Power series with sum-product Taylor coefficients and their resurgence algebra.

Jean Ecalle (Orsay, CNRS) , Shweta Sharma (Orsay)

Abstract. *The present paper is devoted to power series of SP type, i.e. with coefficients that are syntactically sum-product combinations. Apart from their applications to analytic knot theory and the so-called "Volume Conjecture", SP-series are interesting in their own right, on at least four counts : (i) they generate quite distinctive resurgence algebras (ii) they are one of those relatively rare instances when the resurgence properties have to be derived directly from the Taylor coefficients (iii) some of them produce singularities that unexpectedly verify finite-order differential equations (iv) all of them are best handled with the help of two remarkable, infinite-order integral-differential transforms,* mir *and* nir.

Contents

1 Introduction.

1.1 Power series with coefficients of sum-product type.

The notion of SP series.
Sum-product series (or *SP*-series for short) are Taylor series :

$$j(\zeta) := \sum_{n\geq 0} J(n)\ \zeta^n \tag{1}$$

whose coefficients are syntactically of sum-product (SP) type :

$$J(n) := \sum_{\epsilon\leq m<n} \prod_{\epsilon\leq k\leq m} F(\frac{k}{n}) = \sum_{\epsilon\leq m<n} \exp\Big(-\sum_{\epsilon\leq k\leq m} f(\frac{k}{n})\Big) \quad (\epsilon\in\{0,1\}) \tag{2}$$

Summation starts at $\epsilon = 0$ unless $F(0) \in \{0, \infty\}$, in which case it starts at $\epsilon = 1$. It always ends at $n-1$, not n.[1] The two driving functions F and f are connected under $F \equiv \exp(-f)$. Unless stated otherwise, F will be assumed to be meromorphic, and special attention shall be paid to the case when F has neither zeros nor poles, i.e. when f is holomorphic.

The importance of SP-series comes from their analytic properties (isolated singularities of a quite distinctive type) and their frequent occurence in various fields of mathematics (ODEs, knot theory etc).

As for the above definition, it is less arbitrary than may seem at first sight. Indeed, none of the following changes :

(i) changing the grid $\{k/n\}$ to $\{Const\, k/n\}$

(ii) changing the lower summation bounds from 0 or 1 to 2,3 ...

(iii) changing the upper summation bound from $n-1$ to n or $n-2, n-3$ etc or a multiple thereof

(iv) replacing the 0-accumulating products $\prod F(\frac{k}{n})$ by 1-accumulating products $\prod F(\frac{n-k}{n})$ - none of these changes, we claim, would make much difference or even (allowing for minor

[1] This choice is to ensure near-invariance under the change $F(x) \mapsto 1/F(1-x)$. See §3.5

adjustments) take us beyond the class of SP-series.

Special cases of SP series.
For F a polynomial or rational function (resp. a trigonometric polynomial) and for Taylor coefficients $J(n)$ defined by pure products $\prod$ (rather than sum-products $\sum\prod$) the series $j(\zeta)$ would be of hypergeometric (resp. q-hypergeometric) type. Thus the theory of SP-series extends – and bridges – two important fields. But it covers wider ground. In fact, the main impulse for developping it came from knot theory, and we didn't get involved in the subject until Stavros Garoufalidis[2] and Ovidiu Costin[3] drew our attention to its potential.

Overview.
In this first paper, halfway between survey and full treatment[4], we shall attempt five things :

(i) bring out the *main analytic features* of SP-series, such as the dichotomy between their two types of singularities (*outer/inner*), and produce *complete systems of resurgence equations*, which encode in compact form the whole Riemann surface structure.

(ii) *localise* and *formalise* the problem, i.e. break it down into the separate study of a number of *local* singularities, each of which is produced by a specific *non-linear functional transform* capable of a full analytical description, which reduces everything to *formal* manipulations on power series.

(iii) sketch the *general picture* for arbitrary driving functions F and f – pending a future, detailed investigation.

(iv) show that in many instances (f polynomial, F monomial or even just rational) our local singularities satisfy ordinary differential equations, but of a very distinctive type, which accounts for the 'rigidity' of their resurgence equations, i.e. the occurrence in them of essentially *discrete* Stokes constants.[5]

(v) sketch numerous examples and then give a careful treatment, theoretical *and* numerical, of *one special case* chosen for its didactic value (it illustrates all the main SP-phenomena) and its practical relevance to knot theory (specifically, to the knot 4_1).

1.2 The outer/inner dichotomy and the ingress factor.

The outer/inner dichotomy.
Under analytic continuation, SP-series give rise to two distinct types of *singularities*, also

[2] an expert in knot theory who visited Orsay in the fall of 2006.
[3] an analyst who together with S. Garoufalidis has been pursuing an approach to the subject parallel to ours, but distinct : for a comparison, see §12.1.
[4] Two follow-up investigations [SS1],[SS2] are being planned.
[5] contrary to the usual situation, where these Stokes or resurgence constants are free to vary continuously.

referred to as *'generators'*, since under alien derivation they generate the resurgence algebra of our SP-series. On the one hand, we have the *outer generators*, so-called because they never recur under alien derivation (but produce inner generators), and on the other hand we have the *inner generators*, so-called because they recur indefinitely under alien derivation (but never re-produce the outer generators). These two are, by any account, the main types of generators, but for completeness we add two further classes : the *original generators* (i.e. the SP-series themselves) and the *exceptional generators*, which don't occur naturally, but can prove useful as auxiliary adjuncts.

A gratifying surprise : the *mir*-transform.
We shall see that *outer* generators can be viewed as infinite sums of *inner generators*, and that the latter can be constructed quite explicitly by subjecting the driving function F to a chain of nine local transforms, all of which are elementary, save for one crucial step : the *mir*-transform. Furthermore, this *mir*-transform , though resulting from an unpromising mix of complex operations[6], will turn out to be an *integro-differential operator*, of infinite order but with a transparent expression that sheds much light on its analytic properties. We regard this fascinating *mir*-transform, popping out of nowhere yet highly helpful, as the centre-piece of this investigation.

The ingress factor and the cleansing of SP-series.
Actually, rather than directly considering the SP-series $j(\zeta)$ with coefficients $J(n)$, it shall prove expedient to study the slightly modified series $j^{\#}(\zeta)$ with coefficients $J^{\#}(n)$ obtained after division by a suitably defined *'ingress factor'* $Ig_F(n)$ of strictly local character :

$$j^{\#}(\zeta) = \sum J^{\#}(n)\,\zeta^n \qquad \textit{with} \qquad J^{\#}(n) := J(n)/Ig_F(n) \tag{3}$$

This purely technical trick involves no loss of information[7] and achieves two things :

(i) the various *outer* and *inner* generators will now appear as purely *local* transforms of the driving function F viewed as an analytic germ at 0 or at some other suitable base point x_0 (in $[0,1]$ or even outside).

(ii) distinct series $j_{F_i}(\zeta)$ relative to distinct base points x_i (or, put another way, to distinct translates $F_i(x) := F(x + x_i)$ of the same driving function) will lead to exactly the same *inner generators* and so to the same *inner algebra* – which wouldn't be the case but for the pre-emptive removal of Ig_F.

In any case, as we shall see, the ingress factor is a relatively innocuous function and (even when it is divergent-resurgent, as may happen) the effect not only of *removing it* but also, if we so wish, of *putting it back* can be completely mastered.

[6] two Laplace transforms, direct and inverse, with a few violently non-linear operations thrown in.
[7] since information about $j^{\#}$ immediately translates into information about j, and vice versa.

1.3 The four gates to the inner algebra.

We have just described the various types of singularities or 'generators' we are liable to encounter when analytically continuing a SP-series. Amongst these, as we saw, the *inner generators* stand out. They span the *inner algebra*, which is the problem's hard, invariant core. Let us now review the situation once again, but from another angle, by asking: how many *gates* are there for entering the unique *inner algebra* ? There are, in effect, four types:

Gates of type 1: original generators. We may of course enter through an *original generator*, i.e. through a *SP series*, relative to any base point x_0 of our choosing. Provided we remove the corresponding ingress factor, we shall always arrive at the same inner algebra.

Gates of type 2: outer generators. We may enter through an *outer generator*, i.e. through the mechanism of the nine-link chain of section 5, again relative to any base point. Still, when F does have zeros x_i, these qualify as privileged base points, since in that case we can make do with the simpler four-link chain of section 5.

Gates of type 3: inner generators. We may enter through an *inner generator*, i.e. via the mechanism of the nine-link chain of section 4, but only from a base point x_i where f (not F !) vanishes. By so doing, we do not properly speaking *enter* the inner algebra, but rather start *right there*. Due to the ping-pong phenomenon, this inner generator then generates all the other ones. The method, though, has the drawback of introducing a jarring dissymmetry, by giving precedence to *one* inner generator over all others.

Gates of type 4: exceptional generators. We may enter through a *exceptional* or *'mobile' generator*, i.e. once again via the mechanism of the nine-link chain of section 4, but relative to any base point x_0 where f doesn't vanish[8]. It turns out that any such "exceptional generator" generates all the inner generators (– and what's more, *symmetrically* so –), but isn't generated by them. In other words, it gracefully *self-eliminates*, thereby atoning for its parasitical character. Exceptional generators, being 'mobile', have the added advantage that their base point x_0 can be taken arbitrarily close to the base point x_i of any given inner generator, which fact proves quite helpful, computationally and also theoretically.

2 Some resurgence background.

2.1 Resurgent functions and their three models.

The four models: formal, geometric, upper/lower convolutive.
Resurgent 'functions' exist in three/four types of models:

[8] so that the so-called *tangency order* is $\kappa = 0$, whereas for the inner generators it is ≥ 1 and generically $= 1$. See §4.

(i) The *formal model*, consisting of formal power series $\tilde{\varphi}(z)$ of a variable $z \sim \infty$. The tilda points to the quality of being 'formal', i.e. possibly divergent.

(ii) The *geometric models* of direction θ. They consist of sectorial analytic germs $\varphi_\theta(z)$ of the same variable $z \sim \infty$, defined on sectors of aperture $> \pi$ and bisected by the axis $\arg(z^{-1}) = \theta$.

(iii) The *convolutive model*, consisting of *'global microfunctions'* of a variable $\zeta \sim 0$. Each microfunction possesses one *minor* (exactly defined, but with some *information missing*) and many *majors* (defined up to regular germs at the origin, i.e. with some *redundant information*). However, under a frequently fulfilled integrability condition at $\zeta \sim 0$, the *minor* contains all the information, i.e. fully determines the microfunction, in which case all calculations reduce to manipulations on the sole minors. As for the *'globalness'* of our microfunctions, it means that their minors possess the property of "endless analytic continuation" : they can be continued analytically in the ζ-plane along any given (self-avoiding or self-intersecting, whole or punctured[9]) broken line starting from 0 and ending anywhere we like.

Usually, one makes do with a single convolutive model, but here it will be convenient to adduce two of them : the *'upper'* and *'lower'* models. In both, the minor-major relation has the same form ;

$$\begin{array}{llcl} & \textit{minor} & & \textit{major} \\ \textit{upper} & \overset{\frown}{\varphi}(\zeta) & \equiv & \frac{1}{2\pi i}\Big(\overset{\smile}{\varphi}(\zeta\, e^{-\pi i}) - \overset{\smile}{\varphi}(\zeta\, e^{\pi i})\Big) \\ \textit{lower} & \overset{\wedge}{\varphi}(\zeta) & \equiv & \frac{1}{2\pi i}\Big(\overset{\vee}{\varphi}(\zeta\, e^{-\pi i}) - \overset{\vee}{\varphi}(\zeta\, e^{\pi i})\Big) \end{array}$$

while the upper-lower correspondence goes like this :

$$\overset{\wedge}{\varphi}(\zeta) \equiv \partial_\zeta \overset{\frown}{\varphi}(\zeta) \quad ; \quad \overset{\vee}{\varphi}(\zeta) \equiv -\partial_\zeta \overset{\smile}{\varphi}(\zeta) \tag{4}$$

One of the points of resurgent analysis is to resum divergent series of 'natural origin', i.e. to go from the formal model to the geometric one via one of the two convolutive models. Concretely, we go from *formal* to *convolutive* by means of a formal or term-wise Borel transform[10] and from *convolutive* to *geometric* by means of a θ-polarised Laplace transform, i.e. with integration along the half-axis $\arg(\zeta) = \theta$.

$$\begin{array}{lccccccc} & & & \textit{upper} & & & & \textit{lower} \\ \textit{geometric} & \varphi_\theta & & & & \varphi_\theta & & \\ & & \nwarrow \overline{\mathcal{L}}_\theta & & & & \nwarrow \underline{\mathcal{L}}_\theta & \\ \textit{convolutive} & & & \overset{0}{\varphi} & & & & \overset{\diamond}{\varphi} \\ & & \nearrow \overline{\mathcal{B}} & & & & \nearrow \underline{\mathcal{B}} & \\ \textit{formal} & \tilde{\varphi} & & & & \tilde{\varphi} & & \end{array}$$

[9] the broken line may be punctured at a finite number of singularity-carrying points $\zeta_1, \ldots, \zeta_n$, in which case we demand analytic continuability along *all* the 2^n paths that follow the broken line but circumvent each ζ_i to the right or to the left.

[10] thus *upper* (resp. *lower*) Borel takes $\sum a_n\, z^{-n}$ to $\sum a_n \frac{\zeta^n}{n!}$ (resp. $\sum a_n \frac{\zeta^{n-1}}{(n-1)!}$).

Resurgent algebras: the multiplicative structure.
Resurgent functions are stable not just under *addition* (which has the same form in all models) but also under a *product* whose shape varies from model to model:
(i) in the *formal* model, it is the ordinary multiplication of power series.
(ii) in the *geometric* model, it is the pointwise multiplication of analytic germs.
(iii) in the *convolutive* models, it is the *upper/lower* convolution, with distinct expressions for minors[11] and majors:

minor convolution

$$\begin{array}{llcl} \textit{upper}\ \overline{*} & (\overset{\frown}{\varphi}_1 \,\overline{*}\, \overset{\frown}{\varphi}_2)(\zeta) & := & \int_0^\zeta \overset{\frown}{\varphi}_1(\zeta-\zeta_2)\, d\overset{\frown}{\varphi}_2(\zeta_2) \\ \textit{lower}\ \underline{*} & (\overset{\wedge}{\varphi}_1 \,\underline{*}\, \overset{\wedge}{\varphi}_2)(\zeta) & := & \int_0^\zeta \overset{\wedge}{\varphi}_1(\zeta-\zeta_2)\, \overset{\wedge}{\varphi}_2(\zeta_2)\, d\zeta_2 \end{array}$$

major convolution

$$\begin{array}{llcl} \textit{upper}\ \overline{*} & (\overset{\smile}{\varphi}_1 \,\overline{*}\, \overset{\smile}{\varphi}_2)(\zeta) & := & \frac{1}{2\pi i}\int_{I(\zeta,u)} \overset{\smile}{\varphi}_1(\zeta-\zeta_2)\, d\overset{\smile}{\varphi}_2(\zeta_2) \\ \textit{lower}\ \underline{*} & (\overset{\vee}{\varphi}_1 \,\underline{*}\, \overset{\vee}{\varphi}_2)(\zeta) & := & \frac{1}{2\pi i}\int_{I(\zeta,u)} \overset{\vee}{\varphi}_1(\zeta-\zeta_2)\, \overset{\vee}{\varphi}_2(\zeta_2)\, d\zeta_2 \end{array}$$

with $I(\zeta,u) = \left[\frac{1}{2}\zeta + e^{\frac{-\pi i}{2}}u, \frac{1}{2}\zeta + e^{\frac{+\pi i}{2}}u\right]$ and $0 < \frac{\zeta}{u} << 1$. For *major convolution* we *first* fix an auxiliary point u close enough to 0 (so as to steer clear of possible singularities in the convolution factors $\overset{\vee}{\varphi}_i$ or $\overset{\smile}{\varphi}_i$) and *then* calculate the convolution integral for ζ closer still to 0. The resulting integral does depend on u, but only up to a regular germ at 0, which doesn't affect the *class* of the convolution-major.

The upper/lower Borel-Laplace transforms.
For simplicity, let us fix the polarisation $\theta = 0$ and drop the index θ in the geometric model $\varphi_\theta(z)$. We get the familiar formulas, reproduced here just for definiteness:

$$\begin{array}{cccccc} \textit{multiplicative} & & & & \textit{convolutive} & \\ & \textit{upper} & \overset{0}{\varphi}(\zeta) & = & \{\overset{\frown}{\varphi}(\zeta), \overset{\smile}{\varphi}(\zeta)\} & \\ & \overline{\mathcal{B}} \nearrow\swarrow \overline{\mathcal{L}} & & & & \\ \varphi(z) & & & & \downarrow \partial_\zeta \ , \ \downarrow -\partial_\zeta & \\ & \underline{\mathcal{B}} \searrow\nwarrow \underline{\mathcal{L}} & & & & \\ & \textit{lower} & \overset{\diamond}{\varphi}(\zeta) & = & \{\overset{\wedge}{\varphi}(\zeta), \overset{\vee}{\varphi}(\zeta)\} & \end{array}$$

$$\begin{array}{lll} \textit{minors} & \textit{Borel} & \textit{Laplace} \\ \textit{upper} & \overset{\frown}{\varphi}(\zeta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\zeta z}\varphi(z)\frac{dz}{z} & \varphi(z) = \int_0^{+\infty} e^{-z\zeta}\, d\overset{\frown}{\varphi}(\zeta) \\ \textit{lower} & \overset{\wedge}{\varphi}(\zeta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\zeta z}\varphi(z) dz & \varphi(z) = \int_0^{+\infty} e^{-z\zeta}\, \overset{\wedge}{\varphi}(\zeta)\, d\zeta \end{array}$$

[11] minor convolution is possible only under a suitable integrability condition at the origin. That condition is automatically met when *one* (hence *all*) majors verify $\overset{\smile}{\varphi}(\zeta) \to 0$ or $\zeta\, \overset{\vee}{\varphi}(\zeta) \to 0$ as $\zeta \to 0$ uniformly on any sector of finite aperture.

majors	*Laplace*	*Borel*
upper	$\overset{\smile}{\varphi}(\zeta) = \int_c^{+\infty} e^{-\zeta z}\varphi(z)\frac{dz}{z}$	$\varphi(z) = -\frac{1}{2\pi i}\int_{c_*-i\infty}^{c_*+i\infty} e^{z\zeta}\, d\overset{\smile}{\varphi}(\zeta)$
lower	$\overset{\vee}{\varphi}(\zeta) = \int_c^{+\infty} e^{-\zeta z}\varphi(z)dz$	$\varphi(z) = \frac{1}{2\pi i}\int_{c_*-i\infty}^{c_*+i\infty} e^{z\zeta}\ \overset{\vee}{\varphi}(\zeta)\, d\zeta$

Interpretation: Let us assume for definiteness that $0 < \zeta << 1 << z$. In the Borel-Laplace integrals from z to ζ the constant $c >> 1$ has to be taken large enough to leave all singularities of the integrand to its left, i.e. in $\Re(z) < c$. In the Borel integral from $\overset{\smile}{\varphi}(\zeta)$ or $\overset{\vee}{\varphi}(\zeta)$ to $\varphi(z)$, on the other hand, any positive c_*, large or small, will do, but the integrand $\overset{\smile}{\varphi}(\zeta)$ or $\overset{\vee}{\varphi}(\zeta)$ must be suitably chosen in its equivalence class to ensure integrability (which is always possible, for any given c_* or even for all c_* at once).

Monomials in all four models.
The following table covers not only the monomials $J_\sigma(z) := z^{-\sigma}$ but also[12] the whole range of binomials $J_{\sigma,n}(z) := z^{-\sigma}\log^n(z)$ with $\sigma \in \mathbb{C}, n \in \mathbb{N}$.

		minor		*major*
$\sigma \notin \mathbb{Z}$	*upper*	$\overset{\frown}{J}_\sigma(\zeta) = \zeta^\sigma/\Gamma(1+\sigma)$	,	$\overset{\smile}{J}_\sigma(\zeta) = \zeta^\sigma\Gamma(-\sigma)$
	$\nearrow\swarrow$			
$J_\sigma(z) = z^{-\sigma}$		$\downarrow \partial_\zeta$		$\downarrow -\partial_\zeta$
	$\searrow\nwarrow$			
	lower	$\hat{J}_\sigma(\zeta) = \zeta^{\sigma-1}/\Gamma(\sigma)$	,	$\overset{\vee}{J}_\sigma(\zeta) = \zeta^{\sigma-1}\Gamma(1-\sigma)$
....	......	....		....
$s \in \mathbb{N}^+$	*upper*	$\overset{\frown}{J}_s(\zeta) = \frac{1}{s!}\,\zeta^s$	,	$\overset{\smile}{J}_s(\zeta) = (-\zeta)^s\log(\frac{1}{\zeta})$
	$\nearrow\swarrow$			
$J_s(z) = z^{-s}$		$\downarrow \partial_\zeta$		$\downarrow -\partial_\zeta$
	$\searrow\nwarrow$			
	lower	$\hat{J}_s(\zeta) = \frac{1}{(s-1)!}\,\zeta^{s-1}$	,	$\overset{\vee}{J}_s(\zeta) = (-\zeta)^{s-1}\log(\frac{1}{\zeta})$
....	......	....		....
	upper	$\overset{\frown}{J}_0(\zeta) = 1$	,	$\overset{\smile}{J}_s(\zeta) = \log(\frac{1}{\zeta})$
	$\nearrow\swarrow$			
$J_0(z) = 1$		$\downarrow \partial_\zeta$		$\downarrow -\partial_\zeta$
	$\searrow\nwarrow$			
	lower	$\hat{J}_0(\zeta) = 0$	,	$\overset{\vee}{J}_s(\zeta) = \frac{1}{\zeta}$
....	......	....		....
$s \in \mathbb{N}^+$	*upper*	$\overset{\frown}{J}_{-s}(\zeta) = 0$	,	$\overset{\smile}{J}_{-s}(\zeta) = (s-1)!\,\zeta^{-s}$
	$\nearrow\swarrow$			
$J_{-s}(z) = z^s$		$\downarrow \partial_\zeta$		$\downarrow -\partial_\zeta$
	$\searrow\nwarrow$			
	lower	$\hat{J}_{-s}(\zeta) = 0$	,	$\overset{\vee}{J}_{-s}(\zeta) = s!\,\zeta^{-s-1}$

[12] after σ-differentiation

The pros and cons of the upper/lower choices.

Advantages of the lower choice :

(i) $\{\underline{\mathcal{B}}, \underline{\mathcal{L}}, \underline{*}\}$ are more usual/natural choices than $\{\overline{\mathcal{B}}, \overline{\mathcal{L}}, \overline{*}\}$

(ii) the operators $(\partial_z + \omega)^{-1}$ and $(e^{\omega\partial_z} - 1)^{-1}$ which constantly occur in the theory of *singular* differential or difference equations and are ultimately responsible for the frequent occurrence, in this theory, of both divergence and resurgence, turn into *minor* multiplication by $(-\zeta + \omega)^{-1}$ or $(e^{-\omega\zeta} - 1)^{-1}$ in the ζ-plane[13], whereas with the upper choice we would be saddled with the more unwieldy operators $\partial_\zeta^{-1}(-\zeta + \omega)^{-1}\partial_\zeta$ and $\partial_\zeta^{-1}(e^{-\omega\zeta} - 1)^{-1}\partial_\zeta$.

Advantages of the upper choice :

(i) the 'monomial' formulas for J_σ (*supra*) assume a smoother shape, with the simple sign change $-\sigma \mapsto \sigma$ instead of $-\sigma \mapsto \sigma - 1$.

(ii) upper convolution $\overline{*}$ and pointwise multiplication (both in the ζ-plane) have the same unit element, namely $\widehat{\varphi}_0\,(\zeta) \equiv 1$, which is extremely useful when studying *dimorphy phenomena*[14], e.g. the dimorphy of poly- or hyperlogarithms.

In the present investigation, we shall resort to both choices, because :

(i) the lower choice leads to simpler formulas when deriving a function's singularities from its Taylor coefficient asymptotics (see §2.3 *infra*).

(ii) the upper choice is the one naturally favoured by the functional transforms (*nir/mir* and *nur/mur*) that lead to the *inner* and *outer* generators of SP-series (see §4.4-5 and §5.3-5 *infra*).

2.2 Alien derivations as a tool for Riemann surface description.

Resurgent functions are acted upon by a huge range of exotic derivations, the so-called *alien derivations* Δ_ω, with indices ω ranging through the whole of $\mathbb{C}_\bullet := \widetilde{\mathbb{C} - \{0\}}$. In other words, $\arg(\omega)$ is defined *exactly* rather than *mod* 2π. Together, these Δ_ω generate a *free Lie algebra* on $\mathbb{C}_\bullet$. Alien derivations, by pull-back, act on all three models. There being no scope for confusion, the same symbols Δ_ω can be used in each model. *Alien derivations*, however, are linear operators which quantitatively measure the *singularities of minors* in the ζ-plane. To interpret or calculate *alien derivatives*, we must therefore go to (either of) the *convolutive models*, which in that sense enjoy an undoubted primacy.

[13] or *major* multiplication by $(\zeta + \omega)^{-1}$ or $(e^{\omega\zeta} - 1)^{-1}$

[14] i.e. the simultaneous stability of certain function rings unter *two* unrelated products, like pointwise *multiplication* and some form or other of *convolution*.

However, for notational ease, it is often convenient to write down *resurgence equations*[15] in the multplicative models (formal or geometric), the product there being the more familiar multiplication.

For simplicity, in all the following definitions/identities the indices ω are assumed to be on $\mathbb{R}^+ \subset \mathbb{C}_\bullet$. Adaptation to the general case is immediate.

Definition of the operators Δ_ω and $\Delta^\pm_\omega$.

$$\begin{array}{lccc} & \textit{multiplicative} & \textit{convolutive} & \textit{convolutive} \\ \Delta_\omega \;: & \varphi \;\mapsto\; \varphi_\omega & \begin{array}{c} \overset{0}{\varphi} := \{\overset{\frown}{\varphi}, \overset{\smile}{\varphi}\} \;\mapsto\; \overset{0}{\varphi}_\omega := \{\overset{\frown}{\varphi}_\omega, \overset{\smile}{\varphi}_\omega\} \\ \overset{\diamond}{\varphi} := \{\overset{\wedge}{\varphi}, \overset{\vee}{\varphi}\} \;\mapsto\; \overset{\diamond}{\varphi}_\omega := \{\overset{\wedge}{\varphi}_\omega, \overset{\vee}{\varphi}_\omega\} \end{array} & \end{array}$$

$$\begin{array}{lclcl} \overset{\vee}{\varphi}_\omega(\zeta) & := & \sum_{\epsilon_1,\dots,\epsilon_{r-1}} & \lambda_{\epsilon_1,\dots,\epsilon_{r-1}} & \overset{\wedge}{\varphi}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_{r-1} \\ \omega_1 & ,\dots, & \omega_{r-1}\end{smallmatrix}\right)}(\omega - \zeta) \\ \overset{\wedge}{\varphi}_\omega(\zeta) & := & \sum_{\epsilon_1,\dots,\epsilon_r} & \frac{\epsilon_r}{2\pi i}\,\lambda_{\epsilon_1,\dots,\epsilon_{r-1}} & \overset{\wedge}{\varphi}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r \\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)}(\omega + \zeta) \\ \overset{\vee}{\varphi}_\omega(\zeta) & := & \sum_{\epsilon_0,\dots,\epsilon_{r-1}} & \frac{\epsilon_0}{2\pi i}\,\lambda_{\epsilon_1,\dots,\epsilon_{r-1}} & \overset{\vee}{\varphi}^{\left(\begin{smallmatrix}\epsilon_0 & ,\dots, & \epsilon_{r-1} \\ \underline{\omega}_0 & ,\dots, & \underline{\omega}_{r-1}\end{smallmatrix}\right)}(\underline{\omega} + \zeta) \\ \overset{\wedge}{\varphi}_\omega(\zeta) & := & \sum_{\epsilon_0,\dots,\epsilon_r} & \frac{\epsilon_0\,\epsilon_r}{2\pi i}\,\lambda_{\epsilon_1,\dots,\epsilon_{r-1}} & \overset{\vee}{\varphi}^{\left(\begin{smallmatrix}\epsilon_0 & ,\dots, & \epsilon_r \\ \underline{\omega}_0 & ,\dots, & \underline{\omega}_r\end{smallmatrix}\right)}(\underline{\omega} - \zeta) \end{array}$$

$$0 =: \omega_0 < \omega_1 < \omega_2 < \dots < \omega_{r-2} < \omega_{r-1} < \omega_r := \omega \qquad \Big(\, \underline{\omega}_i := -\omega_i \;,\; \forall i \,\Big)$$

The above relations should first be interpreted locally, i.e. for $\zeta/\omega << 1$, and then extended globally by analytic continuation in ζ. Here $\overset{\wedge}{\varphi}^{\left(\begin{smallmatrix}\epsilon \\ \omega\end{smallmatrix}\right)}$ or $\overset{\vee}{\varphi}^{\left(\begin{smallmatrix}\epsilon \\ \omega\end{smallmatrix}\right)}$ denotes the branch corresponding to the left or right circumvention of each intervening singularity ω_i if ϵ_i is $+$ or $-$, and to a branch weightage λ that doesn't depend on the increments ω_i:

$$\lambda_{\epsilon_1,\dots,\epsilon_{r-1}} := \frac{p!\,q!}{r!} \quad \textit{with} \quad p := \sum_{\epsilon_i = +} 1 \;,\; q := \sum_{\epsilon_i = -} 1 \qquad \Big(\sum_{\epsilon_i} \lambda_{\epsilon_1,\dots,\epsilon_{r-1}} \equiv 1\Big)$$

The lateral operators $\Delta^\epsilon_\omega \;:\; \varphi \mapsto \varphi_{\omega_\epsilon}$ (with index $\epsilon = \pm$) are defined by the same formulas as above, but with weights $\lambda_{\epsilon_1,\dots,\epsilon_{r-1}}$ replaced by the much more elementary $2\pi i \lambda^\epsilon_{\epsilon_1,\dots,\epsilon_{r-1}}$:

$$\begin{array}{lcl} \lambda^\epsilon_{\epsilon_1,\dots,\epsilon_{r-1}} & := & 1 \;\; \textit{if} \;\; \epsilon_1 = \epsilon_2 = \dots = \epsilon_{r-1} = \epsilon \; \in \pm 1 \\ & := & 0 \;\; \textit{otherwise} \end{array}$$

Thus, the minor-to-major and minor-to-minor formulas read:

$$\begin{array}{lclcl} \overset{\vee}{\varphi}_{\omega_\epsilon}(\zeta) & := & & 2\pi i & \overset{\wedge}{\varphi}^{\left(\begin{smallmatrix}\epsilon & ,\dots, & \epsilon \\ \omega_1 & ,\dots, & \omega_{r-1}\end{smallmatrix}\right)}(\omega - \zeta) \\ \overset{\wedge}{\varphi}_{\omega_\epsilon}(\zeta) & := & \sum_{\epsilon_r} & \epsilon_r & \overset{\wedge}{\varphi}^{\left(\begin{smallmatrix}\epsilon & ,\dots, & \epsilon & , \epsilon_r \\ \omega_1 & ,\dots, & \omega_{r-1} & , \omega_r\end{smallmatrix}\right)}(\omega + \zeta) \end{array}$$

This settles the action of alien operators in the *lower* convolutive model. Their action in the *upper* model is exactly the same. Their action in the multiplicative models is defined

[15] i.e. any relation $E(\varphi, \Delta_{\omega_1}\varphi, \dots, \Delta_{\omega_n}\varphi) = 0$, linear or not, between a resurgent function φ and one or several of its alien derivatives.

indirectly, by pull-back from the convolutive models (with the same notation Δ_ω holding for all models).
The operators Δ_ω are derivations but the simpler $\Delta^\pm_\omega$ are not.
Indeed, for any two test functions φ_1, φ_2 the identities hold[16] :

$$\begin{aligned}
\Delta_\omega(\varphi_1\,\varphi_2) \;&\equiv\; (\Delta_\omega\varphi_1)\,\varphi_2 + \varphi_1\,(\Delta_\omega\varphi_2)\\
\Delta^\pm_\omega(\varphi_1\,\varphi_2) \;&\equiv\; (\Delta^\pm_\omega\varphi_1)\,\varphi_2 + \varphi_1\,(\Delta^\pm_\omega\varphi_2) + \sum_{\omega_1+\omega_2=\omega}^{\frac{\omega_1}{\omega}>0,\,\frac{\omega_2}{\omega}>0} (\Delta^\pm_{\omega_1}\varphi_1)(\Delta^\pm_{\omega_2}\varphi_2)
\end{aligned}$$

Lateral and median singularities.
The lateral and median operators are related by the following identities :

$$1+\sum_{\omega>0} t^\omega\,\Delta^\pm_\omega \;=\; \exp\Big(\pm 2\pi i\sum_{\omega>0} t^\omega\,\Delta_\omega\Big) \tag{5}$$

$$2\pi i\sum_{\omega>0} t^\omega\,\Delta_\omega \;=\; \pm\log\Big(1+\sum_{\omega>0} t^\omega\,\Delta^\pm_\omega\Big) \tag{6}$$

Interpretation : we first expand *exp* and *log* the usual way, then equate the contributions of each power t^ω from the left- and right-hand sides. Although the above formulas express each $\Delta^\pm_\omega$ as an infinite sum of (finite) Δ_ω-products, and *vice versa*, when applied to any given test function φ the infinite sums actually reduce to a finite number of *non-vanishing* summands[17].

Compact description of Riemann surfaces.
Knowing *all* the alien derivatives (of first and higher orders) of a minor $\overset{\wedge}{\varphi}(\zeta)$ or $\overset{\frown}{\varphi}(\zeta)$ enables one to piece together that minor's behaviour on its *entire Riemann surface* $\mathcal{R}$ from the behaviour of its various alien derivatives on their sole *holomorphy stars*, by means of the general formula :

$$\overset{\wedge}{\varphi}(\zeta_\Gamma) \equiv \overset{\wedge}{\varphi}(\zeta) + \sum_{r\geq 1}\sum_{\omega_i\in\mathbb{C}_\bullet} (2\pi i)^r H_\Gamma^{\omega_1,\dots,\omega_r} t^{\omega_1+\dots\omega_r}\Delta_{\omega_r}\dots\Delta_{\omega_1}\overset{\wedge}{\varphi}(\zeta) \tag{7}$$

Here, ζ_Γ denotes any chosen point on $\mathcal{R}$, reached from 0 by following a broken line Γ in the ζ-plane. Both sums $\sum_r$ and $\sum_{\omega_i}$ are finite[18]. The coefficients $H_\Gamma^\bullet$ are in $\mathbb{Z}$. Unlike in (5)(6), t^ω in (7) should no longer be viewed as the symbolic power of a free variable t, but as an shift operator acting on functions of ζ and changing ζ into $\zeta+\omega$. [19]

To sum up :
(i) alien derivations 'uniformise' everything.
(ii) a full knowledge of a minor's alien derivatives (given for example by a *complete* system

[16] For simplicity, we write the following identities in the *multiplicative* models. When transposing them to the ζ-plane, where they make more direct sense, multiplication must of course be replaced by convolution.
[17] due to the minors having only isolated singularities.
[18] That is to say, when applied to any given resurgent function, they carry only finitely many non-vanishing terms.
[19] with ζ close to 0 and suitably positioned, to ensure that $\zeta+\omega$ be in the holomorphy star of the test function.

of resurgence equations) implies a full knowlege of that minor's Riemann surface.

Strong versus weak resurgence.
"Proper" resurgence equations are relations of the form :

$$E(\varphi, \Delta_\omega \varphi) \equiv 0 \qquad or \qquad E(\varphi, \Delta_{\omega_1}\varphi, \ldots, \Delta_{\omega_n}\varphi) \equiv 0 \tag{8}$$

with expressions E that are typically non-linear (at least in φ) and that may involve arbitrary scalar- or function-valued coefficients. Such equations express unexpected *self-reproduction* properties – that is to say, non-trivial relations between the minor (as a germ at $\zeta = 0$) and its various singularities. Moreover, when the resurgent function φ, in the multiplicative model, happens to be the formal solution of some equation or system $S(\varphi) = 0$ (think for example of a singular differential, or difference, or functional, equation), the resurgence of φ as well as the exact shape of its resurgence equations (8), can usually be derived almost without analysis, merely by letting each Δ_ω act on $S(\varphi)$ in accordance with certain formal rules. Put another way : we can deduce deep *analytic* facts from purely *formal-algebraic* manipulations. What we have here is *full-fledged resurgence* – resurgence at its best and most useful.

But two types of situations may arise which lead to *watered-down* forms of resurgence.

One is the case when, due to severe constraints built into the resurgence-generating problem, the coefficients inside E are no longer free to vary continuously, but must assume discrete, usually entire values : we then speak of *rigid resurgence.*

Another is the case when the expressions E are linear or affine functions of their arguments φ and $\Delta_{\omega_i}\varphi$. The self-reproduction aspect, to which resurgence owes its name, then completely disappears, and makes way for a simple *exchange* or *'ping-pong'* between singularities (in the linear case) with possible 'annihilations' (in the affine case).

Both restrictions entail a severe impoverishment of the resurgence phenomenon. As it happens, and as we propose to show in this paper, SP-series combine these two restrictions : they lead to fairly degenerate resurgence patterns that are both *rigid* and *affine.* Furthermore, as a rule, SP-series verify no useful equation or system $S(\varphi) = 0$ that might give us a clue as to their resurgence properties. In cases such as this, the resurgence apparatus (alien derivations etc) ceases to be a vehicle for *proving things* and retains only its (non-negligible !) *notational value* (as a device for describing Riemann surfaces etc) while the onus of proving the hard analytic facts falls on altogether different tools, like *Taylor coefficient asymptotics*[20] and the *nir/mir*-transforms.[21]

The pros and cons of the $2\pi i$ factor.
On balance, we gain more than we lose by inserting the $2\pi i$ factor into the above definitions of alien derivations. True, by removing it there we would also eliminate it from the identities relating minors to majors (see §2.1), but the factor would sneak back into the J_σ-identities *supra*, thus spoiling the whole set of 'monomial' formulas. Worse still, real-indexed derivations Δ_ω acting on real-analytic derivands φ would no longer produce real-analytic derivatives $\Delta_\omega\varphi$ – which would be particularly damaging in "all-real" settings, e.g. when dealing with chirality 1 knots like 4_1 (see §9 *infra*).

[20] see §2.3
[21] see §4.4,§4.5.

2.3 Retrieving the resurgence of a series from the resurgence of its Taylor coefficients.

SP-series are one of those rare instances where there is no shortcut for calculating the singularities: we have no option but to deduce them from a close examination of the asymptotics of the Taylor coefficients.[22]

The better to respect the symmetry between our series φ and its Taylor coefficients J, we shall view them both as resurgent functions of the variables z resp. n in the multiplicative models and ζ resp. ν in the (lower) convolutive models. The aim then is to understand the correspondence between the triplets:

$$\{\tilde{\varphi}(z), \overset{\diamond}{\varphi}(\zeta), \varphi(z)\} \longleftrightarrow \{\tilde{J}(n), \overset{\diamond}{J}(\nu), J(n)\} \tag{9}$$

and the alien derivatives attached to them.

Retrieving closest singularities.

Let us start with the simplest case, when $\overset{\wedge}{\varphi}$ has a single singularity on the boundary of its disk of convergence, say at ζ_0. We can of course assume ζ_0 to be real positive.

$$\tilde{\varphi}(z) = \sum_{0\leq n}(n+1)!\; J(n)\, z^{-n-1}\;(dv^t) \;\overset{\mathcal{B}}{\mapsto}\; \overset{\wedge}{\varphi}\,(\zeta) = \sum_{0\leq n} J(n)\,\zeta^n\;(cv^t\; on\, |\zeta| < \zeta_0)$$

In order to deduce the closest singularity of $\overset{\wedge}{\varphi}$ from the closest singularity of $\hat{J}$, we first express $J(n)$ as a Cauchy integral on a circle $|\zeta| = |\zeta_0| - \epsilon$. We then deform that circle to a contour $\mathcal{C}$ which coincides with the larger circle $|\zeta| = |\zeta_0| + \epsilon$ except for a slit Γ around the interval $[\zeta_0, \zeta_0 + \epsilon]$ to avoid crossing the axis $[\zeta_0, \infty]$. Then we retain only Γ, thereby neglecting a contribution exponentially small in n. Lastly, we transform Γ into Γ_* (resp. $\underline{\Gamma_*} = -\Gamma_*$) under the change $\zeta = \zeta_0 e^{\nu}$.

$$\begin{aligned}
J(n) &= \frac{1}{2\pi i}\oint \overset{\wedge}{\varphi}\,(\zeta)\,\zeta^{-n-1}\,d\zeta && && (10)\\
&= \frac{1}{2\pi i}\int_{\mathcal{C}} \overset{\wedge}{\varphi}\,(\zeta)\,\zeta^{-n-1}\,d\zeta && (\textit{contour deformation}) &&\\
&= \frac{1}{2\pi i}\int_{\Gamma} \overset{\wedge}{\varphi}\,(\zeta)\,\zeta^{-n-1}\,d\zeta + o(\zeta_0^{-n}) && (\textit{contour restriction}) && (11)\\
&= \frac{e^{-n\nu_0}}{2\pi i}\int_{\Gamma_*} \overset{\wedge}{\varphi}\,(\zeta_0 e^{\nu})\,e^{-n\nu}\,d\nu + o(e^{-n\nu_0}) && (\textit{setting } \zeta := \zeta_0\, e^{\nu} = e^{\nu_0+\nu}) &&\\
&= \frac{e^{-n\nu_0}}{2\pi i}\int_{\Gamma_*} \overset{\vee}{\varphi}_{\zeta_0}\,(\zeta_0 - \zeta_0 e^{\nu})\,e^{-n\nu}\,d\nu + o(e^{-n\nu_0}) && (\textit{always}) && (12)\\
&= \frac{e^{-n\nu_0}}{2\pi i}\int_{\underline{\Gamma_*}} \overset{\vee}{\varphi}_{\zeta_0}\,(\zeta_0 - \zeta_0 e^{-\nu})\,e^{n\nu}\,d\nu + o(e^{-n\nu_0}) && (\textit{always}) && (13)\\
&= e^{-n\nu_0}\int_0^c \overset{\wedge}{\varphi}_{\zeta_0}\,(\zeta_0 e^{\nu} - \zeta_0)\,e^{-n\nu}\,d\nu + o(e^{-n\nu_0}) && (\textit{if } \overset{\diamond}{\varphi}_{\zeta_0} \textit{ integrable}) && (14)
\end{aligned}$$

[22] The present section is based on a private communication (1992) by J.E. to Prof. G.K. Immink. An independent, more detailed treatment was later given by O. Costin in [C2].

Therefore

$$J(n) \equiv e^{-n\nu_0}\ J_{\nu_0}(n) + o(e^{-n\nu_0}) \qquad (\textit{with } \nu_0 := \log(\zeta_0)) \tag{15}$$

where J_{ν_0} denotes the (lower, and if need be, truncated) Borel transform of a resurgent function $\overset{\diamond}{J}_{\nu_0}$ linked to $\overset{\diamond}{\varphi}_{\zeta_0} := \Delta_{\zeta_0}\,\overset{\wedge}{\varphi}$ by:

$$\begin{aligned}
\overset{\diamond}{J}_{\nu_0}(\nu) &= \{\hat{J}_{\nu_0}(\nu), \overset{\vee}{J}_{\nu_0}(\nu)\} & \qquad \overset{\diamond}{\varphi}_{\zeta_0}(\nu) &= \{\overset{\wedge}{\varphi}_{\zeta_0}(\nu), \overset{\vee}{\varphi}_{\zeta_0}(\zeta)\} & \\
\hat{J}_{\nu_0}(\nu) &= \overset{\wedge}{\varphi}_{\zeta_0}(\zeta_0\, e^{\nu} - \zeta_0\,) & \qquad \overset{\wedge}{\varphi}_{\zeta_0}(\zeta) &= \hat{J}_{\nu_0}(\,\log(1 + \tfrac{\zeta}{\zeta_0})) & \qquad (\textit{minors}) \\
\overset{\vee}{J}_{\nu_0}(\nu) &= \overset{\vee}{\varphi}_{\zeta_0}(\zeta_0 - \zeta_0\, e^{-\nu}) & \qquad \overset{\vee}{\varphi}_{\zeta_0}(\zeta) &= \overset{\vee}{J}_{\nu_0}(-\log(1 - \tfrac{\zeta}{\zeta_0})) & \qquad (\textit{majors})
\end{aligned}$$

Retrieving distant singularities.

The procedure actually extends to farther-lying singularities. In fact, if $\hat{J}$ is endlessly continuable, so is $\overset{\wedge}{\varphi}$, and the former's resurgence pattern neatly translates into the latter's, under a set of linear but non-trivial formulas Here, however, we shall only require knowledge of those singularities of $\overset{\wedge}{\varphi}$ which lie on its (0-centered, closed) star of holomorphy. All the other singularities will follow under repeated alien differentiation.

3 The ingress factor.

We must first describe the asymptotics of the "product" part (for $m = n$) of our "sum-product" coefficients. This involves a trifactorisation:

$$\prod_{0 \le k \le n} F(\frac{k}{n}) =: P_F(n) \sim \tilde{I}g_F(n)\, e^{-\nu_*\, n}\, \tilde{E}g_F(n) \quad \textit{with} \quad \nu_* = \int_0^1 f(x)dx \tag{16}$$

with

(i) an *ingress* factor $\tilde{I}g_F$ resummable to Ig_F and purely local at $x = 0$.

(ii) an *exponential* factor $e^{-\nu_*\, n}$, global on $[0, 1]$

(iii) an *egress* factor $\tilde{E}g_F$ resummable to Eg_F and purely local at $x = 1$.

The non-trivial factors (ingress/egress) may be divergent-resurgent (hence the tilda) but, at least for holomorphic data F, they always remain fairly elementary. They often vanish (when F is *even* at 0 or 1) and, even when divergent, they can always be resummed in a canonical way. Lastly, as already hinted, it will prove technically convenient to factor out the first of these (ingress), thereby replacing the original SP-series $j(\zeta)$ by its 'cleansed' and more regular version $j^{\#}(\zeta)$.

