A Tale of Three Structures : the Arithmetics of Multizetas , the Analysis of Singularities , the Lie algebra ARI.

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Abstract. Two new, fast-developing, but at first sight completely disconnected subjects have turned out to be governed by a common underlying structure. These two subjects are : the specific singularities, Stokes phenomena and resurgence patterns exhibited by *singularly perturbed systems* ; and the phenomenon of *dimorphy* (existence of a double product) displayed not only by the so-called *multizeta values* but by a host of other basic transcendental constants. As for the unifying structure, it is the novel Lie algebra ARI which, together with its group GARI and a number of related constructions, is a fascinating object in its own right.

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1 Introduction.

1.1 Multizetas and dimorphy.

For positive integer-valued arguments s_i , the sums

$$\zeta(s_1,\ldots,s_r) \;:=\; \sum\nolimits_{n_1>n_2>\cdots>n_r>0} n_1^{-s_1} n_2^{-s_2} \ldots n_r^{-s_r} \tag{1}$$

define important, presumably transcendental constants, the so-called *multiple zeta values* (MZV) or *multizetas* for short. They first appeared in Euler's Collected Works [Eu]. They are also of frequent occurence in the theory of difference equations [1] – which may be taken as an excuse for discussing them at this venue [2]. They are in fact truly ubiquitous and have a way of cropping up in the most improbable contexts, like:
1) holomorhic dynamics and holomorphic invariants – since 1975 ([E1,E2])
2) knot theory - of all places! – since 1987 ([BL,Ar])
3) number theory – since 1985 ([Z])
4) integration in Feynman diagrams – since 1985 ([Bro])
5) the Grothendieck-Teichmüller group – since 1989 ([D])

Their chief claim to attention, however, is the phenomenon of *dimorphy*, namely the existence of *two natural encodings* and *two multiplication rules.* Dimorphy actually seems to extend to nearly all 'natural' transcendental constants, but it finds its simplest and most starling expression in the $\mathbb{Q}$-ring of multizetas, and that is where it should be studied first.

1.2 Singular and singularly perturbed systems.

Singular Systems (SS for short) are systems of local differential equations:

$$S(t,f,\ldots,f^{(r)})=0 \;\; with \;\; t\sim 0 \qquad (\;\; z=t^{-1} \; or \; h(t) \; ; \; z\sim\infty) \tag{2}$$

[1] they enter as principal ingredient into the definition of the Stokes and resurgence constants associated with the solutions of difference equations.

[2] This survey is based on a talk given at Groningen University, a famous centre for difference equations.

with local analytic data, but with a formal solution (expressed in terms of power series or more general expansions like transseries) that is generically divergent. This divergence, however, being of natural origin, is usually *resummable* [3] and *resurgent* in one or several *critical times* $z = h(t)$. It is also acted upon by exotic derivations, the so-called *alien derivations* $\mathbf{\Delta}_\omega$. Lastly, it possesses non-trivial *holomorphic invariants* $\mathbb{A}_\omega$ of infinite depth, i.e. not calculable from jets of finite order. All these aspects are subsumed in one single equation, the so-called First Bridge Equation.

Singularly Perturbed Systems (SPS for short) are differential systems with an infinitesimal perturbation parameter ϵ, whose solution(s), when expanded in power series of ϵ, generically diverge(s). As a typical example, we may take differential equations of the form

$$S(t, f, \ldots, f^{(r-1)}) + \epsilon f^{(r)} = 0 \ \ with \ \ \epsilon \sim 0 \ \ (\ x = \epsilon^{-1} \ or \ h(\epsilon) \ ; \ x \sim \infty) \tag{3}$$

with the parameter ϵ sitting in front of the highest order derivative [4]. The system undergoes a qualitative change when ϵ vanishes, since the number of free parameters in the general solution drops by at least one unit [5]. We should therefore expect neither regularity nor convergence from ϵ expansions of its solution, but at most resummability and resurgence with respect to some critical $x = h(\epsilon)$.

This is indeed often the case : parametric divergence//resurgence (in ϵ//x) is roughly dual to equational divergence//resurgence (in t//z), but definitely more complex. Instead of one Bridge Equation (per critical time), we now have two, the Second and Third Bridge Equations. And, what is even more important for our present concerns, the singularities ω in the Borel plane(s) which are associated with parametric divergence//resurgence, are made up of *two* ingredients - *two* distinct sets of constants, the u_i and v_i, which interact in a quite specific way and provide the announced link-up with *dimorphy.*

1.3 The common underlying structure: ARI/GARI

In their raw form, as scalar numbers, the multizetas and the more general *dimorphic* constants are not tractable enough. They have to be replaced by suitable generating functions – one for each of the two different encodings.

"Suitable" here means three things : the two multiplication rules on scalars should translate into simple operations on the generating functions;

[3]barring the occurrence of small denominators, but even then there exists a notion of *compensated solution* ([E7]), which restores resummability.

[4]in the case of a DE, or in a corresponding position in the case of PDE.

[5]again, in the case of a DE

and the change from one scalar encoding to the other should translate into a simple correspondence between generating functions.

Then, ideally, the *irreducibles*, i.e. the ultimate building blocks into which one hopes to break down the dimorphic scalars, should correspond to remarkable, elementary special functions (say : polynomials), out of which the generating functions can then be 'assembled'. Moreover, these polynomials may reasonably be expected to carry far more structure (and yield more easily to enumeration and classification) than the scalar irreducibles, which are 'dumb', inert numbers.

That programme turns out to be feasible :
(i) with the algebra ARI and its group GARI as general framework,
(ii) with the basic involution *swap* connecting the two encodings,
(iii) with the *bialternals* (remarkable polynomials constrained by a 'double symmetry') in one-to-one correspondence with the irreducibles,
(iv) with the ARI machinery for the canonical generation of all bialternals.

ARI was already there, ready for service. It had been discovered, ten years earlier, in the course of investigations into Singularly Perturbed Systems and their strange singularity patterns. As pointed out, these singularities result from the interplay of two totally distinct sets of constants, the u_i and v_i, which mix according to the ARI operations.

1.4 Overall scheme.

We shall roughly follow the chronological order. We shall first (§2) review Singular Systems with their single Bridge Equation and their 'pure' singularities; and then move on (§3) to Singularly Perturbed Systems with their two Bridge Equations and their 'mixed' singularities, obtained from the u_i and v_i . Then (§4) we shall try to set the general framework of dimorphy, which involves both dimorphic functions (functions stable under two distinct products) and dimorphic constants (constants whose one and only product can be calculated in two completely distinct ways, leading to unexpected relations). At this point it will be opportune (§5) to introduce the ARI/GARI operations and to catalogue some basic facts. Armed with the ARI/GARI apparatus, we shall then (§6) return to dimorphy in the prototypal case of multizetas, and derive the main facts about these : free generation; existence of canonical irreducibles of two types, even-lengthed and odd-lengthed; the mechanism behind canonicity; the explicit decomposition of multizetas into irreducibles; the special status of $\zeta(2) = \pi^2$. Lastly (§7), to show the versatility of the new ARI/GARI structure, we shall mention some of its natural extensions (the superalgebra SUARI etc) and sketch further applications, notably to 'universal' dimorphy and its natural framework, the field $\mathbb{N}\mathbb{a}$ of *natural* real numbers. Pictorially :

$$
\begin{array}{ccccc}
\textit{Singular Systems} & & & \textit{General Dimorphy} & \\
\downarrow & & \nearrow & \downarrow & \\
\textit{Sing}^{ly}\textit{ Perturbed Systems} & & \nearrow & \textit{Multizeta Arithmetics} & \\
\searrow\nwarrow & \nearrow & & \nearrow\swarrow & \\
 & \textit{ARI/GARI} & & & \\
\swarrow & \downarrow & \searrow & & \\
\dots & \dots & \dots & &
\end{array}
$$

1.5 Some conventions. Reminders about moulds.

Abbreviations :
Throughout, we shall write $P(t) := 1/t$; $Q_c(t) := c/\tan(c\,t)$ and use the following shorthand for sums and differences :

$$u_{12} = u_1 + u_2,\ \ u_{123} = u_1 + u_2 + u_3,\ \dots,\ \ v_{1:2} := v_1 - v_2,\ \ etc \tag{4}$$

Moulds :
Moulds are *functions of a variable number of variables*: they depend on sequences $\boldsymbol{\omega} := (\omega_1, \dots, \omega_r)$ of arbitrary length r. The sequences are systematically written in boldface , with upper indexation when such is called for, and with the product denoting concatenation: e.g. $\boldsymbol{\omega} = \boldsymbol{\omega}^1.\boldsymbol{\omega}^2$. The elements ω_i which make up these sequences are written in normal print, with lower indexation. The sequences themselves are affixed to the moulds as upper indices $A^\bullet = \{A^{\boldsymbol{\omega}}\}$, since moulds are meant to be contracted

$$A^\bullet, B_\bullet \mapsto < A^\bullet, B_\bullet > := \sum A^{\boldsymbol{\omega}}\, B_{\boldsymbol{\omega}}$$

with dual objects (often differential operators or elements of an associative algebra), the so-called co-moulds $B_\bullet = \{B_{\boldsymbol{\omega}}\}$, which carry their own indices in lower position.

Basic mould operations :
Moulds may be *added, multiplied, composed.*
Mould addition is what you expect : components of equal length get added.
Mould multiplication (*mu* or $\times$) is associative, but non-commutative :

$$C^\bullet = \mathrm{mu}(A^\bullet, B^\bullet) = A^\bullet \times B^\bullet \Longleftrightarrow C^{\mathbf{w}} = \sum_{\mathbf{w}=\mathbf{w^1}.\mathbf{w^2}} A^{\mathbf{w^1}} B^{\mathbf{w^2}} \tag{5}$$

(This includes the trivial decompositions $\mathbf{w} = \mathbf{w}.\emptyset$ and $\mathbf{w} = \emptyset.\mathbf{w}$).
The third basic operation, mould composition, won't be required here.

Main symmetry types :
Most useful moulds fall into a few basic *symmetry types.*
A mould $A^\bullet$ is said to be *symmetral* (resp. *alternal*) iff :

$$\sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega^1},\boldsymbol{\omega^2})} A^{\boldsymbol{\omega}} = A^{\boldsymbol{\omega^1}} A^{\boldsymbol{\omega^2}} \quad (resp.\ 0) \qquad \forall\, \boldsymbol{\omega^1} \neq \emptyset\, , \forall\, \boldsymbol{\omega^2} \neq \emptyset \tag{6}$$

A mould $A^\bullet$ is said to be *symmetrel* (resp. *alternel*) iff :

$$\sum_{\boldsymbol{\omega}\in \mathrm{she}(\boldsymbol{\omega^1},\boldsymbol{\omega^2})} A^{\boldsymbol{\omega}} = A^{\boldsymbol{\omega^1}} A^{\boldsymbol{\omega^2}} \quad (resp.\ 0) \qquad \forall\, \boldsymbol{\omega^1} \neq \emptyset\, , \forall\, \boldsymbol{\omega^2} \neq \emptyset \tag{7}$$

Here $\mathrm{sha}(\boldsymbol{\omega^1},\boldsymbol{\omega^2})$ (resp. $\mathrm{she}(\boldsymbol{\omega^1},\boldsymbol{\omega^2})$) denotes the set of all sequences $\boldsymbol{\omega}$ obtained from $\boldsymbol{\omega^1}$ and $\boldsymbol{\omega^2}$ under ordinary (resp. contracting) shuffling. In a contracting shuffle, two adjacent indices ω_i and ω_j stemming from $\boldsymbol{\omega^1}$ and $\boldsymbol{\omega^2}$ respectively may coalesce into $\omega_{ij} := \omega_i + \omega_j$.
The definition of *symmetril/ alternil* is like that of symmetrel/alternel except that the contractions $\omega_i + \omega_j$ get replaced by $\omega_i \otimes \omega_j$ with:

$$A^{\dots,\omega_i\otimes\omega_j,\dots} := \omega_{i:j}^{-1}\, A^{\dots,\omega_i,\dots} + \omega_{j:i}^{-1}\, A^{\dots,\omega_j,\dots} \quad with \quad \omega_{i:j} := \omega_i - \omega_j \tag{8}$$

2 Singular systems and equational resurgence.

2.1 Resummation - monocritical or polycritical

Consider a fairly representative case of polycritical Singular System :

$$SS: \quad \frac{1}{p_i} t^{1+p_i} \dot{f}_i + \lambda_i f_i = b_i(t,f) \quad with \quad 1 \le i \le \nu\ ,\ b_i \in \mathbb{C}\{t,f\} \tag{9}$$

with unknown $f = (f_1, \dots, f_\nu)$, variable $t \sim 0$, $\dot{f}_i := df_i/dt$ and local-analytic data b_i. Assume further that SS can be brought to the normal form :

$$SS^{nor}: \qquad \frac{1}{p_i} t^{1+p_i} \dot{f}_i + \lambda_i f_i = 0 \quad with \qquad 1 \le i \le \nu \tag{10}$$

under a formal (entire) change of variable *and* unknown :

$$f_i = h_i(t, f^{nor}) \in \mathbb{C}[[t, f^{nor}]] \quad ; \quad f_i^{nor} = k_i(t,f) \in \mathbb{C}[[t,f]] \tag{11}$$

This is not always the case (there may be unremovable mixed terms left on the right-hand side of SS^{nor}) but the general case is no different as far as analytic difficulties are concerned.
Our normal system SS^{nor} is immediately integrable and by plugging its solution $f_i^{nor} := u_i \ exp(\lambda_i t^{-p_i})$ back into $h_i(t,.)$ we get the general, formal solution of SS. That solution is generically divergent, but resummable via a complex process, known as accelero-summation, which involves :

- identifying the so-called "critical times", which in the present instance are the $z_{[p_i]} := t^{-p_i}$ for all r distinct values of the p_i, and ordering them $z_1 \prec \cdots \prec z_r$ from slower to faster
- changing from the original variable t to the slowest critical time z_1 and subjecting the formal solution of SS to the formal Borel transform $z_1 \mapsto \zeta_1$.
- successively performing the $r{-}1$ acceleration transforms $z_i \mapsto z_{i+1}$ (like the Laplace transform, they entail an integration from 0 to ∞, but with a kernel of faster-than-exponential decrease at infinity)
- performing the Laplace transform $\zeta_r \mapsto z_r$ with respect to the fastest critical time (which makes sense, because in the ζ_r-plane we get, for the first time, exponential growth at infinity) and reverting to the original variable t

See [Braa,E3,E5,E8]. Pictorially :

$$\begin{array}{cccccccccc} & & & (formal) & & (geometric) & & & \\ (multiplic.) & z_1 & \longleftarrow & t & \dots & t & \longleftarrow & z_r & \\ Borel & \downarrow & & & & & & \uparrow & Laplace \\ (convolut.) & \zeta_1 & \longrightarrow & \zeta_2 & \dots & \zeta_{r-1} & \longrightarrow & \zeta_r & \\ & & accel. & & & & accel. & & \end{array}$$

Only in the monocritical case ($r{=}1$) can we dispense with acceleration. However, whether mono- or polycritical, the resummation process is always *polarising*, because it involves integrating from 0 to ∞ in the various ζ_i-planes and dodging the singularities which are generically present there : remember that the ζ-singularities reflect the fact of z-divergence ! So ultimately the solution f produced by resummation depends on the angles $\theta_1, \dots, \theta_r$ of the r rays chosen for integration.

If however SS is *real* and if we insist, legitimately enough, on getting a *real* solution, we must each time integrate on the positive real half-axis (i.e.

take $\theta_1 = \cdots = \theta_r = 0$) and then there can be no question of dodging the singularities on $\mathbb{R}^+$, because that would saddle us with imaginary parts. Nor can we 'ride rough-shod' over these real-axis singularities : what we must do is take finely honed averages (so-called *well-behaved averages* [6] like the *organic* or *Brownian* or *Catalan* etc... average) of the integrand's various determinations over $\mathbb{R}^+$ (see [Me,EM]).

2.2 Invariant Analysis. The First Bridge Equation.

The singular points ω in each given Borel ζ_q-plane are generated [7] by the corresponding multipliers λ_i

$$\Omega_q = \{\omega \; ; \; \omega = \sum_{p_j = q \; , \; n_j \geq -1} n_j \; \lambda_j\} \tag{12}$$

and the (usually ramified) singularities located there matter on two accounts : they are responsible for the divergence in the first place; and they carry important, chart-invariant information about the system SS. So it is essential to analyse them. This is accomplished by the First Bridge Equation :

$$\mathrm{BE}_1 : \qquad \mathbf{\Delta}_{\varpi} \; f = \mathbb{A}_{\varpi} \; f \tag{13}$$

with $\varpi = \binom{\omega}{q}$ and $\omega \in \Omega_q$ and

$$\mathbb{A}_{\varpi} = u^{n(\varpi)} \; \{\sum_{p_j \geq q} A^j_{\varpi}(u) \; u_j \; \frac{\partial}{\partial u_j} + \sum_{p_j < q} A^j_{\varpi}(u) \; \frac{\partial}{\partial u_j}\} \tag{14}$$

The operators $\mathbf{\Delta}_{\varpi}$ on the left-hand side of (13) are known as *alien derivations*. They measure the singularities over ω in the ζ_q-plane and the Bridge Equation tells us that their effect on f is the same as that of ordinary differential operators – namely the $\mathbb{A}_{\varpi}$ which stand on the right-hand side (13) and which are of the form (14), with scalar coefficients $A^j_{\varpi}(u) \in \mathbb{C}[[u]]$ that depend only on the 'earlier' parameters u_i , i.e. parameters associated with the *slower* critical times.

These ordinary differential operators are entirely determined by, and calculable from, the Bridge Equation, and they have the outstanding property

[6] They must verify two easy algebraic conditions (respecting realness and convolution) and a difficult analytic condition (ensuring convergence).

[7] their distribution (on the corresponding Riemann surface) is of course discrete, but their projection Ω_q on $\mathbb{C}$ may not be.

of being both *analytic* and *holomorphic* invariants [8]. Their set $\{\mathbb{A}_{\varpi}\}$, for all values of q and all $\omega \in \Omega_q$, even constitutes a *complete* and *free* system of *holomorphic* invariants [9].

2.3 Invariant Synthesis. Plain and twisted monomials.

The preceding section dealt with invariant analysis, i.e. the change from SS to $\{\mathbb{A}_{\varpi}\}$. But what about the reverse change, or invariant synthesis: constructing a singular analytic system with prescribed analytic-holomorphic invariants $\{\mathbb{A}_{\varpi}\}$? For simplicity, let us first discuss the case of a monocritical system SS with all p_i equal to 1. So there is only one critical time $z := 1/t$, only one set $\Omega := \Omega_1$, and the indices $\varpi := \binom{\omega}{q}$ of alien derivations simplify to ω.

One goes from f to f^{nor} and back:

$$f_i^{nor}(t,u) \;=\; \Theta\, f_i(t,u) \qquad ; \qquad f_i(t,u) \;=\; \Theta^{-1}\, f_i^{nor}(t,u) \tag{15}$$

by applying a formal automorphism (a substitution operator) $\Theta^{\pm 1}$ which admits two quite distinct types of expansions . The first pair reads:

$$\Theta \;=\; 1 + \sum_{r\geq 1} (-1)^r \sum_{\omega_j} \mathscr{V}^{\omega_1,\ldots,\omega_r}_{\sigma_1,\ldots,\sigma_r}(z)\;\; \mathbb{B}^{\sigma_r}_{\omega_r} \ldots \mathbb{B}^{\sigma_1}_{\omega_1} \tag{16}$$

$$\Theta^{-1} \;=\; 1 + \sum_{r\geq 1} \sum_{\omega_j} \mathscr{V}^{\omega_1,\ldots,\omega_r}_{\sigma_1,\ldots,\sigma_r}(z)\;\; \mathbb{B}^{\sigma_1}_{\omega_1} \ldots \mathbb{B}^{\sigma_r}_{\omega_r} \tag{17}$$

and involves

(i) differential operators $\mathbb{B}^{\sigma}_{\omega}$ similar in form to the $\mathbb{A}_{\omega}$ but non-invariant and, unlike the $\mathbb{A}_{\omega}$, elementarily [10] linked to the Taylor coefficients of SS:

(ii) elementary resurgence monomials $\mathscr{V}^{\omega_1,\ldots,\omega_r}_{\sigma_1,\ldots,\sigma_r}(z)$ that are ∂-friendly, meaning that they behave simply under ordinary differentiation:

$$\partial_z\, \mathscr{V}^{\,\omega_1,\ldots,\omega_r}_{\,\sigma_1,\ldots,\sigma_r}(z) = -\mathscr{V}^{\,\omega_1,\ldots,\omega_{r-1}}_{\,\sigma_1,\ldots,\sigma_{r-1}}(z)\;\; e^{\omega_r z}\, z^{-1-\sigma_r} \tag{18}$$

and less simply under alien differentiation.

