APPENDIX A

Results from Riemannian geometry

Recall the following equation satisfied by the hessian of the distance function to a point outside the cut-locus of that point:

Lemma A.1. *Let (M^n, g) be a Riemannian manifold. Let p in M and let $r_p := d_g(p, \cdot)$. Denote by $h = h_r$ the second fundamental form of the geodesic spheres $S_g(p, r)$. Then for X and Y tangent to $S_g(p, r)$,*

$$\nabla^g_{\nabla^g r_p} h_r(X, Y) = -\operatorname{Rm}(g)(X, \nabla^g r_p, \nabla^g r_p, Y) - h_r \circ h_r(X, Y),$$

at point where r_p is smooth. In particular, the mean curvature H_r of $S_g(p, r)$ satisfies:

$$\nabla^g_{\nabla^g r_p} H_r = -\operatorname{Ric}(g)(\nabla^g r_p, \nabla^g r_p) - |h_r|^2_g \leq -\operatorname{Ric}(g)(\nabla^g r_p, \nabla^g r_p) - \frac{H_r^2}{n-1}.$$

PROOF. The second equation for H_r is simply obtained by tracing the first equation holding on h_r. For the sake of readability, let is remove the dependence of r_p on the point p and the Levi-Civita connection on the metric g:

$$\begin{aligned}
\nabla_{\nabla r} h_r(X, Y) &= \nabla_{\nabla r}(h_r(X, Y)) - h_r(\nabla_{\nabla r} X, Y) - h_r(X, \nabla_{\nabla r} Y)\\
&= g(\nabla_{\nabla r} \nabla_X \nabla r, Y) + g(\nabla_X \nabla r, \nabla_{\nabla r} Y) - g(\nabla_{\nabla_{\nabla r} X} \nabla r, Y) - g(\nabla_X \nabla r, \nabla_{\nabla r} Y)\\
&= -g([\nabla_X, \nabla_{\nabla r}]\nabla r, Y) + g(\nabla_{[X, \nabla r]} \nabla r, Y) - g(\nabla_{\nabla_X \nabla r} r, Y)\\
&= -\operatorname{Rm}(X, \nabla r, \nabla r, Y) - h_r(\nabla_X \nabla r, Y)\\
&= -\operatorname{Rm}(X, \nabla r, \nabla r, Y) - h_r \circ h_r(X, Y).
\end{aligned}$$

Here we have used the definition of the curvature tensor in fourth line together with the vanishing $\nabla_{\nabla r} \nabla r = 0$

□

Define for $k \in \mathbb{R}$ the following function:

$$f_k(r) := \begin{cases} \frac{\sin(\sqrt{k} t)}{\sqrt{k}} & \text{pour } t \in (0, \pi/\sqrt{k}) \text{ si } k > 0,\\ t & \text{pour } t > 0 \text{ si } k = 0,\\ \frac{\sinh(\sqrt{-k} t)}{\sqrt{-k}} & \text{pour } t > 0 \text{ si } k < 0. \end{cases}$$

Theorem A.2. *Let (M^n, g) satisfies $k \leq K_g \leq K$. Then for X orthogonal to $\nabla^g r_p$,*

$$\frac{f_K'(r)}{f_K(r)} g(X, X) \leq \nabla^{g,2} r_p(X, X) \leq \frac{f_k'(r)}{f_k(r)} g(X, X).$$

This inequality holds at points where r_p is smooth.

Before proving this theorem, we recall the following comparison estimate for Ricatti equation:

Lemma A.3. *Let $y : (0, a) \to \mathbb{R}$ differentiable be such that $-K \leq y'(r) + y^2(r) \leq -k$ on $(0, a)$. If $y(r) = r^{-1} + o(1)$ as r goes to 0 then:*

$$\frac{f_K'(r)}{f_K(r)} \leq y(r) \leq \frac{f_k'(r)}{f_k(r)},$$

as long as $f_K(r) > 0$ and $f_k(r) > 0$.

PROOF. Let us prove the upper bound only as the proof for the lower bound is analogous. Define $y_k(r) := \frac{f_k'(r)}{f_k(r)}$ and observe by a direct computation that y_k satisfies

$$y_k'(r) + y_k^2(r) = -k,\ r \in (0,a), \quad y_k(r) = \frac{1}{r} + O(r),$$

In particular, the function $z(r) := y(r) - y_k(r)$ satisfies:

$$v'(r) \leq -(y_k(r) + y(r))v(r),\ r \in (0,a), \quad v(r) = O(r).$$

Grönwall lemma ensures that

$$v(r) \leq \exp\left(-\int_s^r (y_k(u) + y(u))\, du\right) v(s), \quad 0 < s < r.$$

By sending s to 0 and since $v(s) = o(1)$, the result follows.

□

We are now in a position to prove Theorem A.2:

PROOF OF THEOREM A.2. Let us prove the upper bound only as the proof for the lower bound is analogous.

Let X be a unit vector tangent to $S_g(p,r)$ and parallel transport it along a geodesic to p. Recall that $h = \nabla^{g,2} r_p$. Then applying Lemma A.1 to X gives:

$$\nabla^g_{\nabla^g r_p}(h(X,X)) \leq -k - h \circ h(X,X).$$

Observe that by Cauchy-Schwarz inequality, $h \circ h(X,X) \geq |h(X,X)^2$ since $|X|_g = 1$. Therefore,

$$\nabla^g_{\nabla^g r_p}(h(X,X)) \leq -k - |h(X,X)|^2.$$

Applying Lemma A.3 then lets us conclude. Indeed, $2\nabla^{g,2} r_p = \mathcal{L}_{\nabla^g r_p} g$ and g coincides with Euclidean metric up to first order in geodesic coordinates which implies in particular, $\nabla^{g,2} r_p = (r_p^{-1} + o(1))g$.

□

Theorem A.4 (Volume comparison theorem: lower bounds)**.** *Let (M^n, g) be a complete Riemannian manifold with bounded positive sectional curvature, i.e. $K_g \leq K$ for some K. Then for each point $p \in M$,*

$$\mathrm{vol}_g\, B_g(p,r) \geq \mathrm{vol}_{\mathbb{M}^n(K)}\, B_{\mathbb{M}^n(K)}(r), \quad 0 \leq r \leq \mathrm{inj}_g(p),$$

where $B_{\mathbb{M}^n(K)}(r)$ denotes a geodesic ball of radius r in the n-dimensional simply connected space of constant sectional curvature K.

PROOF. According to Theorem A.2, in geodesic coordinates, $\partial_r g_r \geq 2\frac{f_K'(r)}{f_K(r)} g_r$ where $\exp_p^* g =: dr^2 + g_r$. By integrating this differential inequality from 0 to r, one gets $g_r \geq f_K^2(r) g_{\mathbb{S}^{n-1}}$, i.e. $\exp_p^* g \geq dr^2 + f_K^2(r) g_{\mathbb{S}^{n-1}} =: g_K$. By integrating on $B_g(p,r)$:

$$\mathrm{vol}_g\, B_g(p,r) = \int_{\mathbb{B}(0_p,r)} d\mu_{\exp_p^* g} \geq \int_0^r f_K^{n-1}(s)\, ds = \mathrm{vol}_{\mathbb{M}^n(K)}\, B_{\mathbb{M}^n(K)}(r),$$

as desired. □