3.1 Bernoulli numbers and polynomials.

For future use, let us collect a few formulas about two convenient variants of the classical Bernoulli numbers B_k and Bernoulli polynomials $B_k(t)$.

The Bernoulli numbers and polynomials.

$$\mathfrak{b}_k \quad := \quad \frac{\mathfrak{b}^*_k(0)}{k!} = \frac{B_{k+1}(1)}{(k+1)!} \qquad (k \in -1+\mathbb{N}) \tag{17}$$

$$\mathfrak{b}(\tau) \quad := \quad \frac{e^\tau}{e^\tau - 1} \;=\; \sum_{k\geq -1} \mathfrak{b}_k\, \tau^k = \tau^{-1} + \frac{1}{2} + \frac{1}{12}\tau - \frac{1}{720}\tau^3 \ldots \tag{18}$$

$$\mathfrak{b}^*_k(\tau) \quad := \quad \mathfrak{b}(\partial_\tau)\, \tau^k \qquad (k \in \mathbb{C}\,,\, k \neq -1) \tag{19}$$

$$\mathfrak{b}^{**}(\tau,\zeta) \;:=\; \sum_{k\geq 0} \mathfrak{b}^*_k(\tau)\, \frac{\zeta^k}{k!} \;=\; \mathfrak{b}(\partial_\tau)\, e^{\tau\zeta} \;=\; \frac{e^{\tau\zeta} e^\zeta}{e^\zeta - 1} - \frac{1}{\zeta} \tag{20}$$

For $k \in \mathbb{N}$, we have $\mathfrak{b}_k = \frac{B_{k+1}}{(1+k)!}$ for the scalars, and the series $\mathfrak{b}^*_k(\tau)$ essentially coincide with the Bernoulli polynomials. For all other values of k, the scalars $\mathfrak{b}_k$ are no longer defined and the $\mathfrak{b}^*_k(\tau)$ become divergent series in decreasing powers of τ.

$$\mathfrak{b}^*_k(\tau) \quad := \quad \sum_{s=-1}^{k} \mathfrak{b}_s\, \tau^{k-s} \frac{k!}{(k-s)!} \quad = \quad \frac{B_{k+1}(\tau+1)}{k+1} \quad (\textit{if } k \in \mathbb{N}) \tag{21}$$

$$\mathfrak{b}^*_k(\tau) \quad := \quad \sum_{s=-1}^{+\infty} \mathfrak{b}_s\, \tau^{k-s} \frac{\Gamma(k+1)}{\Gamma(k+1-s)} \qquad (\textit{if } k \in \mathbb{C} - \mathbb{Z}) \tag{22}$$

$$\mathfrak{b}^*_k(\tau) \;:=\; \frac{\tau^{k+1}}{k+1} + \sum_{s\geq 0} (-1)^s \mathfrak{b}_s\, \tau^{k-s} \frac{(s-k-1)!}{(-k-1)!} \qquad (\textit{if } k \in -2-\mathbb{N}) \tag{23}$$

The Euler-Bernoulli numbers and polynomials.

$$\beta_k \quad := \quad \frac{\beta^*_k(0)}{k!} = \frac{B_{k+1}(\frac{1}{2})}{(k+1)!} \qquad (k \in -1+\mathbb{N}) \tag{24}$$

$$\beta(\tau) \quad := \quad \frac{1}{e^{\tau/2} - e^{-\tau/2}} \;=\; \sum_{k\geq -1} \beta_k\, \tau^{-k} = \tau^{-1} - \frac{1}{24}\tau + \frac{7}{5760}\tau^3 - \ldots \tag{25}$$

$$\beta^*_k(\tau) \quad := \quad \beta(\partial_\tau)\, \tau^k \qquad (k \in \mathbb{C}\,,\, k \neq -1) \tag{26}$$

$$\beta^{**}(\tau,\zeta) \;:=\; \sum_{k\geq 0} \beta^*_k(\tau)\, \frac{\zeta^k}{k!} \;=\; \beta(\partial_\tau)\, e^{\tau\zeta} \;=\; \frac{e^{\tau\zeta}}{e^{\zeta/2} - e^{-\zeta/2}} - \frac{1}{\zeta} \tag{27}$$

For $k \in \mathbb{N}$, the $\beta^*_k(\tau)$ essentially coincide with the Euler-Bernoulli polynomials. For all other values of k, they are divergent series in decreasing powers of τ.

$$\beta^*_k(\tau) \quad := \quad \sum_{s=-1}^{k} \beta_s\, \tau^{k-s} \frac{k!}{(k-s)!} \quad = \quad \frac{B_{k+1}(\tau+\frac{1}{2})}{k+1} \quad (\textit{if } k \in \mathbb{N}) \tag{28}$$

$$\beta^*_k(\tau) \quad := \quad \sum_{s=-1}^{+\infty} \beta_s\, \tau^{k-s} \frac{\Gamma(k+1)}{\Gamma(k+1-s)} \qquad (\textit{if } k \in \mathbb{C} - \mathbb{Z}) \tag{29}$$

$$\beta^*_k(\tau) \;:=\; \frac{\tau^{k+1}}{k+1} + \sum_{s\geq 0} (-1)^s \beta_s\, \tau^{k-s} \frac{(s-k-1)!}{(-k-1)!} \qquad (\textit{if } k \in -2-\mathbb{N}) \tag{30}$$

For all $k \in \mathbb{N}$ we have the parity relations $\beta^*_{2k} = 0$, $\beta^*_k(-\tau) \equiv (-1)^{k+1}\beta^*_k(\tau)$

The Euler-MacLaurin formula.
We shall make constant use of the basic identities $(\forall s \in \mathbb{N})$:

$$\sum_{1\leq k\leq m} k^s \equiv \mathfrak{b}^*_s(m) - \mathfrak{b}^*_s(0) \equiv \beta^*_s(m+\frac{1}{2}) - \beta^*_s(\frac{1}{2}) \equiv \frac{B_{s+1}(m+1) - B_{s+1}(1)}{s+1}$$

and of these variants of the Euler-MacLaurin formula :

$$\sum_{k}^{0\leq\frac{k}{n}\leq\bar{x}} f(\frac{k}{n}) \sim n\int_0^{\bar{x}} f(x)dx + \frac{f(0)}{2} + \frac{f(\bar{x})}{2} + \sum_{1\leq s\ odd} \frac{\mathfrak{b}_s}{n^s}\big(f^{(s)}(\bar{x}) - f^{(s)}(0)\big) \tag{31}$$

$$\sum_{k}^{0\leq\frac{k}{n}\leq\bar{x}} f(\frac{k}{n}) \sim n\int_0^{\bar{x}} f(x)dx + \frac{f(0)}{2} + \frac{f(\bar{x})}{2} + \sum_{1\leq s\ odd} \frac{\mathfrak{b}^*_s(0)}{n^s}\big(\bar{f}_s - f_s\big) \tag{32}$$

where f_s and $\bar{f}_s$ denote the Taylor coefficients of f at 0 and $\bar{x}$.

3.2 Resurgence of the Gamma function.

Lemma 3.1 (Exact asymptotics of the Gamma function) .
The functions Θ, θ defined on $\{\Re(n) > 0\} \subset \mathbb{C}$ by

$$\Theta(n) \equiv e^{\theta(n)} := (2\pi)^{-\frac{1}{2}}\Gamma(n+1)\, n^{-n-\frac{1}{2}}\, e^n \qquad (\theta(n)\ \textit{real if}\ n\ \textit{real}) \tag{33}$$

possess resurgent-resummable asymptotic expansions as $\Re(n) \to +\infty$:

$$\Theta(n) = 1 + \sum_{1\leq s} \Theta_s\, n^{-s} \quad ; \quad \theta(n) = \sum_{0\leq s} \theta_{1+2s}\, n^{-1-2s} \quad (\textit{odd powers}) \tag{34}$$

with explicit lower/upper Borel transforms :

$$\overset{\wedge}{\theta}(\nu) = -\frac{1}{\nu^2} + \frac{1}{2\,\nu}\frac{1}{\tanh(\nu/2)} \tag{35}$$

$$\begin{aligned} \overset{\frown}{\theta}(\nu) &= +\frac{1}{\nu} + \frac{1}{2}\int_0^{\nu/2} \frac{1}{\tanh(t)}\frac{dt}{t} \\ &= \frac{1}{12}\nu - \frac{1}{2160}\nu^3 + \frac{1}{151200}\nu^5 - \frac{1}{8467200}\nu^7 + \frac{1}{431101440}\nu^9 \dots \end{aligned} \tag{36}$$

This immediately follows from Γ's functional equation. We get successively :

$$\begin{aligned}
\frac{\Theta(n+\frac{1}{2})}{\Theta(n-\frac{1}{2})} &= e\,\Big(\frac{n-\frac{1}{2}}{n+\frac{1}{2}}\Big)^n \\
\theta(n+\frac{1}{2})-\theta(n-\frac{1}{2}) &= 1+n\log(n-\frac{1}{2})-n\log(n+\frac{1}{2}) \\
\partial_n\,\frac{1}{n}\big(\theta(n+\frac{1}{2})-\theta(n+\frac{1}{2})\big) &= -\frac{1}{n^2}+\frac{1}{n-\frac{1}{2}}-\frac{1}{n+\frac{1}{2}} \\
-\nu\,\partial_\nu^{-1}\Big((e^{-\nu/2}-e^{\nu/2})\,\overset{\wedge}{\theta}(\nu)\Big) &= -\nu+e^{\nu/2}-e^{-\nu/2} \\
\overset{\wedge}{\theta}(\nu) &= (e^{\nu/2}-e^{-\nu/2})^{-1}\,\partial_\nu\ \nu^{-1}\,(e^{\nu/2}-e^{-\nu/2}) \\
\overset{\wedge}{\theta}(\nu) &= -\frac{1}{\nu^2}+\frac{1}{2\,\nu}\,\frac{1}{\tanh(\nu/2)}
\end{aligned}$$

Laplace summation along $\arg(\nu)=0$ yields the exact values $\theta(n)$ and $\Theta(n)$. The only non-vanishing alien derivatives are (in multiplicative notation) :

$$\Delta_\omega\,\tilde{\theta} = \frac{1}{\omega} \qquad \forall\omega\in 2\pi i\mathbb{Z}^* \tag{37}$$

$$\Delta_\omega\,\tilde{\Theta} = \frac{1}{\omega}\,\tilde{\Theta} \qquad \forall\omega\in 2\pi i\mathbb{Z}^* \tag{38}$$

Using formula (38) and its iterates for crossing the vertical axis in the ν-plane, we can evaluate the quotient of the regular resummations of $\hat{\Theta}(\nu)$ along $\arg(\nu)=0$ and $\arg(\nu)=\pm\pi$, and the result of course agrees with the complement formula[23] :

$$\frac{1}{\Gamma(n)\,\Gamma(1-n)}=\frac{\sin\pi\,n}{\pi} \qquad \forall n\in\mathbb{C} \tag{39}$$

3.3 Monomial/binomial/exponential factors.

In view of definition (16) and formula (31), for a generic input $F:=e^{-f}$ with $F(0),F(1)\neq 0,\infty$ we get the asymptotic expansions :

$$\tilde{Ig}_{_F}(n) = \exp\Big(-\frac{1}{2}f(0)+\sum_{1\leq s\,odd}\frac{\mathfrak{b}_s}{n^s}f^{(s)}(0)\Big) \tag{40}$$

$$\tilde{Eg}_{_F}(n) = \exp\Big(-\frac{1}{2}f(1)-\sum_{1\leq s\,odd}\frac{\mathfrak{b}_s}{n^s}f^{(s)}(1)\Big) \tag{41}$$

and the important parity relation [24]:

$$\{F^{\vDash}(x)=1/F(1-x)\}\Longrightarrow\{1=\tilde{Ig}_{_F}(n)\tilde{Eg}_{_{F^{\vDash}}}(n)=\tilde{Ig}_{_{F^{\vDash}}}(n)\tilde{Eg}_{_F}(n)\} \tag{42}$$

[23] For details, see [E3] pp 243-244.

[24] In §5 it will account for the relation between the two outer generators which are always present in the generic case (i.e. when $F(0)$ and $F(1)\neq 0,\infty$).

But we are also interested in meromorphic inputs F that may have zeros and poles at 0 or 1. Since the mappings $F \mapsto \tilde{I}g_{_F}$ and $F \mapsto \tilde{E}g_{_F}$ are clearly multiplicative and since meromorphic functions F possess convergent Hadamard products:

$$F(x) = c\, x^d\, e^{\sum_{s=1}^{s=\infty} c_s\, x^s} \prod_i \Big((1 - \frac{x}{a_i})^{k_i}\, e^{k_i \sum_{s=1}^{s=K_i} \frac{1}{s} \frac{x^s}{a_i^s}} \Big) \quad (k_i \in \mathbb{Z}, K_i \in \mathbb{N}) \tag{43}$$

we require the exact form of the ingress factors for monomial, binomial and even/odd exponential factors. Here are the results:

$$\begin{array}{ll}
F_{\text{mon}}(x) = c & \tilde{I}g_{\text{mon}}(n) = c^{+\frac{1}{2}} \\[4pt]
F_{\text{mon}}(x) = c\, x^d \quad (d \neq 0) & \tilde{I}g_{\text{mon}}(n) = c^{-\frac{1}{2}}\, (2\pi n)^{\frac{d}{2}} \\[4pt]
F_{\text{bin}}(x) \ = \prod_i (1 - a_i^{-1} x)^{s_i} & \tilde{I}g_{\text{bin}}(n) \ = \prod_i \big(\tilde{\Theta}(a_i n)\big)^{s_i} \\[4pt]
F_{\text{even}}(x) = \exp\big(-\sum_{s\geq 1} f_{2s}^{\text{even}}\, x^{2s}\big) & \tilde{I}g_{\text{even}}(n) = \ 1 \\[4pt]
F_{\text{odd}}(x) \ = \exp\big(-\sum_{s\geq 0} f_{2s+1}^{\text{odd}}\, x^{2s+1}\big) & \tilde{I}g_{\text{odd}}(n) \ = \exp\big(+\sum_{s\geq 0} f_{2s+1}^{\text{odd}}\, \frac{\mathfrak{b}_{2s+1}^*(0)}{n^{2s+1}}\big)
\end{array}$$

Monomial factors.
The discontinuity between the first two expressions of $Ig_{\text{mon}}(n)$ stems from the fact that for $d = 0$ the product in (2) start from $k = 0$ as usual, whereas for $d \neq 0$ it has to start from $k = 1$. The case $d = 0$ is trivial, and the case $d \neq 0$ by multiplicativity reduces to the case $d = 1$. To calculate the corresponding ingress factor, we may specialise the identity

$$\prod_{1\leq k\leq n} F(\frac{k}{n}) \sim \tilde{I}g_F(n)\; e^{-\nu_*\, n}\; \tilde{E}g_F(n) \tag{44}$$

to convenient test functions. Here are the two simplest choices:

test function $F_1(x) = x$	*test function* $F_2(x) = \frac{2}{\pi}\sin(\frac{\pi}{2}x)$	
$\prod_{k=1}^{k=n} F_1(\frac{k}{n}) = \frac{n!}{n^n}$	$\prod_{k=1}^{k=n} F_2(\frac{k}{n}) = 2\,\pi^{-n}\, n^{1/2}$	*by elem. trigon.*
$\tilde{I}g_{F_1}(n) =$ *unknown*	$\tilde{I}g_{F_2}(n) = \tilde{I}g_{F_1}(n)$	*by parity of* $\frac{F_2(x)}{F_1(x)}$
$\nu_* = 1$	$\nu_* = -\log \pi$	
$\tilde{E}g_{F_1}(n) = \tilde{\Theta}(n)$	$\tilde{E}g_{F_2}(n) = 1$	*by parity of* $F_2(1{+}x)$

With the choice F_2, all we have to do is plug the data in the second column into (44) and we immediately get $\tilde{I}g_{F_2}(n) = (2\pi\, n)^{1/2}$ but before that we have to check the first line's elementary trigonometric identity. With the choice F_1, on the other hand, we need to check that the egress factor does indeed coincide with Θ. This readily follows from:

$$F_1(1+x) \ = \ \exp\big(\sum_{1\leq s}(-1)^s \frac{x^s}{s}\big) \quad \Longrightarrow \quad \tilde{E}g_{F_1}(n) = \exp(\tilde{eg}_{F_1}(n)) \quad \textit{with}$$

$$\tilde{eg}_{F_1}(n) \ = \sum_{1\leq s\ odd} (-1)^{s-1} \frac{n^{-s}}{s} \mathfrak{b}_s^*(0) = \sum_{1\leq s\ odd} n^{-s} (s-1)!\, \mathfrak{b}_s \quad \Longrightarrow$$

$$\overset{\wedge}{eg}_{F_1}(\nu) \ = \sum_{1\leq s} \nu^{s-1} \mathfrak{b}_s = \frac{1}{\nu}\Big(\frac{e^\nu}{e^\nu - 1} - \frac{1}{\nu} - \frac{1}{2}\Big) = -\frac{1}{\nu^2} + \frac{1}{2\,\nu}\, \frac{1}{\tanh(\nu/2)} = \overset{\wedge}{\theta}(\nu)$$

We then plug everything into (44) and use formula (33) of §3.2 to eliminate both $n!/n^n$ and $\Theta(n)$.

Binomial factors.
By multiplicativity and homogeneity, it is enough to check the idendity $\tilde{I}g_{F_3}(n) = \tilde{\Theta}(n)$ for the test function $F_3(x) = 1 - x$. But since $F_3 = 1/F_1^{\models}$ with the notations of the preceding para, the parity relations yield:

$$\tilde{I}g_{F_3}(n) \;=\; 1/\tilde{E}g_{1/F_1^{\models}}(n) \;=\; \tilde{E}g_{F_1^{\models}}(n) \;=\; \tilde{\Theta}(n) \qquad (\textit{see above})$$

which is precisely the required identity. We alse notice that:

$$F(x) = (1 - \frac{x}{a})(1 + \frac{x}{a}) \quad \Longrightarrow \quad \tilde{I}g_F(n) = \tilde{\Theta}(an)\tilde{\Theta}(-an) \equiv 1$$

which agrees with the trivialness of the ingress factor for an *even* input F.

Exponential factors.
For them, the expression of the ingress/egress factors directly follows from (32). Moreover, since the exponentials occurring in the Hadamard product (43) carry only polynomials or entire functions, the corresponding ingress/egress factors are actually convergent.

3.4 Resummability of the total ingress factor.

As announced, we shall have to change our SP-series $j(\zeta) = \sum J(n)\,\zeta^n$ into $j^{\#}(\zeta) = \sum J^{\#}(n)\,\zeta^n$, which involves dividing the coefficients $J(n)$, not by the asymptotic series $\tilde{I}g_F(n)$, but by its exact resummation $Ig_F(n)$. Luckily, this presents no difficulty for meromorphic [25] inputs F. Indeed, the contributions to $\tilde{I}g_F$ of the isolated factors in (43) are separetely resummable:
– for the monomial factors F_{mon} this is trivial
– for the binomial factors F_{bin} this follows from $\tilde{\Theta}$'s resummability (see §3.2)
– for the exponential factors F_{exp} this follows from $\log F_{exp}$ being either polynomial or entire.

As for the global $\tilde{I}g_F$, one easily checks that the Hadamard product (43), rewritten as

$$F = F_{mon}\; F_{even}\; F_{odd}\; \prod_i F_{bin,i} \tag{45}$$

translates into a product of resurgent functions:

$$\tilde{I}g_F = \tilde{I}g_{F_{mon}}\; \tilde{I}g_{F_{even}}\; \tilde{I}g_{F_{odd}}\; \prod_i \tilde{I}g_{F_{bin,i}} \tag{46}$$

which converges in all three models (formal, convolutive, geometric – respective to the corresponding topology) to a limit that doesn't depend on the actual Hadamard decomposition chosen in (43), i.e. on the actual choice of the truncation-defining integers K_i.

[25] for simplicity, let us assume that F has no purely imaginary poles or zeros.

3.5 Parity relations.

Starting from the elementary parity relations for the Bernoulli numbers and polynomials :

$$\begin{array}{lllll} \mathfrak{b}_{2s} = 0 \qquad (s \geq 1) & ; & \beta_{2s} = 0 \qquad (s \geq 0) & & \\ \mathfrak{b}_s(\tau) \equiv (-1)^{s+1}\, \mathfrak{b}_s(-\tau-1) & ; & \beta_s(\tau) \equiv (-1)^{s+1}\, \beta(-\tau) & & (\forall s \geq 0) \end{array}$$

and setting

$$\begin{aligned} F^{\vDash}(x) &:= 1/F(1-x) \\ P_F(n) &:= \prod_{m=0}^{m=n} F(\frac{m}{n}) \\ P_F^{\#}(n) &:= \frac{P_F(n)}{Ig_F(n)\; Eg_F(n)} = (\omega_F)^n \quad \textit{with} \quad \omega_F = \exp(-\int_0^1 f(x)dx) \end{aligned}$$

we easily check that :

$$\begin{array}{lll} \tilde{Ig}_{_F}(n)\, \tilde{Eg}_{_{F^{\vDash}}}(n) = 1 & \textit{and} & \tilde{Ig}_{_{F^{\vDash}}}(n)\, \tilde{Eg}_{_F}(n) = 1 \\ J_{F^{\vDash}}(n) = J_F(n)/P_F(n) & \textit{and} & J^{\#}_{F^{\vDash}}(n) = J^{\#}_F(n)/P^{\#}_F(n) \\ j_{F^{\vDash}}(\zeta) \neq j_F(\zeta/\omega_F) & \textit{but} & j^{\#}_{F^{\vDash}}(\zeta) = j^{\#}_F(\zeta/\omega_F) \end{array}$$

4 Inner generators.

4.1 Some heuristics.

Consider a simple, yet typical case. Assume the driving function f to be *entire* (or even think of it as *polynomial*, for simplicity), steadily increasing on the real interval $[0,1]$, with a unique zero at $\bar{x} \in]0,1[$ on that interval, and no other zeros, real or complex, inside the disk $\{|x| \leq |\bar{x}|\}$:

$$0 < \bar{x} < 1 \;\;,\;\; f(0) < 0 \;,\; f(\bar{x}) = 0 \;,\; f(1) > 0 \;,\;\; f'(x) > 0 \; \forall x \in [0,1] \tag{47}$$

As a consequence, the primitive $f^*(x) := \int_0^x f(x')dx'$ will display a unique minimum at $\bar{x}$ and, for any given large n, the products $\prod_{k=0}^{k=m} F(k/n) = \exp(-\sum_{k=0}^{k=m} f(k/n))$ will be maximal for $m \sim n\bar{x}$. It is natural, therefore, to split the Taylor coefficients $J(n)$ of our sum-product series (2) into a *global* but fairly elementary factor $J_{1,2,3}(n)$, which subsumes all the *pre-critical* terms $F(k/n)$, and a purely *local* but analytically more challenging factor $J_4(n)$, which accounts for the contribution of all *near-critical* terms

$F(k/n)$. Here are the definitions:

$$J(n) \;=\; J_{1,2,3}(n)\; J_4(n) \tag{48}$$

$$J(n) \;:=\; \sum_{0\le m\le n} \prod_{0\le k\le m} F(\frac{k}{n}) \tag{49}$$

$$J_{1,2,3}(n) \;:=\; \prod_{0\le k\le \bar{m}} F(\frac{k}{n}) \qquad \mathit{with} \qquad \bar{m} := \mathrm{ent}(n\,\bar{x}) \tag{50}$$

$$J_4(n) \;:=\; \sum_{0\le m\le n} \left(\prod_{0\le k\le m} F(\frac{k}{n}) \;\Big/ \prod_{0\le k\le \bar{m}} F(\frac{k}{n}) \right) \tag{51}$$

$$= \left\{ \begin{array}{c} \cdots + \frac{1}{F}(\frac{\bar{m}-2}{n})\,\frac{1}{F}(\frac{\bar{m}-1}{n})\,\frac{1}{F}(\frac{\bar{m}}{n}) + \frac{1}{F}(\frac{\bar{m}-1}{n})\,\frac{1}{F}(\frac{\bar{m}}{n}) \\ +\frac{1}{F}(\frac{\bar{m}}{n}) + 1 + F(\frac{\bar{m}+1}{n}) \\ +F(\frac{\bar{m}+1}{n})\,F(\frac{\bar{m}+2}{n}) + F(\frac{\bar{m}+1}{n})\,F(\frac{\bar{m}+2}{n})\,F(\frac{\bar{m}+3}{n}) + \ldots \end{array} \right\} \tag{52}$$

The asymptotics of the global factor $J_{1,2,3}(n)$ as $n \to \infty$ easily results from the variant (32) of the Euler-MacLaurin formula and $J_{1,2,3}(n)$ splits into three subfactors:
(i) a factor $J_1(n)$, local at $x = 0$, which is none other than the ingress factor $Ig_F(n)$ studied at length in §3.
(ii) an elementary factor $J_2(n)$, which reduces to an exponential and carries no divergence.
(iii) a factor $J_3(n)$, local at $x = \bar{x}$ and analogous to the 'egress factor' of §3, but with base point $\bar{x}$ instead of 1.

That leaves the really sensitive factor $J_4(n)$, which like $J_3(n)$ is local at $x = \bar{x}$, but far more complex. In view of its expression as the discrete sum (52), we should expect its asymptotics to be described by a Laurent series $\sum_{k\ge 0} C_{k/2}\, n^{-k/2}$ involving both integral and semi-integral powers of $1/n$. That turns out to be the case indeed, but we shall see that there is a way of jettisoning the integral powers and retaining only the semi-integral ones, i.e. $\sum_{k\ge 0} C_{k+1/2}\, n^{-k-1/2}$. To do this, we must perform a little sleight-of-hand and attach the egress factor J_3 to J_4 so as to produce the joint factor $J_{3,4}$. In fact, as we shall see, the gains that accrue from merging J_3 and J_4 go way beyond the elimination of integral powers.

But rather than rushing ahead, let us describe our four factors $J_i(n)$ and their asymptotic expansions $\tilde{J}_i(n)$:

$$\begin{aligned}
J(n) \;&:=\; J_{1,2,3}(n)\; J_4(n) = J_1(n)\; J_2(n)\, J_3(n)\; J_4(n) = J_{1,2}(n)\; J_{3,4}(n) \\
J_{1,2,3}(n) \;&:=\; \prod_{0\le k\le \bar m} F(\frac{k}{n}) = \exp(-\sum_{0\le k\le \bar m} f(\frac{k}{n})) \qquad \textit{with} \qquad \bar m := \mathrm{ent}(n\,\bar x)
\end{aligned}$$

$$\tilde{J}_1(n) := \exp(-\frac{1}{2}\, f_0 + \sum_{1\le s} \frac{\mathfrak{b}^*_s(0)\, f_s}{n^s}) \qquad \textit{with} \quad f_s := \frac{f^{(s)}(0)}{s!} \tag{53}$$

$$\tilde{J}_2(n) := \exp(-n\,\bar\nu) \qquad\qquad \textit{with} \quad \bar\nu := \int_0^{\bar x} f(x)\, dx \tag{54}$$

$$\tilde{J}_3(n) := \exp(-\frac{1}{2}\, \bar f_0 - \sum_{1\le s} \frac{\mathfrak{b}^*_s(0)\, \bar f_s}{n^s}) \qquad \textit{with} \quad \bar f_s := \frac{f^{(s)}(\bar x)}{s!} \tag{55}$$

$$\tilde{J}_4(n) := \sum_{0\le \bar m\le \bar x\, n} \exp\big(\sum_{0\le k\le \bar m} f(\bar x - \frac{k}{n})\big) + \sum_{0\le \bar m\le (1-\bar x)\, n} \exp\big(-\sum_{1\le k\le \bar m} f(\bar x + \frac{k}{n})\big)$$

In the last identity, the first exponential inside the second sum, namely $\exp(\sum_{1\le k\le 0}(\dots))$, should of course be taken as $\exp(0) = 1$. Let us now simplify $\tilde{J}_3$ by using the fact that $\bar f_0 = f(\bar x) = 0$ and let us replace in $\tilde{J}_4$ the finite $\bar m$-summation (up to $\bar x\, n$ or $(1-\bar x)\, n$) by an infinite m-summation, up to $+\infty$, which won't change the *asymptotics* [26] in n:

$$\tilde{J}_3(n) := \exp(-\sum_{1\le s} \frac{\mathfrak{b}^*_s(0)}{n^s}\, \bar f_s) \tag{56}$$

$$\tilde{J}_4(n) := 2 + \sum_{\substack{1\le m\\ \epsilon=\pm 1}} \exp\Big(-\epsilon \sum_{1\le k\le m} f(\bar x + \epsilon \frac{k}{n})\Big) \tag{57}$$

$$\tilde{J}_4(n) := 2 + \sum_{\substack{1\le m\\ \epsilon=\pm 1}} \exp\Big(-\sum_{1\le s} \frac{\epsilon^{s+1}}{n^s} \big(\mathfrak{b}^*_s(m) - \mathfrak{b}^*_s(0)\big)\, \bar f_s\Big) \tag{58}$$

We can now regroup the factors $\tilde{J}_3$ and $\tilde{J}_4$ into $\tilde{J}_{3,4}$ and switch from the Bernoulli-type polynomials $\mathfrak{b}^*_s(m)$ over to their Euler-Bernoulli counterparts $\beta^*_s(m)$. These have the advantage of being odd/even function of m is s is even/odd, which will enable us to replace m-summation on $\mathbb{N}$ by m-summation on $\frac{1}{2} + \mathbb{Z}$, eventually easing the change from m-summation to τ-integration. Using the parity properties of $\beta^*(\tau)$ (see §3.1) we

[26] It will merely change the *transasymptotics* by adding exponentially small summands. In terms of the Borel transforms $\hat{J}_{3,4}(\nu)$ or $\overset{\frown}{J}_{3,4}(\nu)$, it means that their nearest singularities will remain unchanged.

successively find:

$$\tilde{J}_{3,4}(n) = 2\ \exp\Big(-\sum_{1\leq s}\frac{1}{n^s}\mathfrak{b}_s\,\bar{f}_s\Big)+\sum_{\substack{1\leq m\\ \epsilon=\pm 1}}\exp\Big(-\sum_{1\leq s}\frac{\epsilon^{s+1}}{n^s}\mathfrak{b}^*_s(m)\,\bar{f}_s\Big) \tag{59}$$

$$\tilde{J}_{3,4}(n) = \sum_{\substack{0\leq m\\ \epsilon=\pm 1}}\exp\Big(-\sum_{1\leq s}\frac{\epsilon^{s+1}}{n^s}\mathfrak{b}^*_s(m)\,\bar{f}_s\Big) \tag{60}$$

$$\tilde{J}_{3,4}(n) = \sum_{\substack{0\leq m\\ \epsilon=\pm 1}}\exp\Big(-\sum_{1\leq s}\frac{\epsilon^{s+1}}{n^s}\beta^*_s(m+\frac{1}{2})\,\bar{f}_s\Big) \tag{61}$$

$$\tilde{J}_{3,4}(n) = \sum_{\substack{0\leq m\\ \epsilon=\pm 1}}\exp\Big(-\sum_{1\leq s}\frac{1}{n^s}\beta^*_s(\epsilon m+\epsilon\frac{1}{2})\,\bar{f}_s\Big) \qquad (\textit{by parity}!) \tag{62}$$

$$\tilde{J}_{3,4}(n) = \sum_{m\in\frac{1}{2}+\mathbb{Z}}\exp\Big(-\sum_{1\leq s}\frac{1}{n^s}\beta^*_s(m)\,\bar{f}_s\Big) \tag{63}$$

This last identity should actually be construed as:

$$\tilde{J}_{3,4}(n) = \sum_{m\in\frac{1}{2}+\mathbb{Z}}\exp\Big(-\frac{1}{n}\beta^*_1(m)\,\bar{f}_1\Big)\exp_{\#}\Big(-\sum_{2\leq s}\frac{1}{n^s}\beta^*_s(m)\,\bar{f}_s\Big) \tag{64}$$

Here, the first exponential *exp* decreases fast as m grows, since

$$\beta^*_1(m)\,\bar{f}_1 = \frac{1}{2}\,m^2\,\bar{f}_1 = \frac{1}{2}m^2 f'(\bar{x}) > 0$$

The second exponential $exp_{\#}$, on the other hand, should be *expanded* as a power series of its argument and each of the resulting terms $m^{s_1}\,n^{-s_2}$ should be dealt with separately, leading to a string of clearly *convergent* series. We can now replace the *discrete* m-summation in (63) by a *continuous* τ-integration: here again, that may change the *transasymptotics* in n, but not the *asymptotics*.[27] We find, using the parity properties of the β^*_s and maintaining throughout the distinction between *exp* (unexpanded) and $exp_{\#}$

[27] Indeed, the summation/integration bounds are $\pm\infty$, with a summand//integrand that vanishes exponentially fast there.

(expanded) :

$$\tilde{J}_{3,4}(n) \;=\; \int_{-\infty}^{+\infty} \exp\Big(-\frac{1}{n}\beta_1^*(\tau)\,\bar{f}_1\Big)\,\exp_{\#}\Big(-\sum_{2\le s}\frac{1}{n^s}\beta_s^*(\tau)\,\bar{f}_s\Big)d\tau \tag{65}$$

$$\tilde{J}_{3,4}(n) \;=\; \sum_{\epsilon=\pm 1}\int_{0}^{+\infty} \exp\Big(-\frac{1}{n}\beta_1^*(\epsilon\tau)\,\bar{f}_1\Big)\,\exp_{\#}\Big(-\sum_{2\le s}\frac{1}{n^s}\beta_s^*(\epsilon\tau)\,\bar{f}_s\Big)d\tau \tag{66}$$

$$\tilde{J}_{3,4}(n) \;=\; \sum_{\epsilon=\pm 1}\int_{0}^{+\infty} \exp\Big(-\frac{1}{n}\beta_1^*(\tau)\,\bar{f}_1\Big)\,\exp_{\#}\Big(-\sum_{2\le s}\frac{\epsilon^{s+1}}{n^s}\beta_s^*(\tau)\,\bar{f}_s\Big)d\tau \tag{67}$$

$$\begin{aligned}\tilde{J}_{3,4}(n) \;=\; 2\int_{0}^{+\infty} &\exp\Big(-\frac{1}{n}\beta_1^*(\tau)\,\bar{f}_1\Big)\,\exp_{\#}\Big(-\sum_{3\le s\ odd}\frac{1}{n^s}\beta_s^*(\tau)\,\bar{f}_s\Big)\\ &\times\ \cosh_{\#}\Big(-\sum_{2\le s\ even}\frac{1}{n^s}\beta_s^*(\tau)\,\bar{f}_s\Big)\,d\tau\end{aligned} \tag{68}$$

$$\tilde{J}_{3,4}(n) \;=\; 2\Big[\int_{0}^{+\infty} \exp\Big(-\frac{1}{n}\beta_1^*(\tau)\,\bar{f}_1\Big)\,\exp_{\#}\Big(-\sum_{2\le s}\frac{1}{n^s}\beta_s^*(\tau)\,\bar{f}_s\Big)d\tau\Big]_{\substack{demi\\ integ.\\ part}} \tag{69}$$

The notation in (69) means that we retain only the demi-integral powers of n^{-1} in $\big[\ldots\big]$. In view of the results of §2.3 about the correspondence between singularities and Taylor coefficient asymptotics, the Borel transform $\stackrel{\wedge}{\ell i}\ (\nu) := \stackrel{\wedge}{J}_{3,4}\ (\nu)$ of $\tilde{J}_{3,4}(n)$, or rather its counterpart $\stackrel{\wedge}{Li}\ (\zeta) := \stackrel{\wedge}{\ell i}\ (\log(1+\frac{\zeta}{\omega_F}))$ in the ζ-plane, must in our test-case (47) describe the closest singularities of the *sum-product* function $j(\zeta)$ or rather its 'cleansed' variant $j^{\#}(\zeta)$.

Singularities such as Li shall be referred to as *inner generators* of the resurgence algebra. They differ from the three other types of generators (*original, exceptional, outer*) first and foremost by their stability: unlike these, they self-reproduce indefinitely under alien differentiation. Another difference is this: inner generators (minors and majors alike) tend to carry only *demi-integral*[28] powers of ζ or ν, as we just saw, whereas the other types of generators tend to carry only integral powers (in the minors) and logarithmic terms (in the majors).

So far, so good. But what about the two omitted factors $J_1(n)$ and $J_2(n)$? The second one, $J_2(n)$, which is a mere exponential $exp(-n\bar{\nu})$, simply accounts for the location $\bar{\zeta} = e^{\bar{\nu}}$ at which Li is seen in the ζ-plane. As for the ingress factor $J_1(n)$, *keeping it* (i.e. merging it with $J_{3,4}(n)$) would have rendered Li dependent on the ingress point $x = 0$, whereas *removing it* ensures that Li (and by extension the whole *inner algebra*) is totally independent of the 'accidents' of its construction, such as the choice of ingress point in the x-plane.

As for the move from $\{\mathfrak{b}_s^*\}$ to $\{\beta_s^*\}$, apart from easing the change from summation to integration, it brings another, even greater benefit: by removing the crucial coefficient

[28] in our test-case, i.e. for a driving function f with a simple zero at $\bar{x}$. For zeros of odd order $\tau > 1$ (τ has to be odd to produce an extremum in f^*) we would get ramifications of order $(-\tau + 2s)/(\tau + 1)$ with $s \in \mathbb{N}$, which again rules out entire powers. See §4.2-5 and also §6.1

β_0 in (25) (which vanishes, unlike $\mathfrak{b}_0$ in (18)), it shall enable us to express the future *mir*-transform as a purely integro-differential operator. [29]

One last remark, before bringing these heuristics to a close. We have chosen here the simplest possible way of producing an *inner generator*, namely directly from the *original generator* i.e. the sum-product series itself. To do this, rather stringent assumptions on the driving function f had to be made[30]. However, even when these assumptions are not met, the *original generator* always produces so-called *outer generators* (at least one, but generally two), which in turn always produce *inner generators*. So these two types – outer and inner – are a universal feature of sum-product series.

4.2 The long chain behind *nir//mir*.

Let us now introduce two non-linear functional transforms central to this investigation. The *nir*-transform is directly inspired by the above heuristics. It splits into a chain of subtransforms, all of which are elementary, save for one : the *mir*-transform.

Both *nir* and *mir* depend on a coherent choice of scalars β_k and polynomials $\beta_k^*(\tau)$. The *standard choice*, or Euler-Bernoulli choice, corresponds to the definitions (24) - (30). It is the one that is relevant in most applications to analysis and SP-series. However, to gain a better insight into the β-dependence of *nir//mir*, it is also useful to consider the *non-standard choice*, with free coefficients β_k and accordingly redefined polynomials $\beta_k^*(\tau)$:

$$
\begin{array}{ll}
\textit{standard choice} & \textit{non-standard choice} \\
\beta(\tau) := \dfrac{1}{e^{\tau/2}-e^{-\tau/2}} = \displaystyle\sum_{-1\le k} \beta_k\,\tau^k & \beta(\tau) := \displaystyle\sum_{-1\le k} \beta_k\,\tau^k \\
\beta_k^*(\tau) := \beta(\partial_\tau)\,\tau^k = \dfrac{B_{k+1}(\tau+\frac{1}{2})}{k+1} & \beta_k^*(\tau) := \beta(\partial_\tau)\,\tau^k = \displaystyle\sum_{s=-1}^{s=k} \beta_s\,\tau^{k-s}\,\frac{\Gamma(k+1)}{\Gamma(k+1-s)}
\end{array}
$$

As we shall see, even in the non-standard case it is often necessary to assume that $\beta_{-1}=1$ and $\beta_0=0$ (like in the standard case) to get interesting results. The further coefficients, however, can be anything.

The long, nine-link chain:

$$
\begin{array}{ccccccc}
\overset{\mathbf{nir}}{\rightarrow} & \rightarrow & \rightarrow & \rightarrow & \overset{\mathbf{nir}}{\rightarrow} & H \\
\uparrow & & & & \downarrow & \uparrow^{9} \\
\uparrow & f^* & \overset{3}{\rightarrow} & g^* & \downarrow & h' \\
\uparrow & \uparrow^{2} & & \downarrow^{4} & \downarrow & \uparrow^{8} \\
\overset{\mathbf{nir}}{\longleftarrow} & f & & g & \overset{\mathbf{nir}}{\longrightarrow} & h \\
 & \uparrow^{1} & & \downarrow^{5} & & \uparrow^{7} \\
 & F & & \not{g} & \longrightarrow\!\overset{6}{}\!\longrightarrow & \hbar \\
 & & & & \mathbf{mir} &
\end{array}
\qquad
\begin{array}{rcl}
H & = & \widehat{Li} \ \textit{or}\ \widehat{Le} \\
h' & = & \widehat{\ell i} \ \textit{or}\ \widehat{\ell e} \\
h & = & \overset{\frown}{\ell i} \ \textit{or}\ \overset{\frown}{\ell e}
\end{array}
$$

[29] Indeed, if it β_0 didn't vanish, that lone coefficient would suffice to ruin nearly all the basic formulae (*infra*) about *mir* and *nir*.

[30] like (47)

Details of the nine steps:

$$
\begin{array}{cccccc}
\stackrel{1}{\rightarrow} & : & \textit{precomposition} & : & F \to f & \textit{with} & f(x) := -\log F(x) \\
\stackrel{2}{\rightarrow} & : & \textit{integration} & : & f \to f^* & \textit{with} & f^*(x) := \int_0^x f(x_0)\, dx_0 \\
\stackrel{3}{\rightarrow} & : & \textit{reciprocation} & : & f^* \to g^* & \textit{with} & f^* \circ g^* = id \\
\stackrel{4}{\rightarrow} & : & \textit{derivation} & : & g^* \to g & \textit{with} & g(y) := \frac{d}{dy} g^*(y) \\
\stackrel{5}{\rightarrow} & : & \textit{inversion} & : & g \to \not{g} & \textit{with} & \not{g}(y) := 1/g(y) \\
\stackrel{6}{\rightarrow} & : & \boldsymbol{mir}\ \textit{functional} & : & \not{g} \to \hbar & \textit{with} & \textit{int.}-\textit{diff. expression below} \\
\stackrel{7}{\rightarrow} & : & \textit{inversion} & : & \hbar \to h & \textit{with} & h(\nu) := 1/\hbar(\nu) \\
\stackrel{8}{\rightarrow} & : & \textit{derivation} & : & h \to h' & \textit{with} & h'(\nu) := \frac{d}{d\nu} h(\nu) \\
\stackrel{9}{\rightarrow} & : & \textit{postcomposition} & : & h' \to H & \textit{with} & H(\zeta) := h'\big(\log(1+\frac{\zeta}{\omega})\big) \\
\stackrel{2\ldots7}{\rightarrow} & : & \boldsymbol{nir}\ \textit{functional} & : & g \to h & \textit{with} & \textit{see } \S 4.3 \textit{ infra}
\end{array}
$$

“Compact” and “layered” expansions of *mir*.

The ‘sensitive’ part of the nine-link chain, namely the *mir*-transform, is a non-linear integro-differential functional of infinite order. Pending its detailed description in §4-5, let us write down the general shape of its two expansions: the ‘compact’ expansion, which merely isolates the r-linear parts, and the more precise ‘layered’ expansion, which takes the differential order into account. We have:

$$\frac{1}{\hbar} := \frac{1}{\not{g}} + \sum_{1\leq r\,\in\, odd} \mathbb{H}_r(\not{g}) = \frac{1}{\not{g}} + \sum_{1\leq r\,\in\, odd} \partial^{1-r}\, \mathbb{D}_r(\not{g}) \quad (\text{“}\textit{compact}\text{”})$$

$$\frac{1}{\hbar} := \frac{1}{\not{g}} + \sum_{\substack{1\leq r\,\in\, odd \\ \frac{1}{2}(r+1)\leq s\leq r}} \mathbb{H}_{r,s}(\not{g}) = \frac{1}{\not{g}} + \sum_{\substack{1\leq r\,\in\, odd \\ \frac{1}{2}(r+1)\leq s\leq r}} \partial^{-s}\, \mathbb{D}_{r,s}(\not{g}) \quad (\text{“}\textit{layered}\text{”})$$

with r-linear, purely differential operators $\mathbb{D}_r, \mathbb{D}_{r,s}$ of the form

$$\mathbb{D}_r(\not{g}) := \sum_{\substack{\sum n_i = r \\ \sum i\, n_i = r-1}} {}^*\mathrm{Mir}^{n_0,n_1,\ldots,n_{r-1}} \prod_{0\leq i\leq r-1} (\not{g}^{(i)})^{n_i} \qquad (\text{“}\textit{compact}\text{”})$$

$$\mathbb{D}_{r,s}(\not{g}) := \sum_{\substack{\sum n_i = r \\ \sum i\, n_i = s}} \mathrm{Mir}^{n_0,n_1,\ldots,n_s} \prod_{0\leq i\leq s} (\not{g}^{(i)})^{n_i} \qquad (\text{“}\textit{layered}\text{”})$$

and connected by:

$$\partial\ \mathbb{D}_r(\not{g}) = \sum_{\frac{1}{2}(r+1)\leq s\leq r} \partial^{r+s}\ \mathbb{D}_{r,s}(\not{g}) \qquad (\forall r \in \{1,3,5...\})$$

The β-dependence is of course hidden in the definition of the differential operators $\mathbb{D}_r, \mathbb{D}_{r,s}$: cf §4.5 *infra*. All the information about the *mir* transform is thus carried by the two rational-valued, integer-indexed moulds ${}^*Mir$ and Mir.