[8]Here, the near-homonyms carry totally distinct meanings: *analytic invariants* are invariant under analytic changes of coordinates-and-unknown, whereas *holomorphic invariants* are invariants that depend holomorphically on the system SS, i.e. on its Taylor coefficients.

[9]though they do not always provide a *complete* system of *analytic* invariants.

[10]without the intervention of transcendental constants.

But we may also consider another, for the time being purely formal, pair of expansions, which reads:

$$\Theta = 1 + \sum_{r\geq 1} (-1)^r \sum_{\omega_j} \mathcal{U}^{\omega_1,\ldots,\omega_r}(z) \;\; \mathbb{A}_{\omega_r} \ldots \mathbb{A}_{\omega_1} \tag{19}$$

$$\Theta^{-1} = 1 + \sum_{r\geq 1} \; \sum_{\omega_j} \mathcal{U}^{\omega_1,\ldots,\omega_r}(z) \;\; \mathbb{A}_{\omega_1} \ldots \mathbb{A}_{\omega_r} \tag{20}$$

and involves

(i) the differential operators $\mathbb{A}_\omega$ themselves

(ii) elementary resurgence monomials $\mathcal{U}^{\omega_1,\ldots,\omega_r}(z)$ that are Δ-friendly, meaning that they behave simply under alien differentiation:

$$\Delta_{\omega_0} \mathcal{U}^{\omega_1,\ldots,\omega_r}(z) = \mathcal{U}^{\omega_2,\ldots,\omega_r}(z) \;\; if \;\; \omega_0 = \omega_1 \; (resp. \;\; 0 \;\; if \;\; \omega_0 \neq \omega_1) \tag{21}$$

and less so under ordinary differentiation.

Now, it is easy to check that if we start from a set of invariants $\mathbb{A}_\omega$, then construct Θ^{-1} according to formula (20), then calculate f from f^{nor} according to formula (15), we end up with an f that *formally* verifies the Bridge Equation (13) with respect to the prescribed invariants $\mathbb{A}_\omega$.

Thus, our problem – invariant synthesis – is solved, provided the power series f really verifies a differential system of type (9), with analytic data $b_i(f,t)$ on the right-hand side. Calculating the $b_i(f,t)$ as *formal power series* is easy enough (by applying ∂_z to $\Theta^{\pm 1}$) and it can be shown – this of course is the tricky part – that these series are *automatically* local-analytic if and only if two conditions are fulfilled:

(i) the operators $\mathbb{A}_\omega$ must of course be true holomorphic invariants (of some system SS), which imposes conditions on their ω-dependance (it bounds the ω-growth of closely related operators $\mathbb{A}_\omega^\pm$)

(ii) the resurgent monomials $\mathcal{U}^{\omega_1,\ldots,\omega_r}(z)$ involved in the construction of f, on top of verifying the simple resurgence equations (21), must also be *'well-behaved'*, i.e. satisfy another, far more technical condition [11], which constrains their growth as functions of the sequence $\omega_1, \ldots, \omega_r$.

Now, the simplest and most commonly used Δ-friendly monomials $\mathcal{U}^\bullet(z)$,

[11] similar to that imposed on the *well-behaved averages* mentioned at the end of § 2.1

shorn of their exponential factor [12] , may be defined by the integrals [13]

$$\mathcal{U}^{\omega_1,\ldots,\omega_r}(z) \;:=\; S.P.A. \int_0^\infty \frac{\exp(-\sum \omega_j\, y_j)}{(y_r-y_{r-1})\ldots(y_2-y_1)(y_1-z)} dy_1 \ldots dy_r \quad (22)$$

But they are not, alas, *well-behaved*. However, the closely related family of monomials obtained by introducing a positive constant c – the so-called *'twist'* – which ensures the rapid decrease of the integrand [14] near the origin:

$$\mathcal{U}_c^{\omega_1,\ldots,\omega_r}(z) \;:=\; S.P.A. \int_0^\infty \frac{\exp(-\sum(\omega_j\, y_j + c^2\, \bar{\omega}_j\,/y_j))}{(y_r-y_{r-1})\ldots(y_2-y_1)(y_1-z)} dy_1 \ldots dy_r \quad (23)$$

are indeed *well-behaved*, while still obeying the resurgence rules (21).

The long and the short of it is that, starting from any genuine set $\{\mathbb{A}_\omega\}$ of holomorphic invariants and following the above procedure, but with the plain monomials $\boldsymbol{\mathcal{U}}^{\bullet}(z)$ replaced by their twisted [15] equivalents $\boldsymbol{\mathcal{U}}^{\bullet}_c(z)$, we can produce an *analytic* system SS with the prescribed invariants. This solves *invariant synthesis*, elegantly and *canonically* (upto the choice of c), at least fot mono-critical systems.

But the Δ-friendly resurgence monomials, plain or twisted, have their counterparts in the polycritical case also, and are still given by integrals quite analogous to (22) and (23). This takes care of polycritical harmonic synthesis for systems like SS and even more general ones.

Since, furthermore, the Δ-friendly resurgence monomials are easily and explicitely differentiable with respect to their indices ω_i, and that too under *closed rules*, which never lead outside the family, the objects synthesised according to the above procedure lend themselves extremely well to the study of iso-monodromic, iso-resurgent, iso-Galois, etc... deformations.

2.4 'Display' and 'restriction'. Independence theorems.

Consider a resurgent function f, mono- or polycritical, like, say, the formal solution of the above system SS with all parameters u_i taken to be 0. Two

[12]the difference between $\mathbb{\Delta}/\Delta$, $\mathbb{V}/\mathcal{V}$, $\mathbb{U}/\mathcal{U}$ etc is simply the presence/absence of an exponential factor. Thus: $\mathbb{\Delta}_{\omega_0} \equiv \exp(-\omega_0\, z)\, \Delta_{\omega_0}$ and dually :
$\mathbb{U}^{\omega_1,\ldots,\omega_r}(z) \equiv \exp(\omega_{1..r}\; z)\, \mathcal{U}^{\,\omega_1,\ldots,\omega_r}(z) \quad ; \quad \mathbb{V}^{\omega_1,\ldots,\omega_r}_{\sigma_1,\ldots,\sigma_r}(z) \equiv \exp(\omega_{1..r}\; z)\, \mathcal{V}^{\,\omega_1,\ldots,\omega_r}_{\sigma_1,\ldots,\sigma_r}(z)$

[13]with integration along the rays $\arg(\omega_j\;\, y_j) = 0$ and with S.P.A. standing for Standard Path Averaging, which means that we must assign suitable weights to the various integration paths, arising from the choice of side (right/left) when z_i overtakes z_{i+1}

[14]with integration along the rays $\arg(\omega_j\;\, y_j) = \arg(\overline{\omega}_j\;/y_j) = 0$ and the same S.P.A. rules

[15]for any large enough twist c, with a lower bound c_0 depending only on $\{\mathbb{A}_\omega\}$

important objects, the *display* and the *restriction*, can be attached to f:

$$\mathrm{display}(f) \;:=\; f \;+\; \sum_{r\geq 1}\sum_{q\geq 1}\sum_{\varpi_j} \mathbb{Z}^{\varpi_1,\ldots,\varpi_r}\; \mathbb{\Delta}_{\varpi_r}\ldots\mathbb{\Delta}_{\varpi_1} f \tag{24}$$

$$\mathrm{restrict}(f) \;:=\; \mathrm{display}(f)\,|_{\,t=0} \tag{25}$$

The 'pseudovariables' $\mathbb{Z}^{\varpi_1,\ldots,\varpi_r}$ are symbols dual to the $\mathbb{\Delta}_{\varpi}$. Then span a commutative algebra PSEUDO dual to the associative bialgebra ALIEN of alien derivations [16]. They multiply according to the shuffle product but remain inert under (ordinary) differentiation and postcomposition.

In other words, we form the *display* by adding the pseudovariables, and then we get the *restriction* by erasing the *'true'* variable. The importance of the *display* and *restriction* comes from two facts:

(i) both carry in compact form all the analytic-holomorphic invariants of f and by way of consequence all the information about the ramifications, singularities etc of its various Borel transforms, relative to all *critical times.*

(ii) both are algebra morphisms, and indeed more than that: they commute with the full structure $+, \times, \circ, \partial, \Delta$. Therefore, any relation R, expressible in terms of these operations and involving one or several, simply or multiply resurgent functions f,g,h, etc..., is automatically verified by their *displays* and their *restrictions*:

$$R\quad(\,t;\; f, g, h, \ldots\,) \;\equiv\; 0 \tag{26}$$

$$\Downarrow$$

$$R\quad(\,t;\; \mathrm{display}(f), \mathrm{display}(g), \mathrm{display}(h), \ldots\,) \;\equiv\; 0 \tag{27}$$

$$\Downarrow$$

$$R\quad(\,0;\; \mathrm{restrict}(f), \mathrm{restrict}(g), \mathrm{restrict}(h), \ldots\,) \;\equiv\; 0 \tag{28}$$

This is extremely useful for the description and handling of obstructions to analyticity, and for establishing independence or transcendence results. Typically, if one wishes to show that a number of resurgent functions $f, g, h, \ldots$ verify no relation $R(f, g, h, \ldots) \equiv 0$ other than the ones which define $f, g, h, \ldots$ (say, systems like SS) or those finitely deducible therefrom, one may replace $f, g, h,$ in R by their *displays* or, what is often enough, by

[16] the co-commutative co-product of ALIEN induces by duality the commutative product of PSEUDO, and the associative, non-commutative product of ALIEN induces a natural action of ALIEN on PSEUDO

their *restrictions*, and then project identity (27) or (28) onto each of the pseudovariables *actually present* there. This immediately yields a huge number of often patently incompatible constraints, and so proves the impossibility of $R(f, g, h, \dots) \equiv 0$.

Indeed, although such independence and transcendence results are usually regarded as falling under the jurisdiction of Galois theory, the deeper obstructions [17] really stem from the holomorphic invariants $\mathbb{A}_\omega$ and are best handled with the help of the *display/restriction* machinery.

Summing up the whole section, we may say that the situation for Singular Systems is... singularly satisfactory, since we can rely on a strong, unitary framework ('equational resurgence', acceleration, well-behaved averaging, alien calculus, Bridge Equation, display/restriction, etc) capable of dealing with all the main aspects: resummation, invariant analysis/synthesis, independence/transcendence.

We shall see in a moment that Singularly Perturbed Systems also lend themselves to a unitary treatment, but display a whole series of new features, and a quite distinctive type of resurgence : 'co-equational resurgence'.

3 Singularly perturbed systems and co-equational resurgence.

3.1 Some heuristics: similarities/differences between SS and SPS.

To help us compare Singular and Singularly Perturbed Systems, let us consider two simple instances of the latter category:

$$\dot{f} \;=\; \lambda\, x\, f + \sum_{n+1\geq 0} a_n\, z^{-1}\, f^{n+1} \qquad (\, a_n \in \mathbb{C}\,;\; \sum a_n\, f^{n+1} \in \mathbb{C}\{f\}) \tag{29}$$

$$\dot{f} \;=\; \lambda\, x\, f + a_0(z) \qquad\qquad (a_0(z) \in \mathbb{C}\{z^{-1}\}) \tag{30}$$

with $\dot{f} \;=\; \partial_z f$
$z = 1/t =$ *unknown* $\sim \infty$
$x = 1/\epsilon =$ *singular pertubation parameter* $\sim \infty$

[17] *deeper* in the double sense of being more *potent* obstructions, and also of being *hidden* deep below the the surface: recall that the invariants $\mathbb{A}_\omega$ are not visible on finite jets; they always depend on the infinite tail-end of power series.

Clearly, due to an obvious homogeneity, the general solution of (29) is of the form $f(z,x) \in \mathbb{C}[[1/xz, u\exp(xz)]]$, and so the dependence in x is exactly like the one in z, namely divergent and resurgent [18].

But equation (30) presents us with a totally different situation: if we expand its solution into formal power series, first of $1/z$, then of $1/x$, and subject both series to the corresponding formal Borel transform, we get quite distinct results:

$$\text{Borel}_z \;:\; z^{-n} \mapsto \frac{\zeta^{n-1}}{(n-1)!} \quad ; \quad f(z,x) \mapsto -\frac{\hat{a}(\zeta)}{\lambda x + \zeta} \tag{31}$$

$$\text{Borel}_x \;:\; x^{-n} \mapsto \frac{\xi^{n-1}}{(n-1)!} \quad ; \quad f(z,x) \mapsto -\frac{1}{\lambda}\, a(z + \frac{\xi}{\lambda}) \tag{32}$$

Here $\hat{a}(\zeta)$, being the Borel transform of $a(z) \in \mathbb{C}\{1/z\}$, is an entire function and as such singularity-free. Thus the z-Borel transform of f has only *one* singular point $\zeta_0 := -\lambda\, x$, which is 'fixed' and 'universal' , whereas the x-Borel transform may have any number of singularities $\xi_i := \lambda\,(v_i - z)$ – as many as $a(z)$ has singularities v_i – and these are 'accidental', of 'no particular type' and, due to the presence of z, 'mobile'.

3.2 The Second and Third Bridge Equations.

These two extreme examples (29) and (30) suggest that there must be far-going similarities as well as sharp differences between the divergence/resurgence behaviour of Singular and Singurlarly Perturbed Systems. To clarify the matter, let us consider a Singularly Perturbed System SPS that covers both cases (29) and (30), and yet remains relatively elementary – in particular, monocritical:

$$SPS: \qquad \dot{f} \;=\; \lambda\, x\, f + \sum_{1+n\geq 0} a_n(z)\, f^{1+n} \tag{33}$$

$$\begin{aligned} z &= \textit{variable}\;; \quad z \;=\; 1/t \;\sim \infty; \quad f = f(z; x, u_0)\;;\; \dot{f} := \partial_z f \\ x &= \textit{perturbation parameter}\;; \quad x \;=\; 1/\epsilon \;\sim \infty \\ u_0 &= \textit{integration parameter}\;; \quad u_0 \;\sim 0 \end{aligned}$$

with coefficients $a_n(z)$ regular-analytic at infinity but singular at points $v_1, v_2, \ldots$ away from infinity, and with a right-hand side local-analytic in f and $1/z$:

$$a_n(z) \in \mathbb{C}\{1/z\} \quad ; \quad \sum_{1+n\geq 0} a_n(z)\, f^{1+n} \in \mathbb{C}\{1/z, f\}$$

[18] since for a fixed x equation (29) reduces to a special, monocritical case of singular system SS

If we fix x, the above SPS becomes a mere SS, and its formal solution, as a function of z, satisfies the first First Bridge Equation, with *holomorphic invariants* $\mathbb{A}_\omega$ that are entire functions of x :

$$\begin{array}{lll} \mathrm{BE}_1 & \boldsymbol{\Delta}_\omega\, f = \mathbb{A}_\omega f & (34) \\ \textit{with} & \mathbb{A}_\omega = A_\omega(x) u_0^{n+1} \partial_{u_0} \quad \textit{and} \qquad \omega = n\lambda x \in \Omega_1 := \mathbb{U}.x & \end{array}$$

If however we fix z and expand the solution f in decreasing powers of x, we find that f verifies the so-called Second Bridge Equation :

$$\begin{array}{lll} \mathrm{BE}_2 & \boldsymbol{\Delta}_\omega\, f = \mathbb{P}_\omega f & (35) \\ \textit{with} & \mathbb{P}_\omega = P_\omega(x) u_0^{n+1} \partial_{u_0} \quad \textit{and} \qquad \omega = n\lambda(v_i - z) \in \Omega_2 := \mathbb{U}.(\mathbb{V} - z) & \end{array}$$

which resembles the first, but for two important differences : the active alien derivations have now more complicated indices ω ; and whereas the operators $\mathbb{A}_\omega$ in (34) were entire functions of the perturbation parameter x, the new operators $\mathbb{P}_\omega$ are formal power series of $1/x$, with a divergence/resurgence of their own, which is described by a Third Bridge Equation :

$$\begin{array}{lll} \mathrm{BE}_3 & \boldsymbol{\Delta}_\omega\, \mathbb{P}_{\omega_1} = \sum\limits_{\omega+\omega_1=\omega_2+\omega_3} [\mathbb{P}_{\omega_2}, \mathbb{P}_{\omega_3}] & (36) \\ \textit{with} & \mathbb{P}_\omega \ \textit{as above} \quad \textit{and} \qquad \omega = n\lambda(v_i - v_j) \in \Omega_3 := \mathbb{U}.(\mathbb{V} - \mathbb{V}) & \end{array}$$

3.3 The sets $\mathbb{U}/\mathbb{V}$ and $\nleq_1/\nleq_2/\nleq_3$.

Let us pause awhile and reflect on the nature of the three sets $\nleq_1, \nleq_2, \nleq_3$ of indices attached to the three Bridge Equations. They are themselves constructed from two sets $\mathbb{U}$ and $\mathbb{V}$, which play a symmetric role but could not be more different as to their shape and origin.

The set $\mathbb{U}$ is simple and carries an additive structure, since it is spanned by the 'multipliers' λ_i of the system, with integers $n_i \geq -1$ as coefficients. For our special SPS, there is only one multiplier, namely λ, and $\mathbb{U}$ is made up of all $u = n.\lambda$ with $n = -1, 1, 2, 3 \ldots$ [19]

The set $\mathbb{V}$, on the other hand, is made up of the singularities v_i of the coefficients $a_n(z)$ of the Singularly Perturbed System. These can be 'anything' and carry no *a priori* structure. Moreover, whereas the u_i depend only on a *finite jet* of the Singularly System (via the multipliers λ_i), the v_i depend on the full, untruncated system.

[19] the integration parameter, also noted u_0 to conform to usage, has of course nothing to do with the elements of $\mathbb{U}$.

In a nutshell:

$$\begin{array}{ll|c|lcl}
\mathbb{U} : Universal, Rigid, Structured & & & \Omega_1 & = & \mathbb{U}.x \\
\mathbb{U} = \{u_1, u_2, u_3, \dots\} & & & & & \\
 & & \Longrightarrow & \Omega_2 & = & \mathbb{U}.(\mathbb{V} - z) \\
\mathbb{V} = \{v_1, v_2, v_3, \dots\} & & & & & \\
\mathbb{V} : Variable, Random, Amorphous & & & \Omega_3 & = & \mathbb{U}.(\mathbb{V} - \mathbb{V})
\end{array}$$

$$\begin{array}{rl}
z = 1/t = & variable \ \sim \infty \ \ (involved\ in\ equational\ resurgence) \\
x = 1/\epsilon = & perturbation\ \ parameter \ \sim \infty \ \ (inv.\ in\ coeq.\ resurg.)
\end{array}$$

3.4 The u/v combinatorics.

To get a more precise idea of the way in which the two sets $\mathbb{U}$ and $\mathbb{V}$ combine to produce the singularities ω which govern parametric resurgence, let us break down the solution f of our typical SPS into parts that are r-linear in the coefficients a_n, successively for $r = 1, 2, 3 \dots$ To do this, we require elementary resurgence monomials $\mathcal{V}^\bullet$ that are defined like the $\mathcal{W}^\bullet$ in (18) but without exponential factors $\exp(\omega_r\, z)$ and with all lower indices $\sigma_i \equiv 0$.

