $p=\frac{n+4}{n+3}$ to get the desired estimate holding on $\nabla K(0,1,\cdot,\cdot)*R_1$. This reasoning fails for $\nabla^2 K(0,1,\cdot,\cdot)*R_2$.

Let us consider the unique "mild" solution to $\partial_t h-\Delta h=\nabla R_2$, $h|_{t=0}=0$ with $R_2\in L^2(\mathbb{R}^n\times\mathbb{R}_+)$ given by Duhamel's principle. Then by integration by parts, once the previous equation has been multiplied accross by h, one gets:

$$\|\nabla h\|_{L^2(\mathbb{R}^n\times\mathbb{R}_+)}\leq c\|R_2\|_{L^2(\mathbb{R}^n\times\mathbb{R}_+)}.$$

Therefore, the linear operator (denoted by T here) sending R_2 to ∇h is a continuous linear operator from $L^2(\mathbb{R}^n\times\mathbb{R}_+)$ to $L^2(\mathbb{R}^n\times\mathbb{R}_+)$ with singular kernel $\nabla^2 K$. If we equip $\mathbb{R}^n\times\mathbb{R}_+$ with the parabolic metric $d_{par}(x,t),(y,s)):=|x-y|+\sqrt{|t-s|}$ then [**Ste93**, Chapter 1, Section 5, Theorem 3] and [**Ste93**, Section 7.4] ensures that T is bounded as an operator from $L^p(\mathbb{R}^n\times\mathbb{R}_+)$ to $L^p(\mathbb{R}^n\times\mathbb{R}_+)$ for every $p\in(1,+\infty)$ which ends the proof of the claim. □

□

4. Exercises

Exercise 8.16. *Prove Proposition 8.5 for M non-compact and if $g_i(t)$, $i = 1, 2$ are two complete solutions to DeTurck Ricci flow with background metric g_b satisfying $\sup_M |\mathrm{Rm}(g_b)|_{g_b} < +\infty$.*

Exercise 8.17. *Prove Lemma 8.1.*

Exercise 8.18. *Prove Lemma 8.3.*

Exercise 8.19. *Prove Lemma 8.10.*

Exercise 8.20. *Prove Lemma 8.13.*

Exercise 8.21. *([**Eva10**]) Let Ω be a bounded domain of $\mathbb{R}^n$ with smooth boundary. Let u_1 and u_2 be two C^2 solutions on $\overline{\Omega} \times [0, T]$ with $T > 0$ to the heat equation satisfying the same boundary condition $g : \partial\Omega \times [0, T] \to \mathbb{R}$ assumed to be continuous:*

$$\begin{cases} \partial_t u_i - \Delta u_i = 0, & \text{on } \Omega \times]0, T[, \\ u_i = g, & \text{on } \partial\Omega \times [0, T]. \end{cases} \tag{4.1}$$

We assume that $u_1(x, T) = u_2(x, T)$ for every $x \in \Omega$. Consider the L^2 norm of the difference $w := u_2 - u_1$:

$$e(t) := \int_\Omega w^2(x, t)\, dx, \quad t \in [0, T].$$

(i) *Show that $\dot{e}(t) := \frac{d}{dt} e(t) = -2 \int_\Omega \|\nabla w\|^2(x, t)\, dx$ for $t \in]0, T[$.*

(ii) *Show also that $\ddot{e}(t) := \frac{d^2}{dt^2} e(t) = 4 \int_\Omega (\Delta w)^2(x, t)\, dx$ for $t \in]0, T[$.*

(iii) *Show that $(\dot{e}(t))^2 \leq e(t)\ddot{e}(t)$ for $t \in]0, T[$.*

(iv) *Assume that for $t \in]0, T[$, $e(t) > 0$.*

  (a) *Show that $0 \leq e((1 - \tau)t_1 + \tau t_2) \leq e(t_1)^{1-\tau} e(t_2)^\tau$ for $\tau \in [0, 1]$ and $0 < t_1 \leq t_2 < T$.*

  (b) *Conclude.*