4.3 The *nir* transform.

Integral expression of *nir*.
Starting from f we define $f^{\Uparrow\beta^*}$ and $f^{\uparrow\beta^*}$ as follows:

$$\begin{aligned} f(x) &= \sum_{k\geq\kappa} f_k\, x^k && (\kappa \geq 1\ ,\ f_\kappa \neq 0) \\ f^{\Uparrow\beta^*}(n,\tau) &:= \beta(\partial_\tau)\, f(\frac{\tau}{n}) && \textit{with}\ \ \partial_\tau := \frac{\partial}{\partial\tau} \qquad (70) \\ &:= \sum_{k\geq\kappa} f_k\ n^{-k}\, \beta^*_k(\tau) && \qquad (71) \\ &:= f_\kappa\, \frac{n^{-\kappa}\,\tau^{\kappa+1}}{\kappa+1} + f^{\uparrow\beta^*}(n,\tau) && \qquad (72) \end{aligned}$$

These definitions apply in the standard and non-standard cases alike. Recall that in the standard case $\beta^*_k(\tau) = \frac{B_{k+1}(\tau+\frac{1}{2})}{k+1}$ is an even//odd function of τ for k odd//even, with leading term $\frac{\tau^{k+1}}{k+1}$.

The *nir*-transform $f \mapsto h$ is then defined as follows:

$$\begin{aligned} h(\nu) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(n\,\nu)\,\frac{dn}{n}\int_0^\infty \exp^{\#}(-f^{\Uparrow\beta^*}(n,\tau))\,d\tau \qquad (73) \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(n\,\nu)\,\frac{dn}{n}\int_0^\infty \exp\big(-f_\kappa\,\frac{n^{-\kappa}\,\tau^{\kappa+1}}{\kappa+1}\big)\,\exp_{\#}\big(-f^{\uparrow\beta^*}(n,\tau)\big)\,d\tau \end{aligned}$$

where $\exp_{\#}(X)$ (resp. $\exp^{\#}(X)$) denotes the exponential expanded as a power series of X (resp. of X minus its leading term). Here, we first perform term-by-term, ramified Laplace integration in τ:

$$\int_0^\infty \exp\big(-f_\kappa\,\frac{n^{-\kappa}\,\tau^{\kappa+1}}{\kappa+1}\big)\,\tau^p d\tau = (f_\kappa)^{-\frac{p+1}{\kappa+1}}\ \ (\kappa+1)^{\frac{p+1}{\kappa+1}-1}\ \ \Gamma\Big(\frac{p+1}{\kappa+1}\Big)\ \ n^{\frac{\kappa(p+1)}{\kappa+1}}$$

(with the main determination of $(f_\kappa)^{-\frac{p}{\kappa+1}}$ when $\Re(f_\kappa) > 0$) and then term-by-term (upper) Borel integration in n:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c-i\infty} e^{n\nu}\ n^{-q}\ \frac{dn}{n}\ =\ \frac{\nu^q}{q!}$$

Lemma 4.1 (The *nir*-transform preserves convergence)
Starting from a (locally) convergent f, the $\boldsymbol{\tau}$-integration in (73) usually destroys convergence, but the subsequent n*-integration always restores it. This holds not only in the standard case, but also in the non-standard one, provided $\beta(\tau)$ has positive convergence radius.*

4.4 The reciprocation transform.

Let us first examine what becomes of the nine-link chain in the simplest non-standard case, i.e. with $\beta(\tau) := \tau^{-1}$.

Lemma 4.2 (The simplest instance of *nir*-transform)
For the choice $\beta(\tau) := \tau^{-1}$, the pair $\{h, \hbar\}$ coincides with the pair $\{g, \cancel{g}\}$. In other words, mir *degenerates into the identity, and* nir *essentially reduces to changing the germ f^* into its functional inverse g^* ("reciprocation").*

Since $g = h$, the third column in the 'long chain' becomes redundant here, and the focus shifts to the first two columns, to which we adjoin a new entry $\cancel{f} := 1/f$ for the sake of symmetry. Lagrange's classical inversion formula fittingly describes the involutions $f^* \leftrightarrow g^*$ and $f \leftrightarrow g$, and the simplest way of proving the above lemma is indeed by using Lagrange's formula. On its own, however, that formula gives no direct information about the involution $\cancel{f} \leftrightarrow \cancel{g}$ or the cross-correspondences $\cancel{f} \leftrightarrow g$ and $f \leftrightarrow \cancel{g}$ which are highly relevant to an understanding of the nine-link chain, including in the general case, i.e. for an arbitrary $\beta(\tau)$. So let us first redraw the nine-link chain in the "all-trivial case" $\{\beta(\tau) = \tau^{-1}\,, \kappa = 0\,, f(0) = g(0) = 1\}$ and then proceed with a description of the three afore-mentioned correspondences.

$$
\begin{array}{cccccccc}
\overset{\textbf{nir}}{\rightarrow} & \rightarrow & \rightarrow & \rightarrow & \overset{\textbf{nir}}{\rightarrow} & H & \multicolumn{2}{c}{(\textbf{Trivial case}: \quad \boldsymbol{\beta(\tau) = \tau^{-1}})} \\
\uparrow & & & \downarrow & \uparrow^{9} & & & \\
\uparrow & f^* & \overset{3}{\rightarrow} & g^* & \downarrow & g' & f^*(x) = x + a^*(x) & g^*(y) = y + b^*(y) \\
\uparrow & \uparrow^{2} & & \downarrow^{4} & \downarrow & \uparrow^{8} & & \\
\xleftarrow{\textbf{nir}} & f & & g & \xleftarrow{\textbf{nir}} \;\; \xrightarrow{\textbf{nir}} & g & f(x) = 1 + a(x) & g(y) = 1 + b(y) \\
\nearrow^{1} & \uparrow & & \downarrow^{5} & & \uparrow^{7} & & \\
F & \cancel{f} & & \cancel{g} & \xrightarrow[\textbf{mir=id}]{6} & \cancel{g} & \cancel{f}(x) = 1 + \cancel{a}(x) & \cancel{g}(y) = 1 + \cancel{b}(y)
\end{array}
$$

Lemma 4.3 (Three variants of Lagrange's inversion formula)
The entries a, b, $\cancel{\text{a}}$, $\cancel{\text{b}}$ *in the above diagram are connected by:*

$$a = \sum_{r \geq 1} b_{<r>} \quad \textit{with} \quad b_{<r>} = \sum_{n_1+\cdots+n_r=r} M^{n_1,\ldots,n_r}\, b^{[n_1,\ldots,n_r]} \tag{74}$$

$$a = \sum_{r \geq 1} \cancel{b}_{\{r\}} \quad \textit{with} \quad \cancel{b}_{\{r\}} = \sum_{n_1+\cdots+n_r=r} P^{n_1,\ldots,n_r}\, \cancel{b}^{[n_1,\ldots,n_r]} \tag{75}$$

$$\cancel{a} = \sum_{r \geq 1} \cancel{b}_{[r]} \quad \textit{with} \quad \cancel{b}_{[r]} = \sum_{n_1+\cdots+n_r=r} Q^{n_1,\ldots,n_r}\, \cancel{b}^{[n_1,\ldots,n_r]} \tag{76}$$

with differentially neutral [31] *and symmetral* [32] *integro-differential expressions $\varphi^{[n_1,\ldots,n_r]}$ defined as follows:*

$$\varphi^{[n_1,\ldots,n_r]}(t) := \int_{0<t_1<..<t_r<t} \varphi^{(n_1)}(t_1)\ldots\varphi^{(n_r)}(t_r)\; dt_1\ldots dt_r \tag{77}$$

[31] indeed, since $n_1 + \cdots + n_r = r$, we integrate as many times as we differentiate.

[32] meaning that for any two sequences $\mathbf{n}' = (n'_i)$ and $\mathbf{n}'' = (n''_i)$, we have the multiplication rule $\varphi^{[\mathbf{n}']}\varphi^{[\mathbf{n}'']} \equiv \sum \varphi^{[\mathbf{n}]}$ with a sum running through all $\mathbf{n} \in$ *shuffle*$(\mathbf{n}', \mathbf{n}'')$.

and with scalar moulds $M^\bullet, P^\bullet, Q^\bullet$ easily inferred from the relations :

$$\begin{aligned}
\sum_{\|\bullet\|=r} M^\bullet\, b^{[\bullet]} &= \frac{(-1)^r}{r!}\partial^r (I\, b)^r && (78)\\
\sum_{\|\bullet\|=r} P^\bullet\, \cancel{b}^{[\bullet]} &= \partial_R \cancel{b} I_L \ldots \partial_R \cancel{b} I_L \qquad (r \;\; \textit{times}) && (79)\\
1+\sum_{\bullet} Q^\bullet\, \cancel{b}^{[\bullet]} &= \Big(1+\sum_{\bullet} P^\bullet\, \cancel{b}^{[\bullet]}\Big)^{-1} && (80)
\end{aligned}$$

Remark 1 : In (78), ∂ as usual stands for differentiation and $I = \partial^{-1}$ for integration from 0. In (79), ∂_R denotes the differentiation operator *acting on everything to its right* and $I_L = \partial_L^{-1}$ denotes the integration operator (with integration starting, again, from 0) *acting on everything to its left.*

Thus we find :

$$\begin{aligned}
a &= b_{<1>} + b_{<2>} + b_{<3>} + \ldots\\
b_{<1>} &= -b^{[1]}\\
b_{<2>} &= +b^{[0,2]} + b^{[1,1]}\\
b_{<3>} &= -b^{[0,0,3]} - 4\, b^{[0,1,2]} - 4\, b^{[1,0,2]} - 3\, b^{[0,2,1]} - 15\, b^{[1,1,1]}\\
\ldots
\end{aligned}$$

$$\begin{aligned}
a &= \cancel{b}_{\{1\}} + \cancel{b}_{\{2\}} + \cancel{b}_{\{3\}} + \ldots\\
\cancel{b}_{\{1\}} &= +\cancel{b}^{[1]}\\
\cancel{b}_{\{2\}} &= +\cancel{b}^{[0,2]} + \cancel{b}^{[1,1]}\\
\cancel{b}_{\{3\}} &= +\cancel{b}^{[0,0,3]} + 2\, \cancel{b}^{[0,1,2]} + \cancel{b}^{[1,0,2]} + \cancel{b}^{[0,2,1]} + \cancel{b}^{[1,1,1]}\\
\ldots
\end{aligned}$$

$$\begin{aligned}
\cancel{a} &= \cancel{b}_{[1]} + \cancel{b}_{[2]} + \cancel{b}_{[3]} + \ldots\\
\cancel{b}_{[1]} &= -\cancel{b}^{[1]}\\
\cancel{b}_{[2]} &= +\cancel{b}^{[1,1]} - \cancel{b}^{[0,2]}\\
\cancel{b}_{[3]} &= -\cancel{b}^{[1,1,1]} - \cancel{b}^{[0,0,3]} + \cancel{b}^{[0,2,1]} + \cancel{b}^{[1,0,2]}\\
\ldots
\end{aligned}$$

Remark 2 : The coefficients $M^\bullet, P^\bullet, Q^\bullet$ verify the following identities, all of which are elementary, save for the last one (involving $\sum |Q^\bullet|$) :

$$\begin{aligned}
\sum_{\|\bullet\|=r} (-1)^r\, M^\bullet &= \sum_{\|\bullet\|=r} |M^\bullet| = r^r && (81)\\
\sum_{\|\bullet\|=r} P^\bullet &= \sum_{\|\bullet\|=r} |P^\bullet| = r! && (82)
\end{aligned}$$

$$\sum_{\|\bullet\|=r\geq 2} Q^\bullet = 0 \quad , \quad \sum_{\|\bullet\|=r\geq 2} |Q^\bullet| = \frac{(2\,r-1)!}{(r-1)!\, r!} \qquad (83)$$

Remark 3 : a in terms of b is an elementary consequence of Lagrange's formula for functional inversion, but a in terms of $\cancel{b}$ and $\cancel{a}$ in terms of $\cancel{b}$ are not.

Remark 4 : The formulas (74) through (76) involve only *sublinear* sequences $\mathbf{n} = \{n_1, \ldots, n_r\,;\, n_i \geq 0\}$, i.e. sequences verifying :

$$n_1 + \cdots + n_i \leq i \qquad \forall i \qquad \textit{and} \quad n_1 + \cdots + n_r = r \tag{84}$$

The number of such series is exactly $\frac{(2\,r)!}{r!(r+1)!}$ (Catalan number), which puts them in one-to-one correspondence with r-node binary trees. Moreover, these sublinear sequences are stable under *shuffling* and this establishes a link with the 'classical product' on binary trees[33].

Remark 5 : The various $\varphi^{[n_1,\ldots,n_r]}$, even for sublinear sequences $[n_1, \ldots, n_r]$, are not linearly independent, but this does not detract from the canonicity of the expansions in (74),(75),(76) because the induction rules (78),(79),(80) behind the definition of $M^\bullet, P^\bullet, Q^\bullet$ unambiguously define a privileged set of coefficients.

4.5 The *mir* transform.

Lemma 4.4 (Formula for *mir* in the standard case)
The mir transforms $\cancel{g} \mapsto \hbar$ is explicitely given by :

$$\frac{1}{\hbar(\nu)} = \Big[\frac{1}{\cancel{g}(\nu)}\ \exp_{\#}\Big(-\sum_{r\geq 1} \beta_r\, I^r\, (\cancel{g}(\nu)\partial_\nu)^r\, \cancel{g}(\nu)\Big)\Big]_{I=\partial_\nu^{-1}} \tag{85}$$

Mind the proper sequence of operations :
– *first*, we expand the blocks $(\cancel{g}(\nu)\partial_\nu)^r\, \cancel{g}(\nu)$.
– *second*, we expand $\exp_{\#}(\ldots)$, which involves taking the suitable powers of the formal variable I (with "I" standing for *"integration"*).
– *third* , we divide by $\cancel{g}(\nu)$.
– *fourth*, we move each I^r to the left-most position[34].
– *fifth*, we replace each I^r by the operator ∂_ν^{-r} which stands for n successive formal integrations from 0 to ν.
– *sixth*, we carry out these integrations.

Lemma 4.5 (The integro-differential components $\mathbb{D}_{r,s}$ of *mir*)
The mir *functional admits a canonical expansion :*

$$\frac{1}{\hbar} \;:=\; \frac{1}{\cancel{g}} + \sum_{\substack{1\leq r\,\in\,odd\\ \frac{1}{2}(r-1)\leq s\leq r}} \mathbb{H}_{r,s}(\cancel{g}) \;=\; \frac{1}{\cancel{g}} + \sum_{\substack{1\leq r\,\in\,odd\\ \frac{1}{2}(r-1)\leq s\leq r}} \partial^{-s}\, \mathbb{D}_{r,s}(\cancel{g}) \tag{86}$$

[33] In fact, that product is of recent introduction : see Loday, Ronco, Novelli, Thibon, Hivert in [LR] and [HNT].

[34] this means that all the powers of $\cancel{g}$, $\cancel{g}'$, $\cancel{g}''$ etc must be put to the right of I^r

with r-linear differential operators $\mathbb{D}_{r,s}$ of total order d :

$$\mathbb{D}_{r,s}(\not{g}) \ := \sum_{\substack{\sum n_i = r \\ \sum i\, n_i = s}} \mathrm{Mir}^{n_0,n_1,\dots,n_s} \prod_{0\le i\le s} (\not{g}^{(i)})^{n_i} \tag{87}$$

and coefficients $\mathrm{Mir}^{n_0,n_1,\dots,n_s} \in \frac{1}{s!}\,\mathbb{Z}[\beta_1,\beta_2,\beta_3,\dots]$ *which are themselves homogeneous of "degree"* $\mathrm{r}+1$ *and "order"* s *if to each β_i we assign the "degree"* $\mathrm{i}+1$ *and "order"* i.

For the standard choice $\beta(\tau) := \frac{1}{e^{\tau/2}-e^{-\tau/2}}$, we have $0 = \beta_2 = \beta_4 = \dots$, and so we get only integro-differential components $\mathbb{D}_{r,s}$ which have all *odd* degrees $r = 1,3,5\dots$. Thus:

$$\begin{aligned}
\mathbb{D}_{1,1} &= +\frac{1}{24}\big(\not{g}'\big)\\
\mathbb{D}_{3,2} &= +\frac{1}{1152}\big(\not{g}\,\not{g}'^{\,2}\big)\\
\mathbb{D}_{3,3} &= -\frac{7}{5760}\big(\not{g}'^{\,3} + \not{g}^2\,\not{g}''' + 4\,\not{g}\,\not{g}'\,\not{g}''\big)\\
\mathbb{D}_{5,3} &= +\frac{1}{82944}\big(\not{g}^2\,\not{g}'^{\,3}\big)\\
\mathbb{D}_{5,4} &= -\frac{7}{138240}\big(\not{g}\,\not{g}'^{\,4} + \not{g}^3\,\not{g}'\,\not{g}''' + 4\,\not{g}^2\,\not{g}'^{\,3}\big)\\
\mathbb{D}_{5,5} &= +\frac{31}{967680}\big(\not{g}'^{\,5} + \not{g}^4\,\not{g}^{(5)} + 11\,\not{g}^3\,\not{g}'\,\not{g}^{(4)} + 32\,\not{g}^2\,\not{g}'^{\,2}\,\not{g}'''\\
&\qquad\qquad +15\,\not{g}^3\,\not{g}''\,\not{g}''' + 26\,\not{g}\,\not{g}'^{\,3}\,\not{g}'' + 34\,\not{g}^2\,\not{g}'\,\not{g}''^{\,2}\big)
\end{aligned}$$

However the *mir* formula has a wider range:

Lemma 4.6 (Formula for *mir* in the non-standard case)
The formula (85) and (86) for mir *remains valid if we replace $\beta(\tau) := \frac{1}{e^{\tau/2}-e^{-\tau/2}}$ by any series of the form $\beta(\tau) := \sum_{n\ge -1} \beta_n\, \tau^n$ subject to $\beta_{-1} = 1, \beta_0 = 0$. The even-indexed coefficients β_{2n} need not vanish. When they don't, the expansion (86) may involves homogeneous components $\mathbb{H}_{r,s}$ of any degree* r*, odd or even.*

Dropping the condition $\beta_{-1} = 1$ would bring about only minimal changes, but allowing a non-vanishing β_0 would deeply alter and complicate the shape of the *mir* transform: it would cease to be a purely integro-differential functional. We must therefore be thankful for the parity phenomenon (see §4.1 *supra*) responsible for the occurence, in the *nir* integral, of the Bernoulli polynomials with shift $1/2$ rather than 1.

Lemma 4.7 (Alternative interpretation for the *mir* formula)
The procedure implicit in formula (85) can be rephrased as follows:

(i) *Form* $h(w,y) := \sum_{r\ge 1} \frac{w^r}{r!}\,(\not{g}(y)\partial_y)^r.y$

(ii) *Form* $k(w,y) := \sum_{r\ge 1} \beta_r \frac{w^r}{r!}\,(\not{g}(y)\partial_y)^r.\not{g}(y)$

(iii) *Interpret $\not{g}(y)\partial_y$ as an infinitesimal generator and $h^{\circ w}(y) = h(w,y) = g^*(w{+}f^*(y))$ as the corresponding group of iterates: $h^{\circ w_1}\circ h^{\circ w_2} = h^{\circ(w_1+w_2)}$.*

(iv) *Interpret* $k(w,y)$ *as the Hadamard product, with repect to the* w *variable, of* $\beta(w)$ *and* $\partial_w h(w,y)$.

(v) *Calculate the convolution exponential* $K(w,y) := \exp_\star(-k(w,y))$ *relative to the unit-preserving convolution* $\star$ *acting on the* w *variable.*

(vi) *Integrate* $\int_0^\nu K(\nu-\nu_1,\nu_1)\,(\cancel{g}(\nu_1))^{-1}\,d\nu_1 =: \ell(\nu)$. $\square$

4.6 Translocation of the *nir* transform.

If we set $\eta := \int_0^\epsilon f(x)\,dx$ and then wish to compare:
(i) $f(x)$ and its translates ${}^\epsilon f(x) = e^{\epsilon\partial_x} f(x) = f(x+\epsilon)$
(ii) $h(\nu)$ and its translates ${}^\eta h(\nu) = e^{\eta\partial_\nu} h(\nu) = h(\nu+\eta)$
there are *a priori* four possibilities to choose from:

$$\begin{aligned}
&\textit{choice 1}: \quad (e^{\eta\partial_\nu}\,\mathrm{nir} - \mathrm{nir}\; e^{\epsilon\partial_x})\, f \quad \textit{as a function of} \quad (\epsilon, f)\\
&\textit{choice 2}: \quad (e^{\eta\partial_\nu}\,\mathrm{nir} - \mathrm{nir}\; e^{\epsilon\partial_x})\, f \quad \textit{as a function of} \quad (\eta, f)\\
&\textit{choice 3}: \quad (\mathrm{nir} - e^{-\eta\partial_\nu}\,\mathrm{nir}\; e^{\epsilon\partial_x})\, f \quad \textit{as a function of} \quad (\epsilon, f)\\
&\textit{choice 4}: \quad (\mathrm{nir} - e^{-\eta\partial_\nu}\,\mathrm{nir}\; e^{\epsilon\partial_x})\, f \quad \textit{as a function of} \quad (\eta, f)
\end{aligned}$$

In the event, however, the best option turns out to be choice 3. So let us define the finite (resp. infinitesimal) increments ∇h (resp. $\delta_m h$) accordingly:

$$\nabla h(\epsilon,\nu) \;=\; \sum (\delta_m h)(\nu)\,\epsilon^m \;:=\; \mathrm{nir}(f)(\nu) - \mathrm{nir}({}^\epsilon f)(\nu-\eta) \tag{88}$$

$$\textit{with} \qquad {}^\epsilon f(x) := f(x+\epsilon) \quad \textit{and} \quad \eta := \int_0^\epsilon f(x)\,dx \tag{89}$$

Going back to §4.3, we can calculate $\mathrm{nir}(f)(\nu)$ by means of the familiar double integral (73), and then $\mathrm{nir}({}^\epsilon f)(\nu-\eta)$ by using that same double integral, but after carrying out the substitutions:

$$\nu \;\mapsto\; \nu-\eta = \nu - \sum_{k\geq 0} f^{(k)}(0)\,\frac{\epsilon^{k+1}}{(k+1)!} = \nu - \sum_{k\geq 0} f_k\,\frac{\epsilon^{k+1}}{k+1} \tag{90}$$

$$f(x) \;\mapsto\; {}^\epsilon f(x) = \sum_{k_1,k_2\geq 0} f_{k_1+k_2}\,\frac{(k_1+k_2)!}{k_1!k_2!}\,\epsilon^{k_1}\,x^{k_2} \tag{91}$$

$$f^{\Uparrow\beta}(n,\tau) \;\mapsto\; {}^\epsilon f^{\Uparrow\beta}(n,\tau) = \sum_{k_1,k_2\geq 0} f_{k_1+k_2}\,\frac{(k_1+k_2)!}{k_1!k_2!}\,\epsilon^{k_1}\,n^{-k_2}\,\beta^*_{k_2}(\tau) \tag{92}$$

Singling out the contribution of the various powers of ϵ, we see that each infinitesimal increment $\delta_m h(\nu)$ is, once again, given by the double *nir*-integral, the only difference being that the integrand must now be multiplied by an elementary factor $D_m(n,\tau)$ polynomial in n^{-1} and τ. Massive cancellations occur, which wouldn't occur under any of the other choices 1, 2 or 4, and we can then regroup all the *infinitesimal* increments $\delta_m h(\nu)$ into one global and remarkably simple expressions for the *finite* increment:

Lemma 4.8 (The finite increment ∇h : compact expression.)
Like nir *itself, its finite increment is given by a double integral:*

$$\nabla h(\epsilon,\nu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n\nu}\,\frac{dn}{n}\int_0^{\epsilon\, n} \exp_{\#}\Big(-\beta(\partial_\tau)f(\frac{\tau}{n})\Big)\,d\tau \tag{93}$$

but with truncated Laplace integral and with $\exp_{\#}$ *instead of* $\exp^{\#}$.

The presence in (93) of $exp_{\#}$ instead of $exp^{\#}$ means that we must now expand *everything* within *exp*, including the leading term (unlike in (73)). So we no longer have proper Laplace integration here. Still, due to the truncation $\int_0^{\epsilon n}(\dots)d\tau$ of the integration interval, the integral continues to make sense, at least term-by-term. Due to the form ϵn of the upper bound, it yields infinitely many summands n^{-s}, with positive *and* negative s. However, the second integration $\int_{c-i\infty}^{c+i\infty}(\dots)\frac{dn}{n}$ kill off the n^{-s} with negative s, and turns those with positive s into $\frac{\nu^s}{s!}$. If we correctly interpret and carefully execute the above procedure, we are led to the following analytical expressions for the increment:

Lemma 4.9 (The finite increment ∇h : analytical expression.)
We have:

$$\nabla h(\epsilon,\nu) \;:=\; \sum_{s\geq 1}\frac{(-1)^s}{s!}\sum_{\substack{p_i\geq 0\,,\,p_i\geq q_i\\ q_i\geq -1\,,\,q_i\neq 0}}\frac{\epsilon^m}{m}\,\frac{\nu^n}{n!}\prod_{i=1}^{i=s}\Big(f_{p_i}\beta_{q_i}\frac{p_i!}{(p_i-q_i)!}\Big) \tag{94}$$

$$\textit{with}\qquad m:=1+\sum_i p_i-\sum_i q_i\quad,\quad n:=-1+\sum_i q_i \tag{95}$$

Equivalently, we may write:

$$\nabla h(\epsilon,\nu) \;:=\; \sum_{\substack{m\geq 1\\ n\geq 1}}\delta_{m,n}(f,\beta)\;\epsilon^m\;\nu^n \tag{96}$$

with

$$\delta_{m,n}(f,\beta) \;=\; \sum_{s=1}^{m+n}\frac{(-1)^s}{m\,n!\,s!}\sum_{\substack{m_i\geq 0\,,\,m_i\geq -n_i\\ n_i\neq 0\,,\,n_i\geq -1}}\;\sum_{\substack{\sum m_i=m-1\\ \sum n_i=n+1}}\prod_{i=1}^{i=s}\Big(f_{m_i+n_i}\,\beta_{n_i}\,\frac{(m_i+n_i)!}{m_i!}\Big)$$

Let us now examine the *infinitesimal* increments $\delta_m h$ of (88). Their *analytical* expression clearly follows from (96), but they also admit very useful *compact* expressions. To write these down, we require two sets of power series, the $f^{\sharp m}$ and their upper Borel transforms $\overset{\frown}{f^{\sharp m}}$. These series enter the τ-expansion of $f^{\Uparrow\beta}$:

$$f^{\Uparrow\beta}(n,\tau) = f^{\sharp 0}(n) + \tau\,f^{\sharp 1}(n) + \tau^2\,f^{\sharp 2}(n) + \dots \tag{97}$$

As a consequence, they depend bilinearly on the coefficients of f and β:

$$f^{\sharp 0}(n) := \sum_{0\leq p} p!f_p\beta_p\,n^{-p} \qquad \overset{\frown}{f^{\sharp 0}}(\nu) := \sum_{0\leq p} f_p\beta_p\,\nu^p \qquad (r=0)$$

$$f^{\sharp m}(n) := \sum_{m-1\leq p}\frac{p!}{m!}f_p\beta_{p-m}\,n^{-p} \qquad \overset{\frown}{f^{\sharp 0}}(\nu) := \sum_{m-1\leq p}\frac{p!}{m!}f_p\beta_{p-m}\,\nu^p \quad (r\geq 1)$$

We also require the 'upper' variant $\overline{*}$ of the finite-path convolution:

$$(A\overline{*}B)(t) := \int_0^t A(t-t_1)\,dB(t_1) = \int_0^t B(t-t_1)\,dA(t_1) \tag{98}$$

$$1\overline{*}1 \equiv 1 \quad , \quad \frac{(.)^p}{p!}\overline{*}\frac{(.)^q}{q!} \equiv \frac{(.)^{p+q}}{(p+q)!} \tag{99}$$

along with the corresponding convolution exponential $exp_{\overline{*}}$:

$$\exp_{\overline{*}} A := 1 + A + \frac{1}{2}A\overline{*}A + \frac{1}{6}A\overline{*}A\overline{*}A + \dots \tag{100}$$

Lemma 4.10 (Infinitesimal increments $\delta_m h$: compact expression.) *The infinitesimal increments $\delta_m h$, as defined by the ϵ-expansion $\nabla h(\epsilon,\nu) = \sum_{0\le m} \epsilon^m\,(\delta_m h)(\nu)$, admit the compact expressions:*

$$\begin{aligned}
\delta_1 h &= \partial_\nu \exp_{\overline{*}}(-f^{\sharp 0}) &(101)\\
\delta_2 h &= \frac{1}{2}\,\partial_\nu^2\Big(\big(-f^{\sharp 1}\big) \star \exp_\star(-f^{\sharp 0})\Big) &(102)\\
\delta_3 h &= \frac{1}{3}\,\partial_\nu^3\Big(\big(-f^{\sharp 2} + \frac{1}{2}\,(-f^{\sharp 1})\overline{*}(-f^{\sharp 1})\big)\overline{*}\exp_{\overline{*}}(-f^{\sharp 0})\Big) &(103)\\
\dots & \quad \dots\\
\delta_m h &= \frac{1}{m}\,\partial_\nu^m\Big(\big(\sum_{\substack{\sum i\,k_i = m-1\\ i\ge 1}} \prod \frac{(-f^{\sharp\, i})^{\overline{*}k_i}}{k_i!}\big)\overline{*}\exp_{\overline{*}}(-f^{\sharp 0})\Big) &(104)
\end{aligned}$$

Lemma 4.11 (The increments in the non-standard case)
The above expressions for $\delta_m h$ and ∇h remain valid even if we replace $\beta(\tau) := \frac{1}{e^{\tau/2}-e^{-\tau/2}}$ by any series of the form $\beta(\tau) := \sum_{n\ge -1} \beta_n\,\tau^n$ subject only to $\beta_{-1}=1, \beta_0 = 0$.

Lemma 4.12 (Entireness of $\delta_m h$ and ∇h)
For any polynomial or entire input f, each $\delta_m h(\nu)$ is an entire function of ν and $\nabla h(\epsilon,\nu)$ is an entire function of (ϵ,ν). This holds not only for the standard choice $\beta(\tau) := \frac{1}{e^{\tau/2}-e^{-\tau/2}}$ but also for any series $\beta(\tau) := \sum_{n\ge -1} \beta_n\,\tau^n$ with positive convergence radius[35].

This extremely useful lemma actually results from a sharper statement:

Lemma 4.13 (∇h bounded in terms of f and β.)
If $f(x) \prec \frac{A}{1-ax}$ and $\beta(\tau) \prec \frac{B}{1-b\tau}$ then $\nabla h(\epsilon,\nu) \prec \frac{Const}{1-2a\epsilon\nu}\exp\left(\frac{2AB}{b}\frac{\epsilon}{(1-ab\nu)}\right)$.

Here, of course, for any two power series $\{\varphi,\psi\}$, the notation $\varphi \prec \psi$ is short-hand for "ψ dominates φ", i.e. $|\varphi_n| \le \psi_n\ \forall n$. Under the assumption $f(x) \prec \frac{A}{1-ax}$ and $\beta(\tau) \prec \frac{B}{1-b\tau}$ we get:

$$\begin{aligned}
\overset{\frown}{f^{\#0}}\,(\nu) \prec K_0(\nu) &:= \frac{AB}{1-ab\,\nu}\\
\overset{\frown}{f^{\#m}}\,(\nu) \prec K_m(\nu) &:= \frac{AB}{ab}\,\frac{a^m}{m!}\,\frac{\nu^{m-1}}{1-ab\,\nu}
\end{aligned}$$

[35]subject as usual to $\beta_{-1}=1, \beta_0 = 0$.

After some easy majorisations, this leads to:

$$\begin{aligned}
\delta_m h(\nu) \;&\prec\; \partial_\nu^m \sum_{\substack{1\leq s\leq m\\ m_1+\dots+m_s=m}} \frac{1}{s!}\big(K_{m_1}\overline{*}K_{m_2}\overline{*}\dots K_{m_s}\big)(\nu)\\
&\prec\; \sum_{1\leq s\leq m} \frac{Const}{s!}\Big(\frac{AB}{ab}\Big)^s \frac{(2a)^r\nu^{r-s}}{(1-ab\nu)^s}
\end{aligned}$$

and eventually to:

$$\begin{aligned}
\nabla h(\epsilon,\nu) \;&\prec\; \frac{Const}{s!}\Big(\frac{AB}{ab}\Big)^s \frac{1}{(1-ab\nu)^s} \sum_{s\leq m}(2a\,\epsilon)^m\,\nu^{m-s}\\
&\prec\; \frac{Const}{1-2a\epsilon\nu}\exp\Big(\frac{2AB}{b}\frac{\epsilon}{(1-ab\nu)}\Big)
\end{aligned}$$

In the standard case we may take $B=1, b=\frac{1}{2\pi}$ so that the bound becomes:

$$\nabla h(\epsilon,\nu) \prec \frac{Cons}{1-2a\,\epsilon\,\nu}\exp\Big(\frac{4\pi A\,\epsilon}{1-\frac{a}{2\pi}\,\nu}\Big)$$

Since *Const* is independent of a, A, this immediately implies that $\nabla h(\epsilon,\nu)$ is bi-entire (in ϵ and ν) if $f(x)$ is entire in x.[36]

4.7 Alternative factorisations of *nir*. The *lir* transform.

The *nir* transform and its two factorisations.
In some applications, two alternative factorisations of the *nir*-transform are preferable to the one corresponding to the nine-link chain of §4.2. Graphically:

$$\begin{array}{ccccc}
f & \longrightarrow & \stackrel{\textbf{nir}}{\longrightarrow} & \longrightarrow & h
\end{array}$$

$$\begin{array}{ccccc}
f^* & \stackrel{\textbf{rec}}{\longrightarrow} & g^* & \stackrel{\textbf{imir}}{\longrightarrow} & h^* \\
\uparrow & & \downarrow & & \downarrow \\
f & & g & \dots & h \\
 & & \downarrow & & \uparrow \\
 & & \cancel{g} & \stackrel{\textbf{mir}}{\longrightarrow} & \hbar
\end{array}
\qquad
\begin{array}{c}
(f^* := \partial^{-1} f \;\;,\;\; g^* := \partial^{-1} g) \\ \\ \\ \\
(\cancel{g} := 1/g \quad , \quad \hbar := 1/g)
\end{array}$$

$$\begin{array}{ccccc}
f^* & \stackrel{\textbf{ilir}}{\longrightarrow} & q^* & \stackrel{\textbf{rec}}{\longrightarrow} & h^* \\
\uparrow & & \uparrow & & \downarrow \\
f & \stackrel{\textbf{lir}}{\longrightarrow} & q & & h
\end{array}
\qquad
(q^* := \partial^{-1} q \;\;,\;\; h^* := \partial^{-1} h)$$

In the first alternative, we go by *imir* ("integral" *mir*) from the indefinite integral g^* of g to the indefinite integral h^* of h, rather than from $\cancel{g}$ to $\hbar$. In the second alternative, the middle column (g, g^*) gets replaced by (q, q^*) with q^* denoting the functional inverse

[36] and *only* if f is entire - but this part is harder to prove and not required in practice.

of h^*. In that last scenario, the non-elementary factor-transform becomes *lir* (from f to q) or *ilir* (from f^* to q^*). We get get for these transforms expansions similar to, but in some respects simpler than, the expansions for *mir*:

$$\begin{aligned}
\text{mir} &: \quad \cancel{g} \to \hbar \quad \textit{with} \quad \frac{1}{\hbar} = \frac{1}{\cancel{g}} + \sum_{1 \le r \; odd} \mathbb{H}_r(\cancel{g}) \\
\text{lir} &: \quad f \to q \quad \textit{with} \quad q = f + \sum_{3 \le r \; odd} \mathbb{Q}_r(f)
\end{aligned}$$

$$\begin{aligned}
\text{imir} &: \quad g^* \to h^* \quad \textit{with} \quad h^* = g^* + \sum_{1 \le r \; odd} \mathbb{IH}_r(\cancel{g}) \\
\text{ilir} &: \quad f^* \to q^* \quad \textit{with} \quad q^* = f^* + \sum_{3 \le r \; odd} \mathbb{IQ}_r(f)
\end{aligned}$$

Each term on the right-hand sides is a polynomial in the $f^{(i)}$ and the following integro-differential expressions:

$$\begin{aligned}
f_m^{(d)\{s_1,\ldots,s_r\}} &:= (I_R \bullet f)^{s-d} \bullet (f^{-1} f^{(s_1)} \ldots f^{(s_r)}) && (105) \\
&= (I_R \bullet f)^{m-r} \bullet I_R \bullet (f^{(s_1)} \ldots f^{(s_r)}) && (106) \\
&= I_R \bullet f \bullet I_R \bullet f \ldots I_R \bullet f \bullet I_R \bullet (f^{(s_1)} \ldots f^{(s_r)})
\end{aligned}$$

with $d \ge -1, m \ge r, s_1, \ldots, s_r \ge 1$ and $1{+}m{+}d{=}r{+}s$. Here $I_R := \partial^{-1} = \int_0^{\cdots}$ denotes the integration operator that starts from 0 and acts on *everything standing on the right.* The 'monomial' $f_m^{(d)\{s_1,\ldots,s_r\}}$ has total degree m (i.e. it is m-linear in f) and total differential order d. The notation is slightly redundant since $1{+}m{+}d \equiv r{+}s \equiv \sum(1+s_i)$ but very convenient, since it makes it easy to check that each summand in the expression of $\mathbb{H}_r(f)$ (resp. $\mathbb{IH}_r(f)$) has global degree r and global order 0 (resp. -1). The operators $\mathbb{IH}_r$ and $\mathbb{IQ}_r$ are simpler and in a sense more basic than the $\mathbb{H}_r$ and $\mathbb{Q}_r$.

Proof: Let us write the two reciprocal (formal) functions h^* (known) and q^* (unknown) as sums of a leading term plus a perturbation:

$$\begin{aligned}
h^*(x) &= g^*(x) + \mathbb{IH}(x) \\
q^*(x) &= f^*(x) + \mathbb{IQ}(x)
\end{aligned}$$

The identity $id = h^* \circ q^*$ may be expressed as:

$$\begin{aligned}
id &= (g^* + \mathbb{IH}) \circ (f^* + \mathbb{IQ}) \\
&= id + \mathbb{IH} \circ f^* + \sum_{1 \le r} \frac{1}{r!} (\mathbb{IQ})^r \Big(\partial^r (g^* + \mathbb{IH})\Big) \circ f^*
\end{aligned}$$

But h^* may be written as

$$h^*(x) = (x + \mathbb{JH}) \circ g^*(x) \tag{107}$$

and the identity $id = h^* \circ q^*$ now becomes:

$$0 = \mathbb{JH} + \sum_{1 \le r} \frac{1}{r!} (\mathbb{IQ})^r \; (f^{-1} \bullet \partial)^r \bullet (x + \mathbb{JH}) \tag{108}$$

The benefit from changing $\mathbb{IH}$ into $\mathbb{JH}$ is that we are now handling direct functions of f. Indeed, in view of the argument in §4.2 we have:

$$\mathbb{JH} = \sum_{1\leq r, 1\leq s_i} (-1)^r \Big(\prod_{i=1}^{i=r} \beta_i\Big) (I_R {\scriptstyle\bullet} f)^{1+\sum s_i} {\scriptstyle\bullet} \Big(f^{-1} \prod_{i=1}^{i=r} f^{(s_i)}\Big) \tag{109}$$

The right-hand side turns out to be a linear combination of monomials (105) of order $d = -1$:

$$\mathbb{JH} = \sum_{1\leq r, 1\leq s_i} (-1)^r \Big(\prod_{i=1}^{i=r} \beta_i\Big)\; f_{r+\sum s_i}^{(-1)\{s_1,\ldots,s_r\}} \tag{110}$$

If we now adduce the obvious rules for differentiating these monomials:

$$\begin{aligned}(f^{-1}{\scriptstyle\bullet}\, \partial)^\delta \,{\scriptstyle\bullet}\, f_m^{(d)\{s_1,\ldots,s_r\}} &= f_{m-\delta}^{(d+\delta)\{s_1,\ldots,s_r\}} && (\mathit{if}\ \delta \leq m-r)\\ &= f^{-1}\, f^{(s_1)} \ldots f^{(s_1)} && (\mathit{if}\ \delta = 1+m-r)\\ &= (f^{-1}{\scriptstyle\bullet}\, \partial)^{\delta+r-m-1} {\scriptstyle\bullet}\, f^{-1}\, f^{(s_1)} \ldots f^{(s_1)} && (\mathit{if}\ \delta \geq 2+m-r)\end{aligned}$$

we see at once that the identity (108) yields an inductive rule for calculating, for each m, the m-linear part $\mathbb{IQ}_m$ of $\mathbb{IQ}$. At the same time, it shows that any such $\mathbb{IQ}_m$ will be exactly of global differential order -1, and *a priori* expressible as a polynomial in $f^{-1}, f, f^{(1)}, f^{(2)}, f^{(3)} \ldots$ and finitely many monomials $f_\mu^{(\delta)\{s_1,\ldots,s_\rho\}}$. The only point left to check is the *non-occurence* of negative powers of f, which would seem to result from the above differentiation rules, but actually cancel out in the end result.

4.8 Application: kernel of the *nir* transform.

For any input f of the form $p\, log(x) + Reg(x)$ with $p \in \mathbb{Z}$ and *Reg* a regular analytic germ, the image h of f under *nir* is also a regular analytic germ:

$$nir \quad : \quad f(x) = p\, \log(x) + \mathrm{Reg}_1(x) \;\mapsto \mathrm{Reg}_2(x)$$

The singular part of h, which alone has intrinsic significance, is thus 0. In other words, germs f with logarithmic singularities that are *entire* multiples of $log(x)$ belong to the kernel of *nir* and produce *no inner generators*. This important and totally non-trivial fact is essential when it comes to describing the inner algebra of SP series j_F constructed from a meromorphic F. It may be proven (see [SS1]) either by using the alternative factorisations of the *nir* transform mentioned in the preceding subsection, or by using an exceptional generator $f(x - x_0)$ with base-point x_0 arbitrarily close to 0. An alternative proof, valid in the special case when $Reg_1 = 0$ and relying on the existence in that case of a simple ODE for the *nir*-transform, shall be given in §6.6-7 below.

4.9 Comparing/extending/inverting *nir* and *mir*.

Lemma 4.14 (The case of generalised power-series f)
The nir *transform can be extended to generalised power series*

$$f(x) := \sum_{k_i \geq m} f_{k_i}\, x^{k_i} \quad \big(k_i \uparrow +\infty\ ;\ k_i \in \mathbb{R} \dot{-} \{-1\}\big) \tag{111}$$

in a consistent manner (i.e. one that agrees with mir *and ensures that $\ell(\nu)$ converges whenever $f(x)$ does) by replacing in the double* nir*-integral (73) the polynomials $\beta^*_k(\tau)$ by the Laurent-type series:*

$$\beta^*_k(\tau) := \sum_{s=-1}^{+\infty} \beta_k\, \tau^{k-s} \frac{\Gamma(k+1)}{\Gamma(k+1-s)} = \frac{\tau^{k+1}}{k+1} + \sum_{s=1}^{+\infty}(\dots) \tag{112}$$

As usual, this applies both to the standard and non-standard [37] *choices of β.*

We may also take advantage of the identity $f^{\Uparrow\beta} := \beta(\partial_\tau) f(\frac{\tau}{n})$ to *formally* extend the *nir*-transform to functions f derived from an F with a zero//pole of order p at $x = 0$:

$$F(x) = e^{-f} = x^p\, e^{-w(x)} \quad \textit{with} \quad p \in \mathbb{Z}^* \quad \textit{with} \quad w(.)\ \textit{regular at 0}$$

That formal extension would read:

$$\begin{aligned} h(\nu) &\overset{\text{formally}}{=} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(n\,\nu)\,\frac{dn}{n}\int_0^\infty n^{-p\,\tau} \exp\Big(p\,\lambda(\tau) - \beta(\partial_\tau)\, w(\frac{\tau}{n})\Big)\, d\tau \\ \tilde{\lambda}(\tau) &= \beta(\partial_\tau)(\log\tau) = \tau\log\tau - \tau + \sum_{0\le s}\beta_{2\,s+1}\,(2\,s)!\,\tau^{-2\,s-1} \end{aligned}$$

with λ denoting the Borel-Laplace resummation of the divergent series λ. However, in the above formula for $h(\nu)$, the first integration (in τ) makes no sense at infinity[38] and one would have to exchange the order of integration (first n, then τ), among other things, to make sense of the formula and arrive at the correct result, namely that the *nir*-transform turns functions of the form $f(x) = p\log x + \mathit{Reg}$ into *Reg*[39]. In other words, *there is no inner generator attached to the corresponding base point $x = 0$.* But it would be difficult to turn the argument into a rigorous proof, and so the best approaches remain the ones just outlined in the preceding subsection.

Directly extending *nir* to even more general test functions f would be possible, but increasingly difficult and of doubtful advantage. Extending *mir*, on the other hand, poses no difficulties.

Lemma 4.15 (Extending *mir*'s domain)
The mir *transform $\not{g} \to \hbar$ extends, formally* and *analytically, to general transserial inputs $\not{g}$ of infinitesimal type, i.e. $\not{g}(y) = o(y)$, $y \sim 0$, and even to those with moderate growth $\not{g}(y) = O(y^{-\sigma})$, $\sigma > 0$.*

We face a similar situation when investigating the behaviour of $h(\nu)$ *over* $\nu = \infty$[40] for inputs f of the form:

$$f(x) = \textit{polynomial}(x) \quad \textit{or} \quad \textit{polynomial}(x) + \sum_{i=1}^{i=N} p_i \log(x - x_i) \quad (p_i \in \textit{doZ}^*)$$

[37]For non-standard choices, the series $\beta(\tau) := \sum \beta_s\, \tau^s$ has to be convergent if *nir* is to preserve convergence.

[38]even when interpreted term-by-term, i.e. after expanding $\mathit{exp}\big(-\beta(\partial_\tau)(w(\frac{\tau}{n})\big)$.

[39] as long as $p \in \mathbb{Z}$.

[40] under the change $\zeta = e^\nu$, this behaviour at $-\infty$ in the ν-plane translates into the behaviour over 0 in the ζ-plane.