The linear part involves only terms of the form:

$$S^{w_1}(x) \equiv \mathcal{V}^{\boldsymbol{\omega}^1}(x) \qquad with \qquad \boldsymbol{\omega}^1 = (\omega_1) = (u_1 v_1) \tag{37}$$

The bilinear part involves only terms of the form:

$$S^{w_1,w_2}(x) \equiv \mathcal{V}^{\boldsymbol{\omega}^1}(x) + \mathcal{V}^{\boldsymbol{\omega}^2}(x) - \mathcal{V}^{\boldsymbol{\omega}^3}(x) \tag{38}$$

with

$$\boldsymbol{\omega}^1 = (u_1 v_1, u_2 v_2) \quad ; \quad \boldsymbol{\omega}^2 = (u_{12} v_2, u_1 v_{1:2}) \quad ; \quad \boldsymbol{\omega}^3 = (u_{12} v_1, u_2 v_{2:1})$$

The trilinear part involves only terms of the form:

$$S^{w_1,w_2,w_3}(x) \equiv \sum_{1 \le i \le 15} \epsilon_i \ \mathcal{V}^{\boldsymbol{\omega}^i}(x) \tag{39}$$

with

$$
\begin{array}{ll}
\epsilon_1 = +1 & \boldsymbol{\omega}^1 = (u_1 v_1,\ u_2 v_2,\ u_3 v_3) \\
\epsilon_2 = +1 & \boldsymbol{\omega}^2 = (u_1 v_1,\ u_{23} v_3,\ u_2 v_{2:3}) \\
\epsilon_3 = -1 & \boldsymbol{\omega}^3 = (u_1 v_1,\ u_{23} v_2,\ u_3 v_{3:2}) \\
\epsilon_4 = +1 & \boldsymbol{\omega}^4 = (u_{12} v_2,\ u_1 v_{1:2},\ u_3 v_3) \\
\epsilon_5 = -1 & \boldsymbol{\omega}^5 = (u_{12} v_1,\ u_2 v_{2:1},\ u_3 v_3) \\
\epsilon_6 = +1 & \boldsymbol{\omega}^6 = (u_{12} v_2,\ u_3 v_3,\ u_1 v_{1:2}) \\
\epsilon_7 = -1 & \boldsymbol{\omega}^7 = (u_{12} v_1,\ u_3 v_3,\ u_2 v_{2:1}) \\
\epsilon_8 = +1 & \boldsymbol{\omega}^8 = (u_{123} v_1,\ u_{23} v_{2:1},\ u_3 v_{3:2}) \\
\epsilon_9 = -1 & \boldsymbol{\omega}^9 = (u_{123} v_1,\ u_{23} v_{2:1},\ u_2 v_{2:3}) \\
\epsilon_{10} = +1 & \boldsymbol{\omega}^{10} = (u_{123} v_1,\ u_3 v_{3:1},\ u_2 v_{2:3}) \\
\epsilon_{11} = -1 & \boldsymbol{\omega}^{11} = (u_{123} v_2,\ u_1 v_{1:2},\ u_3 v_{3:2}) \\
\epsilon_{12} = -1 & \boldsymbol{\omega}^{12} = (u_{123} v_2,\ u_3 v_{3:2},\ u_1 v_{1:3}) \\
\epsilon_{13} = +1 & \boldsymbol{\omega}^{13} = (u_{123} v_3,\ u_1 v_{1:3},\ u_2 v_{2:3}) \\
\epsilon_{14} = -1 & \boldsymbol{\omega}^{14} = (u_{123} v_3,\ u_{12} v_{1:3},\ u_2 v_{2:1}) \\
\epsilon_{15} = +1 & \boldsymbol{\omega}^{15} = (u_{123} v_3,\ u_{12} v_{2:3},\ u_1 v_{1:2})
\end{array}
$$

And so on and so forth. The complexity rapidly increases, since the r-linear parts involve expressions:

$$
S^{w_1,\ldots,w_r}(x) \equiv \sum_{1 \le i \le r^*} \epsilon_i \ \mathcal{V}^{\boldsymbol{\omega}^i}(x) \quad \textit{with} \quad r^* = r!! = 1.3.5\ldots(2r-1) \tag{40}
$$

with more and more terms in them, more and more contorted indices ω_i, and properties – like the symmetrality [20] of the mould $S^\bullet$ – which are anything but obvious. So there is a crying need for a machinery that contrives the proper combinations and predicts/explains the main properties of the objects produced. This machinery is the new Lie algebra ARI.

4 Dimorphic monomials and monics.

4.1 The general setting.

This section purports to show, on some of the simplest examples, how dimorphy – *functional and numerical* – works. The discussion shall be very

[20] see §1.5

sketchy. A thorough treatment [21] would require heaps of formulas, obscure the main articulations, and get us mired in depressing details.

In all such constructions, one first produces special functions, with as much structure on them as possible, and then one derives special numbers, in as natural a manner as possible. In the present case, the special 'functions' appear in no less than six distinct representations or 'models':

$(1_z, 1_\zeta)$ *They are primarily radial-analytic function germs:*
the $\varphi(z)$ are defined at ∞, along the real axis, in the z-plane
the $\widehat{\varphi}(\zeta)$ are defined at 0, along the real axis, in the ζ-plane

$(2_z, 2_\zeta)$ *They are also formal objects:*
Our germs are 'analysable', i.e. they can be completely formalised, without loss of information, as formal series (or transseries in more general contexts) denoted by the same symbols but topped by a tilda, which describe their asymptotic (or more generally transasymptotic) behaviour.

$(3_z, 3_\zeta)$ *Lastly, they are global functions:*
In the present instance, our germs possess global anlytic continuations over the whole z-sphere (where they tend to be slightly ramified, at 0 and ∞) and over the whole ζ-sphere (where they tend to be heavily ramified, at various locations ω_0).

z − model		**z − model**	**transf.**	**ζ−model**		**ζ−model**
Variable	z	$\sim \infty$	*Borel*	$0 \sim$	ζ	*Variable*
Elements	$\varphi(z)$	*germs at* ∞	$\longrightarrow$	*germs at* 0	$\widehat{\varphi}(\zeta)$	*Elements*
Product I	$\times$	*multiplicat.*	$\longleftarrow$	*convol. at* 0	$*$	*Product I*
Product II	✡	*convol. at* ∞	*Laplace*	*multiplicat.*	$\times$	*Product II*

The correspondence between the z- and ζ-representations is via the familiar, mutually inverse Borel/Laplace transforms, but here it is essential to adopt the following, slightly unusual normalisations:

$$\textit{Borel} \quad : \quad \varphi(z) \mapsto \hat{\varphi}(\zeta) := \int_{-i\infty}^{+i\infty} \exp(z\zeta)\, \varphi(z) \frac{dz}{z} \quad (\textit{not } \varphi(z)dz\ !) \tag{41}$$

$$\textit{Laplace} \quad : \quad \hat{\varphi}(\zeta) \mapsto \varphi(z) := \int_{0}^{+\infty} \exp(-z\zeta)\, d\hat{\varphi}(\zeta) \quad (\textit{not } \hat{\varphi}(\zeta)d\zeta\ !) \tag{42}$$

[21] which can be had in [E10]

Dimorphy for our functions manifests as closure under two distinct products :

(i) a primary product (Product I) which is pointwise multiplication in the z-plane and the finite convolution $*$ in the ζ-plane.

(ii) a secondary product (Product II) which is pointwise multiplication in the ζ-plane and the infinite convolution $\circledast$ in the z-plane.

But due to our slightly non-standard definition of Borel/Laplace, we must adopt slightly non-standard definitions for the two convolutions :

$$\varphi_3 = \varphi_1 \circledast \varphi_2 \iff \varphi_3(z) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \varphi(z_1)\, \varphi(z_2) (\frac{1}{z_1} + \frac{1}{z_2}) dz_1 \quad (z_1 + z_2 = z)$$

$$\hat{\varphi}_3 = \hat{\varphi}_1 * \hat{\varphi}_2 \iff \hat{\varphi}_3(\zeta) = \int_0^{+\infty} \hat{\varphi}(\zeta_1)\, d\hat{\varphi}(\zeta_2) \quad (\zeta_1 + \zeta_2 = \zeta)$$

4.2 The biresurgence algebra . Exotic derivations.

The biresurgence algebra BIRES is defined by demanding that $\widehat{\varphi}(\zeta)$, as a global function, should have only logarithmic singularities [22], meaning that near each singular point ω_0 on its Riemann surface, $\widehat{\varphi}(\omega_0 + \zeta_0)$ should decompose as a polynomial in $\log(\zeta_0)$ with regular-analytic coefficients :

$$\varphi \in \text{BIRES} \iff \hat{\varphi}(\omega_0 + \zeta_0) \in \mathbb{C}[\log \zeta_0] \otimes \mathbb{C}\{\zeta_0\} \;\; \textit{at all singular } \omega_0$$

Such elements are closed under Product I and Product II, giving BIRES a double algebra structure [23].

Now here comes the interesting part: attached to our two products we have two systems $\{\Delta_{\omega_0}\,,\ \omega_0 \in \mathbb{C}_\bullet\}$ and $\{\nabla_{\omega_0}\,,\ \omega_0 \in \mathbb{C}_\bullet\}$ of *exotic derivations*. They are linear operators of BIRES into itself. They carry indices ω_0 that range through the Riemann surface $\mathbb{C}_\bullet$ of $\log(\zeta)$. And they measure the singularities of $\widehat{\varphi}$ in such a way as to be *derivations* (meaning that they verify the Leibniz rule) with respect to Product I and II respectively.

The operators Δ_{ω_0}, which act as derivations with respect to Product I, are the now well-established *alien derivations*.

The operators ∇_{ω_0}, which act as derivations with respect to Product II, are the less familiar *foreign derivations*.

[22] plus the usual condition of integrability at 0 and (at most) exponential growth in the ζ-plane, to ensure the existence of the Laplace-transform

[23] Here the term *bigebra* springs to mind, but in this context it would be rather confusing, because it usually refers to the simultaneous existence of an (associative) *product* and a (co-commutative) *co-product*. What we have here, however, is two (commutative) *products*. For distinction, one might perhaps venture the name *digebra* or simply : *double algebra*.

These are huge derivation spaces : the Δ_{ω_0} (resp. ∇_{ω_0}) are bound by no *a priori* relations. Both races of derivations generate a *free* Lie algebra – despite their acting on functions of *one* variable !

Generally speaking, Product II is very much 'second' in importance to Product I, and its derivations ∇_{ω_0} are also very much 'second' to the Δ_{ω_0}. The ∇_{ω_0} are not only less commonly used, but also less versatile : their definition is heavily dependent on the logarithmic structure of the singularities (in the ζ-plane), whereas the Δ_{ω_0} can handle *any* singularities whatsoever. But for the purpose of studying dimorphy and in the restrictive framework of BIRES, Product I with its alien derivations and Product II with its foreign derivations, should be viewed as 'symmetric' and 'equal'.

Actually, we shall have to do, not with BIRES as such, but with the subalgebra (for both products) BIRES_0 of functions which (in the ζ-plane) have a *finite ramification degree*:

$$\begin{aligned}\{\widehat{\varphi}\ \textit{has ramif. degree}\ \leq d\} &\iff \{\Delta_{\omega_{d+1}}\ldots\Delta_{\omega_2}\Delta_{\omega_1}\varphi\equiv 0\ ,\ \ \forall\omega_i\in\mathbb{C}_\bullet\}\\ &\iff \{\nabla_{\omega_{d+1}}\ldots\nabla_{\omega_2}\nabla_{\omega_1}\varphi\equiv 0\ ,\ \ \forall\omega_i\in\mathbb{C}_\bullet\}\end{aligned}$$

BIRES_0 itself has interesting subalgebras, two of which – the hyperlogarithmic and hyperzetaic algebras – shall retain our attention.

One last point calls for clarification : how does *functional dimorphy* (stability under two distinct products) translate into *numerical dimorphy* (two different ways of calculating the one and only product on $\mathbb{C}$) ?

If we define the monics by *point evaluation* (especially *antipodal evaluation*, see §4.5, §4.6) of our monomials on the z- or ζ-spheres, then stability under Product I or Product II immediately carries over to the monics, and the only thing left to do is to establish the existence of a simple conversion rule between the two sets of monics.

If on the other hand we derive the monics by *exotic differentiation* of our monomials, the situation is not so different, only tidier : the two multiplication rules for Product I and II still induce definite symmetries in the monics, and the universal conversion laws between the system $\{\Delta_\omega\}$ and the system $\{\nabla_\omega\}$ automatically induces a conversion rule between the monics.

But the second method offers subtle advantages. Not only is the conversion law implicit in the procedure, but its nature also becomes more understandable. Indeed, recall that both races of exotic derivations 'measure', each in its own way, the singularities in the ζ-plane. But one race is geared to Product I, which is a convolution in the ζ-plane and as such *adds* the singular points of the two factors (i.e $(\omega_1,\omega_2)\mapsto\omega_1+\omega_2$), whereas the other race is geared to Product II, which is pointwise multiplication in the ζ-plane, and as such *leaves the singular points in place.*

So, for the two systems of monics produced in this way, and indexed by sequences $u_1, u_2, ..$ and $v_1, v_2, ..$ inherited from the singularities of the monomials, we should expect the conversion law to reduce to an addition of the u_i or a substraction of the v_i. Such will indeed be the case in the simple examples that follow (one must of course go to more complex situations to really appreciate the efficiency of the present machinery).

4.3 The subalgebra of hyperlogarithmic monomials.

The v-encoding:
In the **v**-encoding, our hyperlogarithmic monomials are defined by:

$$\mathrm{Ligg}^{v_1,\dots,v_r}(z) := (-1)^r\,(\partial_z + v_r)^{-1} z^{-1} \dots (\partial_z + v_2)^{-1} z^{-1} (\partial_z + v_1)^{-1} z^{-1} \quad (43)$$

$$\widehat{\mathrm{Ligg}}^{v_1,\dots,v_r}(\zeta) := \int_0^{\zeta} \frac{d\zeta_r}{\zeta_r - v_r} \dots \int_0^{\zeta_3} \frac{d\zeta_2}{\zeta_2 - v_2} \int_0^{\zeta_2} \frac{d\zeta_1}{\zeta_1 - v_1} \quad (44)$$

Formula (43) is relative to the z-plane. Each factor $(\partial_z + v_i)^{-1}$ in it may be interpreted either as the integration operator $\exp(-v_i z)(\int dz)\exp(v_i z)$ acting on everything to its right, in which case the formula yields sectorial function germs at ∞, or again it may be interpreted as the formal operator $\sum_{n \geq 0} (-1)^n v_i^{-1-n} \partial_z^n$ acting on everything to its right, in which case the formula yields the (common) asymptotic expansion of these germs.
Formula (44) is merely the transposition of (43) in the ζ-plane. It yields germs at $\zeta = 0$ with ramified extensions over the whole ζ-plane.

The u-encoding:
It is derived from the **v**-encoding under the simple conversion rules:

$$\mathrm{Lagg}^{u_1,u_2,\dots,u_r}(z) \overset{\textit{essentially}}{:=} \mathrm{Ligg}^{u_1,u_{12},\dots,u_{12\dots r}}(z) \quad (45)$$

$$\mathrm{Ligg}^{v_1,v_2,\dots,v_r}(z) \overset{\textit{essentially}}{:=} \mathrm{Lagg}^{v_1,v_{2:1},\dots,v_{r:r-1}}(z) \quad (46)$$

The caveat 'essentially' means that some simple corrective terms have to be added when several singularities lie on the same half-ray.

Biresurgence and dimorphy:
All monomials $\mathrm{Lagg}^{\mathbf{u}}$ and $\mathrm{Ligg}^{\mathbf{v}}$ are in the biresurgence algebra BIRES. Moreover, subjecting two monomials $\mathrm{Lagg}^{\mathbf{u}}$ (resp. $\mathrm{Ligg}^{\mathbf{v}}$) to Product I (resp.II) amounts to shuffling their **u**- or **v**-sequences. In other words, both moulds

$\mathrm{Lagg}^\bullet$ and $\mathrm{Ligg}^\bullet$ are equally *symmetral*, but relative to Product I and II respectively. Thus, in the z-plane we get :

$$\textit{Product I}: \qquad \mathrm{Lagg}^{\mathbf{u}'}.\,\mathrm{Lagg}^{\mathbf{u}''} \;\equiv\; \sum_{\mathbf{u}\in\mathrm{sha}(\mathbf{u}',\mathbf{u}'')} \mathrm{Lagg}^{\mathbf{u}} \tag{47}$$

$$\textit{Product II}: \qquad \mathrm{Ligg}^{\mathbf{v}'} \star \mathrm{Ligg}^{\mathbf{v}''} \;\equiv\; \sum_{\mathbf{v}\in\mathrm{sha}(\mathbf{v}',\mathbf{v}'')} \mathrm{Ligg}^{\mathbf{v}} \tag{48}$$

4.4 The subalgebra of hyperzetaic monomials.

The natural but inconvenient encoding :
The hyperlogarithmic monomials, as produced by difference equations or systems, are naturally given in the form :

$$\mathrm{Zegg}^{s_1,\dots,s_r}(z) := \sum_{n_1>\dots>n_r>0} (z+n_1)^{-s_1}\dots(z+n_r)^{-s_r} \tag{49}$$

and they are indeed closed under Product I and II. But this 'natural' **s**-encoding leads to rather clumsy multiplication rules and must make way for the usual **u**- and **v**-encodings.

The approximate u- and v-encodings :
In the z-plane, $\mathrm{Zagg}^{\mathbf{u}}$ and $\mathrm{Zigg}^{\mathbf{v}}$ are meromorphic functions given by the expansions :

$$\mathrm{Zagg}^{u_1,\dots,u_r}(z) \overset{ess.}{:=} \sum_{m_1,..,m_r>0} P(m_1-u_1-z)P(m_{12}-u_{12}-z)..P(m_{1..r}-u_{1..r}-z)$$
$$\mathrm{Zigg}^{v_1,\dots,v_r}(z) \overset{ess.}{:=} \sum_{n_1>\dots>n_r>0} P(n_1-v_1-z)P(n_2-v_2-z)..P(n_r-v_r-z) \tag{50}$$

with the obvious conversion rules :

$$\mathrm{Zagg}^{u_1,\dots,u_r}(z) \overset{ess.}{:=} \mathrm{Zigg}^{u_{12\dots r},u_{12\dots r-1},\dots,u_1}(z) \tag{51}$$
$$\mathrm{Zigg}^{v_1,\dots,v_r}(z) \overset{ess.}{:=} \mathrm{Zagg}^{v_r,v_{r-1:r},\dots,v_{1:2}}(z) \tag{52}$$

and the usual abbreviations : $P(t) := 1/t,\ u_{12} := u_1+u_2,\ v_{1:2} := v_1 - v_2$.

In both definitions (50), *ess* (*essentially*) means that the lower order multipoles, which render the expansions convergent while respecting the symmetries, have been omitted (see [E10]). In (51),(52) *ess* also points to omitted, very elementary corrective terms (see [E10]).

The exact u- and v-encodings :

Lest the reader should feel uneasy with *ess*, here are the exact definitions of the monomials $\mathrm{Zagg}(z)^\bullet/\mathrm{Zigg}(z)^\bullet$ in terms of the monics $\mathrm{Zag}^\bullet/\mathrm{Zig}^\bullet$ which are themselves defined directly in §6.2 *infra*.

$$\mathrm{Zagg}^{u_1,\dots,u_r}(z) \overset{exactly}{:=} \mathrm{Zag}^{z+u_1,\dots,z+u_r} \tag{53}$$

$$\mathrm{Zigg}^{v_1,\dots,v_r}(z) \overset{exactly}{:=} \mathrm{Zig}^{z+v_1,\dots,z+v_r} \tag{54}$$

Biresurgence and dimorphy :
All monomials $\mathrm{Zagg}^{\mathbf{u}}$ and $\mathrm{Zigg}^{\mathbf{v}}$ are in the biresurgence algebra BIRES. Moreover the mould $\mathrm{Zagg}^\bullet$ (resp. $\mathrm{Zigg}^\bullet$) is symmetral (resp. symmetril) relative to Product II (resp. Product I) . Thus in the z-plane we get :

$$\textit{Product II} : \qquad \mathrm{Zagg}^{\mathbf{u}'} \star \mathrm{Zagg}^{\mathbf{u}''} \equiv \sum_{\mathbf{u}\in \mathrm{sha}(\mathbf{u}',\mathbf{u}'')} \mathrm{Zagg}^{\mathbf{u}} \tag{55}$$

$$\textit{Product I} : \qquad \mathrm{Zigg}^{\mathbf{v}'} . \mathrm{Zigg}^{\mathbf{v}''} \equiv \sum_{\mathbf{v}\in \mathrm{shi}(\mathbf{v}',\mathbf{v}'')} \mathrm{Zigg}^{\mathbf{v}} \tag{56}$$

4.5 From monomials to monics.

To produce *monics* (i.e. basic numbers) from our *monomials* (basic functions) we have the choice between :

(i) the straightforward but crude method [24] of *evaluation at special points*, which preseves the symmetry types : *symmetral/el/il etc* remains *symmetral/el/il etc.*

(ii) the subtler method of *exotic differentiation*, which changes the symmetry types (*symmetral/el/il etc* becomes *alternal/el/il etc*) but possesses several decisive advantages :

(a) in settings more general than BIRES, the monomials, which we recall are basically germs at $z = \infty$ and $\zeta = 0$, may fail to possess an analytic continuation (even a multivalued, ramified one) to the whole z- or ζ-plane, and then exotic differentiation spares us the embarrasssement of having to evaluate functions at locations where they are not defined ;

(b) exotic differentiation dispenses us from choosing arbitrary points z_0 or ζ_0 where to do the evaluation ;

[24] there is also the even cruder and messier method of *integration over special domains.*

(c) in the not infrequent case when the intervals $[\infty, z_0]$ or $[0, \zeta_0]$ carry singularities, exotic differentiation also dispenses us from choosing this or that determination at z_0 or ζ_0.

(d) exotic differentiation may seem to involve some arbitrariness of its own, via the choice of the index $\omega_0 \in \mathbb{C}_\bullet$ in Δ_{ω_0} or ∇_{ω_0}, but the objection doesn't stand: for any given monomial, there are usually only finitely many ω_0 that yield a non-zero exotic derivative (these are linear combinations of 'simpler' monomials with monics as coefficients) and only one priviledged ω_0 that yields a *constant* derivative, consisting of a *single monic*. Thus, "choosing the derivation" is not an issue, and ω_0 need not even be mentioned in the definition of the monic.