Then the (86) expansion for $h(\nu)$ still converges in some suitable (ramified) neighbourhood of ∞ to some analytic germ, but the latter is no longer described by a power series (or a Laurent series, as we might expect a infinity) nor even by a (well-ordered) transseries, but by a complex combination both kinds of infinitesimals : small and large.[41]

Lemma 4.16 (Inverting *mir*)
The mir *transform admits a formal inverse* $\mathrm{mir}^{-1} : \hbar \to g\!\!\!/$ *that acts, not just formally but also analytically, on general transserial inputs* $\hbar$ *of infinitesimal type. Like* mir, *this inverse* mir^{-1} *admits well-defined integro-differential components* $ID_{r,s}$ *of degree* r *and order* s, *but these are no longer of the form* $\partial^{-s}D_{r,s}$ *with a neat separation of the differentiations (coming first) and integrations (coming last).*

4.10 Parity relations.

With the standard choice for β, we have the following parity relations for the *nir*-transform :

$$\begin{array}{ll}
F^{\vdash}(x) := 1/F(-x) \quad , \quad f^{\vdash}(x) := -f(-x) & \Longrightarrow \\
 & (\textit{tangency } \kappa = 0) \\
\mathrm{nir}(f^{\vdash})(\nu) = -\mathrm{nir}(f)(\nu) & (\textit{tangency } \kappa = 0) \\
\mathrm{nir}(f^{\vdash}) \;\; \textit{and} \;\; \mathrm{nir}(f) \;\; \textit{unrelated} & (\textit{tangency } \kappa \textit{ even} \geq 2) \\
\mathrm{nir}(f^{\vdash})(\nu) = -\mathrm{nir}(f)(\epsilon_\kappa \nu) \quad \textit{with} \quad \epsilon_\kappa^{\frac{1}{\kappa+1}} = -1 & (\textit{tangency } \kappa \textit{ odd} \geq 1) \\
\Rightarrow h^{\vdash}_{\frac{k}{\kappa+1}} = (-1)^{k-1} h_{\frac{k}{\kappa+1}} \quad \textit{with} \quad : (f, f^{\vdash}) \overset{\mathrm{nir}}{\mapsto} (h, h^{\vdash}) & (\textit{tangency } \kappa \textit{ odd} \geq 1)
\end{array}$$

For the *mir*-transform the parity relation doesn't depend on κ and assumes the elementary form :

$$\mathrm{mir}(-g\!\!\!/) = -\mathrm{mir}(g\!\!\!/)$$

5 Outer generators.

5.1 Some heuristics.

In the heuristical excursus at the beginning of the preceding section, we had chosen the driving function f such as to make the nearest singularity an *inner generator*. We must now hone f to ensure that the nearest singularity be an *outer generator*. For maximal simplicity, let us assume that :

$$0 \leq f(0) \leq +\infty \quad \textit{and} \quad 0 < f(x) < +\infty \quad \textit{for} \quad 0 < x \leq 1 \tag{113}$$

Thus $f^*(x) := \int_0^x f(x')dx'$ will be > 0 on the whole interval $]0,1]$. Since we insist, as usual, on $F := \exp(-f)$ being meromorphic, (113) leaves but three possibilities :

$$\begin{array}{lllll}
\textit{Case 1} & : & 0 = F(0) & ; & f(x) = -p\,\log(x) + \sum_{k=0}^{k=\infty} f_k x^k \\
\textit{Case 2} & : & 0 < F(0) < 1 & ; & f(x) = \sum_{k=0}^{k=\infty} f_k x^k \quad (f_0 > 0) \\
\textit{Case 3} & : & F(0) = 1 & ; & f(x) = \sum_{k=\kappa}^{k=\infty} f_k x^k \quad (f_\kappa > 0\,,\, \kappa \geq 1)
\end{array}$$

[41]When re-interprented as a germ over 0 in the ζ-plane, it typically produces an essential singularity there, with Stokes phenomena and exponential growth or decrease, depending on the sector.

In all three cases, the nearest singularity of $j(\zeta)$ (cf (1)) is located at $\zeta = 1$ and reflects the n-asymptotics of the Taylor coefficients $J(n)$.

$$J(n) := \sum_{m=\epsilon}^{m=n-1} \prod_{k=\epsilon}^{k=m-1} F(\frac{k}{n}) \quad \big(\epsilon \in \{0,1\}\big) \quad \Longrightarrow \quad \tilde{J}(n) := \sum_{k \geq 0} j_k \, n^{-k} \tag{114}$$

Case 1 is simplest.[42] By truncating the $\sum \prod$ expansion at $m = m_0$, we get the exact values of all coefficients j_k up to $k = p\, m_0$.

Case 2 corresponds to tangency 0. Here, finite truncations yield only approximate values. To get the exact coefficients, we must harness the full $\sum \prod$ expansion but we still end up with closed expressions for each j_k .

Case 3 corresponds to tangency $\kappa \geq 1$ case. Here, again, the full $\sum \prod$ expansion must be taken into account, but the difference is that we now get coefficients j_k which, though exact, are no longer neatly expressible in terms of elementary functions.

5.2 The short and long chains behind *nur/mur*.

Let us now translate the above heuristics into precise (non-linear) functionals. For case 1, the definition is straightforward:

The short, four-link chain:

$$F \xrightarrow{1} k \xrightarrow{2} h \xrightarrow{3} h' \xrightarrow{4} H \qquad h = \overset{\frown}{\ell u} \, , \, h' = \overset{\wedge}{\ell u} \, , \, H = \overset{\wedge}{L u}$$

Details of the four steps:

$$\begin{array}{lll}
F(x) & := & F_1\, x + F_2\, x^2 + F_3\, x^3 + \dots \qquad\qquad\qquad\qquad\qquad\qquad (converg^t) \\
\downarrow^1 & & \\
k(n) & := & \Big(F(\tfrac{1}{n}) + F(\tfrac{1}{n})F(\tfrac{2}{n}) + F(\tfrac{1}{n})F(\tfrac{2}{n})F(\tfrac{3}{n}) + \dots)/Ig_F(n) \quad (diverg^t) \\
\| & & \\
k(n) & := & \sum_{1 \leq s} \; k_s \, \frac{1}{n^k} \quad (N.B.\ k \equiv J^{\#} \ as\ in\ (3)) \qquad\qquad\qquad (diverg^t) \\
\downarrow^2 & & \\
h(\nu) & := & \sum_{1 \leq s} \; k_s \, \frac{\nu^s}{s!} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad (converg^t) \\
\downarrow^3 & & \\
h'(\nu) & := & \sum_{1 \leq s} \; k_s \, \frac{\nu^{s-1}}{(s-1)!} \qquad\qquad\qquad\qquad\qquad\qquad\qquad (converg^t) \\
\downarrow^4 & & \\
H(\zeta) & := & h'(\log(1+\zeta))
\end{array}$$

Mark the effect of removing the ingress factor Ig_F after the first step. If

$$F(x) = c_0\, x^d\, F_*(x) \quad with \quad F_*(x) = 1 + \dots \in \mathbb{C}\{x^2\} \ (resp.\ \mathbb{C}\{x\})$$

then, according to the results of §3, removing Ig_F amounts to dividing $k(n)$ by $c_0^{-1/2}(2\pi n)^{d/2}$ and integrating $d/2$ times the functions [43] $h(\nu)$ or $h'(\nu)$. The removal of the ingress factor

[42] Here, we must take $\epsilon = 0$ to avoid an all-zero result.

[43] or more accurately $c_0^{1/2}(2\pi)^{-d/2} h(\nu)$ and $c_0^{1/2}(2\pi)^{-d/2} h'(\nu)$.

thus has three main effects:

(i) as already pointed out, it makes the outer generators independent of the ingress point[44]

(ii) depending on the sign of d, it renders the singularities smoother (for $d > 0$) or less smooth (for $d < 0$), in the ν- or ζ-planes alike.

(iii) depending on the parity of d, it leads in the Taylor expansions of the minors $\overset{\frown}{\ell u}(\nu) := h(\nu)$ and $\overset{\wedge}{\ell u}(\nu) := h'(\nu)$ either to integral powers of ν (for d even) or to strictly semi-integral powers (for d odd). This means that the corresponding majors $\overset{\smile}{\ell u}$ and $\overset{\vee}{\ell u}$ and, by way of consequence, the inner generators themselves, will carry *logarithmic* singularities (for d even) or strictly semi-integral powers (for d odd).[45].

Time now to deal with the cases 2 and 3 (i.e. $F(0) \neq 0$). These cases lead to a nine-link chain quite similar to that which in §4.2 did service for the *inner generators*, but with the key steps *nir* and *mir* significantly altered into *nur* and *mur*:

The long, nine-link chain:

$$
\begin{array}{ccccccc}
\overset{\mathbf{nur}}{\to} & \to & \to & \to & \overset{\mathbf{nur}}{\to} & H & \qquad H = \overset{\wedge}{Lu} \\
\uparrow & & & & \downarrow & \uparrow^{9} & \\
\uparrow & f^* & \overset{3}{\to} & g^* & \downarrow & h' & \qquad h' = \overset{\wedge}{lu} \\
\uparrow & \uparrow^{2} & & \downarrow^{4} & \downarrow & \uparrow^{8} & \\
\overset{\mathbf{nur}}{\longleftarrow} & f & & g & \overset{\mathbf{nur}}{\longrightarrow} & h & \qquad h = \overset{\frown}{lu} \\
 & \uparrow^{1} & & \downarrow^{5} & & \uparrow^{7} & \\
 & F & & \not{g} & \longrightarrow\!\overset{6}{}\!\longrightarrow & \hbar & \\
 & & & & \mathbf{mur} & &
\end{array}
$$

Details of the nine steps:

$\overset{1}{\to}$	:	*precomposition*	:	$F \to f$	*with*	$f(x) := -\log F(x)$
$\overset{2}{\to}$	:	*integration*	:	$f \to f^*$	*with*	$f^*(x) := \int_0^x f(x_0)\, dx_0$
$\overset{3}{\to}$	:	*reciprocation*	:	$f^* \to g^*$	*with*	$f^* \circ g^* = id$
$\overset{4}{\to}$	:	*derivation*	:	$g^* \to g$	*with*	$g(y) := \frac{d}{dy} g^*(y)$
$\overset{5}{\to}$	:	*inversion*	:	$g \to \not{g}$	*with*	$\not{g}(y) := 1/g(y)$
$\overset{6}{\to}$	:	***mur*** *functional*	:	$\not{g} \to \hbar$		*see §5.4 infra*
$\overset{7}{\to}$	:	*inversion*	:	$\hbar \to h$	*with*	$h(\nu) := 1/\hbar(\nu)$
$\overset{8}{\to}$	:	*derivation*	:	$h \to h'$		$h'(\nu) := \frac{d}{d\nu} h(\nu)$
$\overset{9}{\to}$	:	*postcomposition*	:	$h' \to H$	*with*	$H(\zeta) := h'(\log(1+\zeta)$
$\overset{2...7}{\to}$	:	***nur*** *functional*	:	$g \to h$	*with*	*see §5.3 infra*

[44] just as was the case with the inner generators.

[45] Removing the ingress factor has exactly the opposite effect on *inner generators*: these generically carry semi-integral powers for d even and logarithmic singularities for d odd.

5.3 The *nur* transform.

Integral-serial expression of *nur*.

Starting from f we define $f^{\Uparrow\mathfrak{b}^*}, f^{\uparrow\mathfrak{b}^*}$ and $f^{\Uparrow\beta^*}, f^{\uparrow\beta^*}$ as follows:

$$\begin{aligned}
f(x) &= \sum_{k\geq\kappa} f_k\, x^k \qquad (\kappa \geq 1\ ,\ f_\kappa \neq 0)\\
f^{\Uparrow\mathfrak{b}^*}(n,\tau) &:= \mathfrak{b}(\partial_\tau) f(\frac{\tau}{n}) = \sum_{k\geq\kappa} f_k\; n^{-k}\; \mathfrak{b}_k^*(\tau) \;=:\; f_\kappa\, \frac{n^{-\kappa}\,\tau^{\kappa+1}}{\kappa+1} + f^{\uparrow\mathfrak{b}^*}(n,\tau)\\
f^{\Uparrow\beta^*}(n,\tau) &:= \beta(\partial_\tau) f(\frac{\tau}{n}) = \sum_{k\geq\kappa} f_k\; n^{-k}\; \beta_k^*(\tau) \;=:\; f_\kappa\, \frac{n^{-\kappa}\,\tau^{\kappa+1}}{\kappa+1} + f^{\uparrow\beta^*}(n,\tau)
\end{aligned}$$

with the usual definitions in the standard case :

$$\begin{aligned}
\mathfrak{b}_k^*(\tau) &:= \frac{B_{k+1}(\tau+1)}{k+1} = \beta_k^*(\tau+\frac{1}{2})\\
\beta_k^*(\tau) &:= \frac{B_{k+1}(\tau+\frac{1}{2})}{k+1}
\end{aligned}$$

where B_k stands for the k^{th} Bernoulli polynomial. Recall that $\mathfrak{b}_k^*(m)$ is a polynomial in m of degree $k+1$, with leading term $\frac{m^{k+1}}{k+1}$. Then the *nur*-transform is defined as follows:

$$nur \quad : \quad f \mapsto h \tag{115}$$

$$h(\nu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(n\,\nu)\,\frac{dn}{n} \sum_{m=0}^{\infty} \exp^{\#}(-f^{\Uparrow\mathfrak{b}^*}(n,m)) \tag{116}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(n\,\nu)\,\frac{dn}{n} \sum_{m=0}^{\infty} \exp\bigl(-f_\kappa\,\frac{n^{-\kappa}\,m^{\kappa+1}}{\kappa+1}\bigr)\,\exp_{\#}\bigl(-f^{\uparrow\mathfrak{b}^*}(n,m)\bigr)$$

or equivalently (and preferably):

$$h(\nu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(n\,\nu)\,\frac{dn}{n} \sum_{m\in\frac{1}{2}+\mathbb{N}}^{\infty} \exp^{\#}(-f^{\Uparrow\beta^*}(n,m)) \tag{117}$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(n\,\nu)\,\frac{dn}{n} \sum_{m\in\frac{1}{2}+\mathbb{N}}^{\infty} \exp\bigl(-f_\kappa\,\frac{n^{-\kappa}\,m^{\kappa+1}}{\kappa+1}\bigr)\,\exp_{\#}\bigl(-f^{\uparrow\beta^*}(n,m)\bigr)$$

where $exp_{\#}(X)$ denotes the usual exponential function, but expanded as a power series of X. Similarly, $exp^{\#}(X)$ denotes the exponential expanded as a power series of X *minus* the leading term of X, which remains within the exponential. An unmarked $exp(X)$, on the other hand, should be construed as the usual exponential function.
The analytical expressions vary depending on the tangency order κ. Indeed, after expanding $exp_{\#}(...)$, we are left with the task of calculating individual sums of type:

$$\begin{aligned}
S_{0,k}(f_0) &= \textstyle\sum_{m\in\mathbb{N}} m^k\; \exp(-f_0\, m) && \textit{in case 2} \quad (\kappa=0, f_0>0)\\
S_{\kappa,k}(f_\kappa) &= \textstyle\sum_{m\in\mathbb{N}} m^k\; \exp\bigl(-f_\kappa\, \frac{n^{-\kappa}\,m^{\kappa+1}}{\kappa+1}\bigr) && \textit{in case 3} \quad (\kappa\geq 1, f_\kappa>0)
\end{aligned}$$

or of type:

$$\begin{array}{lcll} Z_{0,k}(f_0) &=& \sum_{m\in\frac12+\mathbb{N}} m^k\ \exp(-f_0\, m) & \textit{in case 2}\quad (\kappa=0, f_0>0)\\ Z_{\kappa,k}(f_\kappa) &=& \sum_{m\in\frac12+\mathbb{N}} m^k\ \exp\big(-f_\kappa\ \frac{n^{-\kappa}\, m^{\kappa+1}}{\kappa+1}\big) & \textit{in case 3}\quad (\kappa\geq 1, f_\kappa>0)\end{array}$$

Since we assumed f_κ to be positive in all cases, convergence is immediate and precise bounds are readily found. However, only for $\kappa=0$ do the sums $S_{\kappa,k}$, $Z_{\kappa,k}$ admit closed expressions for all k. For the former sums we get:

$$S_{0,k}(\alpha)=\frac{L_k(a)}{(1-a)^{k+1}}\ \ \textit{with}\ \ a:=e^{-\alpha}\ \ ;\ \ L_k(a):=tr_k\Big((1-a)^{k+1}\sum_{0\leq s\leq 2k} s^k\, a^s\Big)$$

where tr_k means that we truncate after the k^{th} power of a, which leads to self-symmetrical polynomials of the form:

$$L_k(a)=a+(1+k+2^k)\,a^2+\cdots+(1+k+2^k)\,a^{k-1}+a^k\ \ \textit{with}\ \ L_k(1)=k!$$

For the latter sums we get the generating function:

$$\sum_{0\leq k} Z_{0,k}(\alpha)\ \frac{\sigma^k}{k!}=\frac{1}{e^{\frac12(\alpha-\sigma)}-e^{-\frac12(\alpha-\sigma)}}=\frac{1}{\alpha-\sigma}+\textit{Regular}(\alpha-\sigma) \tag{118}$$

Hence:

$$Z_{0,k}(\alpha)=\frac{k!}{\alpha^{k+1}}+\textit{Regular}(\alpha) \tag{119}$$

Let us now justify the above definition of *nur*. For a tangency order $\kappa\geq 0$ and a driving function $f(x):=\sum_{s\geq\kappa} f_s\,x^s$ as in the cases 2 or 3 of §5.1, our Taylor coefficients $J(n)$ will have the following asymptotic expansions, *before* and *after* division by the ingress factor $Ig_F(n)$:

$$\tilde J(n)\ :=\ \sum_{0\leq m}\exp\Big(-\sum_{0\leq k\leq m} f(\frac{k}{m})\Big) \tag{120}$$

$$=\ \sum_{0\leq m}\exp\Big(-(m+1)f_0-\sum_{1\leq s} n^{-s}\big(\mathfrak{b}^*_s(m)-\mathfrak{b}^*_s(0)\big)\,f_s\Big) \tag{121}$$

$$\tilde{Ig}_F(n)\ :=\ \exp\big(-\frac12 f_0+\sum_{1\leq s} n^{-s}\mathfrak{b}^*_s(0)f_s\big) \tag{122}$$

$$\tilde J(n)/\tilde{Ig}_F(n)\ =\ \sum_{0\leq m}\exp\Big(-(m+\frac12)\,f_0-\sum_{1\leq s} n^{-s}\mathfrak{b}^*_s(m)\,f_s\Big) \tag{123}$$

$$=\ \sum_{0\leq m}\exp\Big(-\sum_{0\leq s} n^{-s}\mathfrak{b}^*_s(m)\,f_s\Big) \tag{124}$$

Of course, the summand $\frac12 f_0$ automatically disappears when the tangency order κ is >0. But, whatever the value of κ, the hypothesis $f_\kappa>0$ ensures the convergence of the m-summation[46] in (124), which yields, in front of any given power n^{-s}, a well-defined, finite

[46] after factoring $\exp(-\sum_{\kappa\leq s}(...))$ into $\exp(-\sum_{\kappa=s}(...))\,\exp_{\#}(-\sum_{\kappa<s}(...))$ and expanding the second factor as a power series of $(\sum_{\kappa<s}(...))$.

coefficient. If we then suject the right-hand side of (124), term-wise, to the (upper) Borel transform $n \to \nu$, we are led straightaway to the above definition of the *nur*-transform $f(x) \mapsto h(\nu)$.

5.4 Expressing *nur* in terms of *nir*.

Lemma 5.1 (Decomposition of *nur*.)
The nur-transforms reduces to an alternating sum of nir-transforms:

$$\mathrm{nur}(f) = \sum_{p\in\mathbb{Z}} (-1)^p \,\mathrm{nir}(2\pi i\, p + f) \tag{125}$$

It suffices to show that this holds term-by-term, i.e. for the coefficient of each monomial ν^n on the left- and right-hand sides of (125). For $\kappa = 0$ for instance, this results from the identities:

$$\sum_{m\in\frac{1}{2}+\mathbb{Z}} m^k \exp(-f_0\, m) = \sum_{p\in\mathbb{Z}} (-1)^p \frac{(k+1)!}{(2\pi i\, p + f_0)^{k+1}} \tag{126}$$

which are a direct consequence of Poisson's summation formula[47]. The same argument applies for $\kappa > 0$.

As a consequence of the above lemma, we see that whereas the *nir*-transform depends on the exact determination of $logF$, the *nur*-transform depends only on the determination of $F^{1/2}$. This was quite predictable, in view of the interpretation of *nur*.[48].

5.5 The *mur* transform.

Since in this new nine-link chain (of §5.2) all the steps but *mur* are elementary, and the composite step *nur* has just been defined, that indirectly determines *mur* itself, just as knowing *nir* determined *mir* in the preceding section. There are, however, two basic differences between *mur* and *mir*.

(i) *Analytic difference*: whereas the singularities of a *mir*-transform were *mir*-transforms of singularities (reflecting the essential closure of the inner algebra), the singularities of *mur*-transforms are *mir*-transforms, (not *mur*-transforms!) of singularities (reflecting the non-recurrence of outer generators under alien derivation).

(ii) *Formal difference*: unlike *mir*, *mur* doesn't reduce to a purely integro-differential functional. It does admit interesting, if complex, expressions[49] but we needn't bother with them, since the whole point of deriving an exact analytical expression for *mir* was to account for the closure phenomenon just mentioned in (i) but which no longer applies to *mur*.

[47] Decompose the left-hand side of (126) as $\sum_{m\in\frac{1}{2}+\mathbb{Z}} = \sum_{m\in\frac{1}{2}\mathbb{Z}} - \sum_{m\in\mathbb{Z}}$ and formally apply Poisson's formula separately to each sum.

[48] The square root of F comes from our having replaced j_F by $j_F^{\#}$, i.e. from dividing by the ingress factor, which carries the term $e^{-f_0/2} = F(0)^{1/2}$

[49] somewhat similar to the expression for the generalised (non-standard) *mir*-transform when we drop the condition $\beta_0 \neq 0$.

5.6 Translocation of the *nur* transform.

Like with *nir*, it is natural to 'translocate' *nur*, i.e. to measure its failure to commute with translations. To do this, we have the choice, once again, between four expressions (where $\eta := \int_0^\epsilon f(x)\,dx$)

$$\begin{array}{llll} \textit{choice 1}: & (\mathrm{nur}\ e^{\epsilon\partial_x} - e^{\eta\partial_\nu}\ \mathrm{nur})\, f & \textit{as a function of} & (\epsilon, f) \\ \textit{choice 2}: & (\mathrm{nur}\ e^{\epsilon\partial_x} - e^{\eta\partial_\nu}\ \mathrm{nur})\, f & \textit{as a function of} & (\eta, f) \\ \textit{choice 3}: & (\mathrm{nur} - e^{-\eta\partial_\nu}\ \mathrm{nur}\ e^{\epsilon\partial_x})\, f & \textit{as a function of} & (\epsilon, f) \\ \textit{choice 4}: & (\mathrm{nur} - e^{-\eta\partial_\nu}\ \mathrm{nur}\ e^{\epsilon\partial_x})\, f & \textit{as a function of} & (\eta, f) \end{array}$$

but whichever choice we make (let us think of choice 3, for consistency) two basic differences emerge between *nir*'s and *nur*'s translocations:

(i) *Analytic difference*: the finite or infinitesimal increments $\nabla h(\epsilon,\nu)$ or $\delta h_m(\nu)$ defined as in §4.6 but with respect to *nur*, are no longer entire functions of their arguments, even when the driving function f is entire or polynomial. The reason for this is quite simple: with the *nir*-transform, to a shift ϵ in the x-plane there answers a well-defined shift $\eta = \int_0^\infty f(x)dx$ in the ν-plane, calculated from a well-defined determination of $f = -\log F$, but this no longer holds with the *nur*-transform, whose construction involves *all* determinations of f.

(ii) *Formal difference*: these increments still admit exact analytical expansions somewhat similar to (94) and (104) but the formulas are now more complex[50] and above all less useful. Indeed, the main point of these formulas in the *nir* version was to establish that the increments $\nabla h(\epsilon,\nu)$ or $\delta h_m(\nu)$ were entire functions of ϵ and ν, but with *nur* this is no longer the case, as was just pointed out.

5.7 Removal of the ingress factor.

As we saw, changing j_F into $j_F^\#$ brings rather different changes to the construction of the *inner* and *outer* generators: for the *inner* generators it means merging the critical stationary factor J_4 with the *egress* factor Eg_F ; for the *outer* generators it means pruning the critical stationary factor J of the *ingress* factor Ig_F. Nonetheless, the end effect is exactly the same: the parasitical summands $\mathfrak{b}_s^*(0)$ vanish from (58) and (124) alike.

5.8 Parity relations.

$$F^{\vdash}(x) := 1/F(-x) \quad , \quad f^{\vdash}(x) := -f(-x) \qquad \Longrightarrow$$

$$\mathrm{nur}(f^{\vdash})(\nu) = -\mathrm{nur}(f)(\nu) \qquad (\textit{tangency}\ \kappa = 0)$$

[50] with twisted equivalents of the convolution (100), under replacement of the factorials by q-factorials.

6 Inner generators and ordinary differential equations.

In some important instances, namely for *all* polynomial inputs f and *some* rational inputs F, the corresponding inner generators happen to verify ordinary differential equation of a rather simple type – *linear homogeneous with polynomial coefficients* – but often of high degree. These ODEs are interesting on three accounts
(i) they lead to an alternative, more classical derivation of the properties of these inner generators
(ii) they yield a precise description of their behaviour *over* ∞ in the ν-plane, i.e. *over* 0 in the ζ-plane.
(iii) they stand out, among similar-looking ODEs, as leading to a *rigid* resurgence pattern, with essentially *discrete* Stokes constants, insentitive to the *continuously* varying parameters.

6.1 "Variable" and "covariant" differential equations.

As usual, we consider four types of shift operators $\beta(\partial_\tau)$, relative to the choices

$$\begin{aligned}
&\textit{trivial choice} && \beta(\tau) := && \tau^{-1} && (127)\\
&\textit{standard choice} && \beta(\tau) := && \left(e^{\tau/2}-e^{-\tau/2}\right)^{-1} = \tau^{-1}-\frac{1}{24}\tau+\dots && (128)\\
&\textit{odd choice} && \beta(\tau) := && \tau^{-1}+\sum_{s\geq 0}\beta_{2\,s+1}\,\tau^{2\,s+1} && (129)\\
&\textit{general choice} && \beta(\tau) := && \tau^{-1}+\sum_{s\geq 0}\beta_{s}\,\tau^{s} && (130)
\end{aligned}$$

We then apply the *nir*-transform to a driving function f such that $f(0)=0$, with special emphasis on the case $f'(0)\neq 0$:

$$f(x) \;:=\; \sum_{1\leq s\leq r} f_s\, x^s \qquad (131)$$

$$\varphi(n,\tau) \;:=\; \beta(\tau)\, f(\frac{\tau}{n}) = \frac{1}{2}\frac{\tau^2}{n}\, f_1+\dots \quad \in \mathbb{C}[n^{-1},\tau] \qquad (132)$$

$$\varphi(n,\tau) \;:=\; \varphi^{+}(n,\tau)+\varphi^{-}(n,\tau) \quad \textit{with} \quad \varphi^{\pm}(n,\pm\tau)\equiv\pm\varphi^{\pm}(n,\tau) \qquad (133)$$

$$k(n) \;:=\; \Big[\int_0^{\infty}\exp^{\#}(\varphi(n,\tau))\;d\tau\Big]_{\text{singular}} \qquad (134)$$

$$\;:=\; \int_0^{\infty}\exp^{\#}(\varphi(n,\tau))\;\cosh_{\#}(\varphi^{-}(n,\tau))\;d\tau \quad (\textit{if}\;\; f_1\neq 0) \qquad (135)$$

$$\hat{k}(\nu) \;:=\; \Big[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k(n)\,e^{\nu\,n}\;dn\Big]_{\text{formal}} \;= h'(\nu) \qquad (136)$$

$$\widehat{k}(\nu) \;:=\; \Big[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k(n)\,e^{\nu\,n}\;\frac{dn}{n}\Big]_{\text{formal}} \;= h(\nu) \qquad (137)$$

But the case $f(0) \neq 0$ also matters, because it corresponds the so-called "exceptional" or "movable" generators. In that case the *nir*-transform produces no fractional powers. So we set:

$$f(x) := \sum_{0 \leq s \leq r} f_s\, x^s \tag{138}$$

$$k^{\text{total}}(n) \;:=\; \int_0^{\infty} \exp^{\#}(\varphi(n,\tau))\; d\tau \tag{139}$$

$$\hat{k}{}^{\text{total}}(\nu) \;:=\; \Big[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k^{\text{total}}(n)\, e^{\nu\, n}\; dn\Big]_{\text{formal}} \tag{140}$$

$$\overset{\frown}{k}{}^{\text{total}}(\nu) \;:=\; \Big[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k^{\text{total}}(n)\, e^{\nu\, n}\; \frac{dn}{n}\Big]_{\text{formal}} \tag{141}$$

The above definitions also extend to the case $f(0) = 0$. The *nir*-transform then produces a mixture of entire and fractional powers, and the index *total* affixed to k signals that we take them all.
For polynomial inputs, both k^{total} and k along with their Borel transforms verify remarkable linear-homogeneous ODEs. The ones verified by k^{total} are dubbed *variable* because there is no simple description of how they change when the base point changes in the x-plane (i.e. when the driving function undergoes a shift from f to ${}^{\epsilon}\!f$). The ODEs verified by k, on the other hand, deserve to be called *covariant*, for two reasons:
(i) when going from a proper base-point x_i to another proper base-point x_j (*proper* means that $f(x_i) = 0, f(x_j) = 0$), these covariant ODEs verified by $\hat{k}(\nu)$ simply undergo a shift $\nu = \int_{x_i}^{x_j} f(x)dx$ in the ν-plane.
(ii) there is a unique extension of the covariant ODE even to non-proper base-points x_i (i.e. when $f(x_i) \neq 0$), under the same formal covariance relation as above. That extension, of course, doesn't coincide with the *variable* ODE.[51]

"Variable" and "covariant" linear-homogeneous polynomial ODEs:
They are of the form:

$$\textit{variable ODE}: \quad P_v(n, -\partial_n)\, k^{\text{total}}(n) = 0 \;\Leftrightarrow\; P_v(\partial_\nu, \nu)\; \hat{k}{}^{\text{total}}(\nu) = 0 \tag{142}$$

$$\textit{covariant ODE}: \quad P_c(n, -\partial_n)\, k(n) = 0 \;\Leftrightarrow\; P_c(\partial_\nu, \nu)\; \hat{k}(\nu) = 0 \tag{143}$$

with polynomials

$$P_v(n,\nu) \;=\; \sum_{0\leq p\leq d}\sum_{0\leq q\leq \delta} \mathrm{dv_{p,q}}\; \mathrm{n^p}\, \nu^{\mathrm{q}} \tag{144}$$

$$P_c(n,\nu) \;=\; \sum_{0\leq p\leq d}\sum_{0\leq q\leq \delta} \mathrm{dc_{p,q}}\; \mathrm{n^p}\, \nu^{\mathrm{q}} \tag{145}$$

of degree d and δ in the non-commuting variables n and ν: $\quad [n,\nu] = 1$.
The covariance relation reads:

$$P_c^{\,{}^{\epsilon}\!f}(n, \nu - \eta) \equiv P_c^{f}(n,\nu)\;\forall \epsilon \quad \textit{with} \quad {}^{\epsilon}\!f(x) = f(x+\epsilon) \quad \textit{and} \quad \eta := \int_0^{\epsilon} f(x)dx \tag{146}$$

[51] For a proper base-point, on the other hand, the variable ODEs, though still distinct from the covariant ones, are *also* verified by k.

Existence and calculation of the variable ODEs.
For any $s \in \mathbb{N}$ let φ_s, ψ_s denote the polynomials in (n^{-1}, τ) characterised by the identities:

$$
\begin{aligned}
\partial_n^s\, k^{\text{total}}(n) \;&=\; \int_0^\infty \varphi_s(n,\tau)\ \exp^{\#}(\varphi(n,\tau))\ d\tau && (147)\\
\mathit{with} \quad & \varphi_s(n,\tau) \in \mathbb{C}[\partial_n\varphi, \partial_n^2\varphi, \dots, \partial_n^s\varphi] \in \mathbb{C}[n^{-1},\tau] && (148)\\
\int_0^\infty d_\tau^s\big(\tau^k\ \exp^{\#}(\varphi(n,\tau))\big) \;&=\; \int_0^\infty \psi_s(n,\tau)\ \exp^{\#}(\varphi(n,\tau))\ d\tau\ = 0 && (149)\\
\mathit{with} \quad & \psi_s(n,\tau) = \tau^s\, \partial_\tau\varphi(n,\tau) + s\,\tau^{s-1} \in \mathbb{C}[n^{-1},\tau] && (150)
\end{aligned}
$$

For δ, δ' large enough, the first polynomials $\{\varphi_s\,;\ s \le \delta\}$ and $\{\psi_s\,;\ s \le \delta'\}$ become linearly dependent on $\mathbb{C}[n^{-1}]$ or, what amounts to the same, on $\mathbb{C}[n]$. So we have relations of the form:

$$
0 = \sum_{0 \le s \le \delta} A_s(n)\,\varphi_s(n,\tau) + \sum_{0 \le s \le \delta'} B_s(n)\,\psi_s(n,\tau) \quad \mathit{with} \quad A(n), B(n) \in \mathbb{C}[n] \tag{151}
$$

and to each such relation there corresponds a linear ODE for k^{total}:

$$
\Big(\sum_{0 \le s \le \delta} A_s(n)\,\partial_n^s\Big)\ k^{\text{total}}(n) \ = \ 0 \tag{152}
$$

Existence and calculation of the covariant ODEs for $f(0) = 0$.
For each $s \in \mathbb{N}$ let $\varphi_s^{\pm}$ and $\psi_s^{\pm\pm}, \psi_s^{\pm\mp}$ denote the polynomials in (n^{-1}, τ) characterised by the identities:

$$
\begin{aligned}
\partial_n^s\ k(n) \ &= \ \int_0^\infty \Big(\varphi_s^+(n,\tau)\cosh(\varphi^-(n,\tau)) + \varphi_s^-(n,\tau)\sinh(\varphi^-(n,\tau))\Big)\, e^{\varphi^+(n,\tau)}\, d\tau\\
\mathit{with} \quad & \varphi_s^{\pm}(n,\tau) \in \mathbb{C}[\partial_n\varphi^+, \dots, \partial_n^s\varphi^+, \partial_n\varphi^-, \dots, \partial_n^s\varphi^-] \in \mathbb{C}[n^{-1},\tau]
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty d_\tau^s\big(\tau^k\ e^{\varphi^+(n,\tau)}\cosh(\varphi^-(n,\tau))\big) \ &= \ \int_0^\infty d_\tau^s\big(\tau^k\ e^{\varphi^+(n,\tau)}\sinh(\varphi^-(n,\tau))\big) \ = \ 0\\
&\mathit{with}\\
d_\tau^s\big(\tau^k\ e^{\varphi^+(n,\tau)}\cosh(\varphi^-(n,\tau))\big) \ &= \ +\varphi_s^{++}(n,\tau)\ e^{\varphi^+(n,\tau)}\cosh(\varphi^-(n,\tau)d\tau\\
&\quad\ +\varphi_s^{+-}(n,\tau)\ e^{\varphi^+(n,\tau)}\sinh(\varphi^-(n,\tau)d\tau\\
d_\tau^s\big(\tau^k\ e^{\varphi^+(n,\tau)}\sinh(\varphi^-(n,\tau))\big) \ &= \ +\varphi_s^{--}(n,\tau)\ e^{\varphi^+(n,\tau)}\cosh(\varphi^-(n,\tau)d\tau\\
&\quad\ +\varphi_s^{-+}(n,\tau)\ e^{\varphi^+(n,\tau)}\sinh(\varphi^-(n,\tau)d\tau
\end{aligned}
$$

Here again, for δ, δ' large enough, there are going to be dependence relations of the form:

$$
\begin{aligned}
0 \ &= \ \sum_{0 \le s \le \delta} A_s(n)\,\varphi^+(n,\tau) + \sum_{0 \le s \le \delta'} B_s(n)\,\psi^{++}(n,\tau) + \sum_{0 \le s \le \delta'} C_s(n)\,\psi^{++}(n,\tau) && (153)\\
0 \ &= \ \sum_{0 \le s \le \delta} A_s(n)\,\varphi^-(n,\tau) + \sum_{0 \le s \le \delta'} B_s(n)\,\psi^{+-}(n,\tau) + \sum_{0 \le s \le \delta'} C_s(n)\,\psi^{-+}(n,\tau) && (154)\\
\mathit{with} \quad & A(n), B(n), C(n) \in \mathbb{C}[n] && (155)
\end{aligned}
$$

and to each such relation there will corresponds a linear ODE for k :

$$\Big(\sum_{0\leq s\leq\delta} A_s(n)\,\partial_n^s\Big)\ k(n)\ =\ 0 \tag{156}$$

Remark : although the above construction applies, strictly speaking, only to the case of tangency $\kappa = 1$, i.e. to the case $f_0 = 0, f_1 \neq 0$, it is in fact universal. Indeed, if we set $f_0 = f_1 = \cdots = f_{\kappa-1}, f_\kappa \neq 0$ in the covariant ODEs thus found, we still get the correct covariant ODEs for a general tangency order $\kappa > 1$.

Existence and calculation of the covariant ODEs for $f(0) \neq 0$.
There are five steps to follow :
(i) Fix a degree r and calculate $P^f(n,\nu)$ by the above method for an arbitrary f of degree r such that $f(0) = 0$.
(ii) Drop the assumption $f(0) = 0$ but subject f to a shift ϵ such that ${}^\epsilon f(0) = f(\epsilon) = 0$ and apply (i) to calculate $P^{\,\epsilon f}(n,\nu)$ without actually solving the equation $f(\epsilon) = 0$ (keep ϵ as a free variable).
(iii) Calculate the ϵ-polynomial $P^{\,\epsilon f}(n,\nu - f^*(\epsilon))$ with $f^*(x) := \int_0^x f(t)dt$ as usual.
(iv) Divide it by the ϵ-polynomial $f(\epsilon)$ (momentarily assumed to be $\neq 0$) and calculate the remainder P_0 and quotient P_1 of that division :

$$P^{\,\epsilon f}(n,\nu - f^*(\epsilon)) =: P_0^f(n,\nu,\epsilon) + P_1^f(n,\nu,\epsilon)\, f(\epsilon)$$

(v) Use the covariance identity $P^{\,\epsilon f}(n,\nu - f^*(\epsilon)) \equiv P^f(n,\nu)\ \forall\epsilon$ to show that the remainder $P_0^f(n,\nu,\epsilon)$ is actually constant in ϵ. Then set

$$P^f(n,\nu) := P_0^f(n,\nu,0)$$

6.2 ODEs for polynomial inputs f. Main statements.

Dimensions of spaces of variable ODEs:
For $r := \deg(f)$ and for each pair $(x.y.)$ with
$x \in \{v,c\} = \{\mathit{variable}, \mathit{covariant}\}$
$y \in \{t,s,o,g\} = \{\mathit{trivial}, \mathit{standard}, \mathit{odd}, \mathit{general}\}$
the dimension of the corresponding space of ODEs is always of the form:

$$\dim_{\text{x.y.}}(r,d,\delta)\ \ \equiv\ \ \big(d - A_{x.y.}(r)\big)\big(\delta - B_{x.y.}(r)\big) - C_{x.y.}(r) \tag{157}$$

with δ(resp.d) denoting the differential order of the the ODEs in the n-variable (resp. in the ν-variable). Of special interest are the extremal pairs $(\underline{d},\overline{\delta})$ and $(\overline{d},\underline{\delta})$ with

$$\underline{d} = 1 + A_{x.y.}(r) \qquad\qquad \overline{\delta} = 1 + B_{x.y.}(r) + C_{x.y.}(r) \tag{158}$$

$$\overline{d} = 1 + A_{x.y.}(r) + C_{x.y.}(r) \qquad\qquad \underline{\delta} = 1 + B_{x.y.}(r) \tag{159}$$

($\underline{d}$ and $\underline{\delta}$ minimal; $\overline{d}$ and $\overline{\delta}$ co-minimal) because the corresponding dimension is exactly 1.

Dimensions of spaces of variable ODEs:

$$\begin{aligned}
\dim_{\text{v.t.}}(r,d,\delta) &= (d-r)\,(\delta-r-1) - \frac{1}{2}r^2 + \frac{1}{2}r - 1 \\
\dim_{\text{v.s.}}(r,d,\delta) &= (d-r)\,(\delta-r^2-2r+1) - \frac{1}{2}r^2(r+1) && (r\ \textit{even}) \\
&= (d-r)\,(\delta-r^2-2r) - \frac{1}{2}(r^3+r^2-5r+5) && (r\ \textit{odd}) \\
\dim_{\text{v.o.}}(r,d,\delta) &= (d-r)\,(\delta-r^2-2r+1) - \frac{1}{2}r^2(r+1) && (r\ \textit{even}) \\
&= (d-r)\,(\delta-r^2-2r) - \frac{1}{2}(r^3+r^2-3r+3) && (r\ \textit{odd} \neq 3) \\
\dim_{\text{v.g.}}(r,d,\delta) &= (d-r)\,(\delta-r^2-2r) - \frac{1}{2}r^2(r+1)
\end{aligned}$$

Dimensions of spaces of covariant ODEs:

$$\begin{aligned}
\dim_{\text{c.t.}}(r,d,\delta) &= (d-r+1)\,(\delta-r+1) - \frac{1}{2}(r-1)(r-2) \\
\dim_{\text{c.s.}}(r,d,\delta) &= (d-r+1)\,(\delta-r^2-r+1) - \frac{1}{2}r^2(r-1) && (r\ \textit{even}) \\
&= (d-r+1)\,(\delta-r^2-r+1) - \frac{1}{2}(r^2-5)(r-1) && (r\ \textit{odd}) \\
\dim_{\text{c.o.}}(r,d,\delta) &= (d-r+1)\,(\delta-r^2-r+1) - \frac{1}{2}r^2(r-1) && (r\ \textit{even}) \\
&= (d-r+1)\,(\delta-r^2-r+1) - \frac{1}{2}(r^2-3)(r-1) && (r\ \textit{odd} \neq 3) \\
\dim_{\text{c.g.}}(r,d,\delta) &= (d-r+1)\,(\delta-r^2-r+1) - \frac{1}{2}r^2(r-1)
\end{aligned}$$

Tables of dimensions for low degrees $r = \deg(f)$ **:**

degree r	*variable trivial*	*variable standard*	*variable odd*	*variable general*
1	$(\underline{d},\overline{\delta})$	$(\underline{d},\overline{\delta})$	$(\underline{d},\overline{\delta})$	$(\underline{d},\overline{\delta})$
1	$(2,2)$	$(2,4)$	$(2,4)$	$(2,5)$
2	$(3,4)$	$(3,14)$	$(3,14)$	$(3,15)$
3	$(4,7)$	$(4,28)$	$(4,28)$	$(4,34)$
4	$(5,11)$	$(5,64)$	$(5,64)$	$(5,65)$
5	$(6,16)$	$(6,100)$	$(6,104)$	$(6,111)$
6	$(7,22)$	$(7,174)$	$(7,174)$	$(7,175)$
7	$(8,29)$	$(8,244)$	$(8,250)$	$(8,260)$
8	$(9,37)$	$(9,368)$	$(9,368)$	$(9,369)$
9	$(10,46)$	$(10,484)$	$(10,492)$	$(10,505)$
10	$(11,56)$	$(11,670)$	$(11,670)$	$(11,671)$
...	...	...	...	...

degree r	*variable trivial*	*variable standard*	*variable odd*	*variable general*
1	$(\overline{d}, \underline{\delta})$	$(\overline{d}, \underline{\delta})$	$(\overline{d}, \underline{\delta})$	$(\overline{d}, \underline{\delta})$
1	$(3, 1)$	$(2, 4)$	$(2, 4)$	$(3, 4)$
2	$(5, 2)$	$(9, 8)$	$(9, 8)$	$(9, 9)$
3	$(8, 3)$	$(16, 16)$	$(16, 16)$	$(22, 16)$
4	$(12, 4)$	$(45, 24)$	$(45, 24)$	$(45, 25)$
5	$(17, 5)$	$(70, 36)$	$(74, 36)$	$(81, 36)$
6	$(23, 6)$	$(133, 48)$	$(133, 48)$	$(133, 49)$
7	$(30, 7)$	$(188, 64)$	$(194, 64)$	$(204, 64)$
8	$(38, 8)$	$(297, 80)$	$(297, 80)$	$(297, 81)$
9	$(47, 9)$	$(394, 100)$	$(402, 100)$	$(415, 100)$
10	$(57, 10)$	$(561, 120)$	$(561, 120)$	$(561, 121)$
...	...	...	...	...

degree r	*covariant trivial*	*covariant standard*	*covariant odd*	*covariant general*
1	$(\underline{d}, \overline{\delta})$	$(\underline{d}, \overline{\delta})$	$(\underline{d}, \overline{\delta})$	$(\underline{d}, \overline{\delta})$
1	$(1, 1)$	$(1, 2)$	$(1, 2)$	$(1, 2)$
2	$(2, 2)$	$(2, 8)$	$(2, 8)$	$(2, 8)$
3	$(3, 4)$	$(3, 16)$	$(3, 21)$	$(3, 21)$
4	$(4, 7)$	$(4, 44)$	$(4, 44)$	$(4, 44)$
5	$(5, 11)$	$(5, 70)$	$(5, 80)$	$(5, 80)$
6	$(6, 16)$	$(6, 132)$	$(6, 132)$	$(6, 132)$
7	$(7, 22)$	$(7, 188)$	$(7, 203)$	$(7, 203)$
8	$(8, 29)$	$(8, 296)$	$(8, 296)$	$(8, 296)$
9	$(9, 37)$	$(9, 394)$	$(9, 414)$	$(9, 414)$
10	$(10, 46)$	$(10, 560)$	$(10, 560)$	$(10, 560)$
...	...	...	...	...

degree r	*covariant trivial*	*covariant standard*	*covariant odd*	*covariant general*
1	$(\overline{d}, \underline{\delta})$	$(\overline{d}, \underline{\delta})$	$(\overline{d}, \underline{\delta})$	$(\overline{d}, \underline{\delta})$
1	$(1,1)$	$(1,2)$	$(1,2)$	$(1,2)$
2	$(2,2)$	$(4,6)$	$(4,6)$	$(4,6)$
3	$(4,3)$	$(7,12)$	$(7,12)$	$(12,12)$
4	$(7,4)$	$(28,20)$	$(28,20)$	$(28,20)$
5	$(11,5)$	$(45,30)$	$(49,30)$	$(55,30)$
6	$(16,6)$	$(96,42)$	$(96,42)$	$(96,42)$
7	$(22,7)$	$(139,56)$	$(145,56)$	$(154,56)$
8	$(29,8)$	$(232,72)$	$(232,72)$	$(232,72)$
9	$(37,9)$	$(313,90)$	$(321,90)$	$(333,90)$
10	$(46,10)$	$(460,110)$	$(460,110)$	$(460,110)$
...	...	...	...	...