4.6 The $\mathbb{Q}$-rings of hyperlogarithmic monics.

In the case of hyperlogarithmic monomials, *point evaluation* is not totally arbitrary, because there are privileged points where to do the evaluation, namely the antipode 0 of ∞ on the z-sphere and the antipode ∞ of 0 on the ζ-sphere. What we find at the antipodes, however, is logarithmic singularities: polynomials in $\log(z)$ or $\log(\zeta)$, with regular coefficients. Therefore, *evaluation at the antipodes* in this context means *killing the logarithms* and then *evaluating* the regular germ that is left.

By so doing, we get two symmetral monics $\mathrm{Lag}^\bullet$ and $\mathrm{Lig}^\bullet$. The method of *exotic differentiation*, on the other hand, yields two alternal monics $\mathrm{Lan}^\bullet$ and $\mathrm{Lin}^\bullet$. There are simple conversion laws within each pair (dimorphy!) and from pair to pair (equivalence!) and indeed all four systems generate (*linearly*, i.e. span) the same $\mathbb{Q}$-ring of dimorphic monics: that ring depends solely on the lattice (usually $\mathbb{Z}$ or the lattice ${}^r\mathbb{Z}$ generated by all unit roots of order r) where the indices u_i and v_i live ([E10]).

4.7 The $\mathbb{Q}$-rings of hyperzetaic monics.

The hyperzetaic monomials also may be subjected to antipodal evaluation and exotic differentiation. Indeed, if we define $\mathrm{Zagg}^\bullet(z)$ and $\mathrm{Zigg}^\bullet(z)$ directly, as we did in (50) but with all the corrective, lower-order poles which for brevity we omitted there [25], and then read formulas (53)(54) *backwards*, this in effect amounts to deriving the monics $\mathrm{Zag}^\bullet/\mathrm{Zig}^\bullet$ from the monomials by

[25] these missing terms are spelt out in [E10], along with various direct characterisations of $\mathrm{Zagg}^\bullet(z)/\mathrm{Zigg}^\bullet(z)$

antipodal evaluation. [26]

Then there is *exotic differentiation* . It produces the monics $\mathrm{Zan}^\bullet$ (alternal) and $\mathrm{Zin}^\bullet$ (alternil). There is no room for them in this survey, but they are very closely related to the monics $\mathrm{Za}^\bullet$ (symmetral) and $\mathrm{Ze}^\bullet$ (symmetrel), which shall occupy centre-stage in the sequel and which incidentally may also be interpreted as the Taylor coefficients of $\mathrm{Zag}^\bullet/\mathrm{Zig}^\bullet$ at $\mathbf{u} = 0$ or $\mathbf{v} = 0$. See §6.2 *infra.*

In actual fact, what we are going to do is proceed in reverse order – i.e. start from $\mathrm{Za}^\bullet/\mathrm{Ze}^\bullet$, then (78)(79) derive $\mathrm{Zag}^\bullet/\mathrm{Zig}^\bullet$ as their generating 'functions' (which are still to be regarded as 'monics'), then (53)(55) move on to the monomials $\mathrm{Zagg}^\bullet/\mathrm{Zigg}^\bullet$– but that is only for expediency's sake. From the point of view of theory, the scheme outlined in this section – from monomials to monics – is more logical and above all more universal.

4.8 Is dimorphy exhaustive ? Is trimorphy illusory ?

Let us pigeon-hole into one comprehensive table the main creatures encountered in this section :

dimorphic + biresurgent monomials	**dimorphic + transcendent monics**
hyperlogarithmic	**hyperlogarithmic**
$\mathrm{Lagg}^\bullet(z)$ // $\widehat{\mathrm{Lagg}}^\bullet(\zeta)$ I-*symmetral*	$\mathrm{Lag}^\bullet$ // $\mathrm{Lan}^\bullet$ *symmetral//alternal*
$\mathrm{Ligg}^\bullet(z)$ // $\widehat{\mathrm{Ligg}}^\bullet(\zeta)$ II-*symmetral*	$\mathrm{Lig}^\bullet$ // $\mathrm{Lin}^\bullet$ *symmetral//alternal*
hyperzetaic	**hyperzetaic**
$\mathrm{Zagg}^\bullet(z)$ // $\widehat{\mathrm{Zagg}}^\bullet(\zeta)$ II-*symmetral*	$\mathrm{Zag}^\bullet$ // $\mathrm{Za}^\bullet$ *symmetral//symmetral*
$\mathrm{Zigg}^\bullet(z)$ // $\widehat{\mathrm{Zigg}}^\bullet(\zeta)$ I-*symmetril*	$\mathrm{Zig}^\bullet$ // $\mathrm{Ze}^\bullet$ *symmetril//symmetrel*

[26]but each time on the z-sphere, whereas for hyperlogarithms we used alternately the z- and ζ-sphere : no matter how sweeping the overall balance, harmony, symmetries etc which hold sway in this realm of dimorphy, they have their limits too.

This table looks tidy enough, but it leaves out one sensitive aspect: namely the *extensive overlapping* between the hyperlogarithmic and hyperzetaic systems, not of monomials, but of *monics* ! The phenomenon deserves attention, because it seems to take us beyond *dimorphy* into *trimorphy*. Let us try to explain what is at stake on two simple examples – two particular domains of monics.

Domain 1 regroups all hyperlogarithmic monics with indices v_i in $\{0, 1\}$ (and therefore with indices u_i in $\{0, 1, -1\}$). The corresponding hyperlogarithmic monomials are often referred to as *polylogarithms*, but the associated monics are readily seen to be equivalent to the multizetas $\mathrm{Za}^\bullet$ and tend to be assigned to the 'zetaic race'.

Domain 2 regroups all hyperlogarithmic monics with indices u_i and v_i in $\mathbb{Z}$. But they too may be viewed as zetaic monics – namely hyperzetas (more general that the multizetas).

Then we have many variants. Thus, alongside Domain 1 we have the slightly larger Domains 1.r, which regroup the hyperlogarithmic monics with indices v_i either 0 or unit roots of order r. These monics coincide with the 'modulated' multizetas.

Similarly, behind Domain 2 there loom the slightly larger Domains 2.r, which comprise all hyperlogarithmic monics with indices u_i and v_i in the lattice ${}^r\mathbb{Z}$. These monics too dovetail with a class of hyperzetas.

Now, each of these Domains being the intersection of *already dimorphic rings*, we should expect them to be subject to an even richer pattern of constraints. Such indeed is the case. For Domain 2 and its variants, we get, not two, but three sets of seemingly independent relations . For Domain 1 and its variants, we get two full sets of contraints, plus a faint trace of the third set - namely invariance under the 'Hoffman involution' [27].

The question is: *are these three systems of relations independent ?*

For Domain 1 and its variants, it can be shown ([E10]) that the third, 'atrophied' set of relations is actually an (algebraic) consequence of the other two. But what about Domain 2 and its variants? The subject is still in its infancy, but there seem to be quite a few reasons for assuming that, of the three sets of constraints, only two are independent – in other words, that *dimorphy alone is effective, as well as exhaustive, and that trimorphy*

[27] see §6.10

is an illusion. In any case, one would be very hard put to imagine a fully effective trimorphy with all its consequences, and in particular to visualize the corresponding irreducibles.

But the question remains, and much is at stake, because *trimorphy – or should we say its appearance – extends to the immense Domain 3*, which is delineated in §7.4 and encompasses the bulk of all transcendental constants ever encountered in the course of natural operations. Let us stress, however, that this uncertainty does not affect Domain 1 (the multizetas) : here trimorphy is *known* to resolve itself into dimorphy.

5 The overarching structure: ARI/GARI.

5.1 Bimoulds. Swap/Push. Contractions.

A few basic facts about the *mould* formalism have been recalled in §1.5. *Bimoulds* are moulds that depend on double sequences :

$$A^{\bullet} = A^{\mathbf{w}} = A^{w_1,\dots,w_r} = A^{\binom{u_1\ ,\ \dots\ ,\ u_r}{v_1\ ,\ \dots\ ,\ v_r}}$$

and, more crucially, that are subjected to operations which mix up intimately the two sequences [28].

One such operation is the basic involution *swap* :

$$A_*^{\bullet} = \mathrm{swap}(A^{\bullet}) \quad \Longleftrightarrow \quad A_*^{\binom{u_1\ ,\ \dots\ ,\ u_r}{v_1\ ,\ \dots\ ,\ v_r}} = A^{\binom{v_r\ \ ,\ v_{r-1:r}\ \ ,\ \dots\ ,\ v_{2:3}\ ,\ v_{1:2}}{u_{1\dots r}\ ,\ u_{1\dots r-1}\ ,\ \dots\ ,\ u_{12}\ ,\ \ u_1}} \tag{57}$$

Another operation is the *push* :

$$A_*^{\bullet} = \mathrm{push}(A^{\bullet}) \quad \Longleftrightarrow \quad A_*^{\binom{u_1\ ,\ \dots\ ,\ u_r}{v_1\ ,\ \dots\ ,\ v_r}} = A^{\binom{-u_{1..r}\ ,\ u_1\ \ \ u_2\ \ ,\ \dots\ ,\ u_{r-1}}{-v_r\ \ ,\ v_{1:r}\ v_{2:r}\ ,\ \dots\ ,\ v_{r-1:r}}} \tag{58}$$

(We make constant use of the shorthand $u_{12} := u_1{+}u_2$, $v_{1:2} := v_1{-}v_2$ etc).

It is often convenient to represent bimoulds in the so-called 'augmented notation', which consists in adding to any given sequence $\mathbf{w}$ a *redundant* initial term $w_0 = \binom{u_0}{v_0}$. The $\mathbf{u}$-variables are then constrained by the condition $u_0 + u_1 + ..u_r = 0$ and, dually, the $\mathbf{v}$-variables are defined upto addition of a common constant. Thus :

$$A^{\binom{u_1,\dots\ ,u_r}{v_1,\dots\ ,v_r}} \equiv \mathrm{aug}A^{\binom{u_0,\ u_1,\dots\ ,u_r}{v_0,\ v_1,\dots\ ,v_r}} \quad \textit{with} \quad u_0 := -u_{1..r}\ ;\ v_0 := 0. \tag{59}$$

[28] otherwise they ought to be regarded as moulds whose indices $w_i = \binom{u_i}{v_i}$ simply happen to be in $\mathbb{C}^2$ rather than in $\mathbb{C}$.

With the augmented notations, for instance, the *push* reduces to a unit shift on the sequence $\mathbf{w}$.

More operations on bimoulds shall be defined in the sequel, but nearly all of them involve four specific types of *sequence contractions*, denoted by the symbols $\rceil$, $\lceil$, $\rfloor$, $\lfloor$. These are always relative to some given *factorisation* $\mathbf{w} = \mathbf{w}^1\mathbf{w}^2 \ldots \mathbf{w}^s$ of the total sequence. The contraction rules are immediately apparent from the following example. Relative to the factorisation :

$$\mathbf{w} = \ldots \mathbf{a}.\mathbf{b}\ldots = \ldots (\begin{smallmatrix} u_3 \,,\, u_4 \,,\, u_5 \\ v_3 \,,\, v_4 \,,\, v_5 \end{smallmatrix})(\begin{smallmatrix} u_6 \,,\, u_7 \,,\, u_8 \,,\, u_9 \\ v_6 \,,\, v_7 \,,\, v_8 \,,\, v_9 \end{smallmatrix})\ldots$$

the symbols $\rceil$, $\lceil$, $\rfloor$, $\lfloor$ signal the following changes :

$$\mathbf{a}\rceil := (\begin{smallmatrix} u_3 \,,\, u_4 \,,\, u_{56789} \\ v_3 \,,\, v_4 \,,\, v_5 \end{smallmatrix}) \qquad \lceil\mathbf{b} := (\begin{smallmatrix} u_{3456} \,,\, u_7 \,,\, u_8 \,,\, u_9 \\ v_6 \,,\, v_7 \,,\, v_8 \,,\, v_9 \end{smallmatrix}) \tag{60}$$

$$\mathbf{a}\rfloor := (\begin{smallmatrix} u_3 \,,\, u_4 \,,\, u_5 \\ v_{3:6} \,,\, v_{4:6} \,,\, v_{5:6} \end{smallmatrix}) \qquad \lfloor\mathbf{b} := (\begin{smallmatrix} u_6 \,,\, u_7 \,,\, u_8 \,,\, u_9 \\ v_{6:5} \,,\, v_{7:5} \,,\, v_{8:5} \,,\, v_{9:5} \end{smallmatrix}) \tag{61}$$

with the usual abbreviations for sums and differences.

Thus we see that the contractor $\rceil$ *adds* to *the* upper-right element of $\mathbf{a}$ *all* upper elements of neighbouring $\mathbf{b}$, whereas the contractor $\rfloor$ *substracts* from *all* lower elements of $\mathbf{a}$ *the* lower-left element of neighbouring $\mathbf{b}$. And *vice versa* for $\lceil$ and $\lfloor$. Indeed, the $\mathbf{u}$-variables are meant to be added together, and the $\mathbf{v}$-variables to be substracted from one another.

5.2 The Lie algebra ARI.

Consider the bilinear product *ari*:

$$C^\bullet = \mathrm{ari}(A^\bullet, B^\bullet) \quad \Longleftrightarrow \quad C^{\mathbf{w}} = \sum_{\mathbf{w}=\mathbf{b}.\mathbf{c}} (A^{\mathbf{b}}B^{\mathbf{c}} - B^{\mathbf{b}}A^{\mathbf{c}})$$
$$+ \sum_{\mathbf{w}=\mathbf{b}.\mathbf{c}.\mathbf{d}} (A^{\lfloor\mathbf{c}}B^{\mathbf{b}\rceil\mathbf{d}} - B^{\lfloor\mathbf{c}}A^{\mathbf{b}\rceil\mathbf{d}}) + \sum_{\mathbf{w}=\mathbf{a}.\mathbf{b}.\mathbf{c}} (A^{\mathbf{a}\lceil\mathbf{c}}B^{\mathbf{b}\rfloor} - B^{\mathbf{a}\lceil\mathbf{c}}A^{\mathbf{b}\rfloor}) \tag{62}$$

with $\mathbf{b} \neq \emptyset, \mathbf{c} \neq \emptyset$ in all three sums (but $\mathbf{a}$ and $\mathbf{d}$ may be empty) .

The ari*-bracket is anti-commutative, verifies the Jacobi identity, and turns the space of all bimoulds such that* $A^\emptyset = 0$ *into a Lie algebra, known as* ARI.

5.3 The Lie group GARI.

Consider the binary law *gari*:

$$C^\bullet = \mathrm{gari}(A^\bullet, B^\bullet) \qquad \Longleftrightarrow \qquad C^{\mathbf{w}} =$$
$$\sum_{\mathbf{w}=\mathbf{a}^1.\mathbf{b}^1.\mathbf{c}^1\ldots\mathbf{a}^s.\mathbf{b}^s.\mathbf{c}^s.\mathbf{a}^{s+1}} A^{\lceil\mathbf{b}^1\rceil\ldots\lceil\mathbf{b}^s\rceil}B^{\mathbf{a}^1\rfloor}\ldots B^{\mathbf{a}^s\rfloor}B^{\mathbf{a}^{s+1}\rfloor}B_\star^{\lfloor\mathbf{c}^1}\ldots B_\star^{\lfloor\mathbf{c}^s} \tag{63}$$

with summation over all $s \geq 1$ and with factor sequences subject only to $\mathbf{b^i} \neq \emptyset$ and $\mathbf{c^i}.\mathbf{a^{i+1}} \neq \emptyset$ (but consecutive factors $\mathbf{c^i}$ and $\mathbf{a^{i+1}}$ may be empty *separately* and the extreme factors $\mathbf{a^1}$,$\mathbf{c^s}$,$\mathbf{a^{s+1}}$ and even the product $\mathbf{c^s}.\mathbf{a^{s+1}}$ may also be empty, *separetly or simultaneously*). Here $B^\bullet_\star$ denotes the inverse invmu$(B^\bullet)$ of $B^\bullet$ relative to the ordinary (associative, non-commutative) product (*mu* or $\times$) on moulds:

$$C^\bullet = \mathrm{mu}(A^\bullet, B^\bullet) = A^\bullet \times B^\bullet \Longleftrightarrow C^{\mathbf{w}} = \sum_{\mathbf{w}=\mathbf{w^1}.\mathbf{w^2}} A^{\mathbf{w^1}} B^{\mathbf{w^2}} \tag{64}$$

This *gari*-product is clearly affine in $A^\bullet$ but severely non-linear in $B^\bullet$.

It is also associative, and turns the set of all bimoulds such that $A^\emptyset = 1$ *into a Lie group, known as* GARI*, whose Lie algebra is* ARI.

5.4 Some properties of ARI/GARI. Allied structures.

Like ordinary moulds, most interesting bimoulds fall into a few basic *symmetry types*. The definition for *symmetral/alternal* and *symmetrel/alternel* is exactly the same as for ordinary moulds, but in the case *symmetril/alternil* the contraction $w_i \otimes w_j$ should be interpreted as:

$$A^{\ldots,w_i\otimes w_j,\ldots} := P(v_{i:j})\, A^{\ldots,\binom{u_{ij}}{v_i},\ldots} + P(v_{j:i}) A^{\ldots,\binom{u_{ij}}{v_j},\ldots} \quad \textit{with} \quad P(t) := 1/t \tag{65}$$

Thus, for a *symmetral* bimould $A^\bullet$ and factor sequences of length 1 and 2 we get:

$$A^{\binom{u_1}{v_1}} A^{\binom{u_2,u_3}{v_2,v_3}} \equiv A^{\binom{u_1,\,u_2,\,u_3}{v_1,\,v_2,\,v_3}} + A^{\binom{u_2,\,u_1,\,u_3}{v_2,\,v_1,\,v_3}} + A^{\binom{u_2,\,u_3,\,u_1}{v_2,\,v_3,\,v_1}}$$

but if $A^\bullet$ is *symmetrel* (resp. *symmetril*) we get additional, 'contracted' terms on the right-hand side, namely $A^{\binom{u_{12},\,u_3}{v_{12},\,v_3}} + A^{\binom{u_2,\,u_{13}}{v_2,\,v_{13}}}$ resp.

$$P(v_{1:2})\, A^{\binom{u_{12},\,u_3}{v_1,\,v_3}} + P(v_{2:1})\, A^{\binom{u_{12},\,u_3}{v_2,\,v_3}} + P(v_{1:3})\, A^{\binom{u_2,\,u_{13}}{v_2,\,v_1}} + P(v_{3:1})\, A^{\binom{u_2,\,u_{13}}{v_2,\,v_3}}$$

The set of all alternal bimoulds is a subalgebra of ARI *.That of all symmetral bimoulds is a subgroup of* GARI.

These are closure properties for moulds with a simple symmetry. But ARI/GARI is specially well-suited for the study of bimoulds with a double symmetry:

The set $\mathrm{ARI}_{\underline{\mathrm{al/al}}}$ *of bialternal even bimoulds (i.e. bimoulds that are alternal and whose swappee is also alternal) constitute an important subalgebra of* ARI*, and similarly the set* $\mathrm{GARI}_{\underline{\mathrm{as/as}}}$ *of bisymmetral even bimoulds is an important subgroup of* GARI.

Here, "even" means that, for any given length r, the component $A^{w_1,\dots,w_r}$ is an even functions of $\mathbf{w}$. Actually, 'evenness' is *almost* a consequence of the double symmetry: thus, it may be shown that a bialternal bimould automatically has even components for all lengths r, except at most for $r = 1$. But to ensure stability under the ARI-bracket, length-one components also have to be even. This subsidiary parity condition is signalled by underlining: e.g. $\underline{\mathrm{al/al}}$ and $\underline{\mathrm{as/as}}$.

Even more important for our purpose is *the subalgebra* $\mathrm{ARI}_{\underline{\mathrm{al/il}}}$ *of alternal bimoulds with an alternil swappee, and the subgroup* $\mathrm{GARI}_{\underline{\mathrm{as/is}}}$ *of symmetral bimoulds with an symmetril swappee.* Here also, the parity condition implies the evenness of length-one components, but is slightly more technical for $r \geq 2$.

The double symmetry has other consequences: it implies invariance under some form or other of idem-potent transformation, like
(i) the *push* for $\mathrm{ARI}_{\underline{\mathrm{al/al}}}$;
(ii) the *spush* for $\mathrm{GARI}_{\underline{\mathrm{as/as}}}$;
(iii) variants of these for $\mathrm{ARI}_{\underline{\mathrm{al/il}}}$ and $\mathrm{GARI}_{\underline{\mathrm{as/is}}}$ (see[E10]).

It also ensures the existence of an *involutive (or group) automorphism*: thus, the involution *swap*, which is no algebra automorphism on ARI as a whole, becomes one when restricted to $\mathrm{ARI}_{\underline{\mathrm{al/al}}}$.