Differential polynomial P in the noncommuting variables (n, ν).
Our differential operators will be written as polynomials $P(n, \nu)$ of degree (d, δ) in the non-commuting variables (n, ν), which are capable of two realisations:

$$(n, \nu) \longrightarrow (n, -\partial_n) \quad \textit{or} \quad (\partial_\nu, \nu)$$

Both realisation are of course compatible with $[n, \nu] = 1$ and the ODE interpretation goes like this:

$$P(n, -\partial_n)\, k(n) = 0 \qquad \Longleftrightarrow \qquad P(\partial_\nu, \nu) \hat{k} (\nu) = P(\partial_\nu, \nu)\, \partial_\nu\, h(\nu) = 0 \tag{160}$$

Compressing the covariant ODEs.
To get more manageable expressions, we can take advantage of the covariance relation to express everything in terms of shift-invariant data. This involves three steps:
(i) Apply the above the ODE-finding algorithm of §6.1 to a centered polynomial $f(x) = \sum_{i=0}^{r-2} f_i\, x^i + f_r\, x^r$.
(ii) Replace the coefficients $\{f_0, f_1, \ldots, f_{r-2}, f_r\}$ by the shift-invariants $\{\mathbf{f}_0, \mathbf{f}_1, \ldots, \mathbf{f}_{r-2}, \mathbf{f}_r\}$ defined in §6.2 *infra.*
(iii) Replace the β-coefficients by the 'centered' $\boldsymbol{\beta}$-coefficients defined *infra.*

Basic polynomials $f(x)$ and $p(\nu)$.

$$\begin{aligned} f(x) \;&=\; f_0 + f_1 x + \ldots f_r x^r \;=\; (x - x_1) \ldots (x - x_r)\, f_r \qquad (161)\\ p(\nu) \;&=\; p_0 + p_1 \nu + \ldots p_r \nu^r \;=\; (\nu - \nu_1) \ldots (\nu - \nu_r)\, p_r \qquad (162)\\ \textit{with} \quad \nu_i \;&=\; f^*(x_i) \;=\; \int_0^{x_i} f(x)\, dx = \sum_{0 \le s \le r} f_s\, \frac{x_i^{s+1}}{s+1} \qquad (163) \end{aligned}$$

The polynomials $p(\nu)$ are usually normalised by the condition $p_r = 1$ and their zeros ν_i are exactly the images under f^* of the zeros x_i of the input polynomial f. Since these ν_i correspond to the singular points of the inner generators in the ν-plane, we should expect the polynomials $p(\nu)$ to be a crucial ingredient of our ODEs. This is indeed the case – they will appear, predictably enough, as coefficients of the leading derivative.[52]

Basic symmetric functions $x_s^*, x_s^{**}, \nu_s^*, \nu_s^{**}$.

$$x_1^* := \sum_{1\le i\le r} x_i \quad , \quad x_2^* := \sum_{1\le i<j\le r} x_i\, x_j \quad , \ldots , \quad x_r^* := x_1 \ldots x_r \tag{164}$$

$$\nu_1^* := \sum_{1\le i\le r} \nu_i \quad , \quad \nu_2^* := \sum_{1\le i<j\le r} \nu_i\, \nu_j \quad , \ldots , \quad \nu_r^* := \nu_1 \ldots \nu_r \tag{165}$$

$$x_s^{**} := \sum_{1\le i\le r} x_i^s \qquad\qquad (\forall s \in \mathbb{N}) \tag{166}$$

$$\nu_s^{**} := \sum_{1\le i\le r} \nu_i^s \qquad\qquad (\forall s \in \mathbb{N}) \tag{167}$$

The change from the x-data to the ν-data goes like this:

$$\{f_s\} \longrightarrow \{x_s^*\} \xrightarrow{i} \{x_s^{**}\} \xrightarrow{ii} \{\nu_s^{**}\} \xrightarrow{iii} \{\nu_s^*\} \longrightarrow \{p_s\}$$

$$(i) \qquad \sum_{1\le s\le\infty} \frac{1}{s}\frac{x_s^{**}}{x^s} \equiv -\log\Big(1 + \sum_{1\le s\le r} (-1)^r \frac{x_s^*}{x^s}\Big)$$

$$(ii) \qquad \nu_s^{**} \equiv \sum_{s\le t\le(r+1)s} f_{s,t}^*\, x_t^{**} \qquad \textit{with} \qquad \sum_{s\le t\le(r+1)s} f_{s,t}^*\, x^t := (f^*(x))^s$$

$$(iii) \qquad 1 + \sum_{1\le s\le r} (-1)^r \frac{\nu_s^*}{\nu^s} \equiv \exp\Big(-\sum_{1\le s\le\infty} \frac{1}{s}\frac{\nu_s^{**}}{\nu^s}\Big)$$

Centered polynomials. Invariants.

$$\begin{aligned} x_0 &:= \frac{1}{r}(x_1 + \cdots + x_r) \;=\; -\frac{1}{r}\frac{f_{r-1}}{f_r} \\ \nu_0 &:= f^*(x_0) = \int_0^{x_0} f(x)\,dx \;=\; \sum_{0\le s\le r} f_s\, \frac{x_0^{s+1}}{s+1} \\ \underline{\nu}_0 &:= \frac{1}{r}(\nu_1 + \cdots + \nu_r) \;=\; -\frac{1}{r}\frac{p_{r-1}}{p_r} \qquad\qquad (\nu_0 \ne \underline{\nu}_0 \;\; \textit{in general}) \end{aligned}$$

$$\mathbf{f}(x) := f(x + x_0) = \sum_{0\le s\le r} \mathbf{f}_s\, x^s \quad (\mathbf{f}_{r-1} = 0)$$

$$\mathbf{p}(\nu) := p(\nu + \nu_0) = \sum_{0\le s\le r} \mathbf{p}_s\, \nu^s \qquad\qquad \mathbf{P}(\nu) := P(\nu + \nu_0)$$

$$\underline{\mathbf{p}}(\nu) := p(\nu + \underline{\nu}_0) = \sum_{0\le s\le r} \underline{\mathbf{p}}_s\, \nu^s \quad (\underline{\mathbf{p}}_{r-1} = 0) \qquad \underline{\mathbf{P}}(\nu) := P(\nu + \underline{\nu}_0)$$

[52] With the notations 160, this means that $p(\nu)$ is going to accompany the highest power of n in the non-commutative polynomial $P(n, \nu)$.

Centered β-coefficients :

$$\beta(\tau) = \tau^{-1} + \sum_{0\le k} \beta_k\, \tau^k = \tau^{-1}\left(1 + \sum_{1\le k} \frac{b_k}{k!}\tau^k\right) \tag{168}$$

$$1 + \sum_{2\le k} \frac{\mathbf{b}_k}{k!}\tau^k = \left(1 + \sum_{1\le k} \frac{b_k}{k!}\tau^k\right)\left(1 + \sum_{1\le k} \frac{(-b_1)^k}{k!}\tau^k\right) \tag{169}$$

$$\begin{aligned}
\mathbf{b}_1 &= 0 \;=\; 0\\
\mathbf{b}_2 &= b_2 - b_1^2 \;=\; 2\,\beta_1 - \beta_0^2\\
\mathbf{b}_3 &= b_3 - 3\,b_1b_2 + 2\,b_1^3 \;=\; 6\,\beta_2 - 6\,\beta_0\beta_1 + 2\,\beta_0^3\\
\mathbf{b}_4 &= b_4 - 4\,b_1b_3 + 6\,b_1^2b_2 - 3\,b_1^4 \;=\; 24\,\beta_3 - 24\,\beta_0\beta_2 + 12\,\beta_0^2\beta_1 - 3\,\beta_0^4\\
\mathbf{b}_5 &= b_5 - 5\,b_1b_4 + 10\,b_1^2b_3 - 10\,b_1^3b_2 + 4\,b_1^5\\
&= 120\,\beta_4 - 120\,\beta_0\beta_3 + 60\,\beta_0^2\beta_2 - 20\,\beta_0^3\beta_1 + 4\,\beta_0^5
\end{aligned}$$

Invariance and homogeneousness under $f(\bullet) \mapsto \lambda f(\gamma \bullet + \epsilon)$.
Invariance under $f(\bullet) \mapsto f(\bullet + \epsilon)$.

$$\begin{array}{rlll}
(x,n,\nu) & \overset{\partial_\epsilon}{\mapsto} \quad (1,0,-f_0) & & \\
\partial_\epsilon x_i = -1 & (1 \le i \le r) & & \partial_\epsilon x_0 = -1\\
\partial_\epsilon \nu_i = -f_0 & (1 \le i \le r) & & \partial_\epsilon \nu_0 = \partial_\epsilon \underline{\nu}_0 = -f_0\\
\partial_\epsilon f_s = (1+s) f_{1+s} & (0 \le s < r) & \partial_\epsilon f_r = 0 & \partial_\epsilon \mathbf{f}_s = 0 \;\; (0 \le s \le r)\\
\partial_\epsilon p_s = (1+s) p_{1+s} f_0 & (0 \le s < r) & \partial_\epsilon p_r = 0 & \partial_\epsilon \mathbf{p}_s = 0 \;\; (0 \le s \le r)
\end{array}$$

Homogeneousness under $f(\bullet) \mapsto f(\gamma\bullet)$.

$$\begin{aligned}
(x,n,\nu) &\mapsto (\gamma^{-1}x, \gamma n, \gamma^{-1}\nu)\\
(f_s, \mathbf{f}_s) &\mapsto (\gamma^s f_s, \gamma^s \mathbf{f}_s)\\
(p_s, \mathbf{p}_s) &\mapsto (\gamma^{s-r} p_s, \gamma^{s-r}\mathbf{p}_s)
\end{aligned}$$

Homogeneousness under $f(\bullet) \mapsto \lambda f(\bullet)$.

$$\begin{aligned}
(x,n,\nu) &\mapsto (x, \lambda^{-1}n, \lambda\nu)\\
(f_s, \mathbf{f}_s) &\mapsto (\lambda f_s, \lambda \mathbf{f}_s)\\
(p_s, \mathbf{p}_s) &\mapsto (\lambda^{r-s} p_s, \lambda^{r-s}\mathbf{p}_s)\\
\beta_{s-1} &\mapsto \lambda^{-s}\beta_{s-1}
\end{aligned}$$

6.3 Explicit ODEs for low-degree polynomial inputs f.

To avoid glutting this section, we shall restrict ourselves to the standard choice for β and mention only the covariant ODEs. [53] Concretely, for all values of the f-dregree r up to 4 we shall write down a complete set of *minimal* polynomials $P_{(d_i,\delta_i)}(n,\nu)$, of degrees

[53] But we keep extensive tables for all 8 cases $(v,c) \times (t,s,o,g)$ at the disposal of the interested reader.

(d_i, δ_i) in (n, ν), that generate all the other convariant polynomials by non-commutative pre-multiplication by covariant polynomials in (n, ν).[54]. For each r, the sequence

$$(\underline{d}, \overline{\delta})^1, \ldots, (d_i, \delta_i)^{m_i}, \ldots, (\overline{d}, \underline{\delta})^1$$

indicates the degrees (d_i, δ_i) of all minimal spaces with their dimensions m_i, i.e. the number of polynomials in them. For the extreme cases, right and left, that dimension is always 1.

Input f of degree 1.
Invariant coefficients: $\mathbf{f}_1 := f_1$
Covariant shift: $\nu_0 := -\frac{1}{2}\frac{f_0^2}{f_1}$
First leading polynomial (shifted): $\mathbf{p}(\nu) = p(\nu + \nu_0) = \nu$
Second leading polynomial: $\mathbf{q}(n) = n^2$
Covariant differential equations: $(1, 2)$

$$\mathbf{P}_{(1,2)}(n, \nu) = P_{(1,2)}(n, \nu+\nu_0) = n^2\,\nu + \frac{1}{2}\,n - \frac{1}{24}\,\mathbf{f}_1$$

Variable differential equations: $(2, 4)$

Input f of degree 2.
Invariant coefficients:

$$\mathbf{f}_0 = f_0 - \frac{1}{4}\frac{f_1^2}{f_2} \;=\; -\frac{1}{4}\,(x_1 - x_2)^2\, f_2 \quad , \quad \mathbf{f}_2 = f_2$$

Covariant shift:

$$\nu_0 = -\frac{1}{2}\frac{f_0 f_1}{f_2} + \frac{1}{12}\frac{f_1^3}{f_2^2} \;=\; -\frac{1}{12}\,(x_1 + x_2)(x_1^2 - 4x_1x_2 + x_2^2)\, f_2$$

Leading scalar factor:

$$\mathbf{f}_0 = -\frac{1}{4}\,(x_1 - x_2)^2\, f_2;$$

First leading polynomial (shifted)

$$\mathbf{p}(\nu) = p(\nu + \nu_0) = \frac{4}{9}\frac{\mathbf{f}_0^3}{\mathbf{f}_2} + \nu^2$$

Second leading polynomial:

$$\mathbf{q}(n) = \frac{1}{6}\frac{\mathbf{f}_2}{\mathbf{f}_0}\,n^6 + n^8$$

Covariant differential equations: $(2, 8), (3, 7)^2, (4, 6)$

$$\mathbf{P}_{(2,8)}(n, \nu) = P_{(2,8)}(n, \nu+\nu_0) = n^8\,\mathbf{f}_0\,\mathbf{p}(\nu) + n^7\,\mathbf{f}_0\,\nu + n^6\,(\frac{1}{6}\mathbf{f}_2\,\nu^2 + \frac{5}{27}\mathbf{f}_0^3 + \frac{8}{9}\mathbf{f}_0)$$
$$-n^5\,(\frac{1}{6}\mathbf{f}_2\,\nu) + n^4\,(\frac{1}{54}\mathbf{f}_0^2\,\mathbf{f}_2 - \frac{2}{27}\mathbf{f}_2) - n^2\,(\frac{1}{972}\mathbf{f}_0\,\mathbf{f}_2^2) - \frac{1}{583}\,\mathbf{f}_2^3$$

[54] In fact, all *variable* ODEs can also be expressed as suitable combinations of the minimal *covariant* ODEs.

$$\mathbf{P}_{(3,7)}(n,\nu) = P_{(3,7)}(n,\nu{+}\nu_0) = n^7\,\mathbf{f}_0\,\mathbf{p}(\nu) + n^6(-\frac{1}{32}\mathbf{f}_2\,\nu^3 - \frac{1}{72}\mathbf{f}_0^3\,\nu + \mathbf{f}_0\,\nu)$$
$$+n^5\,(\frac{7}{32}\mathbf{f}_2\,\nu^2 + \frac{8}{9}\mathbf{f}_0{+}\frac{2}{9}\mathbf{f}_0^3) + n^4\,(-\frac{1}{288}\mathbf{f}_2\,\mathbf{f}_0^2\nu - \frac{41}{288}\mathbf{f}_2\,\nu)$$
$$+n^3\,(\frac{7}{432}\mathbf{f}_0^2\,\mathbf{f}_2 - \frac{1}{18}\mathbf{f}_2) - n\,(\frac{11}{7776}\mathbf{f}_0\,\mathbf{f}_2^2) + \frac{1}{31104}\,\mathbf{f}_2^3\,\nu$$

$$\mathbf{P}^\dagger_{(3,7)}(n,\nu) = P^\dagger_{(3,7)}(n,\nu{+}\nu_0) = n^7\,\mathbf{f}_0\,\nu\,\mathbf{p}(\nu) + n^6\,(-\frac{5}{3}\mathbf{f}_0\,\nu^2 - \frac{32}{27}\frac{\mathbf{f}_0^4}{\mathbf{f}_2})$$
$$+n^5\,(-\frac{7}{9}\mathbf{f}_0\,\nu + \frac{1}{9}\mathbf{f}_0^3\,\nu) + n^4\,(\frac{2}{27}\mathbf{f}_0^3 - \frac{16}{27}\mathbf{f}_0) + n^2\,(\frac{1}{81}\mathbf{f}_0^2\,\mathbf{f}_2) - n\,(\frac{1}{972}\mathbf{f}_0\,\mathbf{f}_2^2\nu) - \frac{4}{729}\,\mathbf{f}_0\,\mathbf{f}_2^2$$

$$\mathbf{P}_{(4,6)}(n,\nu) = P_{(4,6)}(n,\nu{+}\nu_0) = n^6\,\mathbf{f}_0\,(\nu^2 + \frac{416}{3}\frac{\mathbf{f}_0}{\mathbf{f}_2})\,\mathbf{p}(\nu)$$
$$+n^5\,(-13\,\mathbf{f}_0\,\nu^3 + \frac{416}{3}\frac{\mathbf{f}_0^2}{\mathbf{f}_2}\,\nu - \frac{56}{9}\frac{\mathbf{f}_0^4}{\mathbf{f}_2}\,\nu) + n^4\,(\frac{356}{9}\mathbf{f}_0\,\nu^2 + \frac{1}{9}\mathbf{f}_0^3\,\nu^2 + \frac{3328}{27}\frac{\mathbf{f}_0^2}{\mathbf{f}_2} + \frac{1024}{27}\frac{\mathbf{f}_0^4}{\mathbf{f}_2})$$
$$+n^3\,(-\frac{148}{9}\mathbf{f}_0\,\nu - \frac{26}{27}\mathbf{f}_0^3\,\nu) + n^2\,(\frac{158}{81}\mathbf{f}_0^3 - \frac{16}{3}\mathbf{f}_0) + n\,(\frac{1}{81}\mathbf{f}_0^2\,\mathbf{f}_2\nu) - \frac{1}{972}\,\mathbf{f}_0\,\mathbf{f}_2^2\,\nu^2 - \frac{161}{729}\mathbf{f}_0^2\,\mathbf{f}_2$$

Variable differential equations : $(3,14), (4,11)^2, (5,10)^3, (6,9)^2, (9,8)$

Input f of degree 3.
Invariant coefficients :

$$\begin{aligned}
\mathbf{f}_0 &= f_0 - \frac{1}{3}\frac{f_1\,f_2}{f_3} + \frac{2}{27}\frac{f_2^3}{f_3^2} = \frac{1}{27}(x_1+x_2-2\,x_3)(x_2+x_3-2\,x_1)(x_3+x_1-2\,x_2)f_3\\
\mathbf{f}_1 &= f_1 - \frac{1}{3}\frac{f_2^2}{f_3} = -\frac{1}{3}(x_1^2+x_2^2+x_3^2-x_1x_2-x_2x_3-x_3x_1)f_3\\
\mathbf{f}_3 &= f_3
\end{aligned}$$

Covariant shift :

$$\begin{aligned}
\nu_0 &= -\frac{1}{3}\frac{f_0f_2}{f_3}/+\frac{1}{18}\frac{f_1f_2^2}{f_3^2} - \frac{1}{108}\frac{f_2^4}{f_3^3} = -\frac{1}{108}(x_1+x_2+x_3)(x_1^3+x_2^3+x_3^3+\\
&\quad 24\,x_1x_2x_3 - 3\,x_1x_2^2 - 3\,x_1^2x_2 - 3\,x_3x_1^2 - 3\,x_3^2x_1 - 3\,x_3x_2^2 - 3\,x_3^2x_2)f_3
\end{aligned}$$

Leading scalar factor :

$$\mathbf{a} := 4\,\mathbf{f}_1^3 + 27\,\mathbf{f}_0^2\,\mathbf{f}_3 = -(x_1-x_2)^2\,(x_2-x_3)^2\,(x_3-x_2)^2\,f_3^3$$

First leading polynomial (shifted)

$$\mathbf{p}(\nu) = p(\nu+\nu_0) = \frac{1}{32}\frac{\mathbf{f}_0^2\mathbf{f}_1^3}{\mathbf{f}_3^2} + \frac{27}{64}\frac{\mathbf{f}_0^4}{\mathbf{f}_3} + \Big(\frac{9}{8}\frac{\mathbf{f}_0^2\mathbf{f}_1}{\mathbf{f}_3} + \frac{1}{16}\frac{\mathbf{f}_1^4}{\mathbf{f}_3^2}\Big)\nu + \frac{1}{2}\frac{\mathbf{f}_1^2}{\mathbf{f}_3}\nu^2 + \nu^3$$

Second leading polynomial :

$$\mathbf{q}(n) = \frac{9}{4}\,\frac{\mathbf{f}_1\,\mathbf{f}_3^2}{\mathbf{a}}\,n^{12} + \frac{81}{4}\,\frac{\mathbf{f}_3^2}{\mathbf{a}}\,n^{13} + 6\,\frac{\mathbf{f}_1^2\,\mathbf{f}_3}{\mathbf{a}}\,n^{14} + 3\,\frac{\mathbf{f}_1\,\mathbf{f}_3}{\mathbf{a}}\,n^{15} + n^{16}$$

Covariant differential equations: $(3,16), (4,14)^2, (5,13)^2, (7,12)$

$$\mathbf{P}_{(3,16)}(n,\nu) = P_{(3,16)}(n,\nu+\nu_0) = \ \mathbf{a}\,\mathbf{p}(\nu) + O(n^{15})\,O(\nu^3)$$

$$\begin{aligned}
\mathbf{P}_{(4,14)}(n,\nu) \quad &= P_{(4,14)}(n,\nu+\nu_0) \quad &= \mathbf{f}_1\,\mathbf{a}\,\mathbf{b}\,n^{14}\,p(\nu) + O(n^{13})\,O(\nu^4) \\
\mathbf{P}^{\dagger}_{(4,14)}(n,\nu) \quad &= P^{\dagger}_{(4,14)}(n,\nu+\nu_0) \quad &= \mathbf{a}\,\mathbf{b}\,n^{14}\nu\,p(\nu) + O(n^{13})\,O(\nu^4)
\end{aligned}$$

with the following invariant coefficient $\mathbf{b}$:

$$\begin{aligned}
\mathbf{b} \ :=\ & 2097152\,\mathbf{f}_1^{12} - 766779696\,\mathbf{f}_1^3\,\mathbf{f}_3^3 - 520497152\,\mathbf{f}_1^9\,\mathbf{f}_3 - 36074005128\,\mathbf{f}_0^4\,\mathbf{f}_1^3\,\mathbf{f}_3^3 \\
&+ 1428879744\,\mathbf{f}_1^6\,\mathbf{f}_3^2 - 1314579456\,\mathbf{f}_0^6\mathbf{f}_1^3\,\mathbf{f}_3^3 + 1099865088\,\mathbf{f}_0^4\,\mathbf{f}_1^6\,\mathbf{f}_3^2 \\
&+ 205963264\,\mathbf{f}_1^9\,\mathbf{f}_0^2\,\mathbf{f}_3 - 8872609536\,\mathbf{f}_0^2\,\mathbf{f}_1^6\,\mathbf{f}_3^2 + 73222472421\,\mathbf{f}_0^4\,\mathbf{f}_3^4 \\
&+ 20602694736\,\mathbf{f}_0^2\,\mathbf{f}_1^3\,\mathbf{f}_3^3 + 5971968\,\mathbf{f}_0^6\,\mathbf{f}_1^6\,\mathbf{f}_3^2 + 884736\,\mathbf{f}_0^4\,\mathbf{f}_1^9\,\mathbf{f}_3 - 5165606520\,\mathbf{f}_0^2\,\mathbf{f}_3^4
\end{aligned}$$

$$\begin{aligned}
\mathbf{P}_{(5,13)}(n,\nu) &= P_{(5,13)}(n,\nu+\nu_0) \ &= n^{13}\,(\mathbf{f}_1^2\,\mathbf{c}_1 - 180\,\mathbf{f}_3\,\mathbf{c}_2\,\nu)\,\mathbf{p}(\nu) + O(n^{12})\,O(\nu^5) \\
\mathbf{P}^{\dagger}_{(5,13)}(n,\nu) &= P^{\dagger}_{(5,13)}(n,\nu+\nu_0) \ &= n^{13}\,(\mathbf{c}_3\,\nu + 180\,\mathbf{f}_1\,\mathbf{f}_3\,\mathbf{c}_1\,\nu^2)\,\mathbf{p}(\nu) + O(n^{12})\,O(\nu^5)
\end{aligned}$$

with the following invariant coefficients $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$:

$$\begin{aligned}
\mathbf{c}_1 \ :=\ & 917290620205793280\,\mathbf{f}_0^2\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 + 78717609050112\,\mathbf{f}_0^6\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 \\
&+ 4163751641088\,\mathbf{f}_0^4\,\mathbf{f}_1^{15}\,\mathbf{f}_3 + 50281437903388672\,\mathbf{f}_1^{15}\,\mathbf{f}_3 \\
&+ 1581069280739328\,\mathbf{f}_0^2\,\mathbf{f}_1^{15}\,\mathbf{f}_3 + 17755411807125504\,\mathbf{f}_0^4\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 \\
&+ 99407759207731200\,\mathbf{f}_0^6\,\mathbf{f}_1^9\,\mathbf{f}_3^3 + 5640800181652267776\,\mathbf{f}_0^4\,\mathbf{f}_1^9\,\mathbf{f}_3^3 \\
&+ 344140580192256\,\mathbf{f}_0^8\,\mathbf{f}_1^9\,\mathbf{f}_3^3 + 117266695506065704 32\,\mathbf{f}_0^6\,\mathbf{f}_1^6\,\mathbf{f}_3^4 \\
&+ 3265897813810421 76\,\mathbf{f}_0^8\,\mathbf{f}_1^6\,\mathbf{f}_3^4 - 4984963478435306 88\,\mathbf{f}_1^6\,\mathbf{f}_3^4 \\
&+ 16926659444736\,\mathbf{f}_0^{10}\,\mathbf{f}_1^6\,\mathbf{f}_3^4 - 85405328111733120\,\mathbf{f}_0^8\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
&- 1691608028258304\,\mathbf{f}_0^{10}\,\mathbf{f}_1^3\,\mathbf{f}_3^5 - 15390509185018432260\,\mathbf{f}_0^6\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
&+ 98766738625551624\,\mathbf{f}_0^8\,\mathbf{f}_3^6 - 743253702832987 8624\,\mathbf{f}_0^2\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
&+ 10331678048256\,\mathbf{f}_1^{18} - 27319961213550950355\,\mathbf{f}_0^4\,\mathbf{f}_3^6 \\
&+ 1500717585045441600\,\mathbf{f}_0^2\,\mathbf{f}_3^6 + 226960375516131600\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
&- 882586223845816 32\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 - 8253051882421560660\,\mathbf{f}_0^4\,\mathbf{f}_1^6\,\mathbf{f}_3^4 \\
&- 1478991931831367424\,\mathbf{f}_0^2\,\mathbf{f}_1^9\,\mathbf{f}_3^3 - 450793967617700928\,\mathbf{f}_3^3\,\mathbf{f}_1^9 \\
&+ 3821964710670454374\,\mathbf{f}_0^6\,\mathbf{f}_3^6 - 17792355610879876332\,\mathbf{f}_0^4\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
&- 5761805211034236864\,\mathbf{f}_0^2\,\mathbf{f}_1^6\,\mathbf{f}_3^4
\end{aligned}$$

$$
\begin{aligned}
\mathbf{c}_2 \;:=\; & 346056266653347168\,\mathbf{f}_0^2\,\mathbf{f}_1^3\,\mathbf{f}_3^5 - 561701191680\,\mathbf{f}_1^{18} \\
& - 12492140160024576\,\mathbf{f}_0^2\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 - 1568573227008\,\mathbf{f}_0^6\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 \\
& - 225501511680\,\mathbf{f}_0^4\,\mathbf{f}_1^{15}\,\mathbf{f}_3 - 812740325605376\,\mathbf{f}_1^{15}\,\mathbf{f}_3 \\
& - 83614219370496\,\mathbf{f}_0^2\,\mathbf{f}_1^{15}\,\mathbf{f}_3 - 1675724924436480\,\mathbf{f}_0^4\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 \\
& - 7670187447717888\,\mathbf{f}_0^6\,\mathbf{f}_1^9\,\mathbf{f}_3^3 - 50653665706906368\,\mathbf{f}_0^4\,\mathbf{f}_1^9\,\mathbf{f}_3^3 \\
& - 313456656384\,\mathbf{f}_0^8\,\mathbf{f}_1^9\,\mathbf{f}_3^3 - 23099325980303808\,\mathbf{f}_0^6\,\mathbf{f}_1^6\,\mathbf{f}_3^4 \\
& + 24015789981646272\,\mathbf{f}_1^6\,\mathbf{f}_3^4 + 28152325951488\,\mathbf{f}_0^8\,\mathbf{f}_1^6\,\mathbf{f}_3^4 \\
& - 64268410079232\,\mathbf{f}_0^8\,\mathbf{f}_1^3\,\mathbf{f}_3^5 - 24631997881011588\,\mathbf{f}_0^6\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
& + 1250788627474992675\,\mathbf{f}_0^4\,\mathbf{f}_3^6 + 308789626552560\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
& + 1527911696305152\,\mathbf{f}_1^{12}\,\mathbf{f}_3^2 + 60223210699403700\,\mathbf{f}_0^4\,\mathbf{f}_1^6\,\mathbf{f}_3^4 \\
& + 19863849419539200\,\mathbf{f}_0^2\,\mathbf{f}_1^9\,\mathbf{f}_3^3 + 18320806414937664\,\mathbf{f}_1^9\,\mathbf{f}_3^3 \\
& + 3706040377703682\,\mathbf{f}_0^6\,\mathbf{f}_3^6 + 945719068633781580\,\mathbf{f}_0^4\,\mathbf{f}_1^3\,\mathbf{f}_3^5 \\
& + 263746805911956288\,\mathbf{f}_0^2\,\mathbf{f}_1^6\,\mathbf{f}_3^4
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{c}_3 \;:=\; & 188945409245184\,\mathbf{f}_1^{21} - 4265434334643431940864\,\mathbf{f}_0^8\,\mathbf{f}_1^6\,\mathbf{f}_3^5 \\
& + 10475616970801152\,\mathbf{f}_0^8\,\mathbf{f}_1^{12}\,\mathbf{f}_3^3 - 640977986368479564057 6\,\mathbf{f}_0^2\,\mathbf{f}_1^{12}\,\mathbf{f}_3^3 \\
& + 164358597993366475238 4\,\mathbf{f}_0^4\,\mathbf{f}_1^{12}\,\mathbf{f}_3^3 + 31374503650787328\,\mathbf{f}_0^{10}\,\mathbf{f}_1^9\,\mathbf{f}_3^4 \\
& + 3977644977060256658880\,\mathbf{f}_0^6\,\mathbf{f}_1^9\,\mathbf{f}_3^4 + 1290160568497752489024\,\mathbf{f}_1^9\,\mathbf{f}_3^4 \\
& + 1205510242413389496768\,\mathbf{f}_0^2\,\mathbf{f}_1^9\,\mathbf{f}_3^4 - 48405699843949469920500\,\mathbf{f}_0^4\,\mathbf{f}_1^9\,\mathbf{f}_3^4 \\
& + 129087554262282601920\,\mathbf{f}_1^{12}\,\mathbf{f}_3^3 - 2806159668393991403 52\,\mathbf{f}_1^{15}\,\mathbf{f}_3^2 \\
& + 99734439900921569 28\,\mathbf{f}_0^6\,\mathbf{f}_1^{12}\,\mathbf{f}_3^3 - 1492256344300529883948\,\mathbf{f}_0^4\,\mathbf{f}_1^6\,\mathbf{f}_3^5 \\
& + 914039610015744\,\mathbf{f}_0^{12}\,\mathbf{f}_1^6\,\mathbf{f}_3^5 - 1229190332794475682146 04\,\mathbf{f}_0^6\,\mathbf{f}_1^6\,\mathbf{f}_3^5 \\
& + 2957452975344634 6752\,\mathbf{f}_0^8\,\mathbf{f}_1^9\,\mathbf{f}_3^4 + 132978955491577036 80\,\mathbf{f}_0^{10}\,\mathbf{f}_1^6\,\mathbf{f}_3^5 \\
& + 303119924027553952960 32\,\mathbf{f}_0^2\,\mathbf{f}_1^6\,\mathbf{f}_3^5 - 247635025475393075644 26\,\mathbf{f}_0^6\,\mathbf{f}_1^3\,\mathbf{f}_3^6 \\
& + 169418141509536645120\,\mathbf{f}_0^2\,\mathbf{f}_1^{15}\,\mathbf{f}_3^2 - 9086255417996490204 00\,\mathbf{f}_1^6\,\mathbf{f}_3^5 \\
& + 5603533051087967184\,\mathbf{f}_0^{10}\,\mathbf{f}_3^7 - 6783756426653340902 4\,\mathbf{f}_0^{10}\,\mathbf{f}_1^3\,\mathbf{f}_3^6 \\
& + 361388350697811710760 0\,\mathbf{f}_0^8\,\mathbf{f}_3^7 - 410258669811797597400 00\,\mathbf{f}_0^4\,\mathbf{f}_3^7 \\
& + 581286688880237080992900\,\mathbf{f}_0^6\,\mathbf{f}_3^7 - 1221065965233634266720 0\,\mathbf{f}_0^2\,\mathbf{f}_1^3\,\mathbf{f}_3^6 \\
& + 231937500459010111367085\,\mathbf{f}_0^4\,\mathbf{f}_1^3\,\mathbf{f}_3^6 - 1297732672262124504518 4\,\mathbf{f}_0^8\,\mathbf{f}_1^3\,\mathbf{f}_3^6 \\
& + 1049410426382106624\,\mathbf{f}_0^4\,\mathbf{f}_1^{15}\,\mathbf{f}_3^2 - 100601484577357824\,\mathbf{f}_0^{12}\,\mathbf{f}_1^3\,\mathbf{f}_3^6 \\
& + 27598162056708096\,\mathbf{f}_0^2\,\mathbf{f}_1^{18}\,\mathbf{f}_3 + 76397618921472\,\mathbf{f}_0^4\,\mathbf{f}_1^{18}\,\mathbf{f}_3 \\
& + 4008794200000102400\,\mathbf{f}_1^{18}\,\mathbf{f}_3 + 1381995569479680\,\mathbf{f}_0^6\,\mathbf{f}_1^{15}\,\mathbf{f}_3^2
\end{aligned}
$$

$$
\mathbf{P}_{(7,12)}(n,\nu) = P_{(7,12)}(n,\nu+\nu_0) = n^{12}p(\nu)\,(\mathbf{f}_1^4\,\nu^4 + \mathbf{d}_3\,\nu^3 + \mathbf{f}_1^2\,\mathbf{d}_2\,\nu^2 + \mathbf{f}_1\,\mathbf{d}_1\,\nu + \mathbf{d}_0) + O(n^{11})\,O(\nu^7)
$$

with the following invariant coefficients $\mathbf{d}_0, \mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$:

$$\begin{aligned}
\mathbf{d}_0 \;:=\; & +\frac{29859111}{128}\,\mathbf{f}_0^2 + \frac{3664683}{1600}\,\mathbf{f}_0^4 + \frac{29889}{4000}\,\mathbf{f}_0^6 + \frac{81}{10000}\,\mathbf{f}_0^8 + \frac{22240737}{640}\frac{\mathbf{f}_1^3}{\mathbf{f}_3}\\
& + \frac{336626989}{3200}\frac{\mathbf{f}_0^2\,\mathbf{f}_1^3}{\mathbf{f}_3} + \frac{3493333}{24000}\frac{\mathbf{f}_0^4\,\mathbf{f}_1^3}{\mathbf{f}_f 3} + \frac{159}{5000}\frac{\mathbf{f}_1^3\,\mathbf{f}_0^6}{\mathbf{f}_3} + \frac{1969}{60000}\frac{\mathbf{f}_0^4\,\mathbf{f}_1^6}{\mathbf{f}_3^2}\\
& + \frac{242977752829}{15552000}\frac{\mathbf{f}_1^6}{\mathbf{f}_3^2} + \frac{40541647}{1296000}\frac{\mathbf{f}_0^2\,\mathbf{f}_1^6}{\mathbf{f}_3^2} + \frac{15317}{4860000}\frac{\mathbf{f}_0^2\,\mathbf{f}_1^9}{\mathbf{f}_3^3}\\
& + \frac{203363491}{69984000}\frac{\mathbf{f}_1^9}{\mathbf{f}_3^3} + \frac{83521}{1049760000}\frac{\mathbf{f}_1^{12}}{\mathbf{f}_3^4}
\end{aligned}$$

$$\begin{aligned}
\mathbf{d}_1 \;:=\; & -\frac{368631}{160} + \frac{3305043}{800}\,\mathbf{f}_0^2 + \frac{93339}{2000}\,\mathbf{f}_0^4 + \frac{27}{250}\,\mathbf{f}_0^6 - \frac{642277459}{1296000}\frac{\mathbf{f}_1^3}{\mathbf{f}_3} + \frac{123}{500}\frac{\mathbf{f}_0^4\,\mathbf{f}_1^3}{\mathbf{f}_3}\\
& + \frac{10657943}{18000}\frac{\mathbf{f}_0^2\,\mathbf{f}_1^3}{\mathbf{f}_3} + \frac{697}{9000}\frac{\mathbf{f}_0^2\,\mathbf{f}_1^6}{\mathbf{f}_3^2} + \frac{101072021}{1944000}\frac{\mathbf{f}_1^6}{\mathbf{f}_3^2} + \frac{4913}{1458000}\frac{\mathbf{f}_1^9}{\mathbf{f}_3^3}
\end{aligned}$$

$$\mathbf{d}_2 \;:=\; +\frac{361809}{800} + \frac{10467}{200}\,\mathbf{f}_0^2 + \frac{27}{50}\,\mathbf{f}_0^4 + \frac{479929}{2160}\frac{\mathbf{f}_1^3}{\mathbf{f}_3} + \frac{29}{50}\frac{\mathbf{f}_0^2\mathbf{f}_1^3}{\mathbf{f}_3} + \frac{289}{5400}\frac{\mathbf{f}_1^6}{\mathbf{f}_3^2}$$

$$\mathbf{d}_3 \;:=\; -\frac{6561}{40}\,\mathbf{f}_3 - \frac{1347}{20}\,\mathbf{f}_1^3 + \frac{6}{5}\,\mathbf{f}_0^2\,\mathbf{f}_1^3 + \frac{17}{45}\frac{\mathbf{f}_1^6}{\mathbf{f}_3}$$

Variable differential equations :

$$(4,28), (5,22)^2, (6,20)^3, (7,19)^4, (8,18)^3, (10,17)^2, (16,16)$$

Input f of degree 4.
Invariant coefficients :

$$\begin{aligned}
\mathbf{f}_0 &= f0 - \frac{1}{4}\frac{f_1 f_3}{f4} + \frac{1}{16}\frac{f_2\,f_3^2}{f_4^2} - \frac{3}{256}\frac{f_3^4}{f_4^3} = \frac{1}{256}\prod_{i=1}^{i=4}(4x_i - x_1 - x_2 - x_3 - x_4)\\
\mathbf{f}_1 &= f_1 - \frac{1}{2}\frac{f_2 f_3}{f_4} + \frac{1}{8}\frac{f_3^3}{f_4^2} = -\frac{1}{8}(x_1+x_2-x_3-x_4)(x_2+x_3-x_1-x_4)(x_2+x_4-x_1-x_3)\,f_4\\
\mathbf{f}_2 &= f_2 - \frac{3}{8}\frac{f_3^2}{f_4} = -\frac{1}{8}\left(4(x_1^2+x_2^2+x_3^2+x_4^2) - (x_1+x_2+x_3+x_4)^2\right) f_4\\
\mathbf{f}_4 &= f_4
\end{aligned}$$

Covariant shift :

$$\nu_0 = -\frac{1}{4}\frac{f_0 f_3}{f_4} + \frac{1}{32}\frac{f_1 f_3^2}{f_4^2} - \frac{1}{192}\frac{f_2 f_3^3}{f_4^3} + \frac{1}{1280}\frac{f_3^5}{f_4^4}$$

Leading scalar factors:

$$
\begin{aligned}
\mathbf{a} \;&=\; 256\,\mathbf{f}_0^3\,\mathbf{f}_4^2 - 128\,\mathbf{f}_0^2\,\mathbf{f}_2^2\,\mathbf{f}_4 + 16\,\mathbf{f}_0\,\mathbf{f}_2^4 + 144\,\mathbf{f}_0\,\mathbf{f}_2\,\mathbf{f}_1^2\,\mathbf{f}_4 - 4\,\mathbf{f}_1^2\,\mathbf{f}_2^3 - 27\,\mathbf{f}_1^4\,\mathbf{f}_4 \\
&=\; \prod_{1\le i<j\le 4} (x_i - x_j)^2\, f_4^5 \\
\mathbf{b} \;&=\; -1280\,\mathbf{f}_2^6 + 32256\,\mathbf{f}_0\,\mathbf{f}_2^4\,\mathbf{f}_4 - 269568\,\mathbf{f}_0^2\,\mathbf{f}_2^2\,\mathbf{f}_4^2 + 746496\,\mathbf{f}_0^3\,\mathbf{f}_4^3 + 69984\,\mathbf{f}_0\,\mathbf{f}_1^2\,\mathbf{f}_2\,\mathbf{f}_4^2 \\
&\quad -9504\,\mathbf{f}_1^2\,\mathbf{f}_2^3\,\mathbf{f}_4 + 19683\,\mathbf{f}_1^4\,\mathbf{f}_4^2 \\
&=\; \prod_{\substack{1\le i<j\le 4\\ 1\le k<l\le 4}}^{(i,j)\neq(k,l)} \frac{1}{128}\,\Big(5\,(x_i+x_j-x_k-x_l)^2 + (x_i-x_j)^2 - 5\,(x_k-x_l)^2\Big)\, f_4^6
\end{aligned}
$$

First leading polynomial (shifted)

$$
\begin{aligned}
\mathbf{p}(\nu) \;=\;& \frac{12}{125}\frac{\mathbf{f}_0^3\mathbf{f}_1^2\mathbf{f}_2}{\mathbf{f}_4^2} - \frac{27}{2000}\frac{\mathbf{f}_0^2\mathbf{f}_1^4}{\mathbf{f}_4^2} + \frac{256}{625}\frac{\mathbf{f}_0^5}{\mathbf{f}_4} + \frac{16}{2025}\frac{\mathbf{f}_0^3\mathbf{f}_2^4}{\mathbf{f}_4^3} - \frac{128}{1125}\frac{\mathbf{f}_0^4\mathbf{f}_2^2}{\mathbf{f}_4^2} - \frac{1}{675}\frac{\mathbf{f}_0^2\mathbf{f}_1^2\mathbf{f}_2^3}{\mathbf{f}_4^3} \\
&+ \Big(\frac{32}{25}\frac{\mathbf{f}_0^3\mathbf{f}_1}{\mathbf{f}_4} - \frac{56}{225}\frac{\mathbf{f}_0^2\mathbf{f}_1\mathbf{f}_2^2}{\mathbf{f}_4^2} + \frac{21}{100}\frac{\mathbf{f}_0\mathbf{f}_1^3\mathbf{f}_2}{\mathbf{f}_4^2} - \frac{27}{1000}\frac{\mathbf{f}_1^5}{\mathbf{f}_4^2} + \frac{4}{225}\frac{\mathbf{f}_0\mathbf{f}_1\mathbf{f}_2^4}{\mathbf{f}_4^3} - \frac{2}{675}\frac{\mathbf{f}_1^3\mathbf{f}_2^3}{\mathbf{f}_4^3}\Big)\,\nu \\
&+ \Big(\frac{16}{15}\frac{\mathbf{f}_0^2\mathbf{f}_2}{\mathbf{f}_4} + \frac{9}{10}\frac{\mathbf{f}_0\mathbf{f}_1^2}{\mathbf{f}_4} + \frac{11}{60}\frac{\mathbf{f}_1^2\mathbf{f}_2^2}{\mathbf{f}_4^2} - \frac{4}{15}\frac{\mathbf{f}_0\mathbf{f}_2^3}{\mathbf{f}_4^2} + \frac{4}{225}\frac{\mathbf{f}_2^5}{\mathbf{f}_4^3}\Big)\,\nu^2 + \frac{\mathbf{f}_1\mathbf{f}_2}{\mathbf{f}_4}\,\nu^3 + \nu^4
\end{aligned}
$$

Second leading polynomial:

$$
\mathbf{q}(n) = -\frac{2^{14}\,7\,\mathbf{f}_4^{11}}{3^3\,5^6\,\mathbf{ab}}\,n^{20} - \frac{2^{13}\,11}{3^2\,5^5}\,\frac{\mathbf{f}_2\,\mathbf{f}_4^{10}}{\mathbf{a}\,\mathbf{b}}\,n^{22} + \cdots + 8\,\Big(\frac{\mathbf{f}_1\,\mathbf{f}_2^2\,\mathbf{f}_4}{\mathbf{a}} + 12\,\frac{\mathbf{f}_0\,\mathbf{f}_1\,\mathbf{f}_4^2}{\mathbf{a}}\Big)\,n^{43} + n^{44}
$$

Covariant differential equations:

$$
(4,44), (5,32)^2, (6,28)^3, (7,26)^4, (8,24), (10,23)^4, (11,22)^3, (16,21)^2, (28,20)
$$

$$
\mathbf{P}_{(4,44)}(n,\nu) = \mathbf{P}_{(4,44)}(n,\nu{+}\nu_0) = \mathbf{a}\,\mathbf{b}\,n^{44}\,\mathbf{p}(\nu) + O(n^{43})\,O(\nu^4)
$$

Variable differential equations:

$$
\begin{gathered}
(5,64), (6,44)^2, (7,37)^2, (8,34)^4, (9,32)^5, (10,30)^2, \\
(11,29)^2, (13,28)^5, (15,27)^4, (18,26)^2, (25,25)^2, (45,24)
\end{gathered}
$$

6.4 The global resurgence picture for polynomial inputs f.

The covariant ODEs enable us to describe the exact singular behaviour of $\hat{k}(\nu) = h(\nu)$ at infinity in the ν-plane, and by way of consequence all singularities *over* 0 in the ζ-plane. In the ν-plane, the singularities in question consist of linear combinations of rather elementary exponential factors multiplied by series in negative powers of ν. These are always divergent, resurgent, and resummable. The case of radial inputs f (i.e. $f(x) = f_r\,x^r$) is predictably much simpler and deserves special mention. We find:

$$
\Big(\sum_{r+1\le k} c_s(\omega)\,\nu^{-\frac{s}{r+1}}\Big)\quad \exp\Big(\omega\,\nu^{\frac{r}{r+1}}\Big) \qquad\qquad (\textit{for radial } f) \tag{170}
$$

$$
\Big(\sum_{r+1\le k} c_s(\omega)\,\nu^{-\frac{s}{r+1}}\Big)\quad \exp\Big(\omega\,\nu^{\frac{r}{r+1}} + \sum_{s=1}^{r-2}\omega_s\,\nu^{\frac{s}{r+1}}\Big) \qquad\qquad (\textit{for general } f) \tag{171}
$$

The "leading" frequencies ω featuring in the exponential factors depend only on the leading coefficient f_r of f. Via the variable θ thus defined:

$$\theta := \Big(\frac{r+1}{r}\Big)^r \;\frac{\mathbf{f}_r}{\omega^{r+1}} \;=\; \Big(\frac{r+1}{r}\Big)^r \;\frac{f_r}{\omega^{r+1}} \tag{172}$$

the leading frequencies ω correspond, for each degree r, to the roots of the following polynomials $\boldsymbol{\pi}_r(\theta)$ of degree r:

$$\begin{aligned}
\boldsymbol{\pi}_1(\theta) &= -12+\theta\\
\boldsymbol{\pi}_2(\theta) &= -432+\theta^2 = -2^4\,3^3+\theta^2\\
\boldsymbol{\pi}_3(\theta) &= (240+7\,\theta)\,(-30+\theta)^2 = (2^4\,3\times 5+7\,\theta)\,(-2\times 3\times 5+\theta)^2\\
\boldsymbol{\pi}_4(\theta) &= (1749600000-1620000\,\theta^2+343\,\theta^4)\\
&= (2^8\,3^7\,5^5-2^5\,3^4\,5^4\,\theta^2+7^3\,\theta^4)\\
\boldsymbol{\pi}_5(\theta) &= (-1344+31\,\theta)\,(189+\theta)^2\,(42+\theta)^2\\
&= (-2^6\,3\times 7+31\,\theta)\,(3^3\,7+\theta)^2\,(2\times 3\times 7+\theta)^2\\
\boldsymbol{\pi}_6(\theta) &= (-66395327975424+152320630896\,\theta^2-116688600\,\theta^4+29791\,\theta^6)\\
&= (-2^{12}\,3^9\,7^7+2^4\,3^7\,7^6\,37\,\theta^2-2^3\,3^5\,5^2\,7^4\,\theta^4+31^3\,\theta^6)\\
\boldsymbol{\pi}_7(\theta) &= (3840+127\,\theta)\,(-30+\theta)^2\,(24300+1080\,\theta+37\,\theta^2)^2\\
&= (2^8\,3\times 5+127\,\theta)\,(-2\times 3\times 5+\theta)^2\,(2^2\,3^5\,5^2+2^3\,3^3\,5\,\theta+37\,\theta^2)^2
\end{aligned}$$

For a non-standard choice of β and with the 'centered' coefficients $\mathbf{b}_i$ introduced at the end of §6.2, these polynomials $\boldsymbol{\pi}_r(\theta)$ become:

$$\begin{aligned}
\boldsymbol{\pi}_1(\theta) &= 1+\mathbf{b}_2\,\theta\\
\boldsymbol{\pi}_2\,(\theta) &= 1+2\,\mathbf{b}_3\theta+(\mathbf{b}_3^2+4\,\mathbf{b}_2^3)\,\theta^2\\
\boldsymbol{\pi}_3(\theta) &= 1+3\,(\mathbf{b}_4-6\,\mathbf{b}_2^2)\,\theta+3\,(\mathbf{b}_4^2+18\,\mathbf{b}_2\mathbf{b}_3^2-12\,\mathbf{b}_2^2\mathbf{b}_4+27\,\mathbf{b}_2^4)\,\theta^2\\
&\quad+(\mathbf{b}_4^3-27\,\mathbf{b}_3^4+54\,\mathbf{b}_2\mathbf{b}_3^2\mathbf{b}_4-18\,\mathbf{b}_2^2\mathbf{b}_4^2-54\,\mathbf{b}_2^3\mathbf{b}_3^2+81\,\mathbf{b}_2^4\mathbf{b}_4)\,\theta^3\\
\boldsymbol{\pi}_4(\theta) &= 1+4\,(\mathbf{b}_5-30\,\mathbf{b}_2\mathbf{b}_3)\,\theta+2\,(3\,\mathbf{b}_5^2+80\,\mathbf{b}_2\mathbf{b}_4^2+180\,\mathbf{b}_3^2\,\mathbf{b}_4-180\,\mathbf{b}_2\mathbf{b}_3\mathbf{b}_5\\
&\quad+1320\,\mathbf{b}_2^2\mathbf{b}_3^2-720\,\mathbf{b}_2^3\mathbf{b}_4+1728\,\mathbf{b}_2^5)\,\theta^2+4\,(\mathbf{b}_5^3-160\,\mathbf{b}_3\mathbf{b}_4^3\\
&\quad-90\,\mathbf{b}_2\mathbf{b}_3\mathbf{b}_5^2+180\,\mathbf{b}_3^2\mathbf{b}_4\mathbf{b}_5+80\,\mathbf{b}_2\mathbf{b}_4^2\mathbf{b}_5+1120\,\mathbf{b}_2^2\mathbf{b}_3\mathbf{b}_4^2+864\,\mathbf{b}_3^5\\
&\quad-2520\,\mathbf{b}_2\mathbf{b}_3^3\mathbf{b}_4+1320\,\mathbf{b}_2^2\mathbf{b}_3^2\mathbf{b}_5-720\,\mathbf{b}_2^3\mathbf{b}_4\mathbf{b}_5+1728\,\mathbf{b}_2^5\mathbf{b}_5+1280\,\mathbf{b}_2^3\mathbf{b}_3^3\\
&\quad-2880\,\mathbf{b}_2^4\mathbf{b}_3\mathbf{b}_4)\,\theta^3+(\mathbf{b}_5^4-120\,\mathbf{b}_2\mathbf{b}_3\mathbf{b}_5^3+160\,\mathbf{b}_2\mathbf{b}_4^2\mathbf{b}_5^2+360\,\mathbf{b}_3^2\mathbf{b}_4\mathbf{b}_5^2\\
&\quad-640\,\mathbf{b}_3\mathbf{b}_4^3\mathbf{b}_5+256\,\mathbf{b}_4^5-2560\,\mathbf{b}_2^2\mathbf{b}_4^4+3456\,\mathbf{b}_3^5\mathbf{b}_5+5760\,\mathbf{b}_2\mathbf{b}_3^2\mathbf{b}_4^3\\
&\quad+2640\,\mathbf{b}_2^2\mathbf{b}_3^2\mathbf{b}_5^2-1440\,\mathbf{b}_2^3\mathbf{b}_4\mathbf{b}_5^2-2160\,\mathbf{b}_3^4\mathbf{b}_4^2+4480\,\mathbf{b}_2^2\mathbf{b}_3\mathbf{b}_4^2\mathbf{b}_5\\
&\quad-10080\,\mathbf{b}_2\mathbf{b}_3^3\mathbf{b}_4\mathbf{b}_5+3456\,\mathbf{b}_2^5\mathbf{b}_5^2-3200\,\mathbf{b}_2^3\mathbf{b}_3^2\mathbf{b}_4^2+5120\,\mathbf{b}_2^3\mathbf{b}_3^3\mathbf{b}_5\\
&\quad+6400\,\mathbf{b}_2^4\mathbf{b}_4^3-11520\,\mathbf{b}_2^4\mathbf{b}_3\mathbf{b}_4\mathbf{b}_5)\,\theta^4
\end{aligned}$$

Thus, for $r=1$ we get two basic singular summands:

$$\Big(\sum_{2\leq k} c_s(\omega)\,\nu^{-\frac{s}{2}}\Big)\,\exp\Big(\omega\,\nu^{\frac{1}{2}}\Big) \qquad \textit{(for all } f \textit{ of degree 1)} \tag{173}$$

$$\Big(\nu^{-1}-\frac{1}{\omega}\,\nu^{-\frac{1}{2}}\Big)\,\exp\Big(\omega\,\nu^{\frac{1}{2}}\Big) \qquad \textit{(if } f(x)=f_1\,x) \tag{174}$$

with frequencies ω corresponding to the solutions of $\boldsymbol{\pi}_1(\theta) = 0$ i.e. $\boldsymbol{\pi}_1(2\,\frac{\mathbf{f}_1}{\omega^2}) = 0$ i.e. $\omega = (-2\,\mathbf{b}_2 f_1)^{\frac{1}{2}}$.

For $r = 2$ we have $6 = 2 \times 3$ basic summands

$$\Big(\sum_{3\leq k} c_s(\omega)\,\nu^{-\frac{s}{3}}\Big)\exp\Big(\omega\,\nu^{\frac{2}{3}}\Big) \qquad \textit{(for all } f \textit{ of degree 2)} \tag{175}$$

with frequencies ω corresponding to the solutions of $\boldsymbol{\pi}_2(\theta) = 0$ i.e. $\boldsymbol{\pi}_2(\frac{9}{4}\,\frac{\mathbf{f}_2}{\omega^3}) = 0$.

For $r = 3$ we have $12 = 3 \times 4$ basic summands

$$\Big(\sum_{4\leq k} c_s(\omega)\,\nu^{-\frac{s}{4}}\Big)\exp\Big(\omega\,\nu^{\frac{3}{4}}\Big) \qquad \textit{(for all radial } f \textit{ of degree 3)} \tag{176}$$

$$\Big(\sum_{4\leq k} c_s(\omega)\,\nu^{-\frac{s}{4}}\Big)\exp\Big(\omega\,\nu^{\frac{3}{4}} + \omega_1\,\nu^{\frac{1}{4}}\Big) \qquad \textit{(for all } f \textit{ of degree 3)} \tag{177}$$

with main frequencies ω solution of $\boldsymbol{\pi}_3(\theta) = 0$ i.e. $\boldsymbol{\pi}_3(\frac{64}{27}\,\frac{\mathbf{f}_3}{\omega^4}) = 0$, and with secondary frequencies ω_1 dependent on the main ones and given by:

$$\omega_1 = \frac{2}{3}\,\frac{\mathbf{f}_1}{\omega}\,\frac{\big(1 + (\mathbf{b}_4 + 3\,\mathbf{b}_2^2)\,\theta\big)\,\big(\mathbf{b}_2 + (\mathbf{b}_2\mathbf{b}_4 - 3\,\mathbf{b}_3^2 - 9\,\mathbf{b}_2^3)\,\theta\big)}{\big(1 + (\mathbf{b}_4 - 6\,\mathbf{b}_2^2)\,\theta\big)^2 + 9\,\mathbf{b}_2\,\big(2\,\mathbf{b}_3^2 - \mathbf{b}_2^3\big)\,\theta^2} \tag{178}$$

$$\textit{with} \qquad \mathbf{f}_1 = f_1 - \frac{1}{3}\,\frac{f_2^2}{f_3}\ \ ,\ \ \mathbf{f}_3 = f_3\ \ \textit{and}\ \ \theta = \Big(\frac{4}{3}\Big)^3\,\frac{\mathbf{f}_3}{\omega^4} \tag{179}$$

Lastly, for $r = 4$ we have $20 = 4 \times 5$ basic summands

$$\Big(\sum_{5\leq k} c_s(\omega)\,\nu^{-\frac{s}{5}}\Big)\exp\Big(\omega\,\nu^{\frac{4}{5}}\Big) \qquad \textit{(for all radial } f \textit{ of degree 4)} \tag{180}$$

$$\Big(\sum_{4\leq k} c_s(\omega)\,\nu^{-\frac{s}{5}}\Big)\exp\Big(\omega\,\nu^{\frac{4}{5}} + \omega_2\,\nu^{\frac{2}{4}} + \omega_1\,\nu^{\frac{1}{4}}\Big) \qquad \textit{(for all } f \textit{ of degree 4)} \tag{181}$$

with main frequencies ω solution of $\boldsymbol{\pi}_4(\theta) = 0$ i.e. $\boldsymbol{\pi}_4(\frac{625}{256}\,\frac{\mathbf{f}_4}{\omega^5}) = 0$, and with secondary frequencies ω_1, ω_2 that depend on the main ones and vanish *iff* the shift-invariants $\mathbf{f}_1$ resp. $\mathbf{f}_2$ vanish.

6.5 The antipodal exchange for polynomial inputs f.

As noted in the preceding subsection, the behaviour of our *nir*-transforms $h(\nu)$ at infinity in the ν-plane involves elementary exponential factors multiplied by divergent-resurgent power series $\varphi_\omega(\nu) = \sum_{r+1\leq k} c_s(\omega)\,\nu^{-\frac{s}{r+1}}$, which verify simple linear ODEs easily deducible from the frequencies ω and the original ODE verified by $h(\nu)$. Therefore, to resum the $\varphi_\omega(\nu)$, which are local data at infinity, we must subject them to a formal Borel transform, which takes us back to the origin, with a new set of linear ODEs. This kicks off a resurgence ping-pong between 0 and ∞. [55] Before taking a closer look at it, let us state a useful lemma:

Lemma 6.1 (Deramification of linear homogeneous ODEs) .
Let ρ be a positive integer and $\Phi(t)$ any power series in $\mathbb{C}\{t^{\frac{1}{\rho}}\}$ or $\mathbb{C}\{t^{-\frac{1}{\rho}}\}$ that verifies a linear homogeneous differential equation $P^*(t,\partial_t)\,\Phi(t)=0$ of order δ^* and with coefficients polynomial in $t^{\frac{1}{\rho}}$ of degree d^*. Then Φ automatically verifies a new linear homogeneous differential equation $P(t,\partial_t)\,\Phi(t)=0$ of order δ and with coefficients polynomial in t of degree d such that

$$\delta \leq \delta^*\,\rho \quad , \quad d \leq (1+d^*)(1+\delta^*\,(\rho-1))^2$$

Proof: The initial, ramified differential equation, after division by the leading coefficient and deramification of the denominators, can be written uniquely in the form

$$\Phi^{(\delta^*)} = \sum_{0\leq j<\rho}\ \sum_{0\leq s<\delta^*} a_{\delta^*,j,s}\, t^{\frac{j}{\rho}}\, \Phi^{(s)} \tag{182}$$

with unramified coefficients $a_{\delta^*,j,s}$ that are rational in t. Under successive differentiations and eliminations of the derivatives of order larger than δ^* but $\neq i$, we then get a sequence of similar-looking equations:

$$\Phi^{(i)} = \sum_{0\leq j<\rho}\ \sum_{0\leq s<\delta^*} a_{i,j,s}\, t^{\frac{j}{\rho}}\, \Phi^{(s)} \qquad (\forall i, \delta^*\leq i\leq \delta^*\rho) \tag{183}$$

again with unramified coefficients $a_{\delta^*,j,s}$ rational in t. One then checks that there always exists a linear combination of the $(\rho-1)\,\delta^*$ equations (183) with coefficients $L_i(t)$ polynomial in the $a_{i',j',s'}(t)$ and therefore rational in t, that eliminates the (at most) $(\rho-1)\,\delta^*$ terms of the form $t^{\frac{j}{\rho}}$ with $1\leq j<\rho$ and $0\leq s<\delta^*$. After multiplication by a suitable t-polynomial, this yields the required unramified equation $P(t,\partial_t)\Phi(t)=0$. A closer examination of the process shows that the coefficients a are of the form:

$$a_{i,j,s}(t) = \frac{b_{i,j,s}(t)}{t^{i-\delta^*}\,c(t)^{1+i-\delta^*}} \ \textit{with}\ \deg_t(c)\leq d^*\ ,\ \deg_t(b_{i,j,s})\leq (1+i-\delta^*)\,d^*$$

Plugging this into the elimination algorithm, we get the bound

$$\begin{array}{rcl} d & \leq & (1+(\rho-1)\delta^*)\,(d^*+(\rho-1)(d^*+1)\delta^*) \quad \Longleftrightarrow \\ (1+d) & \leq & (1+d^*)\,(1+\delta^*\,\rho^*)^2 \qquad \textit{with} \qquad \rho^* := \rho-1 \end{array}$$

which, barring unlikely simplifications, is probably near-optimal. $\Box$.
Let us now return to the resurgence ping-pong $0 \leftrightarrow \infty$. Graphically, we get the following sequence of transforms:

$$\begin{tikzcd}
(P_1,k_1,n_1) \arrow[rrrr, bend left] \arrow[dr, "12"] & (P_3^*,k_3,n_3) \arrow[r, "33"] & (P_3,k_3,n_3) \arrow[dr, "34"] & (P_5^*,k_5,n_5) \arrow[r, "55"] & (P_5,k_5,n_5) \\
 & (P_2,k_2,\nu_2) \arrow[u, "23"'] & & (P_4,k_4,\nu_4) \arrow[u, "45"'] &
\end{tikzcd}$$

[55] which is quite distinct from the ping-pong between two inner generators associated with two proper base points x_i, x_j in the x-plane.

Step 1: we have the polynomial-coefficient linear ODE $P_1(n_1, \partial_{n_1})\, k_1(n_1) = 0$ with

$$n_1 \equiv n \sim \infty, k_1(n_1) \equiv k(n), P_1(n_1, \partial_{n_1}) \equiv P(n_1, -\partial_{n_1})$$

Arrow 12: we perform the Borel transform from the variable $n_1 = n$ to the conjugate variable $\nu_2 = \nu$. Thus : $n_1^{-s} \mapsto \frac{\nu_2^{s-1}}{\Gamma(s)}, n_1 \mapsto \partial_{\nu_2}, \partial_{n_1} \mapsto -\nu_2$.

Step 2: we have the polynomial-coefficient linear ODE $P_2(\nu_2, \partial_{\nu_2})\, k_2(\nu_2) = 0$ with

$$\nu_2 \equiv \nu \sim 0, k_2(\nu_2) \equiv \hat{k}(\nu), P_2(\nu_2, \partial_{\nu_2}) \equiv P_1(\partial_{\nu_2}, -\nu_2)$$

Arrow 23: we go from 0 to ∞, that is to say, we now solve the above ODE in powers series of negative powers of ν_2. More precisely, for an input f of degree r, we set $n_3 := \nu_2^{\frac{r}{r+1}} =: \nu_2^{\frac{1}{\kappa_3}}$, the new variable n_3 being the "critical resurgence variable" at ∞, and we then solve the ODE in negative powers of n_3.

Step 3*: we have the ramified-coefficient linear ODE $P_3^*(n_3, \partial_{n_3})\, k_3(n_3) = 0$ with

$$n_3^{\kappa_3} \equiv \nu_2 \ \ \textit{but}\ n_3 \sim \infty\,,\ k_3(n_3) \equiv k_2(\nu_2),\ P_3^*(n_3, \partial_{n_3}) \equiv P_2(n_3^{\kappa_3}, \frac{n_3^{\kappa_3 - 1}}{\kappa_3}\partial_{n_3})$$

Arrow 33: since for an input f of degree r, we must take $\kappa_3 = \frac{r+1}{r}$, this leads to a ramification of order r in the coefficients of P_3^*. We then apply the above Lemma 6.1 with $\rho = r$ to deramify P_3^* to P_3.

Step 3: we have the polynomial-coefficient linear ODE $P_3(n_3, \partial_{n_3})\, k_3(n_3) = 0$ with n_3 and k_3 as in step 3* but with a linear homogeneous differential operator P_3 which, unlike P_3^*, is polynomial in n_3.

Arrow 34: we perform the Borel transform from the variable n_3 to the conjugate variable ν_4. Thus : $n_3^{-s} \mapsto \frac{\nu_4^{s-1}}{\Gamma(s)}, n_3 \mapsto \partial_{\nu_4}, \partial_{n_3} \mapsto -\nu_4$.

Step 4: we have the polynomial-coefficient linear ODE $P_4(\nu_4, \partial_{\nu_4})\, k_4(\nu_4) = 0$ with ν_4 conjugate to n_3 and $P_4(\nu_4, \partial_{\nu_4}) \equiv P_3(\partial_{\nu_4}, -\nu_4)$

Arrow 45: we go from 0 to ∞ and from increasing power series of the variable ν_4 to decreasing power series of the variable n_5. For an input f of degree r, we set $n_5 := \nu_4^{\frac{r+1}{r}} =: \nu_4^{\frac{1}{\kappa_5}} = \nu_4^{\kappa_3}$, the new variable n_5 being the "critical resurgrence variable" at ∞.

Step 5*: we have the ramified-coefficient linear ODE $P_5^*(n_5, \partial_{n_5})\, k_5(n_5) = 0$ with

$$n_5^{\kappa_5} \equiv \nu_4 \ \ \textit{but}\ n_5 \sim \infty\,,\ k_5(n_5) \equiv k_4(\nu_4),\ P_5^*(n_5, \partial_{n_5}) \equiv P_4(n_5^{\kappa_5}, \frac{n_5^{\kappa_5 - 1}}{\kappa_5}\partial_{n_5})$$

Arrow 55: since for an input f of degree r, we must take $\kappa_5 = \frac{r}{r+1} = \frac{1}{\kappa_3}$, this leads to a ramification of order $r+1$ in the coefficients of P_5^*. We then apply once again the above Lemma 6.1 with $\rho = r+1$ to deramify P_5^* to P_5.

Step 5: we have the polynomial-coefficient linear ODE $P_5(n_5, \partial_{n_5})\, k_5(n_5) = 0$ with n_5 and k_5 as in step 5* but with a linear homogeneous differential operator P_5 which, unlike P_5^*, is polynomial in n_5.

6.6 ODEs for monomial inputs F.

General meromorphic inputs F, with more than one zero or pole, shall be investigated in §7.2 and §8.3-4 with the usual *nir-mir* approach. Here, we shall restrict ourselves to strictly *monomial* F, i.e. with only one zero *or* pole (but of abitrary order p), for these monomial inputs, and only they, give rise to *nir* transforms that verify linear ODEs with polynomial coefficients. So for now our inputs shall be:

$$f(x) := +p\,\log(1+p\,x) \quad , \quad F(x) := (1+p\,x)^{-p} \qquad (p\in\mathbb{N}^*) \tag{184}$$

$$f(x) := -p\,\log(1-p\,x) \quad , \quad F(x) := (1-p\,x)^{+p} \qquad (p\in\mathbb{N}^*) \tag{185}$$

and we shall set as usual:

$$\begin{aligned}
k(n) \;&:=\; \mathrm{singular}\Big(\int_0^\infty e_{\#}^{-\beta(\partial_\tau)\,f(\frac{\tau}{n})}\;d\tau\Big) && \in \Gamma(1/2)\,n^{1/2}\,\mathbb{Q}[[n^{-1}]]\\
\overset{\frown}{h}(\nu) \;&:=\; \mathrm{formal}\Big(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k(n)\,e^{\nu\,n}\frac{dn}{n}\Big) = h(\nu) && \in \nu^{-1/2}\,\mathbb{Q}\{\nu\}\\
\hat{k}(\nu) \;&:=\; \mathrm{formal}\Big(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k(n)\,e^{\nu\,n}dn\Big) = h'(\nu) && \in \nu^{-3/2}\,\mathbb{Q}\{\nu\}
\end{aligned}$$

Unlike with the polynomial inputs f of §6.2-5, the global *nir* transforms now verify no (variable) polynomial linear-homogeneous ODEs. Only their singular parts, which in the present case ($\forall p$) always consist of semi-entire powers of the variable, do verify (covariant) linear ODEs with polynomial coefficients. These equations depend only on the absolute value $|p|$ and read:

$$\begin{aligned}
\big(n+n\partial_n-\frac{|p|}{2}\big)^{|p|}\,k(n) \;&=\; n^{|p|}\,k(n)\\
\big(\partial_\nu-\nu\,\partial_\nu-\frac{|p|}{2}\big)^{|p|}\,h(\nu) \;&=\; (\partial_\nu)^{|p|}\,h(\nu)\\
\big(\partial_\nu-\nu\,\partial_\nu-1-\frac{|p|}{2}\big)^{|p|}\,\hat{k}(\nu) \;&=\; (\partial_\nu)^{|p|}\,\hat{k}(\nu)
\end{aligned}$$

If we regard n and ν no longer as commutative variables (as in §4 and §5), but as non-commutative ones bound by $[n,\nu]=1$ (as in the preceding sections), our covariant ODEs read:

$$P(n,-\partial_n)\,k(n)=0 \;\;,\;\; \partial_\nu^{-1}\,P(\partial_\nu,\nu)\,\partial_\nu\,h(\nu)=0 \;\;,\;\; P(\partial_\nu,\nu)\,\hat{k}(\nu)=0$$

$$\textit{with}\qquad P(n,\nu) := \big(n-n\,\nu-\frac{|p|}{2}\big)^{|p|}-n^{|p|} = (n-\nu\,n-1-\frac{|p|}{2})^{|p|}-n^{|p|}$$

If we now apply the covariance relation (146) to the shifts (ϵ,η):

$$\epsilon := -1/|p|\;,\;\eta := \int_0^\epsilon f(x)\,dx = 1\;,\;{}^{\epsilon}\!f(x) = |p|\,\log(|p|x)$$

we find a centered polynomial P_* predictably simpler than P:

$$P_*(n,\nu) = P(n,\nu+\eta) = \big(-n\,\nu-\frac{|p|}{2}\big)^{|p|}-n^{|p|}$$

Although our covariant operators $P(n,\nu)$ are now much simpler, and of far lower degree in n, than was the case for polynomial inputs f, their form is actually harder to derive. As for their dependence on $|p|$ rather than p, it follows from the general parity relation for the *nir* transform (cf §4.10), but here it also makes direct formal sense. Indeed, in view of $[n,\nu]=1$, we have the chain of formal equivalences:

$$\begin{aligned}
\{\big(n-n\,\nu-\tfrac{p}{2}\big)^{p}\,k(n)=n^{p}\,k(n)\} &\iff\\
\{k(n)=\big(n-n\,\nu-\tfrac{p}{2}\big)^{-p}\,n^{p}\,k(n)\} &\iff\\
\{k(n)=n^{p}\,\big(n-n\,\nu+\tfrac{p}{2}\big)^{-p}\,k(n)\} &\iff\\
\{n^{-p}\,k(n)=\big(n-n\,\nu+\tfrac{p}{2}\big)^{-p}\,k(n)\} &
\end{aligned}$$

which reflects the invariance of $P(n,\nu)$ [56] under the change $p\mapsto -p$.

From the form of the centered differential operator, it is clear that $h(1-\nu)$ has all its irregular singular points over the unit roots, plus a regular singular point at infinity.

Remark: Although both inputs $f_1(x)=\frac{1}{p}\,x^p-1$ and $f_2(x)=\pm p\log(1\pm p\,x)$ lead to *nir*-transforms $h_1(1-\nu)$ and $h_2(1-\nu)$ with radial symmetry and singular points over the unit roots of order p, there are far-going differences:
(i) h_2 verifies much simpler ODEs than h_1
(ii) conversely, h_1 verifies much simpler resurgence equations than h_2 (see *infra*)
(iii) the singularities of h_1 over ∞ are of divergent-resurgent type (see §6.4-5) whereas those of h_2 are merely ramified-convergent (see §6.7).

Let us now revert to our input (184) or (185) with the corresponding *nir* transform $h(\nu)$ and its linear ODE. That ODE always has very explicit power series solutions at $\nu=0$ and $\nu=\infty$ and, as we shall see, this is what really matters. At $\nu=0$ the solutions are of the form:

$$\begin{aligned}
k(n) &= \sum_{s\in-\frac{1}{2}+\mathbb{N}} k_s\,n^{-s} \quad , \quad h(\nu)=\sum_{s\in-\frac{1}{2}+\mathbb{N}} k_s\,\nu^{s} \qquad (\textit{relevant})\\
k^{\mathrm{en}}(n) &= \sum_{s\in\mathbb{N}} k_s\,n^{-s} \quad , \quad h^{\mathrm{en}}(\nu)=\sum_{s\in\mathbb{N}} k_s\,\nu^{s} \qquad (\textit{irrelevant})
\end{aligned}$$

but only for $p\in\{\pm1,\pm2,\pm3\}$ are the coefficients explicitable.

The case $p=\pm1$.

$$\begin{aligned}
k_{-\frac{1}{2}+r} &= 0 \quad \textit{if} \quad r\geq 1 \quad \textit{and} \quad k_{-\frac{1}{2}}=\Big(\frac{\pi}{2}\Big)^{\frac{1}{2}}\\
h_{-\frac{1}{2}+r} &= 0 \quad \textit{if} \quad r\geq 1 \quad \textit{and} \quad k_{-\frac{1}{2}}=\Big(\frac{1}{2}\Big)^{\frac{1}{2}}
\end{aligned}$$

The case $p=\pm2$.

$$\begin{aligned}
k_{-\frac{1}{2}+r} &= 2^{-5\,r}\,\frac{(2\,r)!(2\,r)!}{r!\,r!\,r!}\,k_{-\frac{1}{2}} \quad \textit{with} \quad k_{-\frac{1}{2}}=\Big(\frac{\pi}{8}\Big)^{\frac{1}{2}}\\
h_{-\frac{1}{2}+r} &= 2^{-3\,r}\,\frac{(2\,r)!}{r!\,r!}\,h_{-\frac{1}{2}} \qquad \textit{with} \quad h_{-\frac{1}{2}}=\Big(\frac{1}{8}\Big)^{\frac{1}{2}}
\end{aligned}$$

[56] or more accurately: the invariance of the relation $P(n,\nu)\,k(n)=0$.

The case $p = \pm 3$. The coefficients of k, h have no simple multiplicative structure, but the entire analogues $k^{\text{en}}, h^{\text{en}}$ are simple superpositions of hypergeometric series.

6.7 Monomial inputs F: global resurgence.

Let us replace the pair (h, P) by (ho, Po) with

$$ho(\nu) := h(1-\nu) \quad ; \quad Po(n,\nu) := (-1)^p \, P(-n, 1-\nu) = (\nu\, n + \frac{p}{2})^p - n^p \tag{186}$$

so as to respect the radial symmetry and deal with a function ho having all its singular points over the unit roots $e_j = \exp(2\pi i j/p)$. At the crucial points $\nu_0 \in \{0, \infty, e_0, \dots, e_{p-1}\}$ the p-dimensional kernel of the operator

$$Po(\partial, \nu+\nu_0) = \big((\nu+\nu_0)\partial_\nu + \frac{p}{2}\big)^p \; - \; \big(\partial_\nu\big)^p \tag{187}$$

is spanned by the following systems of fundamental solutions

$$\begin{array}{llll}
\textit{at } \nu_0 = 0: & ho_s(\nu) & \in \nu^s \, \mathbb{C}\{\nu^p\} & (0 \le s \le p-1) \\
\textit{at } \nu_0 = \infty: & hi_s(\nu) & \in \bigoplus\limits_{0 \le \sigma \le s-1} \Big(\nu^{-p/2} \, \mathbb{C}\{\nu^{-1}\} \, \dfrac{(\log \nu)^\sigma}{\sigma!} \Big) & (0 \le s \le p-1) \\
\textit{at } \nu_0 = e_j: & ha_j(\nu) & \in \nu^{-1/2} \, \mathbb{C}\{\nu\} & \\
 & ha_{j,s}(\nu) & \in \mathbb{C}\{\nu\} & (1 \le s \le p-1)
\end{array}$$

The singular solutions ha_j (normalised in a manner consistent with the radial symmetry) are, up to sign, none other than the *inner generators* whose resurgence properties we want to describe. For $p \ge 3$, their coefficients have no transparent expression, but the coefficients of the hi_s and, even more so, those of the ho_s do possess a very simple multiplicative structure, which allows us to apply the method of coefficient asymptotics in §2.3 to derive the resurgence properties of the ha_j, and that too from 'both sides' — from 0 and ∞. A complete treatment shall be given in [ES] but here we shall only state the result and describe the closed resurgence system governing the behaviour of the ha_j. To that end, we consider their Laplace integrals along any given axis $\arg \nu = \theta$, with the "location factor" $e^{-e_j\, n}$:

$$haa_j^\theta(n) := e^{-e_j\, n} \int_0^{e^{i\theta}\infty} ha_j(\nu) \, d\nu \qquad\qquad (\theta \in \mathbb{R}\,,\; j \in \mathbb{Z}/p\mathbb{Z}) \tag{188}$$

Everything boils down to describing the effect on the system $\{haa_1^\theta, \dots, haa_p^\theta\}$ of crossing a singular axis $\theta_0 = \arg(e_{j_2} - e_{j_1})$, i.e. of going from $\theta_0 - \epsilon$ to $\theta_0 + \epsilon$. The underlying ODE being linear, such a crossing will simply subject $\{haa_1^\theta, \dots, haa_p^\theta\}$ to a linear transformation with constant coefficients. Moreover, since all $ha_j(\nu)$ are in $\nu^{-1/2}\, \mathbb{C}\{\nu\}$, two full turns (i.e. changing θ to $\theta + 4\pi$) ought to leave $\{haa_1^\theta, \dots, haa_p^\theta\}$ unchanged. All the above facts can be derived in a rather straightforward manner by resurgence analysis (see [ES]) but, when translated into matrix algebra, they lead to rather complex matrices and to remarkable, highly non-trivial relations between these. Of course, the relations in question also admit *'direct' algebraic* proofs, but these are rather difficult – and in any case much longer than their *'indirect' analytic* derivation. The long subsection which follows is entirely devoted to this 'algebraic' description of the resurgence properties of the ha_j.

6.8 Monomial inputs F: algebraic aspects.

Some elementary matrices.
Eventually, ϵ will stand for -1 and ϵ^q for $e^{\pi i q}$, $\forall q \in \mathbb{Q}$, but for greater clarity ϵ shall be kept free (unassigned) for a while. We shall encounter both ϵ-carrying matrices, which we shall underline, and ϵ-free matrices. For each p, we shall also require the following elementary square matrices ($p \times p$):
$\mathcal{I}$: identity
$\underline{\mathcal{I}}$: ϵ-carrying diagonal
$\mathcal{J}$: Jordan correction
$\mathcal{P}$: unit shift
$\mathcal{Q}$: twisted unit shift
These are hollow matrices, whose only nonzero entries are:

$$\begin{array}{lclll}
\underline{\mathcal{I}}[i,j] &=& \epsilon^{j/p} & \textit{if} & j=i \\
\mathcal{J}[i,j] &=& 1 & \textit{if} & j=i+1 \\
\mathcal{P}[i,j] &=& 1 & \textit{if} & j=i+1 \mod p \\
\mathcal{Q}[i,j] &=& 1 & \textit{if} & j=i+1 \\
\mathcal{Q}[i,j] &=& -1 & \textit{if} & j=i+1-p
\end{array}
\qquad
\begin{array}{lclll}
&&&& \\
&&&& \\
\mathcal{P}^k[i,j] &=& 1 & \textit{if} & j=i+k \mod p \\
\mathcal{Q}^k[i,j] &=& 1 & \textit{if} & j=i+k \\
\mathcal{Q}^k[i,j] &=& -1 & \textit{if} & j=i+k-p
\end{array}$$

The simple-crossing matrices $\underline{\mathcal{M}}_k, \mathcal{M}_k$.
Let $fr(x)$ resp. $en(x)$ denote the *fractional* resp. *entire* part of any real x:

$$x \equiv \mathrm{fr}(x) + \mathrm{en}(x) \quad \textit{with} \quad x \in \mathbb{R} \ , \ \mathrm{fr}(x) \in [0,1[\ , \ \mathrm{en}(x) \in \mathbb{Z}$$

Fix $p \in \mathbb{N}^*$ and set $e_j := \exp(2\pi i j/p), \forall j \in \mathbb{Z}$. For any $k \in \frac{1}{2}\mathbb{Z}$, it is convenient to denote θ_k the axis of direction $2\pi(\frac{k}{p}+\frac{3}{4})$, i.e. the axis from e_{j_1} to e_{j_2} for any pair $j_1, j_2 \in \mathbb{Z}$ such that $j_1 + j_2 = 2k \mod p$ and $(k < j_1 < j_2)_p^{\mathrm{circ}}$. The matrix $\underline{\mathcal{M}}_k$ corresponding to the (counterclockwise) crossing of the axis θ_k has the following elementary entries:

$$\begin{array}{lclll}
\underline{\mathcal{M}}_k[i,j] &=& 1 & \textit{if} & i=j \\
\underline{\mathcal{M}}_k[i,j] &=& -\epsilon^{\mathrm{fr}(\frac{j-k}{p})-\mathrm{fr}(\frac{i-k}{p})} \frac{p!}{(|i-j|)!(p-|i-j|)!} & \textit{if} & \mathrm{fr}(\frac{i+j-2k}{p})=0 \\
&&& \textit{and} & \mathrm{fr}(\frac{i-k}{p}) > \mathrm{fr}(\frac{j-k}{p}) \\
\underline{\mathcal{M}}_k[i,j] &=& 0 & \textit{otherwise} &
\end{array}$$

Alternatively, we may start from the simpler matrix $\underline{\mathcal{M}}_0$:

$$\begin{array}{lclll}
\underline{\mathcal{M}}_0[i,j] &=& 1 & \textit{if} & i=j \\
\underline{\mathcal{M}}_0[i,j] &=& -\epsilon^{\frac{j-i}{p}} \frac{p!}{(i-j)!(p-i+j)!} & \textit{if} & i>j \textit{ and } i+j=2k \mod p \\
\underline{\mathcal{M}}_0[i,j] &=& 0 & \textit{otherwise} &
\end{array}$$

and deduce the general $\underline{\mathcal{M}}_k$ under the rules:

$$\underline{\mathcal{M}}_k[i,j] = \underline{\mathcal{M}}_0[[i-k]_p, [j-k]_p] \qquad \textit{with} \quad [x]_p := p\,.\,\mathrm{en}(\frac{x}{p})$$

$\underline{\mathcal{M}}_k$ carries unit roots of order $2p$ (hence the underlining) but can be turned into a unit root-free matrix $\mathcal{M}_k$ under a k-independent conjugation:

$$\mathcal{M}_k = \underline{\mathcal{I}} \ \underline{\mathcal{M}} \ \underline{\mathcal{I}}^{-1} \tag{189}$$

with the elementary diagonal matrix $\underline{\mathcal{I}}$ defined above. We may therefore work with the simpler matrices $\mathcal{M}_k$ whose entries are:

$$
\begin{array}{rcllll}
\mathcal{M}_k[i,j] & = & 1 & \textit{if} & i=j \\
\mathcal{M}_k[i,j] & = & -\epsilon^{\mathrm{en}(\frac{j-k}{p})-\mathrm{en}(\frac{i-k}{p})}\frac{p!}{(|i-j|)!(p-|i-j|)!} & \textit{if} & \mathrm{fr}(\frac{i+j-2k}{p})=0 \\
 & & & \textit{and} & \mathrm{fr}(\frac{i-k}{p})>\mathrm{fr}(\frac{j-k}{p}) \\
\mathcal{M}_k[i,j] & = & 0 & \textit{otherwise} &
\end{array}
$$

However, since $\underline{\mathcal{I}}$ and $\mathcal{P}$ do not commute, we go from $\underline{\mathcal{M}}_k$ to $\underline{\mathcal{M}}_{k+1}$ under the *regular* shift $\mathcal{P}$ but from $\mathcal{M}_k$ to $\mathcal{M}_{k+1}$ under the *twisted* shift $\mathcal{Q}$:

$$
\underline{\mathcal{M}}_{k+1}=\mathcal{P}^{-1}\ \underline{\mathcal{M}}_k\ \mathcal{P}\quad,\quad \mathcal{M}_{k+1}=\mathcal{Q}^{-1}\ \mathcal{M}_k\ \mathcal{Q} \tag{190}
$$

The multiple-crossing matrices $\underline{\mathcal{M}}_{k_2,k_1},\mathcal{M}_{k_2,k_1}$.

For any $k_1,k_2\in\frac{1}{2}\mathbb{Z}$ such that $k_2>k_1$ we set :

$$
\underline{\mathcal{M}}_{k_2,k_1}:=\underline{\mathcal{M}}_{k_2}\ \underline{\mathcal{M}}_{k_2-\frac{1}{2}}\ \underline{\mathcal{M}}_{k_2-\frac{2}{2}}\ \dots\ \underline{\mathcal{M}}_{k_1+\frac{3}{2}}\ \underline{\mathcal{M}}_{k_1+\frac{2}{2}}\ \underline{\mathcal{M}}_{k_1+\frac{1}{2}} \tag{191}
$$

$$
\mathcal{M}_{k_2,k_1}:=\mathcal{M}_{k_2}\ \mathcal{M}_{k_2-\frac{1}{2}}\ \mathcal{M}_{k_2-\frac{2}{2}}\ \dots\ \mathcal{M}_{k_1+\frac{3}{2}}\ \mathcal{M}_{k_1+\frac{2}{2}}\ \mathcal{M}_{k_1+\frac{1}{2}} \tag{192}
$$

For $k_2<k_1$ or $k_2=k_1$ we set of course:

$$
\underline{\mathcal{M}}_{k_2,k_1}:=\underline{\mathcal{M}}^{-1}_{k_1,k_2}\quad,\quad \mathcal{M}_{k_2,k_1}:=\mathcal{M}^{-1}_{k_1,k_2}\quad,\quad \underline{\mathcal{M}}_{k,k}:=\mathcal{M}_{k,k}:=\mathcal{I}
$$

thus ensuring the composition rule:

$$
\underline{\mathcal{M}}_{k_3,k_2}\ \underline{\mathcal{M}}_{k_2,k_1}=\underline{\mathcal{M}}_{k_3,k_1}\quad,\quad \mathcal{M}_{k_3,k_2}\ \mathcal{M}_{k_2,k_1}=\mathcal{M}_{k_3,k_1}\quad \big(\forall k_i\in\frac{1}{2}\mathbb{Z}\big)
$$

Since $\underline{\mathcal{M}}_{p+k}\equiv\underline{\mathcal{M}}_k$ and $\mathcal{M}_{p+k}\equiv\mathcal{M}_k$ for all k (p-periodicity), each full-turn matrix $\underline{\mathcal{M}}_{p+k,k}$ or $\mathcal{M}_{p+k,k}$ is conjugate to any other. It turns out, however, that just two of them (corresponding to $k\in\{0,1\}$ if $p=0$ or $1\mod 4$, and to $k\in\{\pm\frac{1}{2}\}$ if $p=2$ or $3\mod 4$) admit a simple or at least tolerably explicit normalisation (i.e. a conjugation to the canonical Jordan form, or in this case, a more convenient variant thereof). That normalisation involves remarkable lower diagonal matrices $\mathcal{L}$ and $\mathcal{R}$. To construct $\mathcal{L}$ and $\mathcal{R}$, however, we require a set of rather intricate polynomials H^δ_d.