5.5 Some remarkable elements of ARI.

Bimoulds with a double symmetry *do matter*– and in more ways than one. But they are rather thin on the ground, and not so easy to construct. So it comes as a relief to know that *most of them*, and in some important cases *all of them*, can be derived from a small set of rather elementary bimoulds, the so-called bielementals $\mathrm{belam}_r^\bullet/\mathrm{belim}_r^\bullet$. These depend only on the component length r and on a two-variable function $\mathrm{xaxi}(w_1) := \mathrm{xa}(u_1)\,\mathrm{xi}(v_1)$ [29], or rather the even part of xaxi. All components of $\mathrm{belam}_r^\bullet/\mathrm{belim}_r^\bullet$ are $\equiv 0$, except the component of length r, which reduces to a simple superposition:

$$\mathrm{belam}_{r,\,\mathrm{xaxi}}^{w_1,\dots,w_r} = \mathrm{belam}_{r,\,\mathrm{xaxi}}^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} :=$$

$$\sum_{\substack{i,j,m,n\in\mathbb{Z}_{r+1}\\ \dots<i\leq m<j\leq n<\dots}} \mathrm{bel}_r^{i,j;\,m,n}\ \mathrm{xa}(u_i+u_{i+1}+\dots+u_{j-1})\ \mathrm{xi}(v_m-v_n)\ \equiv$$

$$\sum_{\substack{i,j,m,n\in\mathbb{Z}_{r+1}\\ \dots<i\leq m<j\leq n<\dots}} \frac{1}{2}\ \mathrm{bel}_r^{i,j;\,m,n}\ \Big(\mathrm{xa}(u_{i\dots j-1})\ \mathrm{xi}(v_{m:n})\ + \mathrm{xa}(u_{j\dots i-1})\ \mathrm{xi}(v_{n:m})\Big) \quad (66)$$

[29]with variables u_1, v_1 in two (possibly different) abelian groups.

with a *swappee*

$$\mathrm{belim}^{\bullet}_{r,\mathrm{xaxi}} := \mathrm{swap}(\mathrm{belam}^{\bullet}_{r,\,\mathrm{xaxi}}) \equiv \mathrm{belam}^{\bullet}_{r,\,\mathrm{xixa}} \tag{67}$$

and with integer coefficients

$$\mathrm{bel}_r^{i,j;\,m,n} \equiv \mathrm{bel}_r^{j,i;\,n,m} := \frac{(-1)^{[m-i]_r+[n-j]_r} \quad [r-1]_r!}{[m-i]_r!\;[n-j]_r!\;[j-m-1]_r!\;[i-n-1]_r!} \tag{68}$$

This calls for a few comments:

The above formulas use the cyclic *augmented notation*: we index the variable u_i, v_i of the r-th component of a bimould on $\mathbb{Z}_{r+1} := \mathbb{Z}/(r+1)\mathbb{Z}$ after adding the two 'redundant' variables $u_0 := -u_{1\dots r}$ et $v_0 := 0$. The inequalities under the $\sum$ sign are of course relative to the cyclic order on $\mathbb{Z}_{r+1}$ and, for any $k \in \mathbb{Z}_{r+1}$, $[k]_r$ denotes the representative of k in $\{0, 1, \dots r\}$.

Formula (67) shows that the involution *swap* leaves bielementals unchanged, apart from swapping xa and xi. But the main facts are these:

(i) all bimoulds $\mathrm{belam}^{\bullet}_{r,\,\mathrm{xaxi}}$ are *bialternal*
(ii) they vanish for *odd* or (iff $r \geq 2$) *semi-constant* functions *xaxi*
(iii) they are non-zero for *even* functions *xaxi* constant in neither variable
(iv) they generate *most other* bialternals under the *ari*-bracket [30].

5.6 Further remarkable elements of ARI.

As usual, we set: $P(t) := 1/t$ and $Q_c(t) := c/\tan(c\,t)$ for some $c \in \mathbb{C}$.
The identities

$$\mathrm{pa}_1^{w_1} := P(u_1) \quad ; \quad \mathrm{pa}_r^{w_1,\dots,w_r} := P(u_{12\dots r})\,(\mathrm{pa}_{r-1}^{w_1,\dots,w_{r-1}} - \mathrm{pa}_{r-1}^{w_2,\dots,w_r}) \tag{69}$$

$$\mathrm{pi}_r^{w_1,\dots,w_r} := (v_1 + v_2 + \dots v_r)\, P(v_1)\, P(v_{1:2})\, P(v_{2:3}) \dots P(v_{r-1:r})\, P(v_r) \tag{70}$$

define (the former by induction, the latter directly) two series of rather peculiar bimoulds, the $\mathrm{pa}^{\bullet}_r$ and $\mathrm{pi}^{\bullet}_r$, which depend each on one set of variables – the u_i or v_i, and have only one non-zero component, that of length r. The $\mathrm{pa}^{\bullet}_r$ and $\mathrm{pi}^{\bullet}_r$ are *alternal*, and although not *bialternal*, they still possess a double symmetry of sorts, since they are exchanged, not under the involution

[30] for the precise statements, see §6.5 and also§7.3

swap, but under another important, if less general involution : the *slap* [31]. Moreover they self-reproduce under the ARI-bracket:

$$\text{ari}(\text{pa}^\bullet_{r_1}, \text{pa}^\bullet_{r_2}) = (r_1 - r_2)\, \text{pa}_{r_1+r_2} \tag{71}$$
$$\text{ari}(\text{pi}^\bullet_{r_1}, \text{pi}^\bullet_{r_2}) = (r_1 - r_2)\, \text{pi}_{r_1+r_2} \tag{72}$$

which means that the subalgebras ARI_{pa} and ARI_{pi} of ARI generated by the bimoulds $\text{pa}^\bullet_r$ or $\text{pi}^\bullet_r$ are each isomorphic to the algebra Diff_t spanned by the differential operators $t^{n+1}\partial_t$.

5.7 Some remarkable elements of GARI.

By Lie exponentiation, the algebra isomorphisms just mentioned induce group isomorphisms between each of the subgroups:
(i) $\text{GARI}_{\text{pa}} := \text{expari}(\text{ARI}_{\text{pa}})$
(ii) $\text{GARI}_{\text{pi}} := \text{expari}(\text{ARI}_{\text{pi}})$
and the group:
(iii) $\text{Diffeo}_t := \exp(\text{Diff}_t)$
of formal, identity-tangent diffeomorphisms $t \mapsto t + O(t^2)$ of $\mathbb{C}_{,0}$ unto itself.

Of special interest are the images $\text{par}^\bullet \in \text{GARI}_{\text{pa}}$ and $\text{pil}^\bullet \in \text{GARI}_{\text{pi}}$ of the diffeomorphism $f \in \text{Diffeo}_{\text{t}}$ defined by $f(t) := 1 - \exp(-t)$. Like all bimoulds in GARI_{pa} and GARI_{pi} , $\text{par}^\bullet$ and $\text{pil}^\bullet$ are symmetral, but the remarkable and unexpected thing is that their *swappees* $\text{pir}^\bullet := \text{swap}(\text{par}^\bullet)$ and $\text{pal}^\bullet := \text{swap}(\text{pil}^\bullet)$ are symmetral too.

The bisymmetral pairs $\text{pal}^\bullet/\text{pil}^\bullet$ *and* $\text{par}^\bullet/\text{pir}^\bullet$ *thus defined are central to the theory.* They *do not* fulfill the parity condition and so *do not* belong to $\text{GARI}_{\underline{\text{as/as}}}$. Indeed, upto rescaling and under suitable additional conditions ("eupolarity"), they are the only bisymmetral (bi)moulds [32] that depend on one set of variables only ($\mathbf{u}$ or $\mathbf{v}$) and whose r-th component is homogeneous of degree -r.

Of the two pairs, $\text{pal}^\bullet/\text{pil}^\bullet$ is the more important by far. It has a 'eu-trigonometric' counterpart $\text{tal}^\bullet/\text{til}^\bullet$, obtained by replacing P by Q_c and adding suitable corrective terms that involve only even powers of c – see [E10].

These bisymmetral bimoulds enjoy an incredible number of properties and sit at the hub of a galaxy of some sixty 'special bimoulds', which are investigated in [E10] and whose applications far outstrip multizeta theory.

[31] wich acts as an automorphism, but only on the subalgebra $\text{ARI}_{\text{eupol}} \subset \text{ARI}$ consisting of so-called *eupolar* bimoulds, which are particular rational functions of $\mathbf{u}$ or $\mathbf{v}$

[32] being constant in one series of variables, bimoulds like $\text{pal}^\bullet/\text{pil}^\bullet$, $\text{par}^\bullet/\text{pir}^\bullet$ etc are often referred to as "moulds".

5.8 Further remarkable elements of GARI.

The *scramble* is a general bimould transform defined by:

$$A^\bullet \mapsto B^\bullet = \text{scramble}(A^\bullet) \quad \textit{with} \quad B^\bullet := \sum_{\mathbf{w}^* \in \text{scram}(\mathbf{w})} \epsilon(\mathbf{w},\mathbf{w}^*)\, A^{\mathbf{w}^*} \tag{73}$$

where scram($\mathbf{w}$) is the set of all sequences $\mathbf{w}^* = (\begin{smallmatrix} u_1^* & ,\dots, & u_r^* \\ v_1^* & ,\dots, & v_r^* \end{smallmatrix})$ which have the same length r as $\mathbf{w} = (\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix})$ and are characterised by the property that for each $j \in \{1, \dots, r\}$:

$$u_1^* v_1^* + u_2^* v_2^* + \dots u_j^* v_j^* = \sum_{1 \le i \le j} \Big(\sum_{p_{j,i-1} < p \le p_{j,i}} u_p \Big) v_{q_{j,i}} \tag{74}$$

for some pair $\{p_{j,k}\}, \{q_{j,k}\}$ of intertwined sequences:

$$0 = p_{j,0} < q_{j,1} \le p_{j,1} < q_{j,2} \le p_{j,2} < \dots < q_{j,j} \le p_{j,j} < r$$

There are exactly $r!! := 1.3 \dots (2r-1)$ such sequences $\mathbf{w}^*$. Each u_j^* is a sum of one or several consecutive u_i and each v_j^* is either of the form $v_{j_\star}$, in which case we set $\epsilon(\mathbf{w},\mathbf{w}^*,j) := 1$, or of the form $v_{j_\star} - v_{j_{\star\star}}$, in which case we set $\epsilon(\mathbf{w},\mathbf{w}^*,j) := \text{sign}(j_{\star\star} - j_\star)$. (Mark the inversion). Multiplied together, these signs define the global sign factor $\epsilon(\mathbf{w},\mathbf{w}^*) := \prod_{j=1}^{j=r} \epsilon(\mathbf{w},\mathbf{w}^*,j)$ in the definition of the scramble transform.

In the above definition (73), $A^\bullet$ was assumed to be a bimould, but it could just as well be a mere mould, in which case $A^{\mathbf{w}^*}$ should be interpreted as $A^{u_1^* v_1^*,\dots,u_r^* v_r^*}$. Thus, the *scramble* turns moulds and bimoulds alike into bimoulds.

One of the reasons for the importance of the *scramble* is that it preserves the two basic symmetry types: *if the mould or bimould $A^\bullet$ is alternal (resp. symmetral), so is the bimould $B^\bullet = \text{scramble}(A^\bullet)$* .

Remarkable (bi)moulds tend to have remarkable *'scramblees'*. Thus the symmetral mould $\mathcal{V}^\bullet(z)$, which is central to *equational resurgence* and *Singular Systems*, yields the bimould $S^\bullet(x) := \text{scramble}(\mathcal{V}^\bullet(x))$, which is central to *co-equational resurgence* and *Singularly Perturbed Systems*: see §3.4 *supra.*

Closely linked to the symmetral resurgence monomials $\mathcal{V}^\bullet(z)$ are the alternal hyperlogarithmic monics $V^\bullet = V^\bullet_{\omega_0}$ featuring [33] in the resurgence equations:

$$\Delta_{\omega_0} \mathcal{V}^\bullet(z) = V^\bullet_{\omega_0} \times \mathcal{V}^\bullet(z) \tag{75}$$

[33] The lower index is actually redundant and may be dropped, since $V_{\omega_0}^{\omega_1,\dots,\omega_r} \equiv 0$ unless $\omega_1 + \dots \omega_r = \omega_0$

When scrambled, that mould yields the so-called *tesselation mould* ([E6]) $\mathrm{tes}^\bullet := \mathrm{scramble}(V^\bullet)$, which dominates the geometry of co-equational resurgence in the Borel planes, and possesses many arresting features, like *being locally constant in its two series of variables, the u_i as well as the v_i* (although $\mathrm{tes}^\bullet$ is a superposition of several highly complex functions). Thus for $r = 2$, the tesselation coefficient :

$$\mathrm{tes}^{w_1,w_2} := V^{u_1 v_1, u_2 v_2} + V^{u_{12} v_2, u_1 v_{1:2}} - V^{u_{12} v_1, u_2 v_{2:1}}$$

is, contrary to appearances, locally constant on $\mathbb{C}^4$ and assumes only three distinct values there, namely 0 and ± 1 .

5.9 Basic complexity of ARI/GARI.

The basic complexity of ARI/GARI (as reflected in its main operations, associated structures, fundamental bimoulds, etc) is quite high. Thus, for a given component length r, the inversion *invgari* in GARI or the Lie exponential *expari* of ARI into GARI resolves into a sum of a fast increasing number (marked # in the table below) of terms, each of which fills upto half a line, or more, of small print :

length r	1	2	3	4	5	6	7	8	...
#(invgari)	1	4	20	112	672	4 224	27 459	183 040	...
#(expari)	1	4	21	126	818	5 594	39 693	289 510	...

For $r = 8$, we already get six figure numbers, and spelling out the corresponding formulas in full would take about one hundred pages. This means that one must often rely heavily on automatic computation when *exploring* the fringes and by-paths of ARI/GARI. Fortunately, however, the whole field is so strongly structured, and so harmonious too, offering so many hints and props to intuition, that facts and formulas are easy to guess and, once guessed, quickly yield to rigorous proving. Writing down all these proofs is of course another matter, due to the sheer mass of the facts already unearthed or yet to emerge !

6 The arithmetics of multizetas.

6.1 Formal multizetas $\mathrm{za}^\bullet/\mathrm{ze}^\bullet$.

In the *first encoding* , the *generalised* or *modulated* multizetas are defined by :

$$\mathrm{Ze}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}} := \sum_{n_1 > \dots n_r > 0} n_1^{-s_1} \dots n_r^{-s_r}\ e_1^{n_1} \dots e_r^{n_r} \tag{76}$$

with $s_j \in \mathbb{N}^*$, $\epsilon_j \in \mathbb{Q}/\mathbb{Z}$, $e_j := \exp(2\pi i \epsilon_j)$
The *second encoding* may be directly defined, via the polylogarithms, but it is more expeditious to derive it from the first encoding :

$$\mathrm{Za}^{e_1,0^{(s_1-1)},\ldots,e_r,0^{(s_r-1)}} \overset{essentially}{:=} \mathrm{Ze}^{\binom{\epsilon_r\ ,\ \epsilon_{r-1:r}\ ,\ldots,\ \epsilon_{1:2}}{s_r\ ,\ s_{r-1}\ ,\ldots,\ s_1}} \tag{77}$$

Setting $\epsilon_j \equiv 0, e_j \equiv 1$, we get the *usual* or *plain* multizetas.

On its obvious domain of convergence $\Re(s_1) > 1, \Re(s_{12}) > 2, \ldots,$ the series (76) defines a holomorphic function *with a meromorphic extension to the whole of* $\mathbb{C}^r$ which in turn possesses (see [E10]) :
(i) a remarkable singular locus,
(ii) remarkably simple multipoles described by the *Bernoully mould*,
(iii) a *'parity property'* reminiscent of the *reflexion property* of the Riemann zeta function [34].

From the arithmetical point of view, however, the $\mathbb{Q}$-ring generated by the values of the multizeta function on $\mathbb{Z}^r$ (at regular points or even at singular ones, after canonical removal of the multipole) is no larger than the $\mathbb{Q}$-ring generated (in fact : spanned) by its values on $\mathbb{N}^r$. So we may restrict our attention to the latter.

The multizetas, whether plain or modulated, are eminently 'dimorphic' creatures : they are doubly closed under multiplication, since to the two encodings there correspond two distinct ways of calculating their products. These are the two classical systems of *quadratic relations*, which can be derived in any number of ways. In pithy mould language, with the conventions of §1.5 , they can be enuntiated as follows :

(i) *The mould* $\mathrm{Ze}^\bullet$*, where defined, is symmetrel* [35]*, and there is a unique extension to the divergent case* [36] *that keeps it symmetrel and gives* $\mathrm{Ze}^{\binom{0}{1}} = 0$.

(ii) *The mould* $\mathrm{Za}^\bullet$*, where defined, is symmetral and there is a unique extension to the divergent case that keeps it symmetral and gives* $\mathrm{Za}^0 = \mathrm{Za}^1 = 0$.

But these two extensions do not exactly coincide. There is a slight discrepancy, which calls for some simple corrective terms ([E10]) in the conversion formula (77). Hence the mention "essentially" in the middle of (77).

[34] somewhat confusingly known as 'functional equation'

[35] since the mould $\mathrm{Ze}^\bullet$ has two-storeyed indices $\omega_i = \binom{\epsilon_i}{s_i}$, the contractions $\omega_i + \omega_j$ must of course be interpreted as $\binom{\epsilon_i+\epsilon_j}{s_i+s_j}$

[36] this is the only case ($\epsilon_1 = 0, e_1 = 1, s_1 = 1$) when the series (67) diverges.

All the indications, numerical and theoretical, are that the two sets of 'quadratic relations' do express the totality of algebraic constraints on multizetas. So we may confidently replace the *true multizetas* $\mathrm{Ze}^\bullet, \mathrm{Za}^\bullet$, which at the moment are still largely beyond the reach of arithmetics [37], by their *formal* or *symbolic* counterparts $\mathrm{ze}^\bullet, \mathrm{za}^\bullet$, written in lower-case letters and subject only to
(i) the symmetrelity of $\mathrm{ze}^\bullet$
(ii) the symmetrality of $\mathrm{za}^\bullet$
(iii) the conversion rules (77)
and address the problem of unravelling all the algebraic consequences.

6.2 Generating functions $\mathrm{zag}^\bullet/\mathrm{zig}^\bullet$.

In *scalar* form, the multizetas are rather unwieldy, and it is more convenient to replace them by *generating series*, so tailored as to preserve the simplicity of the two symmetries and the transparency of the conversion rule. The proper definitions are:

$$\mathrm{zig}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{v_1\ ,\dots,\ v_r}} := \sum_{1\le s_j} \mathrm{ze}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}}\ v_1^{s_1-1}\dots v_r^{s_r-1} \tag{78}$$

$$\mathrm{zag}^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} := \sum_{1\le s_j} \mathrm{za}^{e_1,0^{(s_1-1)},\dots,e_r,0^{(s_r-1)}}\ u_1^{s_1-1}u_{12}^{s_2-1}\dots u_{12\dots r}^{s_r-1} \tag{79}$$

In the *formal* case the components of the two new moulds $\mathrm{zag}^\bullet/\mathrm{zig}^\bullet$ are mere power series, but in the *genuine* case, i.e. of for the moulds $\mathrm{Zag}^\bullet/\mathrm{Zig}^\bullet$ built from the *numerical* multizetas, these power series sum up to meromorphic functions with interesting properties, such as verifying simple difference equations ([E10]).

Moreover, we have the implications:

$$\begin{aligned} \mathrm{ze}^\bullet\ \textit{symmetrel} &\iff \mathrm{zig}^\bullet\ \textit{symmetril} \\ \mathrm{za}^\bullet\ \textit{symmetral} &\iff \mathrm{zag}^\bullet\ \textit{symmetral} \end{aligned}$$

and the conversion rule (77) translates into:

$$\mathrm{swap}(\mathrm{zig}^\bullet) \overset{exactly}{=} \mathrm{mu}(\mathrm{zag}^\bullet, \mathrm{mono}^\bullet) \tag{80}$$

Here, $\mathrm{mono}^\bullet$ is an elementary, constant-valued mould: up to rational factors, its values are 'monozetas' $\zeta(s)$, hence its name. So for the generating functions, the conversion rule essentially reduces to the involution *swap*. Remark

[37] despite the trail-blazing work of Apéry and, more recently, T.Rivoal ([A],[C],[R])

that formula (80) uses the primary mould product *mu* (see §1.5). But due to the elementary nature of $\mathrm{mono}^\bullet$, the right-hand side of (80) may also be written as an (exceptionnally commutative) product in GARI. Indeed:

$$\mathrm{mu}(\mathrm{zag}^\bullet, \mathrm{mono}^\bullet) \equiv \mathrm{gari}(\mathrm{zag}^\bullet, \mathrm{mono}^\bullet) \equiv \mathrm{gari}(\mathrm{mono}^\bullet, \mathrm{zag}^\bullet)$$

Thus, studying the formal multizetas boils down to *finding and describing all the symmetral/symmetril pairs of (essential) swappees* $\mathrm{zag}^\bullet/\mathrm{zig}^\bullet$ *with values in the ring of formal power series.*

6.3 Immediate bipartion and arduous tripartition.

As a natural element of GARI, the mould $\mathrm{zag}^\bullet$ splits into two and even three factors:

$$\mathrm{zag}^\bullet = \mathrm{gari}(\mathrm{zag}^\bullet_{\mathrm{I+II}}, \mathrm{zag}^\bullet_{\mathrm{III}}) \qquad (\mathrm{zag}^\bullet_{\mathrm{III}} \in \mathrm{GARI}^{\mathrm{o.l.}}_{\underline{\mathrm{as/is}}}) \tag{81}$$

$$\mathrm{zag}^\bullet = \mathrm{gari}(\mathrm{zag}^\bullet_{\mathrm{I}}, \mathrm{zag}^\bullet_{\mathrm{II}}, \mathrm{zag}^\bullet_{\mathrm{III}}) \qquad (\mathrm{zag}^\bullet_{\mathrm{II}} \in \mathrm{GARI}^{\mathrm{e.l.}}_{\underline{\mathrm{as/is}}}) \tag{82}$$

The factors $\mathrm{zag}^\bullet_{\mathrm{I}}$, $\mathrm{zag}^\bullet_{\mathrm{II}}$, $\mathrm{zag}^\bullet_{\mathrm{I+II}}$ are of type "e.l.", meaning that their components of even/odd length are even/odd functions of $\mathbf{w}$. The factor $\mathrm{zag}^\bullet_{\mathrm{III}}$ on the other hand is of type "o.l.", meaning that its components of even/odd length are odd/even functions of $\mathbf{w}$. Under the algebra isomorphisms (87) (see §6.4 below), bimoulds of type "e.l." (resp. "o.l.") correspond to bimoulds that are (in both cases) even functions of $\mathbf{w}$ but whose only non-zero components have even (resp. odd) lengths [38]. Hence the abbreviations e.l. (even-lengthed) and o.l. (odd-lengthed).

But there is a major difference between (81) and (82). The first factorisation is elementary, immediate, and indisputably canonical, with the $\mathrm{zag}^\bullet_{\mathrm{III}}$ factor given by:

$$\mathrm{gari}(\mathrm{zag}^\bullet_{\mathrm{III}}, \mathrm{zag}^\bullet_{\mathrm{III}}) = \mathrm{gari}(\mathrm{imne}(\mathrm{invgari}(\mathrm{zag}^\bullet)), \mathrm{zag}^\bullet) \tag{83}$$

where *imne* denotes the elementary ARI/GARI automorphism:

$$\mathrm{imne} := \mathrm{impar} \circ \mathrm{neg} : \qquad A^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} \mapsto (-1)^r A^{\binom{-u_1\ ,\dots,\ -u_r}{-v_1\ ,\dots,\ -v_r}} \tag{84}$$

Since all elements of GARI have exactly one square root, (83) determines $\mathrm{zag}^\bullet_{\mathrm{III}}$ and then (81) determines $\mathrm{zag}^\bullet_{\mathrm{I+II}}$ by division.

The difficulty with the second, more precise factorisation (82), which consists in disentangling the factors $\mathrm{zag}^\bullet_{\mathrm{I}}$ and $\mathrm{zag}^\bullet_{\mathrm{II}}$, is not its existence, which

[38]this applies only to the factors $\mathrm{zag}^\bullet_{\mathrm{I}}$ and $\mathrm{zag}^\bullet_{\mathrm{II}}$. See [E10]

again is quite straightforward, but its canonicity : there are infinitely many ways of detaching the 'polar-trigonometric' factor $\mathrm{zag}^\bullet_{\mathrm{I}}$, which carries the π^2-dependence, from $\mathrm{zag}^\bullet_{\mathrm{I+II}}$, and the 'right' choice hinges on the notion of *kwa*-orthogonality: see §6.6, §6.7 below.

These factorisations hold not only in the *formal*, but also in the *genuine* case, and thus lead to two canonical splittings of the $\mathbb{Q}$-ring of multizetas, one immediate, the other more recondite :

$$\mathbb{Z}\mathrm{eta} = \mathbb{Z}\mathrm{eta}_{\mathrm{I+II}} \otimes \mathbb{Z}\mathrm{eta}_{\mathrm{III}} \tag{85}$$

$$\mathbb{Z}\mathrm{eta} = \mathbb{Z}\mathrm{eta}_{\mathrm{I}} \otimes \mathbb{Z}\mathrm{eta}_{\mathrm{II}} \otimes \mathbb{Z}\mathrm{eta}_{\mathrm{III}} \quad \textit{with} \quad \mathbb{Z}\mathrm{eta}_{\mathrm{I}} = \mathbb{Q}[\pi^2] \tag{86}$$

The ring $\mathbb{Z}\mathrm{eta}_{\mathrm{III}}$ (resp. $\mathbb{Z}\mathrm{eta}_{\mathrm{I+II}}$) is generated by all *irreducibles* of odd length (resp. by those of even length, plus the odd man out $\zeta(2) = \pi^2/6$).

6.4 The free generation theorem.

The $\mathbb{Q}$*-ring* $\mathbb{Z}\mathrm{eta}$ *of formal multizetas, as well as the three factor-rings I,II,III, are polynomial rings, that is to say, they are freely generated on* $\mathbb{Q}$ *by a countable system of 'irreducibles'. This holds equally for the* plain *and the more general* modulated *multizetas.*

Although it has been open for the better part of the nineties, this *free generation theorem* is a very simple affair. The only difficulty is to establish the closure under the *ari*-bracket of the space $\mathrm{ARI}_{\underline{\mathrm{al/il}}}$ of all alternal/alternil bimoulds (with the subsidiary 'parity' condition : see §5.4). The neatest proof consists in observing that the space $\mathrm{ARI}_{\underline{\mathrm{al/al}}}$ of bialternal bimoulds is (trivially) a subalgebra of ARI, and in using either of the two explicit isomorphisms:

$$\mathrm{adari}(\mathrm{pal}^\bullet) \;\; \textit{or} \;\; \mathrm{adari}(\mathrm{par}^\bullet) : \qquad \mathrm{ARI}_{\underline{\mathrm{al/al}}} \Rightarrow \mathrm{ARI}_{\underline{\mathrm{al/il}}} \tag{87}$$

where $\mathrm{adari}(\mathrm{pal}^\bullet)$ (resp. $\mathrm{adari}(\mathrm{par}^\bullet)$) denotes the adjoint action in ARI of the bisymmetral mould $\mathrm{pal}^\bullet$ (resp. $\mathrm{par}^\bullet$) constructed in §5.7

Observe that, while these isomophisms make it certain that $\mathrm{ARI}^{\mathrm{ent}}_{\underline{\mathrm{al/il}}}$ *is* a subalgebra, they do not exchange the subalgebras $\mathrm{ARI}^{\mathrm{ent}}_{\underline{\mathrm{al/al}}}$ and $\mathrm{ARI}^{\mathrm{ent}}_{\underline{\mathrm{al/il}}}$ of 'entire-valued' bimoulds (i.e. bimoulds with values in the ring of formal power series). In fact, these two subalgebras *are not* isomorphic.

The *general entire-valued*, symmetral/symmetril pair of swappees $\mathrm{zag}^\bullet/\mathrm{zig}^\bullet$ is then obtained by postcomposition in GARI of a *particular* $\mathrm{zag}^\bullet$ (e.g. the 'genuine', 'numerical' $\mathrm{Zag}^\bullet$ or its first factor $\mathrm{Zag}^\bullet_{\mathrm{I}}$) by the *general* element of $\mathrm{GARI}^{\mathrm{ent}/\#}_{\underline{\mathrm{as/is}}} = \mathrm{expari}(\mathrm{ARI}^{\mathrm{ent}/\#}_{\underline{\mathrm{al/il}}})$. Here, *ent* means *entire-valued* as usual, and # denotes an additional condition which depends on the ring of multizetas that is being considered : thus for the *plain* multizetas, # simply means *constant in the* **v**-*variables*. For the *modulated* multizetas, see §6.5 below.

The problem has thus been completely linearised, and the *free generators* of $\mathbb{Z}$eta, or *irreducibles* (other than π^2), are seen to be in one-to-one correspondence with the generators of $\mathrm{ARI}^{\mathrm{ent}/\#}_{\underline{\mathrm{al/il}}}$ *as a vector space*.

But *as a Lie algebra*, $\mathrm{ARI}^{\mathrm{ent}/\#}_{\underline{\mathrm{al/il}}}$ has far more structure on it than the $\mathbb{Q}$-ring $\mathbb{Z}$eta of formal multizetas, and its *linear generators* may be further analysed – down to *Lie generators*. This is where the exciting work on multizetas actually *begins* !

6.5 Generators and dimensions.

The Broadhurst-Kreimer conjectures.
Let us consider jointly the *plain multizetas* (without unit roots) and the *Eulerian multizetas* (modulated by the unit roots ±1) – the former because of their obvious importance; the latter because, contrary to appearences, they are actually *simpler*. Let $P_{s,r}$ (resp. $E_{s,r}$) be the smallest number of irreducibles of length r and weight s ($:= s_1 + \ldots + s_r$) needed to produce, jointly, a complete system of irreducibles for the plain (resp. Eulerian) multizetas.

Relying on extensive numerical computations and some inspired guesswork, Broadhurst and Kreimer have conjectured that the dimensions $P_{s,r}$ and $E_{s,r}$ for the 'genuine', as opposed to 'formal', multizetas could be read off the generating functions :

$$\prod_{s\geq 3, r\geq 1} (1 - x^s y^r)^{P_{s,r}} \stackrel{?}{=} 1 - \frac{x^3 y}{1-x^2} + \frac{x^{12} y^2 (1-y^2)}{(1-x^4)(1-x^6)} \tag{88}$$

$$\prod_{s\geq 3, r\geq 1} (1 - x^s y^r)^{E_{s,r}} \stackrel{?}{=} 1 - \frac{x^3 y}{(1-x^2)(1-xy)} \tag{89}$$

The two series of relevant algebras.
For any $p \geq 1$ let $\mathbb{Z}_p$ be the subgroup $\{0, 1/p, \ldots, (p-1)/p\}$ of $\mathbb{Q}/\mathbb{Z}$ and let $\mathrm{ARI}^{\underline{\mathrm{ent}/\mathbb{Z}_p}}$ denote the subalgebra of ARI (it is one !) which regroups all bimoulds $A^\bullet$:
(i) with **u**-variables ranging through $\mathbb{C}$

(ii) with $\mathbf{v}$-variables ranging through $\mathbb{Z}_p$
(iii) with values in the ring of formal power series in $\mathbf{u}$
(iv) with the self-correlation constraints:

$$A^{\binom{u_1\ ,\dots,\ u_r}{qv_1\ ,\dots,\ qv_r}} \equiv \sum_{qv_i^*=qv_i} A^{\binom{qu_1\ ,\dots,\ qu_r}{v_1^*\ ,\dots,\ v_r^*}} \qquad (\forall\, q \mid p) \tag{90}$$

and in particular (for q=p):

$$A^{\binom{u_1\ ,\dots,\ u_r}{0\ ,\dots,\ 0}} \equiv \sum_{v_i^*\in\mathbb{Z}_p} A^{\binom{qu_1\ ,\dots,\ qu_r}{v_1^*\ ,\dots,\ v_r^*}} \tag{91}$$

The subalgebras directly relevant to the study of the multizetas modulated by unit roots of order p are $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_p}_{\underline{\mathrm{al/al}}}$ and $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_p}_{\underline{\mathrm{al/il}}}$. For $p = 1$ their elements are simply $\mathbf{v}$-constant bimoulds.

Eulerian multizetas and generators of $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_2}_{\underline{\mathrm{al/al}}}$.
For $r = 1$ and d even we set:

$$\mathrm{bela}^{w_1}_{1,d} = u_1^d\ (\mathit{resp.}\ \ (2^{-d}-1)\,u_1^d)) \quad \mathit{if} \quad v_1 = 0\ (\mathit{resp.}\ \ v_1 = 1/2) \tag{92}$$

and for $r \geq 1$ and d even we set:

$$\mathrm{bela}^{\bullet}_{r,d} := \mathrm{belam}^{\bullet}_{r,\mathrm{xaxi}} \quad \mathit{with} \quad \mathrm{xa}(t) := t^d\ ;\ \mathrm{xi}(0) := 0\ ;\ \mathrm{xi}(1/2) := 1 \tag{93}$$

(i) *All bimoulds* $\mathrm{bela}^{\bullet}_{r,d}$ *(for $r = 1, 2, 3\dots$ and $d = 2, 4, 6\dots$) are non-zero, bialternal, and self-correlated.*
(ii) *They freely generate the algebra* $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_2}_{\underline{\mathrm{al/al}}}$.

Eulerian multizetas and generators of $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_2}_{\underline{\mathrm{al/il}}}$.
(i) *Each bialternal* $\mathrm{bela}^{\bullet}_{r,d}$ *has a canonical counterpart or 'extension'* $\mathrm{bema}^{\bullet}_{r,d}$, *of alternal/alternil type, self-correlated, and with a first non-zero component (i.e. the one of length r) equal to the single non-zero component of* $\mathrm{bela}^{\bullet}_{r,d}$.
(ii) *These* $\mathrm{bema}^{\bullet}_{r,d}$ *freely generate the algebra* $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_2}_{\underline{\mathrm{al/il}}}$.
(iii) *The Eulerian irreducibles correspond one-to-one to the bialternals spanning* $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_2}_{\underline{\mathrm{al/al}}}$. *More precisely, the number $E_{s,r}$ of independent irreducibles of weight s and length r coincides with the dimension of the cell of* $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_2}_{\underline{\mathrm{al/al}}}$ *consisting of bimoulds of length r and total degree $d = s - r$.*
(iv) *This establishes the Broadhurst-Kreimer conjecture (89) for the* formal

Eulerian multizetas [39].

Plain multizetas and generators of $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_1}_{\underline{\mathrm{al/il}}}$.
(i) *For each even* d *there is a canonical pair of alternal/alternil swappees* $\mathrm{ma}^\bullet_d/\mathrm{mi}^\bullet_d$ *with initial components* $\mathrm{ma}^{w_1} := u_1^d$, $\mathrm{ma}^{w_1} := v_1^d$ *and with a total of* d *non-zero components.*
(ii) *It is conjectured that the* $\mathrm{ma}^\bullet_d$ *freely generate* $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_1}_{\underline{\mathrm{al/il}}}$.

This conjecture (formulated in a different, less flexible framework) has been around for quite a while now, but it is rather crude and can be considerably sharpened, by reasoning on bialternals. Indeed:

Plain multizetas and generators of $\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_1}_{\underline{\mathrm{al/al}}}$.
We require three integer sequences α, β, γ (with $\alpha(d) \equiv \beta(d) + \gamma(d-2)$):

$$\sum \alpha(d)\ x^d := x^6\ (1-x^2)^{-1}\ (1-x^4)^{-1} \tag{94}$$
$$\sum \beta(d)\ x^d := x^6\ (1-x^2)^{-1}\ (1-x^6)^{-1} \tag{95}$$
$$\sum \gamma(d)\ x^d := x^8\ (1-x^4)^{-1}\ (1-x^6)^{-1} \tag{96}$$

and three series of bialternals:

$$\begin{array}{rl} \mathrm{ekma}^\bullet_d/\mathrm{ekmi}^\bullet_d & d\ \textit{even} \geq 2 \\ \mathrm{doma}^\bullet_{d,b}/\mathrm{domi}^\bullet_{d,b} & d\ \textit{even} \geq 10,\ 1 \leq b \leq \beta(d) \\ \mathrm{carma}^\bullet_{d,c}/\mathrm{carmi}^\bullet_{d,c} & d\ \textit{even} \geq 8,\ 1 \leq c \leq \gamma(d) \end{array}$$

of total degree d and with a single non-zero component of length respectively 1,2,4 . The definition of the first two pairs is straightforward:

$$\mathrm{ekma}^{w_1}_d := u_1^d\ ;\ \mathrm{ekmi}^{w_1}_d := v_1^d \tag{97}$$

$$\mathrm{doma}^{w_1,w_2}_{d,b} := \mathrm{fa}(u_1,u_2)\ (\mathrm{ga}(u_1,u_2))^{b-1}\ (\mathrm{ha}(u_1,u_2))^{d/2-3b} \tag{98}$$
$$\mathrm{domi}^{w_1,w_2}_{d,b} := \mathrm{fi}(v_1,v_2)\ (\mathrm{gi}(v_1,v_2))^{b-1}\ (\mathrm{hi}(v_1,v_2))^{d/2-3b} \tag{99}$$

with

$$\begin{aligned} \mathrm{fa}(u_1,u_2) &:= u_1u_2(u_1-u_2)(u_1+u_2)(2u_1+u_2)(2u_2+u_1) \\ \mathrm{ga}(u_1,u_2) &:= (u_1+u_2)^2u_1^2u_2^2 \quad ; \quad \mathrm{ha}(u_1,u_2) := u_1^2+u_1u_2+u_2^2 \\ \mathrm{fi}(v_1,v_2) &:= v_1v_2(v_1-v_2)(v_1+v_2)(2v_1-v_2)(2v_2-v_1) \\ \mathrm{gi}(v_1,v_2) &:= (v_1-v_2)^2v_1^2v_2^2 \quad ; \quad \mathrm{hi}(v_1,v_2) := v_1^2-v_1v_2+v_2^2 \end{aligned}$$

[39] Instead of Eulerian multizetas, Broadhurst and Kreimer speak of 'Euler sums'.

The definition of the last pair, $\text{carma}^\bullet/\text{carmi}^\bullet$, is more roundabout. Observe first that the $\text{ekma}_d^\bullet$ are not *free* in ARI, but *bound* (for each degree d) by exactly $\gamma(d)$ independent relations of the form :

$$\sum_{d_1+d_2=d+2} R_c^{d_1,d_2}\, [\text{ekma}_{d_1}^\bullet, \text{ekma}_{d_2}^\bullet] \;=\; 0^\bullet \quad (1 \le c \le \gamma(d)\,,\; R_c^{d_1,d_2} \in \mathbb{Q}) \tag{100}$$

which result from the expansions :

$$[\text{ekma}_{d_1}^\bullet, \text{ekma}_{d_2}^\bullet] = \sum_{1\le b\le\beta(d_1+d_2)} K_{d_1,d_2}^b \;\text{doma}_{d_1+d_2,b}^\bullet \quad (K_{d_1,d_2}^b \in \mathbb{Q}) \tag{101}$$

Next, consider the moulds :

$$\text{vima}_{d,c}^\bullet := \sum_{d_1+d_2=d+2} R_c^{d_1,d_2}\, [\text{ma}_{d_1}^\bullet, \text{ma}_{d_2}^\bullet] \;\neq\; 0^\bullet \tag{102}$$

with $R_c^{d_1,d_2}$ as in (100). By construction :
(a) $\text{vima}_{d,c}^\bullet$ *is of alternal/alternil type*
(b) *its components of length 1,2,3 vanish*
(c) *its (non-vanishing) component of length 4 defines a bialternal mould, which is precisely the sought-after mould* $\text{carma}_{d,c}^\bullet$

Now, the crude conjecture at the end of the last para can be replaced by the much sharper, but also more tractable statements :

(i) *The moulds* $\text{ekma}_d^\bullet$ *are free under the ari-bracket upto the contraints (100). More precisely, the number* $P_{s,r}^*$ *of linearly independent bialternals of length* r *, degree* d *(and weight* $s := d + r$*) generated by the* $\text{ekma}_d^\bullet$ *is given by the BK-like formula:*

$$\prod_{s\ge3,r\ge1} (1 - x^s y^r)^{P_{s,r}^*} = 1 - \frac{x^3 y}{1-x^2} + \frac{x^{12}y^2}{(1-x^4)(1-x^6)} \tag{103}$$

(ii) *The moulds* $\text{carma}_{d,k}^\bullet$ *are free under the ari-bracket. As a consequence, the number* $P_{s,r}^{**}$ *of linearly independent bialternals of length* r *, degree* d *(and weight* $s := d+r$*) generated by the* $\text{carma}_{d,k}^\bullet$ *is given by the BK-like formula :*

$$\prod_{s\ge3,r\ge1} (1 - x^s y^r)^{P_{s,r}^{**}} = 1 - \frac{x^{12}y^4}{(1-x^4)(1-x^6)} \tag{104}$$

(iii) *In combination, the* $\text{ekma}_d^\bullet$ *and* $\text{carma}_{d,k}^\bullet$ *generate the bialternal algebra*

$\mathrm{ARI}^{\mathrm{ent}/\mathbb{Z}_1}_{\underline{\mathrm{al/al}}}$, *freely upto the sole constraints (100). As a consequence, the total number* $P_{s,r}$ *of linearly independent bialternals of length* r *, degree* d *(and weight* $s := d + r$ *) is given by the BK formula (88).*

Unlike in the Eulerian case, the three above statements have been established only upto length $r = 7$ $(\forall d)$ and remain *conjectural* beyond that. But the supporting evidence is overwhelming ([E10]) and and in any case they have the merit of completely removing the weirdness of the artificial-looking corrective term $x^{12}y^2(1-y^2)/((1-x^4)(1-x^6))$ in the BK-formula : the explanation is simply that to each 'missing' bialternal of length 2 (– there just aren't 'enough' $\mathrm{doma}^\bullet/\mathrm{domi}^\bullet$ around –) there answers, under the transparent mechanism (100)+(101)+(102), a 'stop-gap' bialternal of length 4 (– namely the 'unexpected' $\mathrm{carma}^\bullet/\mathrm{carmi}^\bullet$–).