The auxiliary polynomials $H^\delta_d(x,y)$.
These polynomials, of global degree d in each of their two variables x,y, also depend on an integer-valued parameter $\delta\in\mathbb{Z}$. They are d-inductively determined by the following system of difference equations in y, along with the initial conditions for $y=0$:

$$
H^\delta_d(x,y)\ =\ H^\delta_d(x,y-1)+(x-d)\,H^\delta_{d-1}(x,y-1) \tag{193}
$$

$$
H^\delta_d(x,0)\ =\ \frac{(x+\delta+d)!}{(x+\delta)!}\ =\ \prod_{0<d_1\le d}(x+\delta+d_1) \tag{194}
$$

One readily sees that this induction leads to the direct expression :

$$
\begin{aligned}
H_d^\delta(x,y) &= \sum_{d_1=0}^{d} \frac{(x-1-d+d_1)!}{(x-1-d)!}\frac{(x+\delta+d-d_1)!}{(x+\delta)!}\frac{y!}{d_1!(y-d_1)!} \\
&= \sum_{d_1=0}^{d} \frac{1}{d_1!} \prod_{0\le k_1<d_1} (x-d+k_1) \prod_{1\le k_2\le d-d_1} (x+\delta+k_2) \prod_{0\le k_3<d_1} (y-k_3)
\end{aligned} \tag{195}
$$

which is turn can be shown to be equivalent to :

$$
H_d^\delta(x,y) = \sum_{d_1=0}^{d} \left[\!\!\left[\frac{\delta+2d_1}{\delta+d_1}\right]\!\!\right]!! \; \frac{(2d+d_1-y)!}{(d+2d_1-y)!} \prod_{0\le d_2\le d}^{d_2\neq d_1} \frac{(x-d_2)}{(d_1-d_2)} \tag{196}
$$

$$
= \sum_{d_1=0}^{d} \left[\!\!\left[\frac{\delta+2d_1}{\delta+d_1}\right]\!\!\right]!! \prod_{d_1<d_3\le d} (d+d_1+d_3-y) \prod_{0\le d_2\le d}^{d_2\neq d_1} \frac{(x-d_2)}{(d_1-d_2)} \tag{197}
$$

with :

$$
\begin{aligned}
\left[\!\!\left[\frac{a}{b}\right]\!\!\right]!! &:= \frac{a!}{b!} && \textit{if} \quad a,b\in\mathbb{N} \\
&:= (-1)^{a-b}\frac{(-1-b)!}{(-1-a)!} && \textit{if} \quad a,b\in-\mathbb{N}^* \\
&:= 0 && \textit{otherwise}
\end{aligned} \tag{198}
$$

The left normalising matrix $\mathcal{L}$.

$$\begin{array}{llll}
\textit{if} \;\; i<j & (\forall p) & : \mathcal{L}[i,j] = & 0 \\[6pt]
\textit{if} \;\; p=0 & \mod 4 \quad \textit{and} & \ldots & \\
2\,j \le p, & i{+}j \le p{+}2 & : \mathcal{L}[i,j] = & (-1)^{i} \frac{(i-1)!}{(j-1)!(i-j)!} \\
2\,j \le p, & i{+}j > p{+}2 & : \mathcal{L}[i,j] = & (-1)^{j} \frac{(i-1)!}{(2j-3)!(p-2j+2)!}\, H^1_{p-i}(j{-}2,p) \\
2\,j > p & & : \mathcal{L}[i,j] = & (-1)^{j} \frac{(p-j)!}{(p-i)!(i-j)!} \\[6pt]
\textit{if} \;\; p=1 & \mod 4 \quad \textit{and} & \ldots & \\
2\,j \le p{+}1, & i{+}j \le p{+}2 & : \mathcal{L}[i,j] = & (-1)^{i} \frac{(i-1)!}{(j-1)!(i-j)!} \\
2\,j \le p{+}1, & i{+}j > p{+}2 & : \mathcal{L}[i,j] = & -(-1)^{j} \frac{(i-1)!}{(2j-3)!(p-2j+2)!}\, H^1_{p-i}(j{-}2,p) \\
2\,j > p{+}1 & & : \mathcal{L}[i,j] = & (-1)^{j} \frac{(p-j)!}{(p-i)!(i-j)!} \\[6pt]
\textit{if} \;\; p=2 & \mod 4 \quad \textit{and} & \ldots & \\
2\,j \le p, & i{+}j \le p{+}1 & : \mathcal{L}[i,j] = & (-1)^{i} \frac{(i-1)!}{(j-1)!(i-j)!} \\
2\,j \le p, & i{+}j > p{+}1 & : \mathcal{L}[i,j] = & -(-1)^{j} \frac{(i-1)!}{(2j-2)!(p-2j+1)!}\, H^0_{p-i}(j{-}1,p) \\
2\,j > p & & : \mathcal{L}[i,j] = & (-1)^{j} \frac{(p-j)!}{(p-i)!(i-j)!} \\[6pt]
\textit{if} \;\; p=3 & \mod 4 \quad \textit{and} & \ldots & \\
2\,j \le p{-}1, & i{+}j \le p{+}1 & : \mathcal{L}[i,j] = & (-1)^{i-1} \frac{(i-1)!}{(j-1)!(i-j)!} \\
2\,j \le p{-}1, & i{+}j > p{+}1 & : \mathcal{L}[i,j] = & (-1)^{i-1} \frac{(i-1)!}{(2j-2)!(p-2j+1)!}\, H^0_{p-i}(j{-}1,p) \\
2\,j > p{-}1 & & : \mathcal{L}[i,j] = & (-1)^{p-j} \frac{(p-j)!}{(p-i)!(i-j)!}
\end{array}$$

The right normalising matrix $\mathcal{R}$.

$$\begin{array}{llll}
\textit{if} \;\; i<j & (\forall p) & : \mathcal{R}[i,j] = & 0 \\
\textit{if} \;\; p=0 & \bmod 4 \quad \textit{and} & \ldots & \\
2\,j \le p{-}2, \;\; i{+}j \le p & & : \mathcal{R}[i,j] = & (-1)^{i} \frac{(i-1)!}{(j-1)!(i-j)!} \\
2\,j \le p{-}2, \;\; i{+}j > p & & : \mathcal{R}[i,j] = & (-1)^{j} \frac{(i-1)!}{(2j-1)!(p-2j)!}\, H^{-1}_{p-i}(j,p) \\
2\,j > p{-}2 & & : \mathcal{R}[i,j] = & (-1)^{j} \frac{(p-j)!}{(p-i)!(i-j)!} \\
\textit{if} \;\; p=1 & \bmod 4 \quad \textit{and} & \ldots & \\
2\,j \le p{-}1, \;\; i{+}j \le p & & : \mathcal{R}[i,j] = & (-1)^{i} \frac{(i-1)!}{(j-1)!(i-j)!} \\
2\,j \le p{-}1, \;\; i{+}j > p & & : \mathcal{R}[i,j] = & -(-1)^{j} \frac{(i-1)!}{(2j-1)!(p-2j)!}\, H^{-1}_{p-i}(j,p) \\
2\,j > p{-}1 & & : \mathcal{R}[i,j] = & (-1)^{j} \frac{(p-j)!}{(p-i)!(i-j)!} \\
\textit{if} \;\; p=2 & \bmod 4 \quad \textit{and} & \ldots & \\
2\,j \le p{-}2, \;\; i{+}j \le p{-}1 & & : \mathcal{R}[i,j] = & (-1)^{i} \frac{(i-1)!}{(j-1)!(i-j)!} \\
2\,j \le p{-}2, \;\; i{+}j > p{-}1 & & : \mathcal{R}[i,j] = & -(-1)^{j} \frac{(i-1)!}{(2j)!(p-2j-1)!}\, H^{-2}_{p-i}(j{+}1,p) \\
2\,j > p{-}2 & & : \mathcal{R}[i,j] = & (-1)^{j} \frac{(p-j)!}{(p-i)!(i-j)!} \\
\textit{if} \;\; p=3 & \bmod 4 \quad \textit{and} & \ldots & \\
2\,j \le p{-}3, \;\; i{+}j \le p{-}1 & & : \mathcal{R}[i,j] = & (-1)^{i-1} \frac{(i-1)!}{(j-1)!(i-j)!} \\
2\,j \le p{-}3, \;\; i{+}j > p{-}1 & & : \mathcal{R}[i,j] = & (-1)^{i-1} \frac{(i-1)!}{(2j)!(p-2j-1)!}\, H^{-2}_{p-i}(j{+}1,p) \\
2\,j > p{-}3 & & : \mathcal{R}[i,j] = & (-1)^{p-j} \frac{(p-j)!}{(p-i)!(i-j)!}
\end{array}$$

Normalisation identities for the full-turn matrices $\mathcal{M}_{p+k,k}$:

$$\begin{array}{rcll}
\mathcal{L}\;\; \mathcal{M}_{p+1,\,1}\;\; \mathcal{L}^{-1} & = & (-1)^{p-1}(\mathcal{I}+\mathcal{J})^{p} & \textit{if} \;\; p=0 \textit{ or } 1 \quad \bmod 4 \\
\mathcal{R}\;\; \mathcal{M}_{p+0,\,0}\;\; \mathcal{R}^{-1} & = & (-1)^{p-1}(\mathcal{I}+\mathcal{J})^{p} & \textit{if} \;\; p=0 \textit{ or } 1 \quad \bmod 4 \\
\mathcal{L}\;\; \mathcal{M}_{p+\frac{1}{2},+\frac{1}{2}}\;\; \mathcal{L}^{-1} & = & (-1)^{p-1}(\mathcal{I}+\mathcal{J})^{p} & \textit{if} \;\; p=2 \textit{ or } 3 \quad \bmod 4 \\
\mathcal{R}\;\; \mathcal{M}_{p-\frac{1}{2},-\frac{1}{2}}\;\; \mathcal{R}^{-1} & = & (-1)^{p-1}(\mathcal{I}+\mathcal{J})^{p} & \textit{if} \;\; p=2 \textit{ or } 3 \quad \bmod 4
\end{array}$$

with $\mathcal{I}$ denoting the identity matrix and $\mathcal{J}$ the matrix carrying a maximal upper-Jordan side-diagonal:

$$\mathcal{J}[i,j] = 1 \quad \textit{if} \quad j = 1{+}i \quad \textit{and} \quad \mathcal{J}[i,j] = 0 \quad \textit{otherwise}$$

This result obviously implies that *all* full-rotation matrices $\mathcal{M}_{p+k,\,k}$ are also conjugate to $(-1)^{p-1}(\mathcal{I}+\mathcal{J})^{p}$ but the point, as already mentioned, is that only for $k \in \{0,1\}$ or $\{\pm\frac{1}{2}\}$ do we get an explicit conjugation with simple, lower-diagonal matrices like $\mathcal{L},\mathcal{R}$. As for the choice of $(-1)^{p-1}(\mathcal{I}+\mathcal{J})^{p}$ rather than $(-1)^{p-1}\mathcal{I}+\mathcal{J}$ as normal form, it is simply a matter of convenience, and a further, quite elementary conjugation, immediately takes us from the one to the other.

Defining identities for the normalising matrices $\mathcal{L}, \mathcal{R}$.

$$
\begin{aligned}
\mathcal{R} &= (\mathcal{I}+\mathcal{J})\,\mathcal{L}\,\mathcal{Q}^{-1} && (199)\\
\mathcal{R} &= \mathcal{L}\,\mathcal{W} && (200)
\end{aligned}
$$

with the twisted shift matrix $\mathcal{Q}$ defined right at the beginning of §6.8 and with

$$
\begin{aligned}
\mathcal{W} &= \mathcal{M}_{1,0} && \textit{if}\;\; p = 0 \;\; \textit{or} \;\; 1 \quad \bmod 4\\
\mathcal{W} &= \mathcal{M}_{\frac{1}{2},-\frac{1}{2}} && \textit{if}\;\; p = 2 \;\; \textit{or} \;\; 3 \quad \bmod 4
\end{aligned}
$$

The matrix entries of $\mathcal{W}$ are elementary binomial coefficients:

$$
\begin{array}{ll}
\textit{if}\;\; i<j & : \mathcal{W}[i,j]=0\\
\textit{if}\;\; i=j & : \mathcal{W}[i,j]=1\\
\textit{if}\;\; i>j\;\; \textit{and}\ldots & \\
p\in\{0,1\} \bmod 4\;\; \textit{and}\;\; p-i-j\in\{1,2\} & : \mathcal{W}[i,j]=-\frac{p!}{(i-j)!(p-i+j)!}\\
p\in\{2,3\} \bmod 4\;\; \textit{and}\;\; p-i-j\in\{0,1\} & : \mathcal{W}[i,j]=-\frac{p!}{(i-j)!(p-i+j)!}\\
\textit{otherwise} & : \mathcal{W}[i,j]=0
\end{array}
$$

If we now eliminate either $\mathcal{R}$ (resp. $\mathcal{L}$) from the system (199),(200) and express the remaining matrix as a sum of an *elementary part* (which corresponds to the two extreme subdiagonal zones and carries only binomial entries) and a *complex part* (which corresponds to the middle subdiagonal zone and involves the intricate polynomials H_d^δ), we get a linear system which, as it turns out, completely determines $\mathcal{L}^{comp.}$ or $\mathcal{R}^{comp.}$ (viewed as unknown) in terms $\mathcal{L}^{elem.}$ or $\mathcal{R}^{elem.}$ (viewed as known). Thus:

$$
(\mathcal{I}+\mathcal{J})\,(\mathcal{L}^{elem.}+\mathcal{L}^{comp.}) = (\mathcal{L}^{elem.}+\mathcal{L}^{comp.})\,\mathcal{W}\,\mathcal{Q} \tag{201}
$$

To understand just how special the value $\epsilon = -1$ and the case of full-rotation matrices are, let us briefly examine, first, the case of full rotations with unassigned ϵ, then the case of partial rotations with $\epsilon = -1$.

Complement: full rotations with $\epsilon \neq -1$.
Keeping ϵ free and setting $V_p(t,\epsilon) := \det(t\,\mathcal{I} - \mathcal{M}_{p,0})$ we get:

$$
\begin{aligned}
V_2(t,\epsilon) &= (t+1)^2 \;-2^2\;(1+\epsilon)\,t\\
V_3(t,\epsilon) &= (t-1)^3 \;+3^3\;(1+\epsilon)\,t\,\epsilon\\
V_4(t,\epsilon) &= (t+1)^4 \;-2^3\;(1+\epsilon)\,t\,(1+16\,\epsilon+32\,\epsilon^2+14\epsilon\,t+t^2)\\
V_5(t,\epsilon) &= (t-1)^5 \;+5^4\;(1+\epsilon)\,t\,\epsilon\,(1+5\,\epsilon+5\,\epsilon^2+3\,t+t^2)
\end{aligned}
$$

Etc....The only conspicuous properties of the V_p polynomials seem to be:

$$
\begin{aligned}
V_p(t,-1) &= \big(t+(-1)^p\big)^p && (202)\\
V_p(1,\epsilon) &= \epsilon^p\;V_p(1,\epsilon^{-1}) && (203)
\end{aligned}
$$

(202) follows from the short analysis argument given in §6.7, and we have devoted the bulk of the present section (§6.8) to checking it algebraically. As for the self-inversion property (203), it directly follows from the way the simple-crossing matrices $\mathcal{M}_k$ are constructed.

As far as we can see, the V_p polynomials appear to possess only one additional property, albeit a curious one (we noticed it empirically and didn't attempt a proof). It is this: for p prime ≥ 5 and $t = 1$ we have (at least up to $p = 59$):

$$V_p(1,\epsilon) = \det(\mathcal{I} - \mathcal{M}_{p,0}) = p^p\,\epsilon\,(1+\epsilon)\,(1+\epsilon+\epsilon^2)^{\kappa(p)}\,W_p(\epsilon) \tag{204}$$

with $\kappa(p) = 1$ (resp.2) if $p = 2$ (resp.1) $\mod 4$ and some $\mathbb{Q}$-irreducible polynomial $W(\epsilon) \in \mathbb{Z}(\epsilon)$. However, $V_p(t,\epsilon) \neq 0 \mod 1+\epsilon+\epsilon^2$, which reduces the above relation (204) to a mere oddity. [57] More generally, the "semi-periodicity" in k of $\mathcal{M}_{k,0}$ that we noticed for $\epsilon = -1$ has no counterpart for any other value of ϵ, not even for $\epsilon^3 = 1$ or, for that matter, $\epsilon = 1$.

Complement: partial rotations with $\epsilon = -1$.
The partial-rotation matrices $\mathcal{M}_{k_2,k_1}$ with $|k_2 - k_1| \leq \frac{p}{2}$ all share the same trivial characteristic polynomial $(t-1)^p$, but possess increasingly numerous and increasingly large Jordan blocks as $|k_2 - k_1|$ goes from 0 to $\frac{p}{2}$. For $\frac{p+1}{2} < |k_2 - k_1| < p$, the Jordan blocks disappear and the characteristic polynomials become thoroughly unremarkable, apart from being self-inverse (*always* so if p is even, *only when* $k_2 - k_1 \in \mathbb{Z}$ if p is odd). For $|k_2 - k_1| = p$, as we saw earlier in this section, we have one single Jordan block of maximal size, with eigenvalue ∓ 1 depending on the parity of p. That leaves only the border-line case $|k_2 - k_1| = \frac{p+1}{2}$. We have no Jordan blocks then, yet the characteristic polynomials possess a remarkable factorisation on $\mathbb{Z}$:

$$\textit{If} \quad k_2 - k_1 = \pm\frac{p+1}{2} \quad \textit{then} \;:$$

$$(\textit{for}\;\; p\;\; \textit{odd}) \qquad \det(t\mathcal{I} - \mathcal{M}_{k_2,k_1}) = (t-1)\prod_{s=1}^{\frac{p-1}{2}} P_s(p,t)$$

$$(\textit{for}\;\; p = 0 \mod 4) \qquad \det(t\mathcal{I} - \mathcal{M}_{k_2,k_1}) = \prod_{s=1}^{\frac{p}{4}} \big(P_{2s-1}(p,t)\big)^2$$

$$(\textit{for}\;\; p = 2 \mod 4) \qquad \det(t\mathcal{I} - \mathcal{M}_{k_2,k_1}) = P_{\frac{p}{2}}(p,t)\prod_{s=1}^{\frac{p-2}{4}} \big(P_{2s-1}(p,t)\big)^2$$

with polynomials $P_s(p,t) \in \mathbb{Q}[p,t]$ quadratic and self-inverse in t, of degree $2s$ in p, and assuming values in $\mathbb{Z}[t]$ for $p \in \mathbb{Z}$:

$$P_s(p,t) = \big(1-t\big)^2 + \Big(\prod_{i=0}^{s-1}\frac{p-i}{1+i}\Big)^2 t$$

Complement: some properties of the polynomials H_d^δ.
For any fixed $n, d \in \mathbb{N}$ with $n \leq d$, the $H_d^\delta(x,n)$ and $H_d^\delta(n,y)$, as polynomials in x or y, factor into a string of fully explicitable one-degree factors. This immediately follows from the expansions (195),(196),(197). Conversely, the factorisations may be directly derived

[57] True, we have $V_p(1,\epsilon) = \textit{Const} \mod 1+\epsilon+\epsilon^2$, but this is a trivial consequence of $V_p(1,\epsilon)$ being self-inverse in ϵ.

from the induction (193),(194) and then serve to establish the remaining properties. Most zeros (x, y) in $\mathbb{Z}^2$ or $(\frac{1}{2}\mathbb{Z})^2$ can also be read off the factorisation. All the above properties suggest a measure of symmetry between the two variables, under the simple exchange $x \leftrightarrow y$. But there also exists a more recondite symmetry, which is best expressed in terms of the polynomials

$$K_d^\delta(x, y) := H_d^\delta(d - x, \frac{1}{2} + 2d + \delta - \frac{3}{2}x + \frac{1}{2}\, y) \tag{205}$$

under the exchange $y \leftrightarrow -y$. It reads, for $x = n \in \mathbb{N} \cup [0, d]$:

$$\begin{aligned} K_d^\delta(n, y) + K_d^\delta(n, -y) &= 2^{n-1} \left[\!\left[\frac{d+\delta-n}{2d+\delta-2n} \right]\!\right]!! \prod_{0<i<n}^{i\ odd} (y^2 - i^2) \quad (n\ even) \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad (n\ odd) \end{aligned}$$

with the factorial ratio $[\![\ldots]\!]!!$ defined as in (198)

6.9 Ramified monomial inputs F : infinite order ODEs.

If we now let p assume arbitrary complex values α, our *nir*-transform $h(\nu)$ and its centered variant $h_*(\nu) = h(\nu + \nu_*) = h(\nu + 1)$ ought to verify the following ODEs of infinite order

$$Q(\partial_\nu, \nu)\, h(\nu) := \Big((\partial_\nu - \nu\, \partial_\nu - \frac{\alpha}{2})^\alpha - \partial_\nu^\alpha \Big) h(\nu) = 0 \qquad \alpha \in \mathbb{C} \tag{206}$$

$$Q_*(\partial_\nu, \nu)\, h_*(\nu) := \Big((-\nu\, \partial_\nu - \frac{\alpha}{2})^\alpha - \partial_\nu^\alpha \Big) h_*(\nu) = 0 \qquad \alpha \in \mathbb{C} \tag{207}$$

to which a proper meaning must now be attached. This is more readily done with the first, non-centered variant, since

Proposition 6.1 .
The nir*-transform $h_\alpha(\nu)$ of $f_\alpha(x) := \alpha\ \log(1 + \alpha\, x)$ is of the form*

$$h_\alpha(\nu) = -h_{-\alpha}(\nu) = \frac{1}{\sqrt{2}\,\alpha} \sum_{n \in \mathbb{N}} \gamma_{-\frac{1}{2}+n}(\alpha^2)\, \nu^{-\frac{1}{2}+n} \tag{208}$$

with $\gamma_{-\frac{1}{2}+n}(\alpha^2)$ polynomial of degree n in α^2 and it verifies (mark the sign change) an infinite integro-differential equation of the form

$$\Big(\sum_{1 \le k} \partial_\nu^{-k}\ \ S_k(\nu\partial_\nu + \frac{k}{2}, \alpha - k) \Big)\, h_\alpha(-\nu) = 0 \tag{209}$$

with integrations ∂_ν^{-k} from $\nu = 0$ and with elementary differential operators $\mathbb{S}(.,.)$ which, being polynomial in their two arguments, merely multiply each monomial ν^n by a scalar factor polynomial in (n, k, α), effectively yielding an infinite induction for the calculation of the coefficients $\gamma_{-\frac{1}{2}+n}(\alpha^2)$.

Thus the first three coefficients are

$$\gamma_{-\frac{1}{2}}(\alpha^2) = 1 \ \ , \ \ \gamma_{\frac{1}{2}}(\alpha^2) = \frac{1}{12}(\alpha^2-1) \ \ , \ \ \gamma_{\frac{3}{2}}(\alpha^2) = \frac{1}{864}(\alpha^2-1)\,(\alpha^2+23)$$

For $n \geq 1$ all polynomials $\gamma_{-\frac{1}{2}+n}(\alpha^2)$ are divisible by (α^2-1) but this is their only common factor.

Remark: the regular part of the *nir*-transform h_α of f_α has the same shape $\sum_{n\in\mathbb{N}} \alpha^{-1}\,\gamma_n(\alpha^2)$ as the singular part, also with $\gamma_n(\alpha^2)$ polynomial of degree n in α^2, but it doesn't verify the integro-differential equation (206). We'll need the following identies:

$$[\mathbf{d},\mathbf{D}] = \mathbf{d} \qquad (\textit{here} \qquad \mathbf{d} = \partial_\nu \ \ , \ \ \mathbf{D} = \nu\partial_\nu + \frac{\alpha}{2}) \tag{210}$$

$$\begin{aligned}
(\mathbf{d}+\mathbf{D})^\alpha &= \sum_{0\leq k} \qquad S_k(\mathbf{D}+\frac{\alpha-k}{2},\alpha-k)\ \mathbf{d}^{\alpha-k} \\
&= \sum_{0\leq k} \mathbf{d}^{\frac{\alpha-k}{2}}\ S_k(\mathbf{D} \qquad\quad ,\alpha-k)\ \mathbf{d}^{\frac{\alpha-k}{2}} \\
&= \sum_{0\leq k} \mathbf{d}^{\alpha-k}\ S_k(\mathbf{D}-\frac{\alpha-k}{2},\alpha-k)
\end{aligned}$$

The non-commutativity relation $[\mathbf{d},\mathbf{D}] = 1$, combined with the above expansions, yields for the polynomials S_k the following addition equation:

$$S_k(\mathbf{D},\beta_1+\beta_2) = \sum_{k_1+k_2=k} S_{k_1}(\mathbf{D}-\frac{\beta_2-k_2}{2},\beta_1-k_1)\,S_{k_2}(\mathbf{D}+\frac{\beta_1-k_1}{2},\beta_2-k_2)$$

and the difference equation:

$$S_k(\frac{\beta}{2},\beta) \equiv S_k(\frac{\beta+1}{2},\beta-1) \tag{211}$$

That relation, in turn, has two consequences: on the one hand, it leads to a finite expansion (212) of $S_k(\mathbf{D},\beta)$ in powers of $\mathbf{D}$ with coefficients $T_{2k_*}(\beta)$ that are polynomials in β of degree exactly k_* with $2k_* \leq k$. On the other, it can be partially reversed, leading, for entire values of b, to a finite expansion (213) of $T_{2k}(b)$ in terms of some special values of $S_{2k-1}(\,.\,,b)$.

$$S_k(\mathbf{D},\beta) \;=\; \Big(\prod_{i=1}^{k}(\beta+i)\Big) \sum_{k_1+2\,k_2=k}^{k_1,k_2\geq 0} \frac{\mathbf{D}^{k_1}}{k_1!}\ \frac{T_{2k_2}(\beta)}{(2\,k_2)!} \qquad \forall\beta\in\mathbb{C} \tag{212}$$

$$T_{2k}(b) \;=\; \frac{(2k)!\ b!}{(2k+b)!} \sum_{0\leq c\leq b} (c-\frac{b}{2})\ S_{2k-1}(\frac{c-b}{2},c) \qquad \forall b\in\mathbb{N} \tag{213}$$

Together, (212) and its reverse (213) yield an explicit inductive scheme for constructing the polynomials T_{2k}. We first calculate $T_{2k}(b)$ for b whole, via the identity (214) whose terms $S_{2k-1}(\,.\,,b)$ involve only the earlier polynomials $T_{2h}(c)$, with indices $h<k$ and $c\leq b$. The identity reads:

$$\frac{T_{2k}(b)}{(2k)!b!} = \sum_{0\leq c\leq b}^{0\leq h<k} \frac{T_{2h}(c)}{(2h)!c!}\ \frac{(c/2-b/2)^{2k-2h-1}}{(2\,k-2\,h-1)!}\ \frac{(2k+c)!}{(2k+b)!}\ \frac{(c-b/2)}{(c+2\,k)} \tag{214}$$

Then we use Lagrange interpolation (215)-(216) to calculate $T_{2k}(\beta)$ for general complex arguments β :

$$T_{2k}(\beta) = \sum_{1\le b\le k} \Lambda_k(\beta,b)\; T_{2k}(b) \qquad \forall\beta\in\mathbb{C} \tag{215}$$

$$\Lambda_k(\beta,b) := \frac{\beta}{b} \prod_{1\le i\le k}^{i\ne b} \frac{i-\beta}{i-b} \tag{216}$$

First values of the $T_{2\,k}$-polynomials:

$$\begin{aligned}
T_0(\beta) &= 1\\
T_2(\beta) &= \frac{1}{12}\,\beta\\
T_4(\beta) &= \frac{1}{240}\,\beta\,(-2+5\,\beta)\\
T_6(\beta) &= \frac{1}{4032}\,\beta\,(16+42\,\beta+35\,\beta^2)\\
T_8(\beta) &= \frac{1}{34560}\,\beta\,(-4+5\,\beta)(36-56\,\beta+35\,\beta^2)\\
T_{10}(\beta) &= \frac{1}{101376}\,\beta\,(768-2288\,\beta+2684\,\beta^2-1540\,\beta^3+385\,\beta^4)
\end{aligned}$$

Special values of the $T_{2\,k}$-polynomials:

$$\begin{aligned}
T_{2\,k}(2) &= \frac{2}{(2\,k+1)(2\,k+2)}\\
T_{2\,k}(1) &= \frac{1}{(2\,k+1)\,2^{2\,k}}\\
T_{2\,k}(0) &= 0 \quad \textit{if} \quad k\ne 0 \quad \textit{and} \quad T'_{2\,k}(0)=\frac{B_{2\,k}}{2\,k}\\
T_{2\,k}(-1) &= B_{2\,k}(\frac{1}{2})\\
T_{2\,k}(-2) &= -(2\,k-1)\,B_{2\,k}\\
T_{2\,k}(-1-2\,k) &= (-1)^k\,\frac{(2\,k)!}{4^k\,k!}
\end{aligned}$$

with B_n and $B_n(.)$ denoting the Bernoulli numbers and polynomials.

Special values of the S_k-polynomials:

$$\textit{For } k \textit{ odd} : \quad S_k(\mathbf{D},-1-k) = \prod_{-\frac{k}{2}<s<\frac{k}{2}}^{k\in\mathbb{Z}} (\mathbf{D}+s)$$

$$\textit{For } k \textit{ even} : \quad S_k(\mathbf{D},-1-k) = \prod_{-\frac{k}{2}<s<\frac{k}{2}}^{k\in\mathbb{Z}-\frac{1}{2}\mathbb{Z}} (\mathbf{D}+s)$$

Note that in neither case are the bounds $\pm k/2$ reached by s, since the pair $\{k/2, s\}$ always consists of an integer and a half-integer. $S_k(\mathbf{D}, b)$ appears to have no simple factorisation structure except (trivially) for $b = 1$ and $b = 2$ when in view of (212),(213) we have:

$$\begin{aligned} S_k(\mathbf{D}, 1) &= 2^{-k-1}\Big((2\,\mathbf{D}+1)^{k+1} - (2\,\mathbf{D}-1)^{k+1}\Big) \\ S_k(\mathbf{D}, 2) &= 2^{-1}\Big((\mathbf{D}+1)^{k+2} + (\mathbf{D}-1)^{k+2} - 2\,\mathbf{D}^{k+2}\Big) \end{aligned}$$

Since $S_1(n, \alpha-1) = n\,\alpha$, the induction rule for the γ-coefficients reads

$$\begin{aligned} \gamma_{-\frac{1}{2}}(\alpha^2) &= 1 \\ \gamma_{-\frac{1}{2}+n}(\alpha^2) &= \sum_{1\le k\le n} (-1)^{k+1}\,\frac{\Gamma(\frac{1}{2}+n-k)}{\Gamma(\frac{1}{2}+n)}\,\frac{S_{k+1}(n-\frac{k}{2}, a-k-1)}{S_1(n,\alpha-1)}\,\gamma_{-\frac{1}{2}+n-k}(\alpha^2) \\ &= \sum_{1\le k\le n} (-1)^{k+1}\,\frac{\Gamma(\frac{1}{2}+n-k)}{\Gamma(\frac{1}{2}+n)}\,\frac{S_{k+1}(n-\frac{k}{2}, a-k-1)}{n\,\alpha}\,\gamma_{-\frac{1}{2}+n-k}(\alpha^2) \end{aligned}$$

Moreover, since $S(k)(\mathbf{D}, -1) = S(k)(\mathbf{D}, -2) = \cdots = S(k)(\mathbf{D}, -k) = 0$, when α is a positive integer, the above induction involves a constant, finite number of terms, with a sum $\sum$ over $1 \le k \le \alpha - 1$ instead of $1 \le k \le n$, which is consistent which the finite differential equations of §6.6.

6.10 Ramified monomial inputs F: arithmetical aspects.

In this last subsection, we revert to the case of polynomial inputs f and replace the high-order ODEs verified by k by a first-order order differential system, so as to pave the way for a future paper [SS2] devoted to understanding, from a pure ODE point of view, the reasons for the rigidity of the *inner algebra's* resurgence, i.e. its surprising insentivity to the numerous parameters inside f.

The normalised coefficients $\gamma_r, \delta_r, \delta_r^{\mathrm{ev}}$ of the series $h_\alpha, k_\alpha, k_\alpha^{\mathrm{ev}}$, whose definitions we recall:

$$\begin{aligned} h_\alpha(\nu) &= \frac{1}{\sqrt{2}}\frac{1}{\alpha}\sum_{r\in-\frac{1}{2}+\mathbb{N}} \gamma_r(\alpha^2)\,\nu^r \quad \textit{with} \quad \gamma_{-\frac{1}{2}}(\alpha^2) \equiv 1 \\ k_\alpha(n) &= \frac{\sqrt{\pi}}{\sqrt{2}}\frac{1}{\alpha}\sum_{r\in-\frac{1}{2}+\mathbb{N}} \delta_r(\alpha^2)\,n^{-r} \quad \textit{with} \quad \delta_{-\frac{1}{2}}(\alpha^2) \equiv 1 \\ k_\alpha(n)k_\alpha(-n) =: k_\alpha^{\mathrm{ev}}(n) &= \frac{\pi i}{2\alpha^2}\sum_{r\in-1+2\,\mathbb{N}} \delta_r^{\mathrm{ev}}(\alpha^2)\,n^{-r} \quad \textit{with} \quad \delta_{-1}^{\mathrm{ev}}(\alpha^2) \equiv 1 \end{aligned}$$

seem to possess remarkable arithmetical properties, whether we view them
(i) as polynomials in α
(ii) as polynomials in r
(iii) as rational numbers, for α fixed in $\mathbb{Z}$.
These arithmetical properties, at least some of them, do not obviously follow from the

shape of the *nir* transform nor indeed from the above induction. Thus, as *polynomials*, the γ coefficients appear to be exactly of the form:

$$\gamma_{-\frac{1}{2}+r}(\alpha^2) = \frac{6^{-r}}{(2r)!} \frac{(\alpha^2-1)\,\gamma^*_{-\frac{1}{2}+r}(\alpha^2)}{\prod_{5\le p\le r+2} p^{\mu_{r,p}}} \;=\; \frac{1}{(2r)!}\,\frac{a^r}{6^r}\,\gamma^{**}_{-\frac{1}{2}+r}(\alpha^2-1) \tag{217}$$

with the $\gamma^*_{-\frac{1}{2}+r}(\alpha^2)$ irreducible in $\mathbb{Z}[\alpha^2]$ and with on the denominator a product $\prod$ involving only prime numbers between 5 and $r+2$. This at any rate holds for all values of n up to 130. The surprising thing is not the presence of these p in $[5, r+2]$ but rather the fact that their powers $\mu_{r,p}$ seem to obey no exact laws (though they are easily majorised), unlike the powers of 2 and 3 that are *exactly* accounted for by the factor 6^{-r}. But this 2- and 3-adic regularity seems to go much further. It becomes especially striking if we consider the polynomials $\gamma^{**}_{-\frac{1}{2}+r}$ in the rightmost term of (217) after changing to the variable $a := \alpha^2 - 1$. Indeed:

Conjecture 6.1 (2- or 3-adic expansions for the γ as α-polynomials) .
The polynomials γ^{**} defined by

$$\gamma_{-\frac{1}{2}+r}(a+1) \;=:\; \frac{1}{(2r)!}\,\frac{a^r}{6^r}\,\gamma^{**}_{-\frac{1}{2}+r}(a) \qquad , \qquad \gamma^{**}_{-\frac{1}{2}+r} \in \mathbb{Q}[a^{-1}] \tag{218}$$

possess 2- and 3-adic expansions to all orders:

$$\begin{aligned} \gamma^{**}_{-\frac{1}{2}+r}(a) \;&=\; \sum_{0\le j} \lambda_{2,j}(r,a)\,2^j \qquad && \Big(\lambda_{2,j}(r,a) \in \{0,1\}\Big) \\ \gamma^{**}_{-\frac{1}{2}+r}(a) \;&=\; \sum_{0\le j} \lambda_{3,j}(r,a)\,3^j \qquad && \Big(\lambda_{3,j}(r,a) \in \{0,1,2\}\Big) \end{aligned}$$

with coefficients $\lambda_{p,j}$ that in turn depend only on the first j terms of the p-adic expansion of r*. In other words:*

$$\begin{aligned} \lambda_{2,j}(r,a) = \lambda_{2,j}([r_0, r_1, \ldots, r_{j-1}], a) \qquad & \textit{with} \quad r = \sum_{0\le i<j} r_i\,2^i \mod 2^j \\ \lambda_{3,j}(r,a) = \lambda_{3,j}([r_0, r_1, \ldots, r_{j-1}], a) \qquad & \textit{with} \quad r = \sum_{0\le i<j} r_i\,3^i \mod 3^j \end{aligned}$$

Moreover, as a polynomial in a^{-1}, each $\lambda_{2,j}(r,a)$ is of degree j at most.

These facts have been checked up to the p-adic order $j = 25$ and for all r up to 130. Moreover, no such regularity seems to obtain for the other p-adic expansions, at any rate not for $p = 5, 7, 11, 13$.

Conjecture 6.2 (p-adic expansions for the γ as r-polynomials) .
The γ^{**} defined as above verify

$$\gamma^{**}_{-\frac{1}{2}+r}(a) \;= 1 + \sum_{1\le d\le r} a^{-d}\,Q_d(r) \tag{219}$$

with universal polunomials $Q_d(r)$ of degree $3d$ in r and of the exact arithmetical form :

$$
\begin{aligned}
Q_d(r) &= \Big(6^d \prod_{p\, prime \geq 2} p^{-\mu_p(d)}\Big) \Big(Q^*_d(r) \prod_{1\leq i\leq d}(r-i)\Big)\\
Q^*_d(r) &= \sum_{1\leq i\leq 2d} c_{d,i}\, r^i \qquad \textit{with} \qquad (c_{d,1},\ldots,c_{d,2d}) \quad \textit{coprime}\\
\mu_p(d) &= \sum_{0\leq s} \mathrm{en}\Big(\frac{2\,d}{(p{-}3)\,p^s}\Big) \qquad \textit{if} \quad p\geq 5\\
\mu_3(d) &= \sum_{0\leq s} \mathrm{en}\Big(\frac{d}{2\,p^s}\Big)\\
\mu_2(d) &= -\sum_{0\leq s} d_s \qquad\quad \textit{if} \quad d=\sum_{0\leq s} d_s\, 2^s \quad (d_s\in\{0,1\})\\
c_{d,2d} &\in (-1)^d\mathbb{N}^+
\end{aligned}
$$

with $\mathrm{en}(x)$ denoting as usual the entire part of x.

Conjecture 6.3 (Special values of the Q_d) .
If for any $q\in\mathbb{Q}$ we set

$$
\begin{aligned}
\mathrm{pri}(q) &:= q \quad \textit{if} \quad q\in\mathbb{N} \quad \textit{and} \quad q \quad \textit{prime}\\
\mathrm{pri}(q) &:= 1 \qquad\quad \textit{otherwise}
\end{aligned}
$$

then for any $d,s\in\mathbb{N}^*$ we have

$$
Q_d(d{+}s) \in \frac{1}{Q_{d,s}}\mathbb{Z} \quad \textit{with} \quad Q_{d,s} := \prod_{\substack{1\leq j\leq s\\ s<k\leq s+2\,j}} \mathrm{pri}\Big(\frac{d{+}k}{j}\Big) \tag{220}
$$

Moreover, for s fixed and d large enough, the denominator of $Q_d(d{+}s)$ is exactly $Q_{d,s}$. Note that by construction $Q_d(d{+}s)$ is automatically quadratfrei as soon as $d>2\,s\,(s{-}1)$.

Together with the trivial identities $Q_d(s)=0$ for $s\in[0,d]\cup\mathbb{N}$, this majorises *denom*$(Q_d(s)$ for all $s\in\mathbb{N}$. We have no such simple estimates for negative values of s.

Conjecture 6.4 (Coefficients δ^{ev}) .
The normalised coefficients δ_r of k_α (with $r\in -1+2\,\mathbb{N}$) are of the form :

$$
\delta^{\mathrm{ev}}_r(\alpha^2) = \frac{1}{B_r}\,R_r(\alpha^2) = \frac{A_r}{B_r}\,R^*_r(\alpha^2)\prod_{d|r}(\alpha^2-d^2) \tag{221}
$$

where
(i) A_r is of the form $\prod_{p|r}^{p\,\mathrm{prime}} p^{\sigma_{p,r}}$ with $\sigma_{r,p}\in\mathbb{N}$
(ii) B_r is of the form $\prod_{(p,r)=1}^{p\,\mathrm{prime}\leq r+2} p^{\tau_{p,r}}$ with $\tau_{r,p}\in\mathbb{N}$
(iii) $R^*_r(\alpha^2)$ is an irreducible polynomial in $\mathbb{Z}[\alpha^2]$
However, when α takes entire values q, the arithmetical properties of $\delta^{\mathrm{ev}}_r(q^2)$ become more dependent on q than r. In particular;
(iv) $\mathrm{denom}(\delta_r(q^2)) = \prod_{p|q}^{p\,\mathrm{prime}} d^{\kappa_{r,q,p}}$ *with $\kappa_{r,q,p}\in\mathbb{N}$ and $\kappa_{r,p,p}\leq 3$*
(v) $\mathrm{denom}(\delta_r(p^2)) = p^{\kappa_{r,p,p}}$ *with $\kappa_{r,p,p}\leq 3$ (p prime).*
This suggests a high degree of divisibility for $R^*_r(q^2)$ and above all $R_r(q^2)$, specially for q prime. In particular we surmise that :
(vi) $R_r(p^2)\in(p{-}1)!\,\mathbb{Z}$ for p prime.

6.11 From flexible to rigid resurgence.

Let us, in this concluding subsection, revert to the case of polynomial inputs f. We assume the tangency order to be 1. To get rid of the demi-entire powers, we go from k to a new unknown K such that

$$k(n) = n^{\frac{1}{2}}\; K(n) \qquad (k(n) \in n^{\frac{1}{2}}\mathbb{C}[[n^{-1}]]\ ,\ K(n) \in \mathbb{C}[[n^{-1}]]) \tag{222}$$

The new differential equation in the n-plane reads $P(n, -\partial_n)K(n) = 0$ and may be written in the form:

$$\Big(\prod_{i=1}^{r}(\partial_n + \nu_i)\Big)\, K(n) + \sum_{i=0}^{r-1} \theta_i^*(n)\, \partial_n^i K(n) = 0 \quad ; \quad \theta_i^*(n) = O(n^{-1}) \tag{223}$$

This ODE is equivalent to the following first order differential system with r unknowns $K_i^* = \partial_n^i K$ $(0 \le i \le r-1)$:

$$\begin{aligned}
\partial_n K_0^* - K_1^* &= 0\\
\partial_n K_1^* - K_2^* &= 0\\
&\dots\\
\partial_n K_{r-2}^* - K_{r-1}^* &= 0\\
\partial_n K_{r-1}^* + \sum_{i=0}^{r-1}\big(\nu_{r-i}^* + \theta_i^*(n)\big)\, K_i &= 0
\end{aligned} \tag{224}$$

with ν_l standing for the symmetric sum of order l of $\nu_1, \dots, \nu_r$. Changing from the unknowns K_i^* $(0 \le i \le r-1)$ to the unknowns K_i $(1 \le i \le r)$ under the vandermonde transformation

$$K_i^* = \sum_{0 \le j \le r-1} (-\nu_i)^j\, K_j = \sum_{0 \le j \le r-1} (-\nu_i)^j\, \partial_n^j K$$

we arrive at a new differential system in normal form:

$$\partial_n K_i + \nu_i\, K_i + \sum_{i=1}^{r} \theta_{i,j}(n)\, K_j = 0 \qquad (1 \le i \le r) \tag{225}$$

with constants ν_i and rational coefficients $\tau_i(n)$ which, unlike the earlier $\theta_i(n)$, are not merely $O(n^{-1})$ but also, crucially, $O(n^{-2})$. Concretely, the rank-1 matrix $\Theta = [\theta_{i,j}]$ is conjugate to a rank-1 matrix $\Theta^* = [\theta_{i,j}^*]$ with only one non-vanishing (bottom) line, under the vandermonde matrix $V = [v_{i,j}]$:

$$\Theta = V^{-1}\, \Theta^*\, V \quad \textit{with} \quad v_{i,j} = (-\nu_i)^{j-1}\ ,\ \theta_{i,j}^* = 0\ \ \textit{if}\ i < r\ ,\ \theta_{r,j}^* = \theta_{j-1}^* \tag{226}$$

The coefficients $\theta_{i,j}$ have a remarkable structure. They admit a unique factorisation of the form:

$$\theta_{i,j}(n) = \frac{1}{\delta_i}\, \frac{\alpha_j(n)}{\gamma(n)} \tag{227}$$

with factors $\alpha_i, \beta_j, \gamma$ derived from symmetric polynomials α, β, γ of $r-1$ f-related variables (not counting the additional n-variable):

$$\begin{aligned}
\delta_i &= \delta(x_1 - x_i, \ldots, \widehat{x_i - x_i}, \ldots, x_r - x_i) && (228)\\
\alpha_j(n) &= \alpha(x_1 - x_j, \ldots, \widehat{x_j - x_j}, \ldots, x_r - x_j)(n) && (229)\\
\gamma(n) &= \gamma(x_1 - x_i, \ldots, \widehat{x_i - x_i}, \ldots, x_r - x_i)(n) && (230)\\
&= \gamma(x_1 - x_j, \ldots, \widehat{x_j - x_j}, \ldots, x_r - x_j)(n) && (231)
\end{aligned}$$

Let us take a closer look at all three factors:

(i) The γ factor is simply the "second leading polynomial" of §6.2 and §6.3 after division by its leading term $n^{\overline{\delta}}$. Being a direct shift-invariant of f, if may also be viewed as a polynomial in $\mathbf{f}_0, \mathbf{f}_1 \ldots$.

(ii) The δ factor comes from the inverse vandermonde matrix $V^{-1} = [u_{i,j}]$. Indeed:

$$u_{i,j} = \sigma_{r-j}(\nu_1, \ldots, \nu_r)\, \delta_i \qquad \textit{with} \qquad \delta_i = \prod_{1 \le s \le r}^{s \ne i} \frac{1}{\nu_s - \nu_i} = u_{i,r} \tag{232}$$

Moreover:

$$\begin{aligned}
\nu_i &= f^*(x_i) = f_r \int_0^{x_i} \prod_{1 \le j \le r}^{j \ne i} (x - x_j)\, dx\\
\nu_i - \nu_j &= \nu(x_i, x_j\,;\, x_1 \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_r)
\end{aligned}$$

with a function ν antisymmetric (resp. symmetric) in its first two (resp. last $r-2$) variables, and completely determined by the following two identities:

$$\begin{aligned}
\nu(x_1, x_2\,;\, y_1, \ldots, y_{r-2}) &\equiv \nu(x_1 - t, x_2 - t\,;\, y_1 - t, \ldots, y_{r-2} - t) \qquad \forall t\\
\nu(x, -x\,;\, y_1, \ldots, y_{r-2}) &= -\nu(-x, x\,;\, y_1, \ldots, y_{r-2})\\
&= (-1)^{r-1}\, 4\, f_r \sum_{1 \le s \le [\frac{r-2}{2}]} \frac{\sigma_{r-2s}(y_1, \ldots, y_{r-2})}{4\, s^2 - 1}\, t^{2s+1}\\
&= -4\, f_r \int_0^x t_2\, dt_2 \int_0^{t_2} \Big(\prod_{i=1}^{r-2} (y_i + t_1) + \prod_{i=1}^{r-2} (y_i - t_1) \Big)\, dt_1
\end{aligned}$$

where $\sigma_l(y_1, y_2, \ldots)$ denotes the symmetric sum of order l of $y_1, y_2 \ldots$.