6.6 Canonicity and *kwa*-orthogonality.

As soon as a major problem is shown to admit infinitely many solutions, the next question to ask is : is there, among the lot, a clearly distinguished, *canonical* choice ? Despite the notion's inherent vagueness, 'canonicity' makes a world of difference, and can never be taken for granted.

A case in point is the *transexponential growth scale*. Viewed as a self-mapping of $]..,+\infty]$, the exponential map admits infinitely many smooth conjugations to the unit-shift, and as many systems of fractional iterates, none of which enjoys any precedence over the others : all can be shown to be, in some precise sense, 'undistinguishable' at infinity. This indeterminacy tells us something essential about the *fractal* nature of the transexponential growth scale, and leads to a rather curious construction : the *Grand Cantor* ([E5],[H]).

So anyone of inquisitive disposition, upon encountering the multizetas for the first time, hearing of their dimorphy, and learning about their decomposibility into irreducibles, is likely to ask himself, not without some trepidation : are there canonical irreducibles and is there, going with them, a canonical decomposition ? The answer this time is : *yes – there are, and there is.*

The key here is *orthogonality* with respect to some well-chosen quadradic forms.

For any pair of monomials $P_r^{\mathbf{w}} := \prod_1^r u_i^{p_i} v_i^{p_i^*}$, $Q_r^{\mathbf{w}} := \prod_1^r u_i^{q_i} v_i^{q_i^*}$, we set :

$$\begin{aligned}\mathrm{kwai}(P_r^\bullet, Q_r^\bullet) &:= \frac{p_1!\ldots p_r!}{(p_1+\ldots p_r)!}\frac{q_1!\ldots q_r!}{(q_1+\ldots q_r)!} \qquad \textit{if}\;\; p_i = q_i^*,\; p_i^* = q_i \;\; (\forall i)\\ &:= 0 \qquad\qquad \textit{otherwise}\end{aligned}$$

By linearity, this extends (assuming the sums to be finite) to a bilinear form on the space of **w**-polynomial bimoulds :

$$\mathrm{kwai}(A^\bullet, B^\bullet) := \sum_{r\geq 0} \mathrm{kwai}(A^\bullet_r, B^\bullet_r) \tag{105}$$

with the nice property of *swap*- and *push*-invariance :

$$\mathrm{kwai}(A^\bullet, B^\bullet) \equiv \mathrm{kwai}(\mathrm{swap}(A^\bullet), \mathrm{swap}(B^\bullet)) \tag{106}$$
$$\mathrm{kwai}(A^\bullet, B^\bullet) \equiv \mathrm{kwai}(\mathrm{push}(A^\bullet), \mathrm{push}(B^\bullet)) \tag{107}$$

For any pair of **u**-polynomial and **v**-constant (resp. **u**-constant and **v**-polynomial) bimoulds we set :

$$\mathrm{kwa}(A^\bullet, B^\bullet) := \mathrm{kwai}(A^\bullet, \mathrm{swap}(B^\bullet)) \tag{108}$$
$$\textit{resp.}\quad \mathrm{kwi}(A^\bullet, B^\bullet) := \mathrm{kwai}(\mathrm{swap}(A^\bullet), B^\bullet) \tag{109}$$

As quadratic forms, *kwa/kwi* are not *definite-positive*, not even when restricted to the subalgebra $\mathrm{ARI}^{\mathrm{ent}}_{\underline{\mathrm{al/al}}}$ of bialternal polynomials. On that subalgebra, however, they are *non-degenerate*. In other words, if a bialternal $Pa^\bullet/Pi^\bullet$ is *co-bialternal*, that is to say *kwa*-orthogonal to all other bialternals $Qa^\bullet/Qi^\bullet$:

$$\mathrm{kwa}(Pa^\bullet, Qa^\bullet) = 0\,;\ \forall\, Qa^\bullet \quad (\Longleftrightarrow \mathrm{kwi}(Pi^\bullet, Qi^\bullet) = 0\,;\ \forall\, Qi^\bullet) \tag{110}$$

then it is automatically 0.

But each indeterminacy encountered in the search for a canonical decomposition of multizetas into irreducibles is ultimately traceable to a degree of freedom within $\mathrm{ARI}^{\mathrm{ent}}_{\underline{\mathrm{al/al}}}$. *Therefore, imposing co-bialternality removes all theses indeterminacies at one stroke.*

This of course does not end the matter : one must also convince onseself that this is the most judicious choice. But a careful comparison with other possible criteria, in particular with other definitions of orthogonality, leaves little room for hesitation : co-bialternality relative to *kwa* is indeed the right criterion. [40]

6.7 Canonical π^2-dependence .

The sensitive first factor $\mathrm{zag}^\bullet_{\mathrm{I}}$ in §6.3, which carries all the π^2-dependence of multizetas and nothing but the π^2-dependence, can itself be split into three factors :

$$\mathrm{zag}^\bullet_{\mathrm{I}} = \mathrm{gari}(\mathrm{tal}^\bullet, \mathrm{midfactor}^\bullet, \mathrm{invgari}(\mathrm{pal}^\bullet)) \tag{111}$$

[40] A string of arguments in support of this claim are set forth in [E10]

All three factors are described in detail in [E10], but let us point out some of their most salient features.

The right factor invgari(pal$^\bullet$) is the *gari*-inverse of the bisymmetral mould pal$^\bullet$ (see §5.7). But pal$^\bullet$ does not verify the parity condition and its inverse is of symmetral/symmetril (rather than bisymmetral) type.

The left factor is the *eutrigonometric* mould tal$^\bullet$. Roughly, it is derived from the simpler *eupolar* mould pal$^\bullet$ by 'periodisation'. It is indisputably the dominant factor, the one that makes the largest contribution to $\mathrm{zag}_{\mathrm{I}}^\bullet$, but since it is not entire-valued (it has multiple poles at the origin $\mathbf{u} = 0$), it has to be corrected by the two other factors.

The middle factor is even-degreed; all others moulds are alternate-degreed, i.e. their components of even/odd length are polynomials of even/odd degree.

Of course, all these moulds have a double symmetry: pal$^\bullet$, tal$^\bullet$ and the mid-factor are bisymmetral, while $\mathrm{zag}_{\mathrm{I}}^\bullet$ and invgari(pal$^\bullet$) are symmetral/symmetril.

To sum up:

$$\begin{aligned}
\mathrm{zag}_{\mathrm{I}}^\bullet &\in \mathrm{GARI}^{\mathrm{ent}}_{\mathrm{as/is}}\\
\mathrm{invgari}(\mathrm{pal}^\bullet) &\in \mathrm{GARI}_{\mathrm{as/is}}\\
\mathrm{pal}^\bullet \ \textit{and}\ \mathrm{tal}^\bullet &\in \mathrm{GARI}_{\mathrm{as/as}}\\
\mathrm{midfactor}^\bullet &\in \mathrm{GARI}_{\underline{\mathrm{as/as}}} = \mathrm{expari}(\mathrm{ARI}_{\underline{\mathrm{al/al}}})
\end{aligned}$$

Proving the existence of a factorisation (111) is the easy bit. The difficulty is to make it 'canonical'. There is indeed a huge indeterminacy in the definition of the mid-factor, which *a priori* can be postcomposed (in GARI) by any bimould belonging to the group:

$$\mathrm{gari}(\mathrm{invgari}(\mathrm{pal}^\bullet), \mathrm{expari}(\mathrm{ARI}^{\mathrm{ent}}_{\underline{\mathrm{al/il}}}), \mathrm{pal}^\bullet) \tag{112}$$

and carrying only rational coefficients (upto π^2-rescaling). However, imposing co-bialternality on $\mathrm{zag}_{\mathrm{I}}^\bullet$ freezes the situation, unambiguously defines the canonical mid-factor, and even leads to tolerably explicit [41] formulas for its expansion.

[41] for the meaning of "explicit" in this context, see next para.

6.8 Canonical decomposition into irreducibles.

The task of analysing the alternal/alternil, entire-valued, respectively even- and alternate-degreed factors:

$$\mathrm{lozag}^\bullet_{\mathrm{II}} := \mathrm{logari}(\mathrm{zag}^\bullet_{\mathrm{II}}) \quad ; \quad \mathrm{lozag}^\bullet_{\mathrm{III}} := \mathrm{logari}(\mathrm{zag}^\bullet_{\mathrm{III}})$$

essentially reduces to constructing, for each even d, the alterna/alternil, polynomial-valued, alternate-degreed pair $\mathrm{ma}^\bullet_{\mathrm{d}}/\mathrm{mi}^\bullet_{\mathrm{d}}$ mentioned two paras earlier. Here again, *existence* is a direct consequence of the existence of the genuine multizetas (just take the 'genuine' $\mathrm{Lozag}^\bullet_{\mathrm{III}}$ and regroup there all terms with the same weight:= degree+ length) and *uniqueness+canonicity* is ensured by co-bialternality, i.e. by the requirement that $\mathrm{ma}^\bullet_{\mathrm{d}}/\mathrm{mi}^\bullet_{\mathrm{d}}$ be *kwa*-orthogonal to all bialternals.

The interesting part is the explicit construction, for all lengths r from 1 to d, of all the non-vanishing components [42] of $\mathrm{ma}^\bullet_{\mathrm{d}}/\mathrm{mi}^\bullet_{\mathrm{d}}$. There are two (equivalent) inductive schemes for accomplishing this. One would suffice, but both deserve mention, if only to underscore the all-pervading **u/v** symmetry. Here they are:

$$\begin{array}{lcccl} \text{Scheme I} & \{\mathit{alternal} & / & \mathit{alternil}\} & \mathit{upto\ length}\ \ 2\,r'-1 \\ & & \downarrow & & \\ & \{\mathit{alternal} & / & \mathit{alternil}\} & \mathit{upto\ length}\ \ 2\,r' \\ & & \downarrow & & \\ & \{\mathit{alternal} & / & \text{push-}\mathit{inv}^t\} & \mathit{upto\ length}\ \ 2\,r'+1 \\ & & \downarrow & & \\ & \{\mathit{alternal} & / & \mathit{alternil}\} & \mathit{upto\ length}\ \ 2\,r'+1 \end{array}$$

$$\begin{array}{lcccl} \text{Scheme II} & \{\mathit{alternal} & / & \mathit{alternil}\} & \mathit{upto\ length}\ \ 2\,r'-1 \\ & & \downarrow & & \\ & \{\mathit{alternal} & / & \mathit{alternil}\} & \mathit{upto\ length}\ \ 2\,r' \\ & & \downarrow & & \\ & \{\text{push-}\mathit{inv}^t & / & \mathit{alternil}\} & \mathit{upto\ length}\ \ 2\,r'+1 \\ & & \downarrow & & \\ & \{\mathit{alternal} & / & \mathit{alternil}\} & \mathit{upto\ length}\ \ 2\,r'+1 \end{array}$$

In either scheme, the first and second steps (first and second downward arrows) are given by rather complex, but totally explicit formulas. The third step (third downward arrow) involves the addition of a corrective term to restore the missing 'second symmetry'. The corrective term is *a priori* defined upto an arbitrary bialternal, but in fact unambiguously fixed by the

[42] $\mathrm{ma}^\bullet_{\mathrm{d}}/\mathrm{mi}^\bullet_{\mathrm{d}}$ being of alternal/alternil type, its component of length r has degree $d-r+1$

condition of co-bialternality imposed on $\mathrm{ma}^\bullet_{\mathrm{d}}/\mathrm{mi}^\bullet_{\mathrm{d}}$. This third step is definitely more complex, but still 'explicit' in the precise sense that it does not involve the solving of larger and larger linear systems [43]: one relies instead on Plancherel-like formulas attached to *kwa*-orthogonality.

For details, complements, and formulas *ad nauseam*, see [E10].

6.9 Canonical elimination of all '1'. Redistributivity.

As mentioned earlier, bialternal polynomials $Pa^\bullet_{r,d}/Pi^\bullet_{r,d}$ (length r, degree d) automatically possess two additional properties:

(i) *parity*: they are even functions of $\mathbf{w}$.

(ii) *push-invariance*: they are left unchanged by the *push* – which, we recall, acts on each mould component as a unit-shift on sequences $\{w_0, w_1, \ldots, w_r\}$ (with the 'augmented notation', see §5.1)

But they also possess a third important property:

(iii) *redistributivity.*

Redistributivity is best expressed in terms of the *swappee* $Pi^\bullet_{r,d}$ and using the 'augmented notation'. It says that whenever a given value v_{i_0} occurs several times in the string $\{v_0, v_1, \ldots, v_r\}$, $Pi^{\mathbf{w}}_{r,d}$ can be rewritten as a finite sum $\sum_{\mathbf{w}^*} c(v_{i_0}, \mathbf{w}, \mathbf{w}^*) Pi^{\mathbf{w}^*}_{r,d}$, with *universal, whole* coefficients $c(v_{i_0}, \mathbf{w}, \mathbf{w}^*)$ and with new sequences $\mathbf{w}^\bullet$ involving strings $\{v^*_0, v^*_1, \ldots, v^*_r\}$ with the same length and same elements as in the old string $\{v_0, v_1, \ldots, v_r\}$, but in different order and with v_{i_0} occurring only once. In other words, the multiplicity of v_{i_0} can be *redistributed* among the v_i other than v_{i_0}.

Like *push*-invariance, redistributivity is strictly weaker than bialternality, but it deserves to be investigated for its own sake. In fact, a whole string of subalgebras of ARI can be defined by imposing various conditions of *push*-invariance and/or redistributivity, and they play a critical part in the canonical decomposition of multizetas (see [E10]), mainly because (unlike the bialternal subalgebras) they are characterised by a *finite "group of constraints"*, which makes them amenable to Hilbert's theory of invariants and renders the calculation of their dimensions, for each cell (r, d), if not always easy, at least completely algorithmic.

[43] This seems a sensible definition of 'explicit' in the present context, as it maintains a meaningful distinction with 'constructive'.

Lastly, the phenomenon of redistributivity is intimately bound up with the theorem on the *eliminability of all '1'*, which says, in the case of plain multizetas, that each $\mathrm{Ze}^{s_1,\ldots,s_r}$ $(s_1 \geq 1)$ can be expressed, uniquely upto the symmetrelity of $\mathrm{Ze}^\bullet$, as a finite combination over $\mathbb{Q}$ of '1-free' multizetas $\mathrm{Ze}^{s_1^*,\ldots,s_{r^*}^*}$ of equal weight but possibly smaller length (i.e. $s_i^* \neq 1$, $\sum s_i^* = \sum s_i$, $r^* \leq r$).

6.10 From symbols to numbers. Other relations. The canonical rational associator.

Apart from the two sets of *quadratic relations*, which underlie our construction of *symbolic* or *formal* multizetas, the *genuine, numerical* multizetas verify a host of other relations, chief amongst which are:

(i) the two-term *digonal relation* [44] , also known as Hoffman involution
(ii) the five-term *pentagonal relation*, due to Drinfel'd
(iii) the six-term *hexagonal relation*, also due to Drinfel'd

All the indications are that:

$$\{quadratic\ \ relations\} \iff \{gonal\ \ relations\} \tag{113}$$

The implication $\Longrightarrow$ in particular is most probably valid. The digonal relation, at any rate, is already known ([E10]) to be derivable from the quadratic relations. For the hexagonal relation, the proof has not been seriously attempted, but ought to be feasible. For the pentagonal relation, it might perhaps be more elusive.

Be that as it may: *if* the implication $\Longrightarrow$ holds in (113), then the canonical factor $\mathrm{zag}_{\mathrm{I}}$ defined in §6.7 immediately yields (after re-scaling π^2 to 1) a *canonical, rational Drinfel'd associator* [45].

7 Conclusion and further vistas.

7.1 The rich nexus around ARI/GARI .

ARI/GARI does not stand isolated in the mathematical landscape. It is part of a whole nexus of *over-, side-* and *sub-structures*. It is also an area

[44] *di*gonal, not *dia*gonal. This is non-standard nomenclature, but it has has the merit of rhyming with the other *n*-gonal relations, which is only proper, given that all three relations really belong together.
[45] the notion of associator goes back to [D], which also establishes the existence of rational associators, but leaves the question of canonicity open.

of great natural 'biodiversity', with a large number of *operations* acting on ARI/GARI and an even larger population of *special moulds* living there.

Over-structures.
ARI/GARI is part of the larger algebra/group AXI/GAXI of all 'symplectic' derivations/automorphisms of BIM0 (this is the associative algebra of bimoulds, but under the ordinary mould product). 'Symplectic' here means that all operations performed on $\mathbf{w}$-sequences preserve both $\sum u_i\, v_i$ and $\sum du_i \wedge dv_i$, while involving only $\mathbf{u}$-sums and $\mathbf{v}$-differences.

Side-structures.
Within AXI/GAXI, ARI/GARI stands cheek by jowl with two other, rather different structures, ANI/GANI and AMI/GAMI, which, however, must be taken into consideration in order to get a *completely closed system of operations* and especially to control the effect of the *swap* on the *gari*-product. The algebra ARI also possesses a useful superalgebra analogue : SUARI (see §7.3 infra).

Sub-structures.
In practice one seldom works in the global ARI/GARI, but in a host of remarkable subalgebras/subgroups, which range from the *totally free* to the *strongly bound*, and are defined by various
(1) invariance conditions
(2) properties like *redistributivity* (see §6.9) etc
(3) restrictions on the nature of the $\mathbf{w}$-dependence : polynomial; flat (i.e. piece-wise constant); polar; trigonometric; well-articulated; etc.
(4) simple or double symmetries like al/al, al/il, al/ul (the 'mirror image' of al/il), al/iil or al/uul (the 'trigonometric' counterparts), or of course the 'multiplicative' analogues as/as, as/is, as/us, as/iis, as/uus, etc.

Wealth of operations.
Alongside its many natural *non-linear* operations, ARI/GARI also possesses a number of non-trivial and quite distinct *linear involutions* (which after all is only fitting for an apparatus geared to the investigation of dimorphy !) *that act as automorphisms on suitable subalgebras/subgroups*, like :

(1) the *swap* (see §5.1) : on the space of *push*-invariant (bi)moulds
(2) the *slap* (see §5.6,[E10]) : on the space of eupolar (bi)moulds
(3) the *clap* (see [E10]) : on the space of polynomial-valued (bi)moulds
(4) the *flap* (see [E10]) : on the spaces of flat/polar-valued (bi)moulds

The ARI/GARI structure is also essentially invariant under the (quadripotent !) *Fourier transform*, applied to all $\mathbf{u}$- and $\mathbf{v}$-variables.

Bounty of special moulds.
Multizeta theory makes use of all manner of 'special (bi)moulds' - up to one hundred at the moment. This being a survey, only a handful of them could be mentioned here. The rest are described and tabulated in [E10].

But numerous other moulds, which originate in different theories (local dynamics, differential/functional equations, KAM theory, etc) and have been around for quite some time, also seem to fit effortlessly into the ARI/GARI framework. What is more, this 're-housing' often leads to a simpler derivation of their properties. It sheds light in particular on the puzzling parallelism between 'sum-like' and 'difference-like' moulds. Facts such as these bolster one's faith in the future of ARI/GARI.

7.2 Just how new is ARI/GARI ?

I hope I may be forgiven for recalling a few facts about the genesis of ARI/GARI – and for (ab)using the vertical pronoun :

(i) I handled multizetas quite extensively in the mid-seventies, but in the context of holomorphic dynamics and holomorphic invariants, where *dimorphy* is of little or no relevance

(ii) I chanced upon the ARI/GARI structure via the *scramble* transform (§3.4, §5.8) in the late eighties, while investigating Singularly Perturbed Systems ([E6]).

(iii) I returned to the multizetas in August 1999, but this time with the ARI/GARI apparatus in hand and with dimorphy at the centre of attention. *Free generation* quickly followed, but *canonical decomposition* proved more elusive : there seemed at first to be a choice between several, equally appealing decompositions, and it took some close comparing and weighing before one of them emerged as 'clearly canonical'.

So ARI/GARI *is a new structure.* Insidious rumours are afloat, which tend to assimilate ARI to competing constructions, like the Ihara algebra or the so-called renormalisation algebra, *but they have no substance to them whatsoever.* ARI is anterior to both in point of time, vaster than both, and it radically differs from both in that it operates with a *double* series of variables, which makes it pre-adapted to the investigation of dimorphy.

Of these two reductionist pleas, the attempt to equate ARI with the Ihara algebra is particularly preposterous and disingenuous, given that :

(i) ARI almost immediately yielded the free generation theorem for multizetas, which had resisted repeated attempts based on the Ihara algebra.

(ii) the Ihara algebra cannot accommodate any – *not a single one* !!! – of the sixty-odd 'special bimoulds' like $\mathrm{pal}^\bullet/\mathrm{pil}^\bullet$, $\mathrm{tal}^\bullet/\mathrm{til}^\bullet$, etc, which are required to describe the π^2-dependence of multizetas.

(iii) the Ihara algebra cannot handle the modulated multizetas; in particular it has no room for bimoulds like $\mathrm{bela}^\bullet/\mathrm{beli}^\bullet$ which are needed to enumerate the irreducibles attached to the Eulerian multizetas.

(iv) the approach based on the Ihara algebra did not and could not yield the *canonical* irreducibles nor the *canonical* rational associator. In fact, the followers and upholders of that approach seemed blissfully unaware of the existence of any such things.

7.3 SUARI, superalgebra companion to ARI.

Let SUARI := ^{0}SUARI $\oplus$ ^{1}SUARI denote the Lie *super-algebra* defined in this way:

(i) ^{0}SUARI (resp. ^{1}SUARI) consists of all bimoulds whose non-vanishing components have even (resp. odd) lengths r

(ii) the super-bracket is the bilinear form *suari*:

$$\begin{aligned} C^\bullet = \mathrm{suari}(A^\bullet, B^\bullet) \quad \Longleftrightarrow \quad C^{\mathbf{w}} &= \sum_{\mathbf{w}=\mathbf{b}.\mathbf{c}} (A^{\mathbf{b}} B^{\mathbf{c}} - (-1)^{r(\mathbf{b})r(\mathbf{c})} B^{\mathbf{b}} A^{\mathbf{c}}) \\ &+ \sum_{\mathbf{w}=\mathbf{b}.\mathbf{c}.\mathbf{d}} ((-1)^{r(\mathbf{c})r(\mathbf{b}.\mathbf{d})} A^{\lfloor \mathbf{c}} B^{\mathbf{b} \rceil \mathbf{d}} - (-1)^{r(\mathbf{c})r(\mathbf{d})} B^{\lfloor \mathbf{c}} A^{\mathbf{b} \rceil \mathbf{d}}) \\ &+ \sum_{\mathbf{w}=\mathbf{a}.\mathbf{b}.\mathbf{c}} ((-1)^{r(\mathbf{a}.\mathbf{c})r(\mathbf{b})} A^{\mathbf{a} \lceil \mathbf{c}} B^{\mathbf{b} \rfloor} - (-1)^{r(\mathbf{a})r(\mathbf{b})} B^{\mathbf{a} \lceil \mathbf{c}} A^{\mathbf{b} \rfloor}) \end{aligned} \tag{114}$$

which is patterned [46] on the bracket *ari* of ARI but verifies the super-commutator and super-Jacobi identities:

$$\begin{aligned} 0 &\equiv \mathrm{suari}(A^\bullet_{r_1}, A^\bullet_{r_2}) + (-1)^{r_1 r_2}\, \mathrm{suari}(A^\bullet_{r_2}, A^\bullet_{r_1}) \\ 0 &\equiv (-1)^{r_1 r_3}\, \mathrm{suari}(A^\bullet_{r_1}, \mathrm{suari}(A^\bullet_{r_2}, A^\bullet_{r_3})) \\ &+ (-1)^{r_2 r_1}\, \mathrm{suari}(A^\bullet_{r_2}, \mathrm{suari}(A^\bullet_{r_3}, A^\bullet_{r_1})) \\ &+ (-1)^{r_3 r_2}\, \mathrm{suari}(A^\bullet_{r_3}, \mathrm{suari}(A^\bullet_{r_1}, A^\bullet_{r_2})) \end{aligned}$$

[46] $r(\mathbf{w})$ denotes the length of the sequence $\mathbf{w}$

The *swap* and *push* are defined exactly as in ARI, but the relevant simple (resp. double) symmetry is now *super-alternal* (resp. *super-bialternal*, meaning *super-alternal with an super-alternal swappee*). A bimould $A^\bullet$ is said to be super-alternal iff:

$$\sum_{\mathbf{w}=\mathrm{sha}(\mathbf{w^1},\mathbf{w^2})} (-1)^{n(\mathrm{w};\,\mathrm{w}^1,\mathrm{w}^2)}\, A^{\mathbf{w}} = 0 \qquad\qquad \forall \mathrm{w}^1 \neq \emptyset\ ,\ \forall \mathrm{w}^2 \neq \emptyset \tag{115}$$

with the integer $n(\mathrm{w};\,\mathrm{w}^1,\mathrm{w}^2)$ denoting the number of order reversals in the shuffle $\mathbf{w}$ of $\mathbf{w^1}$ and $\mathbf{w^2}$.

Moulds with this simple/double symmetry are stable under *suari*, leading to the following string of super-algebras (with self-explanatory notations):

$$\mathrm{SUARI}_{\underline{\mathrm{sual/sual}}} \;\subset\; \mathrm{SUARI}_{\mathrm{sual}} \;\subset\; \mathrm{SUARI} \tag{116}$$

Like their bialternal models, the super-bialternals automatically possess three series of additional properties, albeit sleightly different ones:

(i) they display alternate parity, i.e. their components of even/odd length are even/odd functions of $\mathbf{w}$ – except for the length-1 component, which is not necessarily odd. But it *has to be* if we want closure under *suari*. This subsidiary condition is signalled by underlining *sual/sual* in (116)

(ii) they are super-*push*-invariant, i.e. $\mathrm{push}(A^\bullet_r) \equiv (-1)^r A^\bullet_r$

(iii) they are redistributive (see §6.9)

The bielementals $\mathrm{belam}^\bullet_r/\mathrm{belim}^\bullet_r$ of ARI (see §5.5,§6.5) have their exact analogue: the super-bielementals $\mathrm{subelam}^\bullet_r/\mathrm{subelim}^\bullet_r$. The definition is much the same except that the integer $[k]_r$ is now replaced everywhere by the integer part $[\![k]\!]_r$ of $[k]_r/2$. The exact transposition goes like this:

$$\mathrm{subelam}^{w_1,\dots,w_r}_{r,\,\mathrm{xaxi}} = \mathrm{subelam}^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}}_{r,\,\mathrm{xaxi}} :=$$

$$\sum_{\substack{i,j,m,n\in\mathbb{Z}_{r+1}\\ \cdots<i\leq m<j\leq n<\cdots}} \mathrm{subel}^{i,j;m,n}_r\ \mathrm{xa}(u_i+u_{i+1}+\cdots+u_{j-1})\ \mathrm{xi}(v_m-v_n)\ \equiv$$

$$\sum_{\substack{i,j,m,n\in\mathbb{Z}_{r+1}\\ \cdots<i\leq m<j\leq n<\cdots}} \frac{1}{2}\ \mathrm{subel}^{i,j;m,n}_r \Big(\mathrm{xa}(u_{i\dots j-1})\ \mathrm{xi}(v_{m:n})\ +(-1)^r\,\mathrm{xa}(u_{j\dots i-1})\ \mathrm{xi}(v_{n:m})\Big)$$

with a *swappee*:

$$\mathrm{subelim}^\bullet_{r,\,\mathrm{xaxi}} := \mathrm{swap}\,(\mathrm{subelam}^\bullet_{r,\,\mathrm{xaxi}}) \equiv \mathrm{subelam}^\bullet_{r,\,\mathrm{xixa}} \tag{117}$$

and with integer coefficients:

$$\mathrm{subel}_r^{i,j;m,n} \equiv (-1)^r \mathrm{subel}_r^{j,i;n,m} :=$$
$$[\![i,j;m,n]\!]_r \; \frac{(-1)^{r\,(i-1)}\,(-1)^{[\![m-i]\!]_r+[\![n-j]\!]_r}\;\;[\![r-1]\!]_r!}{[\![m-i]\!]_r!\;[\![n-j]\!]_r!\;[\![j-m-1]\!]_r!\;[\![i-n-1]\!]_r!}$$

that carry a 'coherence factor' $[\![i,j;m,n]\!]_r$ alternately equal to 1 or 0:

$$[\![i,j;m,n]\!]_r := 1+[\![m-i]\!]_r+[\![n-j]\!]_r+[\![j-m-1]\!]_r+[\![i-n-1]\!]_r-[\![r-1]\!]_r$$

which was absent from the parallel definition of $\mathrm{bel}_r^{i,j;m.n}$ since:

$$[i,j;m,n]_r := 1+[m-i]_r+[n-j]_r+[j-m-1]_r+[i-n-1]_r-[r-1]_r \equiv 1$$

If the length r is even/odd, the super-bielemental $\mathrm{subelam}^{\bullet}_{\mathrm{r,\,xaxi}}$ depends only on the even/odd part of *xaxi*. But the important thing is that *in either case* $\mathrm{subelam}^{\bullet}_{\mathrm{r,\,xaxi}}$ *is super-bialternal.*

So far, the modifications have been fairly straightforward, but this is about to change. Indeed, as Yu. Manin points out ([Ma]), switching from *ordinary* to *super* is not simply a matter of sprinkling the usual formulas with a few *minus* signs here and there. It often entails highly non-trivial changes and may reward us with unexpected insights. In the present instance, the change from ARI to SUARI and more particularly from $\mathrm{ARI}_{\underline{\mathrm{al/al}}}$ to $\mathrm{SUARI}_{\underline{\mathrm{sual/sual}}}$ brings a most welcome simplification, which is this:

Whereas in ARI *the bielementals generate* most *bialternals, in* SUARI *the super-bielementals generate* all *super-bialternals.*

Let us examine the situation in two special cases:

Recall (§6.5) that the algebra of **u**-polynomial, **v**-constant bialternals (of special importance, because they correspond to the plain-multizetaic irreducibles) was (non-freely) generated by two sets of bialternals: the utterly simple ekma$^\bullet$/ekmi$^\bullet$ (of length 1) and the bafflingly complex carma$^\bullet$/carmi$^\bullet$ (of length 4). In complete contrast, the super-algebra of **u**-polynomial, **v**-constant super-bialternals is now (again, non-freely) generated by the *simple system* suekma$^\bullet$/suekmi$^\bullet$ (with length 1 and odd degrees), *there being no need for super-analogues* [47] *of* carma$^\bullet$/carmi$^\bullet$.

As for the case of the self-correlated , **u**-polynomial bialternals with their **v**-variables in $\mathbb{Z}_2$ (which correspond to the Eulero-multizetaic irreducibles),

[47]such analogues, in fact, do not exist.

the situation was already simple in ARI and it remains so in SUARI : we had free generation by the system $\{\,\mathrm{bela}^{\bullet}_{r,d}/\mathrm{beli}^{\bullet}_{r,d}\ ;\ r\geq 1, d\ \mathit{even}\}$ obtained by particularising the bielementals $\mathrm{belam}^{\bullet}/\mathrm{belim}^{\bullet}$; and now we have free generation by the system $\{\,\mathrm{subela}^{\bullet}_{r,d}/\mathrm{subeli}^{\bullet}_{r,d}\ ;\ r\geq 1, r{+}d\ \mathit{even}\}$ obtained by particularising the super-bielementals $\mathrm{subelam}^{\bullet}/\mathrm{subelim}^{\bullet}$.

Much else, but not everything, carries over to SUARI. The various Broadhurst-Kreimer formulas have, predictably enough, their (often simpler) anologues. But at the moment it is unclear whether the sixty-odd special moulds like $\mathrm{pal}^{\bullet}/\mathrm{pil}^{\bullet}$, which are such an outstanding feature of ARI/GARI, possess interesting super-analogues, or for that matter any analogues at all.

7.4 The ring $\mathbb{N}\mathrm{a}$ of naturals and the huge scope of dimorphy.

Strange to say, but the notion of dimorphy is itself 'dimorphous' :

(i) For a space $\mathcal{D}$ of *functions*, dimorphy means closure under two *distinct* products : usually point-wise multiplication and some form or other of convolution.

(ii) For a space $\mathbb{D}$ of *numbers*, dimorphy means being a countable $\mathbb{Q}$-ring and possessing two distinct, *natural* bases $\{\alpha_m\}$ and $\{\beta_n\}$, each with its countable indexation m and n [48], with finite conversion laws :

$$\alpha_m \equiv \sum_n H^n_m\,\beta_n \quad ; \quad \beta_n \equiv \sum_m K^m_n\,\alpha_m \tag{118}$$

and with *two distinct ways* of calculating the *one and only* product on $\mathbb{D}$, which is ordinary number multiplication :

$$\alpha_m\,\alpha_n \equiv \sum_r A^r_{m,n}\,\alpha_r \quad ; \quad \beta_m\,\beta_n \equiv \sum_r B^r_{m,n}\,\beta_r \tag{119}$$

All four sums have to be finite, and the constant H, K, A, B are rational.

Clearly, since one may always concoct artificial bases $\{\alpha_m\}$,$\{\beta_n\}$ to meet the above conditions, the whole emphasis in this notion of numerical dimorphy must lie on the *naturalness* of the two bases . This may seem a rather shaky foundation for a mathematical definition, but we venture to suggest

[48] m an n usually do not range through $\mathbb{N}$, but through more complex sets, like the monoid generated by $\mathbb{N}$.

that in fact it is not: in all known instances of dimorphy, there is no scope for hesitation; the two bases $\{\alpha_m\}$,$\{\beta_m\}$ are clearly *there for all to see*, unmistakably *nature-given*, whereas it often takes a considerable amount of toil to *extract the hidden core* of $\mathbb{D}$, which usually is an algebraically free system $\{\gamma_r\}$ of irreducibles. This third set $\{\gamma_r\}$, typically, lies buried deep below the surface and, at least when taken in canonical form, tends to be found exactly mid-way between the two 'emerging' sets $\{\alpha_m\}$,$\{\beta_n\}$. So, even though it may be argued that numerical dimorphy is, ultimately, pure *māyā*, it is the sort of *māyā* that you must work hard to dispel...

In any case, functional and numerical dimorphy go hand in hand, and the proper framework for their joint investigation would seem to be, not the so-called 'theory of periods' [49], but the twin systems $\mathcal{N}\!a, \mathbb{N}\mathbb{a}$, whose construction, very roughly, goes like this:

(i) We produce the function germs f (of z, at ∞) in $\mathcal{N}\!a$ 'out of nothing', i.e. from $f(z) \equiv 1$, by taking larger and larger closures under the (direct and reverse) operations $+, \times, \partial_z, \circ$, with $\mathbb{Q}$ or $\mathbb{A}$ as scalar field. Very early on in this enlargement process, chronic *divergence* appears in the formal series or transseries $\tilde{f}$ which 'expand' at ∞ the germs f in $\mathcal{N}\!a$.

(ii) We carefully refrain from introducing artificial derivations on $\mathcal{N}\!a$, for fear of compromising the natural character of the construction. Rather, we ask: are there – *already, without our doing* – exotic derivations (i.e. derivations not generated by ∂_z) that act on $\mathcal{N}\!a$ and respect its natural topology? And we find that there is indeed a teeming profusion of them – two systems in fact, the alien derivations Δ_ϖ and the foreign derivations ∇_ϖ . The reason for this plethora of exotic derivations is the omnipresence of *divergence* in $\mathcal{N}\!a$: to analyse this divergence, qualitatively and quantitatively, suitable operators are called for, which are precisely the exotic derivations.

(iii) To produce $\mathbb{N}\mathbb{a}$ from $\mathcal{N}\!a$ (i.e. numbers from function germs, monics from monomials) we *do not* evaluate our germs at given points [50]. Rather, we let the exotic derivations act on these germs, and it turns out that the exotic derivatives of our monomials are expressible as sums of 'simpler' or 'earlier' monomials, with well-defined, generically transcendental scalar coefficients.

(iv) We harvest all these coefficients, declare them to be 'monics', and call $\mathbb{N}\mathbb{a}$ the ring generated by them.

[49] despite all the hullabaloo about 'periods', the constants there are given pell-mell, with no natural indexation, and all the symmetries central to dimorphy are broken, beginning with the sum/integral symmetry.

[50] say, rational or algebraic points

The advantage of *exotic derivation* over *pointwise evaluation* lies not at all in the nature of the constants being produced (they are much the same with both methods) but rather in the orderliness of the procedure, which turns out monics directly in *mould form*, automatically gives them the *right type of indexation*, and tells us to which fundamental *symmetry type* they belong (as moulds).

There are at least four major 'domains of dimorphy', of increasing size, in $\mathbb{N}\mathbb{a}$. They comprise, respectively :

(i) the multizetas

(ii) the general hyperlogarithmic monics (see §4)

(iii) the monics associated with monomials that verify *affine* differential equations, with coefficients in $\mathbb{Q}[z]$ or $\mathbb{A}[z]$

(iv) the monics associated with monomials that verify *'bipolynomial'* [51] differential equations, again with coefficients in $\mathbb{Q}[z]$ or $\mathbb{A}[z]$

The third and (especially) fourth domain of dimorphy are incredibly large and would seem to encompass more or less all constants encountered in 'real life'. In fact, dimorphy appears to extend as far as he sight reaches : the whole of $\mathbb{R}$'s explorable-constructive part seems to be 'dimorphic' to the core.

Needless to say, these constructions certainly admit many variations, and their exhaustive investigation (for ex. extending to Domains 2,3,4 the whole algebraic apparatus developped for Domain 1) would require huge efforts - and might not repay them. Yet, strangely, the central fact about $\mathbb{N}\mathbb{a}$, namely *numerical dimorphy*, is easy enough to establish, at least for these four domains. It directly mirrors the fact of *functional dimorphy*, which follows from the stability of $\mathcal{N}a$ under the generalised 'Borel-Laplace' transform, which itself is but an adaptation of the Fourier transform. So the least we can say is that dimorphy is 'well-connected' ! It is nothing of an accident or localised oddity.

P.S. This section owes much to discussions I had with Joris van der Hoeven.

[51] 'bipolynomial' means that the differential equation may involve not only ordinary products of f, f',f'', etc, but also convolution products of type $\ast$.

7.5 Are there exotic derivations acting on numbers ?

Dimorphy is by no means the end of the story. After identifying the phenomenon and acknowledging its scope, we must raise another question : does dimorphy really exhaust the arithmetico-algebraic structure of $\mathbb{N}\mathfrak{a}$, the 'explorable part' of $\mathbb{R}$? For instance, narrowing the focus to Domain 1 : do the 'quadratic relations' exhaust the set of algebraic constraints on multizetas ? One would assume the answer to be yes, but at the moment the tool-kit of transcendence theory seems woefully inadequate to tackle such questions[52]. That might change, however, if we had at our disposal, for *numbers*, the sort of high-powered machinery that we have for *functions*, namely : exotic derivations. We might then go about disproving the existence of 'undesirable relations' $R(\alpha, \beta, \dots) = 0$ for *numbers* by following the very same scheme which we outlined in §2.4 and which works wonders in the case of *functions*.

To be of any use, *numerical* exotic derivations ought to meet three conditions :
(i) acting on some countable extension $\mathbb{D}$ of $\mathbb{Q}$ or $\mathbb{A}$
(ii) annihilating $\mathbb{Q}$ or $\mathbb{A}$
(iii) acting 'universally', in the sense that their action on any x in $\mathbb{D}$ ought to be deducible from some universal representation of x, according to some universal procedure, without assuming any foreknowledge of the 'origin' or 'personal history' of x (i.e. the relations which define x, or which define the numbers of which it is made) − *unlike* in Galois theory, but *like* in alien calculus, where the action of an alien derivation Δ_ω on a resurgent function f is defined 'universally', without any reference to the 'history' of f [53].

Now, do such numerical derivations exist ? They probably do, in fact some are already known to exist, but the examples constructed so far are not very useful, since they act on rings or fields $\mathbb{D}$ obtained by adding to $\mathbb{Q}$ or $\mathbb{A}$ numbers characterised by certain 'unnatural' *lacunarity* conditions, which typically tend to *exclude* the interesting numbers, i.e. the *naturals* of $\mathbb{N}\mathfrak{a}$. But the search for numerical derivations has not yet begun in earnest and nothing warrants pessimism.

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[52] despite remarkable break-throughs like [Ap] and [R]
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