(iii) The α factor stems from the coefficients θ_s in the differential equation (225). Indeed, in view of (i) and (ii) and with the n-variable implicit:

$$\theta_{i,j} = \frac{1}{\delta_i} \frac{\alpha_j(n)}{\gamma(n)} = \sum_{\substack{1 \le t \le r\\ 1 \le s \le r}} u_{i,t}\, \theta^*_{t,s}\, v_{s,j} = \sum_{1 \le s \le r} u_{i,r}\, \theta^*_{r,s}\, v_{s,j} = \sum_{1 \le s \le r} \frac{1}{\delta_i}\, \theta^*_{s-1}\, (-\nu_j)^{s-1}$$

Hence

$$\alpha_j(n) = \gamma(n) \sum_{0 \le l \le r-1} (-\nu_j)^l\, \theta^*_l(n) \tag{233}$$

We may also insert the covariant shift ν_0 or $\underline{\nu}_0$ and rewrite the differential equation (225) as :

$$\Big(\prod_{i=1}^{r}(\partial_n+\nu_i)\Big)\,K(n)+\sum_{i=0}^{r-1}\theta_i^{\#}(n)\,(\partial_n+\nu_0)^i K(n)=0 \tag{234}$$

$$\Big(\prod_{i=1}^{r}(\partial_n+\nu_i)\Big)\,K(n)+\sum_{i=0}^{r-1}\underline{\theta}_i^{\#}(n)\,(\partial_n+\underline{\nu}_0)^i K(n)=0 \tag{235}$$

with new coefficients $\theta_i^{\#},\underline{\theta}_i^{\#}$ that are not only shift-invariant but also root-symmetric.[58] This leads for the α factors to expressions :

$$\alpha_j(n)=\gamma(n)\sum_{0\le l\le r-1}(\nu_0-\nu_j)^l\,\theta_l^{\#}(n)=\gamma(n)\sum_{0\le l\le r-1}(\underline{\nu}_0-\nu_j)^l\,\underline{\theta}_l^{\#}(n)$$

which have over (229) the advantage of involving only shift-invariants, namely the $\theta_l^{\#}$ or $\underline{\theta}_j^{\#}$ (root-symmetric) and the $\nu_0-\nu_j$ or $\underline{\nu}_0-\nu_j$ (not root-symmetric). To show the whole extent of the *rigidity*, we may even introduce new parameters by taking a non-standard shift operator $\beta(\partial_\tau)$,

$$\beta(t)=t^{-1}+\sum_{0\le k}\beta_k\,t^k=t^{-1}+\sum_{1\le k}b_k\,t^{k-1}\qquad (b_k\equiv\beta_{k-1})$$

but with a re-indexation $b_k=\beta_k$ to do justice to the underlying homogeneity.[59] The case $r=1$ is uninteresting (no ping-pong, there being only one inner generator), and here are the results for $r=2$ and 3.

Input f of degree 2 : $f(x)=(x-x_1)\,(x-x_2)\,f_2$

$$\begin{aligned}
\delta(y_1) &= -\frac{1}{6}\,f_2\,y_1^3\\
\gamma(y_1) &= +36\,f_2^4\,y_1^2+288\,f_2^4\,b_2\,n^{-2}\\
\alpha(y_1) &= +(5\,f_2^4\,y_1^2-3\,f_2^6\,y_1^6\,b_2)\,n^{-2}+8\,(f_2^6\,y_1^5\,b_3-12\,f_2^5\,b_2\,y_1^3)\,n^{-3}\\
&\quad +8\,(9\,f_2^5\,y_1^2\,b_3-7\,f_2^4\,b_2-3\,f_2^6\,y_1^4\,b_2^2)\,n^{-4}\\
&\quad +64\,f_2^6\,y_1^3\,b_2\,b_3\,n^{-5}+24\,\big(12\,f_2^5\,b_2\,b_3+f_2^6\,y_1^2\,(4\,b_2^3+b_3^2)\big)\,n^{-6}\\
&\quad +128\,f_2^6\,b_2\,(4\,b_2^3+b_3^2)\,n^{-8}
\end{aligned}$$

Input f of degree 3 : $f(x)=(x-x_1)\,(x-x_2)\,(x-x_3)\,f_3$

[58] i.e. symmetric with respect to the roots x_i of f.

[59] N.B. the present b_k differ from those in (169).

$\mathbf{y}_1 := y_1 + y_2\,,\ \mathbf{y}_2 := y_1\, y_2$

$$\delta(y_1,y_2) \;=\; \frac{1}{144}\, f_3^2\, \mathbf{y}_2^3\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)$$

$$\begin{aligned}
\gamma(y_1,y_2) \;=\; & -2^6\, 3^3\, f_3^{10}\, \mathbf{y}_1\, \mathbf{y}_2^2\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\,(4\,\mathbf{y}_2 - \mathbf{y}_1^2)\\
& -2^6\, 3^3\, f_3^{9}\, \mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\,(3\,\mathbf{y}_2 - \mathbf{y}_1^2)\, n^{-1}\\
& +2^9\, 3^3\, f_3^{10}\, b_2\, \mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\,(3\,\mathbf{y}_2 - \mathbf{y}_1^2)^2\, n^{-2}\\
& +2^6\, 3^3\,\big(243\, f_3^{9}\, \mathbf{y}_1\, b_2\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\\
& + f_3^{10}\, b_3\,(513\,\mathbf{y}_1^2\,\mathbf{y}_2^2 + 28\,\mathbf{y}_1^6 - 252\,\mathbf{y}_1^4\,\mathbf{y}_2 + 216\,\mathbf{y}_2^3)\big)\, n^{-3}\\
& -2^6\, 3^6\,(3\,\mathbf{y}_2 - \mathbf{y}_1^2)\,\big(33\, f_3^{9}\, b_3 + 4\, f_3^{10}\, b_2^2\, \mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\big)\, n^{-4}\\
& -2^8\, 3^7\, f_3^{10}\, b_2\, b_3\,(3\,\mathbf{y}_2 - \mathbf{y}_1^2)^2\, n^{-5}\\
& +2^6\, 3^9\,\big(9\, f_3^{9}\, b_2\, b_3 - f_3^{10}\, b_3^2\, \mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\big)\, n^{-6}\\
& +2^6\, 3^{12}\, f_3^{10}\, b_3\,(4\, b_2^3 + b_3^2)\, n^{-9}
\end{aligned}$$

For the α factor, we mention only the two lowest and highest powers of n^{-1}:

$$\begin{aligned}
\alpha(y_1,y_2) = & \Big(\frac{1}{27}\, f_3^{11}\, \mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\,(8748\,\mathbf{y}_2^5 - 13851\,\mathbf{y}_1^2\,\mathbf{y}_2^4 + 378\,\mathbf{y}_1^4\,\mathbf{y}_2^3 + 2403\,\mathbf{y}_1^6\,\mathbf{y}_2^2\\
& -600\,\mathbf{y}_1^8\,\mathbf{y}_2 + 40\,\mathbf{y}_1^{10}) + \frac{2}{27}\, f_3^{13}\, b_2\, \mathbf{y}_1\, \mathbf{y}_2^2\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\,(4\,\mathbf{y}_2 - \mathbf{y}_1^2) \times\\
& (9\,\mathbf{y}_2^2 + 6\,\mathbf{y}_1^2\,\mathbf{y}_2 - \mathbf{y}_1^4)\,(81\,\mathbf{y}_2^3 - 36\,\mathbf{y}_1^2\,\mathbf{y}_2^2 + 9\,\mathbf{y}_1^4\,\mathbf{y}_2 - \mathbf{y}_1^6)\Big)\, n^{-2}\\
& +\Big(f_3^{10}\, \mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\,(2835\,\mathbf{y}_2^3 - 675\,\mathbf{y}_1^2\,\mathbf{y}_2^2 - 9\,\mathbf{y}_1^4\,\mathbf{y}_2 + \mathbf{y}_1^6)\\
& +\frac{2}{27}\, f_3^{12}\, b_2\, \mathbf{y}_1\,(9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)\,(66339\,\mathbf{y}_2^6 - 129033\,\mathbf{y}_1^2\,\mathbf{y}_2^5 + 175770\,\mathbf{y}_1^4\,\mathbf{y}_2^4\\
& -119475\,\mathbf{y}_1^6\,\mathbf{y}_2^3 + 38520\,\mathbf{y}_1^8\,\mathbf{y}_2^2 - 5742\,\mathbf{y}_1^{10}\,\mathbf{y}_2 + 319\,\mathbf{y}_1^{12})\\
& -\frac{8}{9}\, f_3^{13}\, b_3\, \mathbf{y}_1^2\, \mathbf{y}_2^2\,(4\,\mathbf{y}_2 - \mathbf{y}_1^2)\,(9\,\mathbf{y}_2^3 - 18\,\mathbf{y}_1^2\,\mathbf{y}_2^2 + 9\,\mathbf{y}_1^4\,\mathbf{y}_2 - \mathbf{y}_1^6) \times\\
& (9\,\mathbf{y}_2 - 2\,\mathbf{y}_1^2)^2\Big)\, n^{-3} + \sum_{4\le s\le 18} (\ldots)\, n^{-s}\\
& +2 \times 3^{14}\, f_3^{13}\, b_3\,(4\, b_2^3 + b_3^2)\,(b_2\, b_4 - 12\, b_2^3 - 3\, b_3^2)\, b_4\,(3\,\mathbf{y}_2 - \mathbf{y}_1^2)\, n^{-19}\\
& +\frac{3^{15}}{2}\, f_3^{13}\, b_3\,(4\, b_2^3 + b_3^2)\,\big(27\,(4\, b_2^3 + b_3^2 - b_2\, b_4)^2 - b_4^3\big)\, n^{-21}
\end{aligned}$$

For a direct, ODE-theoretical derivation of the *rigidity* phenomenon, see [SS2]. General criteria will also be given there for deciding which parameters inside an ODE contribute to the resurgence constants (or Stokes constants) and which don't.

7 The general resurgence algebra for SP series.

We recall the definition of the *raw* and *cleansed* SP series:

$$j_F(\zeta) := \sum_{0\leq n} J_F(n)\,\zeta^n \qquad \textit{with} \qquad J_F(n) := \sum_{0\leq m<n}\prod_{0\leq k\leq m} F(\frac{k}{n}) \tag{236}$$

$$j_F^{\#}(\zeta) := \sum_{0\leq n} J_F^{\#}(n)\,\zeta^n \qquad \textit{with} \qquad J_F^{\#}(n) := J_F(n)/Ig_{{}_F}(n) \tag{237}$$

We also recall that the $\perp$ transform turns the set $\{F, f, f^*, j_F^{\#}\}$ into the set $\{F^{\vDash}, f^{\vDash}, f^{\vDash *}, j^{\#}_{F^{\vDash}}\}$ with:

$$F^{\vDash}(x) = 1/F(1-x) \quad ; \quad f^{\vDash}(x) = -f(1-x) \quad ; \quad f^{\vDash *}(x) = f^*(1-x) - \eta_F \tag{238}$$

$$j^{\#}_{F^{\vDash}}(\zeta) = j_F^{\#}(\frac{\zeta}{\omega_F}) \qquad \textit{with} \quad \omega_F := e^{-\eta_F} \quad \textit{and} \quad \eta_F := \int_0^1 f(x)dx \tag{239}$$

and with f^* denoting as usual the primitive of f that vanishes at 0. We shall now (pending a more detailed investigation in [SS1]) sketch how the various generators arise and how they reproduce under alien differentiation. Piecing all this information together, we shall then get a global description of the Riemann surfaces of our SP functions.

For convenience, let us distinguish two degrees of difficulty:
– first, the case of holomorphic inputs f
– second, the case of meromorphic inputs F
and split the investigation into two phases:
– first, focusing on the auxiliary ν-plane
– second, reverting to the original ζ-plane.

7.1 Holomorphic input f. The five arrows.

7.1.1 From *original* to *outer*.

Let us check, in the four simplest instances, that SP series (our so-called *original* generators) with an holomorphic input f always give rise to two *outer* generators[60]

$$\{\overset{\wedge}{lo}_{\rm in}(\nu), \overset{\wedge}{Lo}_{\rm in}(\zeta) = \overset{\wedge}{lo}_{\rm in}(1+\zeta)\} \quad , \quad \{\overset{\wedge}{lo}_{\rm out}(\nu), \overset{\wedge}{Lo}_{\rm out}(\zeta) = \overset{\wedge}{lo}_{\rm out}(1+\omega_F\zeta)\}$$

located respectively over $\{\nu = 0, \zeta = 1\}$ or $\{\nu = \eta_F, \zeta = 1/\omega_F\}$ and produced under the *nur*-tranform, i.e. by inputting respectively f or $f^{\vDash}$ into the long chain of §5.2
Case 1: $-f^*$ *decreases on* $[0,1]$.
To explain the occurence $Lo_{\rm in}$, apply the argument at the beginning of §5.1. To explain the occurence $Lo_{\rm out}$, the shortest way is to pick $\epsilon > 0$ small enough for $-f^*$ to be decreasing on the whole of $[0, 1+\epsilon]$, and then to form the SP series $jj_F^{\#}(\zeta)$ defined exactly as $j_F^{\#}(\zeta)$ but with a summation ranging over $0 \leq m < (1+\epsilon)\,n$ instead of $0 \leq m < n$.

[60] which exceptionally coalesce into one when $\eta_F = 0, \omega_F = 1$, which may occur only in the cases 3 or 4 *infra*.

Then $jj_F^\#$ clearly has no singularity at $\zeta = 1/\omega_F$. On the other hand, applying once again the argument of §5.1 to the difference $jj_F^\# - j_F^\#$ we see that it has at $\zeta = 1/\omega_F$ a singularity which, up to the dilation factor ω_F, is given by the *nur*-transform of ${}^1f$ with ${}^1f(x) := f(1+x)$. In view of the parity relation of §5.8 it is also equal to *minus* the *nur*-transform of $({}^1f)^{\vdash}$. But $({}^1f)^{\vdash} = f^{\models}$. Hence the result[61].

A trivial - but telling - example corresponds to the choice of a constant input $F(x) \equiv \alpha$ with $0 < \alpha < 1$. We then get :

$$j_F(\zeta) = \frac{\alpha}{1-\alpha}\Big(\frac{1}{1-\zeta} - \frac{1}{1-\alpha\zeta}\Big) \quad ; \quad j_F^\#(\zeta) = \frac{1}{\alpha^{-1/2} - \alpha^{1/2}}\Big(\frac{1}{1-\zeta} - \frac{1}{1-\alpha\zeta}\Big) \tag{240}$$

Case 2: $-f^*$ *increases on* $[0,1]$.

The $\models$ transform turns case 2 into case 1, with f anf $f^{\models}$ exchanged. Hence the result. Again, we have the trivial example of a constant input $F(x) \equiv \alpha$ but now with $1 < \alpha$. We then get the same power series as in (240) but with α changed into $1/\alpha$, which of course agrees with the relation (239) between $j_F^\#$ and $j_{F^{\models}}^\#$.

Case 3: $-f^*$ *decreases on* $[0, x_0]$*, then increases on* $[x_0, 1]$.
Here again, the argument at the beginning of §5.1 takes care of Lo_{in}. To justify Lo_{out}, all we have to do is observe that the $\models$ transform turns case 3 into another instance of that same case 3, while exchanging the roles of Lo_{in} and Lo_{out}.

Case 4: $-f^*$ *increases on* $[0, x_0]$*, then decreases on* $[x_0, 1]$.
Case 4 is exactly the reverse of case 3. The argument about $j_F^\#$ and $jj_F^\#$ (see case 1) takes care of Lo_{out} and then the fact that $\models$ turns case 4 into another case 4 justifies the occurence of Lo_{in}. Case 4, however, presents us with a novel difficulty : the presence for $-f^*$ of a maximum at $x = x_0$ gives rise (see §7.1.2 infra) to an inner generator Li located at a point $\omega'_F = e^{-\eta'_F}$ (with $\eta'_F = \int_0^{x_0} f(x)dx$) that is closer to the origin than both 1 (location of Lo_{in}) and ω_F (location of Lo_{out}). So the method of §5.1 for translating coefficient asymptotics into *nearest* singularity description no longer applies. One must then resort to a suitable *deformation* argument. We won't go into the details, but just mention a simplifying circumstance : from the fact that *inner generators* never produce *outer generators* (under alien differentiation), it follows that the actual manner of pushing Li beyond Lo_{in} and Lo_{out} (i.e. under right or left circumvention) doesn't matter.

7.1.2 From *original* to *inner*.

Case 4: $-f^*$ *decreases on* $[0, x_0]$*, then increases on* $[x_0, 1]$.
When f has a simple zero at x_0, i.e. when the "tangency order"is $\kappa = 1$, we are back to the heuristics of §4.1. When f has a multiple zero (necessarily of odd order, if f^* is to have an extremum there), we have a tangency order $\kappa \in \{3, 5, 7 \dots\}$ and the same argument as in §4.1 points to the existence of a singularity over η'_F in the ν-plane or ω'_F in the ζ-plane, with η'_F, ω'_F as above. In the ν-plane, this singularity is characterised by an upper-minor $\overset{\frown}{li}$ given by :

$$\overset{\frown}{li} := \text{nir}({}^0f) + \text{nir}({}^0f^{\vdash}) \quad \textit{with} \quad {}^0f(x) := f(x_0 + x) \tag{241}$$

[61]Recall that $f^{\vdash}(x) := -f(-x)$ and $f^{\models}(x) := -f(1-x)$

In view of the parity relation (cf §4.10) this implies:

$$\overset{\frown}{li}\,(\nu) = \sum_{0\leq k} h_{\frac{-\kappa+2k}{\kappa+1}}\,\nu^{\frac{-\kappa+2k}{\kappa+1}} \quad \textit{with} \quad h = \mathrm{nir}(f) = \sum_{0\leq k} h_{\frac{-\kappa+k}{\kappa+1}}\,\nu^{\frac{-\kappa+k}{\kappa+1}} \tag{242}$$

Thus, only every second coefficient of h goes into the making of $\overset{\frown}{li}$. Moreover, since κ here is necessarily odd, the ratio $\frac{-\kappa+2k}{\kappa+1}$ can never be an integer. This means that the corresponding *majors*[62] never carry any logarithms, but only fractional powers.

7.1.3 From *outer* to *inner*.

The relevant functional transform here is *nur*, which according to (125) is an infinite superposition of *nir* transforms applied separately to all determinations of $\log f(.)$. To calculate the alien derivatives of $\overset{\frown}{lo}_{\mathrm{in}}$ or $\overset{\frown}{lo}_{\mathrm{out}}$, we must therefore apply the recipe of the next para (§7.1.4) to the various $nir(2\pi i\, k+\log f(0)+\dots)$ or $nir(2\pi i\, k+\log(f(1)+\dots)$. Exceptionnally, if $2\pi i\, k+\log f(0)$ or $2\pi i\, k+\log f(1)$ vanishes for some k, we must also deal with tangency orders $\kappa>0$ and apply the recipe of the para after next (§7.1.5). But in this as in that case, the result will always be *some* inner generator $\overset{\frown}{li}$, and never an outer one.

7.1.4 From *exceptional* to *inner*.

Let $\overset{\frown}{le}$ an exceptional generator with base point x_1. Assume, in other words, that $f(x_1)\neq 0$ and:

$$\overset{\frown}{le} = {}^{\nu_1}h = \mathrm{nir}\,({}^{x_1}f) \quad \textit{with} \quad {}^{x_1}f(x) := f(x_1+x)\ \ ,\ \ \nu_1 := \int_0^{x_1} f(x)dx \tag{243}$$

To calculate the alien derivatives of $\overset{\frown}{le}$, we go back to the long chain §4.2 and decompose the *nir*-transform into elementary steps from 1 to 7. The elementary steps 1,2,4,5,7 neither produce nor destroy singularities. The steps that matter are the reciprocation (step 3) and the *mir*-transform (step 6). The singularities produced by reciprocation are easy to predict. As for the *mir*-transform, its integro-differential expression (85) and the properties of the Euler-Bernoulli numbers[63] show that the closest singularity or singularities of $\overset{\frown}{le}$ [64] necessarily correspond to closest singularity//ies of $\not{g}$ (see Lemma 4.7). Now comes the crucial, non-trivial fact: this one-to-one correspondance between singularities of $\not{g}$ and $\overset{\frown}{le}$ holds also in the large, at least when the initial input f is holomorphic. This is by no means obvious, since the singularities of $\not{g}$ might combine with those of β to produce infinitely many new ones, farther away, under the Hadamard product mechanism . To show that this *doesn't occur*, assume the existence of a point ν_2 in the ν-plane where ${}^{\nu_1}h = \overset{\frown}{le}$ is singular but g is regular. We can then write $\nu_2 = \int_0^{x_2} f(x)dx$

[62]whether $\overset{\smile}{li}, \overset{\vee}{li}$ or $\overset{\vee}{Li}$

[63] more exactly, the fact that the singularities of β are all on $2\pi i\mathbb{Z}^*$.

[64]i.e. those lying on the boundary of the disk of convergence. Recall that for an exceptional generator we have a tangency order $\kappa=0$ and so $\overset{\frown}{le}$ is a regular, unramified germ at the origin.

for some x_2 and then choose x_3 close enough to x_2 to ensure that the exceptional generator ${}^{\nu_3}h$ of base point x_3 is regular at ν_2 [65]. We then use the bi-entireness of the finite *nir*-increment $\nabla h(\epsilon, \nu)$ with $\epsilon = x_3 - x_1$, $\nu = \nu_3 - \nu_1$ to conclude that ${}^{\nu_1}h := nir({}^{x_1}h)$, just like ${}^{\nu_3}h := nir({}^{x_3}h)$, is regular at ν_2.

7.1.5 From *inner* to *inner*. Ping-pong resurgence.

Let $\overset{\frown}{li}_1$ be an inner generator with base point x_1. This means that $f(x_1) = 0$ and:

$$\overset{\frown}{li}_1 = {}^{\nu_1}h = \mathrm{nir}\,({}^{x_1}f) \quad \textit{with} \quad {}^{x_1}f(x) := f(x_1{+}x) \ \ , \ \ \nu_1 := \int_0^{x_1} f(x)dx \tag{244}$$

Assuming once again f to be holomorphic, the same argument as above shows that all singularities of $\overset{\frown}{li}_1$, not just the closest ones, correspond to zeros x_i of f. They are therefore inner generators $\overset{\frown}{li}_2, \overset{\frown}{li}_3, \overset{\frown}{li}_4 \ldots$ with base points $x_2, x_3, x_4 \ldots$ and the resurgence equations between them

$$\Delta_{\nu_q - \nu_p}\ \overset{\frown}{li}_p = \overset{\frown}{li}_q \tag{245}$$

will exactly mirror the resurgence equations between the singularities of g. The only difference is that if $\overset{\frown}{li}_p$ *"sees"* $\overset{\frown}{li}_q$, i.e. if (245) holds, then the converse is automatically true: $\overset{\frown}{li}_q$ *sees* $\overset{\frown}{li}_p$. Exceptional generators, on the other hand, *see* but are not *seen.* [66]

7.1.6 Recapitulation. One-way arrows, two-way arrows.

Let us sum up pictorially our findings for a holomorphic input f :

$$\begin{array}{cccccccc}
\longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \textit{inner} & \longleftarrow & \ldots & \longleftarrow \\
\uparrow & & & \nearrow & \uparrow\downarrow & & & \uparrow \\
\uparrow & & \textit{outer} & & \uparrow\downarrow & & & \uparrow \\
\uparrow & \nearrow & & \searrow & \uparrow\downarrow & & & \uparrow \\
\textit{original} & \longrightarrow & \longrightarrow & \longrightarrow & \textit{inner} & \longleftarrow & \ldots & (\textit{exceptional}) \\
\downarrow & \searrow & & \nearrow & \uparrow\downarrow & & & \downarrow \\
\downarrow & & \textit{outer} & & \uparrow\downarrow & & & \downarrow \\
\downarrow & & & \searrow & \uparrow\downarrow & & & \downarrow \\
\longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \textit{inner} & \longleftarrow & \ldots & \longleftarrow
\end{array}$$

The above picture displays four types of generators:
- one *original* generator, which is none other than the 'cleansed' SP series
- two *outer* generators (*in* and *out*) which may occasionally coalesce

---

[65] by ensuring that ${}^{\nu_3}h$ has ν_2 within its convergence disk.

[66] Regarding the inner generators, one may note that what matters is the geometry in the ν-plane, not in the x-plane. Consider for instance:

$$f(x) := (x - x_1)(x - x_2)(x - x_3) \quad \textit{with} \quad x_1 = 1, x_2 = 2, x_3 = 2 + \epsilon + \epsilon^2\, i \ \ (0 < \epsilon << 1)$$

Then a simple calculation shows that the inner generator $\overset{\frown}{li}_1$ *sees* $\overset{\frown}{li}_2$ but not $\overset{\frown}{li}_3$, although x_1 *sees* x_2 and x_3. (On the other hand, $\overset{\frown}{li}_2$ *sees* both $\overset{\frown}{li}_1$ and $\overset{\frown}{li}_3$.)

- a countable number of *inner* generators: as many as f has zeros
- a continous infinity of *exceptional* generators: any x_i where f doesn't vanish can serve as base point.

The picture also shows five types of arrows linking these generators[67]. All these arrows are one-way, except for those linking pairs of inner generators.

As this "one-way/two-way traffic" suggests, the various generators differ widely as to origin, shape, and function.

The original generator clearly stands apart, not just because it kicks off the whole generation process, but also because it makes (immediate) sense only in the ζ-plane: in the ν-plane it is relegated to infinity.

Directly proceeding from it under the *nur*-transform, we have two outer generators, which in turn generate the potentially more numerous inner generators, this time under the *nir*-transform, relatively in each case to a given determination of $f = -\log F$. To each such determination (corresponding to an additive term $2\pi ik$) there answers a distinct inner algebra Inner_f spanned by K inner generators, with $K := \mathrm{card}\{f^{-1}(0)\}$.

Another way of entering the inner algebras is via exceptional generators, but these are "artificial" in the sense that they never occur naturally, i.e. under analytic continuation of the original generator. They are more in the nature of auxiliary tools[68]. Also, since each exceptional generator results from applying the *nir*-transform to a *given* determination of $f = \log F$, it gives acces to *one* inner algebra Inner_f, unlike the outer generators, which give access to *them all*.

These inner algebras Inner_f are in one-to-one correspondance with $\mathbb{Z}$. Though distinct (and usually disjoint) from each other, they are essentially isomorphic. Each of them is also "of one piece" in the sense that for any pair $\overset{\frown}{li}_p, \overset{\frown}{li}_q \in \mathit{Inner}_f$, there is always a connecting chain li_{n_i} starting at $\overset{\frown}{li}_p$, ending at $\overset{\frown}{li}_q$, and such that any two neighbours li_{n_i} and $li_{n_{i+1}}$ *see* each other.

The emphasis so far has been on the singularities in the ν-plane. Those in the ζ-plane follow, except *over* the origin $\zeta = 0$, where quite specific and severe singularities may also occur (*at* the origin itself, i.e. on the main Riemann leaf, the SP function is of course regular). For a brief discussion of these 0-based singularities and their resurgence properties, see §7.2.1 below.

7.2 Meromorphic input F: the general picture.

Let us briefly review the main changes which take place when we relax the hypothesis about $f := -\log(F)$ being holomorphic and simply demand that F be *meromorphic*.[69]

[67] meaning in each case that the *target* is generated by the *source* under alien differentiation.

[68] as components of the *nur*-transform under the Poisson formula (see 125) and also, as we just saw, as mobile tools for sifting out true singularities from illusory ones (see §7.1.4).

[69] Since F and $F^{\vDash}$ (recall that $F^{\vDash}(x) := 1/F(1-x)$) are essentially on the same footing, it would make little sense to assume one to be *holomorphic* rather than the other. So we must assume *meromorphy*, even *strict* meromorphy, with at least one zero or pole.

7.2.1 Logarithmic/non-logarithmic singularities.

If F has at $x = 0$ a zero or pole of order $d \in \mathbb{Z}^*$, we must replace the $\sum_{0 \leq m < n}$ summation in (2) by $\sum_{0 < m < n}$ for the definition of the SP coefficients $J_F(n)$ to make sense. More significantly, depending of the parity of d, the outer and inner singularities may exchange their logarithmic/non-logarithmic nature. Recall that for the *cleansed* SP function and $d = 0$, the outer generators have purely logarithmic singularities[70] while the inner generators have power-type singularities, with strictly rational (non-entire) powers. That doesn't change when d is $\neq 0$, at least where the *cleansed* SP series are concerned. However, when we revert to the *raw* SP series, i.e. to the position prior to coefficient division by the ingress factor $Ig_F(n) \sim n^{-d/2}\,(c_0 + O(n^{-1}))$, we are faced with a neat dichotomy:
(i) d even: nothing changes.
(ii) d odd: everything gets reversed, with the outer singularities becoming strict rational (semi-integral) powers and the inner singularities becoming purely logarithmic[71]

7.2.2 Welding the inner algebras into one.

The presence of even a single zero or pole in F, no matter where – whether at x=0 or x=1 or elsewhere – suffices to abolish the distinction between the various inner algebras $Inner_f$ attached to the various determinations of $f := -\log(F)$, since f itself now becomes multivalued and assumes the form:

$$f(x) = \sum d_i \log(x - x_i) + holomorphic(x) \tag{246}$$

Everything hinges on $d := g.c.d.(d_1, d_2, \ldots)$. If $d = 1$, then all inner algebras merge into one. If $d > 1$, they merge into d distinct but "isomorphic" copies.

Notice that no such change affects the outer generators, because these are constructed, not from f, but directly from F (in the case of the *raw* SP function) or $F^{1/2}$ (in the case of the *cleansed* SP function).

7.3 The ζ-plane and its violent 0-based singularities.

Converting ν-singularities into ζ-singularities.
So far, we have been describing the outer/inner singularities in the auxiliary ν-plane (more exactly, the ν-Riemann surface) which is naturally adapted to Taylor coefficient asymptotics. To revert to the original ζ-plane, we merely apply the formulas for step 9 in the long chain of §4.2 which convert ν-singularities into ζ-singularities, for majors as well minors. The resurgence equations, too, carry over almost unchanged, with the additive indices ν_i simply turning into multiplicative indices ζ_i. But there is one exception, namely the origin $\zeta = 0$. Under the correspondence $\nu \mapsto \zeta = e^\nu$, the SP function's behaviour *over* $\zeta = 0$ will reflect its behaviour *over* the "point" $\Re(\nu) = -\infty$ on various Riemann leaves. This is the tricky matter we must now look into.

[70] i.e. with majors of type $Reg_1(\zeta) + Reg_2(\zeta)\log(\zeta)$.

[71] At least in the generic case, i.e. for a tangency order $\kappa = 1$. For $\kappa > 1$, the inner singularities involve a mixture of rational powers and logarithms.

Description/expansion of the 0-based singularities.
The SP function itself is regular *at* $\zeta = 0$, i.e. on the main Riemann leaf,[72] but usually not *over* $\zeta = 0$, i.e. on the other leaves. Studying these 0-singularities entirely reduces to studying the 0-singularities of the outer/inner generators $\overset{\wedge}{Lo}$ / $\overset{\wedge}{Li}$, which in turn reduces to investigating the ∞-behaviour of $\overset{\wedge}{lo}$ / $\overset{\wedge}{li}$. This can be done in the standard manner, by going to the long chain of §4.2 and applying the *mir*-transform to $\not{g}$, *but locally at* $-\infty$[73]. The integro-differential expansion for *mir* still converges in this case, but no longer formally so (i.e. coefficient-wise), and it still yields inner generators, but of a very special, quite irregular sort. Pulled back into the ζ-plane, they produce violent singularities *over* $\zeta = 0$, usually with exponentially explosive/implosive radial behaviour, depending on the sectorial neighbourhood of 0.

Resurgence properties of the 0-based singularities.
Fortunately, no detailed *local* description of the 0-based singularities is required to calculate their alien derivatives and, therefore, to obtain a complete system of resurgence equations for our original SP function. Indeed, turning k times around $\zeta = 0$ on *some* leaf amounts to making a $2\pi ik$-shift in the ν-plane, again on *some* leaf. But the effect of that is easy to figure out, especially for an holomorphic input f (in that case, it simply takes us from one inner algebra $Inner_f$ to the next) but also for a general meromorphic input F (for illustrations, see the examples of §8.3, especially examples 8.7 and 8.8. See also §6.6-8.)

7.4 Rational inputs F: the inner algebra.

Let F be a rational function of degree d :

$$F(x) = \prod_{1 \le j \le r} \big(1 - \frac{x}{\alpha_j}\big)^{d_j} \quad \textit{with} \quad d_j \in \mathbb{Z}^* \quad , \quad \delta := \text{g.c.d.}(|d_1|, \ldots, |d_r|) \tag{247}$$

and let $x_0, \ldots, x_{d-1}$ be the zeros (counted with multiplicities) of the equation $F(x) = 1$. We then fix a determination of the the corresponding f :

$$f(x) = -\log(F) = -\sum_{1 \le j \le r} d_j \log\big(1 - \frac{x}{\alpha_j}\big) \tag{248}$$

with its Riemann surface S_f . We denote $X_j^n \subset S_f$ the set of all $x^\star \in S_f$ lying over $x_j \in \mathbb{C}$ and such that $f(x^\star) = 2\pi i\delta$ and select some point $x_0 \in X_0^0$ as base point of S_f. The internal generators will then correspond one-to-one to the points of $\cup_{1 \le j < d} X_j^0$ and be located at points ν_j of the ramified ν-plane, with projections $\dot{\nu}_j$ such that :

$$\dot{\nu}_j - \dot{\nu}_i = \int_{x_i}^{x_j} f(x)dx \qquad (x_i \in X_i^0 \ , \ x_j \in X_j^0 \ ) \tag{249}$$

For two distinct points x'_j, x''_j in the same X_j^0, the above integral is obviously a multiple of $2\pi i\delta$. Therefore, three cases have to be distinguished.

[72] or, if F has a zero/pole of odd order d, it is of the form $\zeta^{d/2}\varphi(\zeta)$, but again with a regular φ.
[73] each time on the suitable leaf, of course.

Case 1. F has only one single zero α_1 (of any multiplicity $d_1 = p$) or again one single pole α_1 (of any multiplicity $d_1 = p$). In that case, we have exactly p sets X_j^0, but each one reduces to a single point, since f has only one single logaritmic singularity. That case ("monomial input F") was investigated in detail in §6.6, §6.7,§6.8.
Case 2: F has one simple zero α_1 and one simple pole α_2. The position is now the reverse: we then have only one set X_0^0,but with a countable infinity of points in it, since f has now two logarithmic singularities, thus allowing integrals (249) with distinct end points x_0', x_0'' both in X_0^0. (See §8.11 below).
Case 3: F has either more than two distinct zeros, or more than two distinct poles, or both. We then have $p+q$ distinct sets X_j^0 , with p (resp. q) the number of distinct zeros (resp. poles). Each such X_j^0 contains a countable number of points x_j, to which there answer, in the ramified ν-plane, distinct singular points ν_j that generate a set $\mathcal{N}_j$ whose projection $\dot{\mathcal{N}_j}$ on $\mathbb{C}$ is of the form $\nu_j + \Omega$, with

$$\Omega \;=\; \Big\{\omega\ ,\ \omega = \sum_{\sum n_j d_j = 0} n_j\, d_j\, \eta_j = -2\pi i \sum_{\sum n_j d_j = 0} n_j\, d_j\, \alpha_j\Big\} \tag{250}$$

$$\eta_j \;=\; -\int_{\mathcal{I}_j} \log(1 - \frac{x}{\alpha_j})\, dx \;=\; 2\pi i\, (x_0 - \alpha_j) \tag{251}$$

and with integration loops $\mathcal{I}_j$ so chosen as to generate the fundamental homotopy group of $\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_r\}$. Each $\mathcal{I}_j$ describes a positive turn round α_j and the choice of the loops' common end-point is immaterial, since changing the end-point merely adds adds a common constant to each η_j, which constant cancels out from the sums ω due to the condition $\sum n_j\, d_j = 0$.

Though Ω usually fails to be discrete as soon as ≥ 3, the sets $\mathcal{N}_j$ are of course always discrete in the ramified ν-plane. In particular, from any given singular point $\nu_i \in \mathcal{N}_i$ only finitely many $\nu_j \in \mathcal{N}_j$ can be *seen* – those namely that correspond to *'simple'* integration paths in (249), i.e. typically paths whose projection on $\mathbb{C}$ is *short* and doesn't *self-intersect.*[74] The other points $\nu_j \in \mathcal{N}_j$ are located on more removed Riemann leaves and therefore *hidden from view* (from ν_i).

8 The inner resurgence algebra for SP series.

8.1 Polynomial inputs f. Examples.

Example 8.1 *:* $f(x) = x^r$ There is only one inner generator $\hat{h}(\nu)$ which, up to to the factor $\nu^{-1/2}$, is an entire function of ν.

Example 8.2 *:* $f(x) = x^r - 1$ There are r inner generators. We have exact radial symmetry, of radius 1 in the x-plane and radius $\eta = 1/(r+1)$ in the ν-plane. Every singular point there sees all the others: we have 'multiple ping-pong', governed by a very simple resurgence system (see [SS1]).

[74] Various examples of such situations shall be given in §8.3, with *simple/complicated* integration paths corresponding to *visible/invisible* singularities. The general situation, with the exact criteria for visibility/invisibility, shall be investigated in [S.S.1]

Example 8.3 *:* $f(x) = \prod_{j=1}^{j=r}(x - x_j)$ Every such configuration, including the case of multiple roots, can be realised by continuous deformations of the radial-symmetric configuration of Example 8.2, and the thing is to keep track of the ν_j-pattern as the x_j-pattern changes. When $\nu_j - \nu_i$ becomes small while $x_j - x_i$ remains large, that usually reflects mutual invisibility of ν_i and ν_j. Thus, if $r = 3$ and $x_1 = 0$, $x_2 = 1$, $x_3 = 1 + \epsilon\, e^{i\theta}$ with $0 < \epsilon \ll 1$, the case $\theta = \pi/2$ with its approximate symmetry $x_2 \leftrightarrow x_3$ corresponds to three mutually visible singularities ν_1, ν_2, ν_3, but when θ decreases to 0, causing x_3 to make a $-\pi/2$ rotation around x_2, the point ν_3 makes a $-3\pi/2$ rotation around ν_2, so that the projection $\dot{\nu}_3$ actually lands on the real interval $[\dot{\nu}_1, \dot{\nu}_2]$. But the new ν_3 has actually moved to an adjacent Riemann leaf and is no longer *visible* from ν_1.

8.2 Holomorphic inputs f. Examples.

Example 8.4 *:* $f(x) = \exp(x)$
To the unique 'zero' $x_0 = -\infty$ of $f(x)$ there answers a unique inner generator $\hat{h(\nu)}$. It is of rather exceptional type, in as far as its local behaviour is described by a *transseries* rather than a series, but the said transseries is still produced by the usual mechanism of the nine-link chain.

Example 8.5 *:* $f(x) = \exp(x) - 1$ *or* $f(x) = \sin^2(x)$
All zeros x_i of $f(x)$ contribute distinct inner generators, identical up to shifts but positioned at different locations ν_j.

Example 8.6 *:* $f(x) = \sin(x)$
Here, the periodic $f(x)$ still has infinitely many zeros but is constant-free (i.e. is itself the derivative of a periodic function). As a consequence, we have just two inner generators, at two distinct locations, like in the case $f(x) = 1 - x^2$ but of course with a more complex resurgence pattern.

8.3 Rational inputs F. Examples.

Example 8.7 *:* $F(x) = (1 - x)$
The inner algebra here reduces to one generator $h(\nu)$ and a fairly trivial one at that, since $h(\nu) = \mathit{const}\, \nu^{-1/2}$, as given by the *semi-entire* part of the *nir*-transform. In contrast, the *entire* part of the *nir*-transform (which lacks intrinsic significance) is, even in this simplest of cases, a highly transcendental function: in particular, it verifies no linear ODE with polynomial coefficients.

Example 8.8 *:* $F(x) = (1 - x)^p$
Under the change $x \to p\, x$, this reduces the case of "monomial F", which was extensively investigated in §6.6, §6.7, §6.8. We have now exactly p internal generators $h_j(\nu)$ located at the unit roots $\nu_j = -e^{2\pi i\, j/p}$ and verifying a simple ODE of order p, with polynomial coefficients. Each singular point ν_j 'sees' all the others, and the resurgence regimen is completely encapsulated in the matrices $\mathcal{M}_{p,q}$ of §6.7, which account for the basic closure phenomenon: a $4\pi i$-rotation (around any base point) leaves the whole picture unchanged.

Example 8.9 $: F(x) = \frac{x^2-\alpha^2}{1-\alpha^2} = \frac{x^2+\beta^2}{1+\beta^2} \;\; , \;\; \alpha = i\,\beta$
The general results of §7.4 apply here, with $x_0 = 0, x_1 = 1$ and the lattice $\Omega = 4\pi i\alpha\,\mathbb{Z} = -4\pi\beta\,\mathbb{Z}$. We have therefore two infinite series of internal generators in the ν-plane, located over $\dot\nu_0 + \Omega$ and $\dot\nu_1 + \Omega$ respectively, where the difference $\dot\nu_1 - \dot\nu_0$ may be taken equal to any determination of $-\int_0^1 \log(F(x))dx$. However, depending on the value of the parameters α, β, each singular point ν_j 'sees' *one, two* or *three* singular points of the 'opposite' series. Let us illustrate this on the three 'real' cases:
Case 1: $0 < \beta$.
The only singularity *seen* (resp. *half-seen*) from ν_0 is ν_1 (resp. ν_1^*) with

$$\begin{aligned} \dot\nu_1 &= \dot\nu_0 + 2\,\eta \\ \dot\nu_1^* &= \dot\nu_0 + 2\,\eta + 4\pi\beta \\ \textit{with} \qquad \eta &= 2 - 2\,\beta \arctan(1/\beta) > 0 \end{aligned}$$

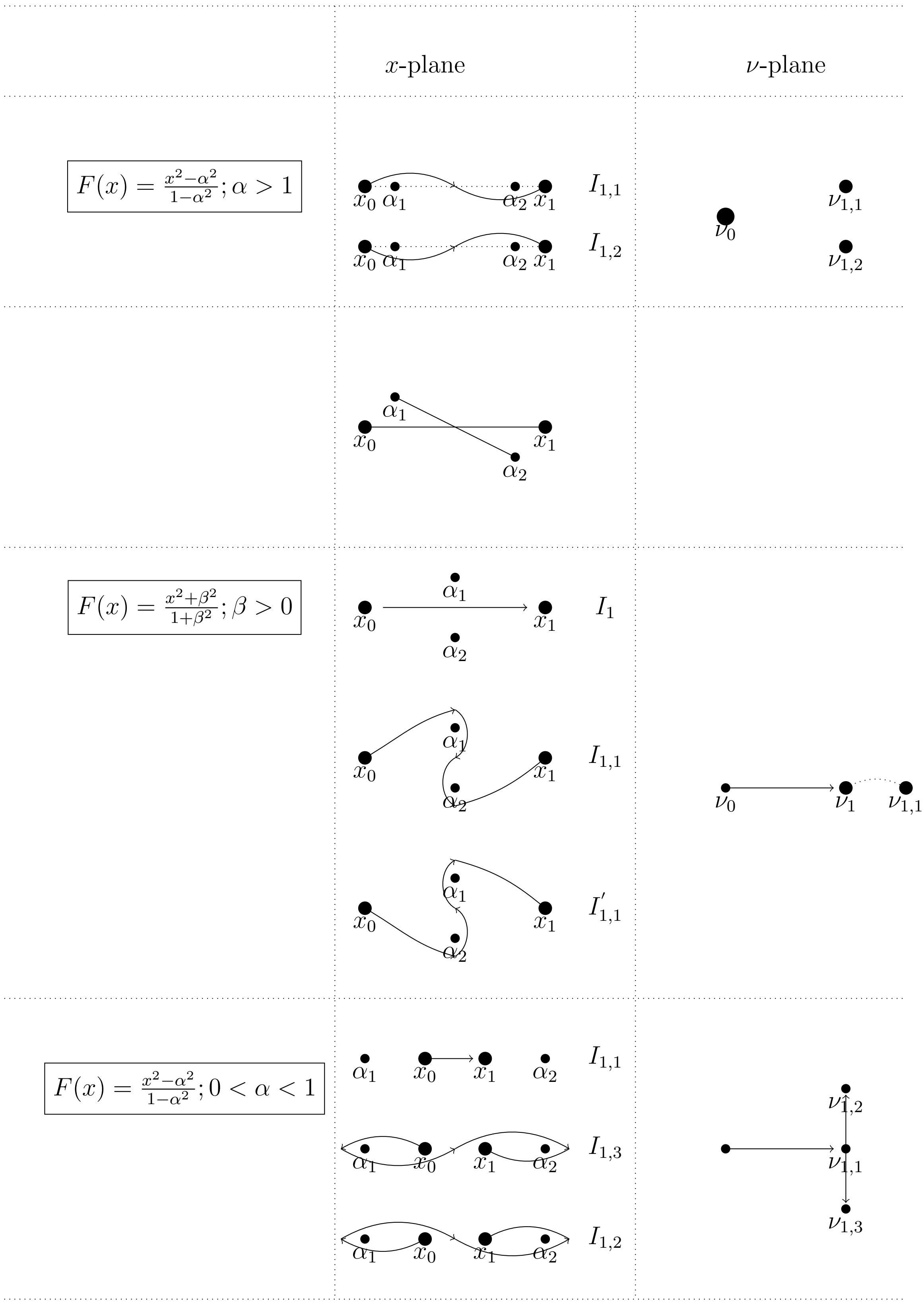
All other singularities above $\dot\nu_1 + \Omega$ lie are on further Riemann leaves. The singularity ν_1 corresponds to the straight integration path $\mathcal{I}_1$ whereas ν_1^* corresponds to either of the equivalent paths $\mathcal{I}_1^*$ and $\mathcal{I}_1^{**}$.
Case 2: $0 < \alpha < 1$.
Only two singularities are *seen* from ν_0, namely ν_1^* and ν_1^{**} of projections :

$$\begin{aligned} \dot\nu_1^* &= \dot\nu_0 + 2\,\eta + 2\,\pi i\alpha \\ \dot\nu_1^{**} &= \dot\nu_0 + 2\,\eta - 2\,\pi i\alpha \\ \textit{with} \qquad \eta &= 2 - \alpha \log\big(\frac{1+\alpha}{1-\alpha}\big) > 0 \end{aligned}$$

They correspond to the integration paths $\mathcal{I}^*$ and $\mathcal{I}^{**}$.
Case 3: $1 < \alpha$.
Three singularities are *seen* from ν_0, namely $\nu_1, \nu_1^*, \nu_1^{**}$ of projections :

$$\begin{aligned} \dot\nu_1 &= \dot\nu_0 + 2\,\eta \\ \dot\nu_1^* &= \dot\nu_0 + 2\,\eta + 2\,\pi i\alpha \\ \dot\nu_1^{**} &= \dot\nu_0 + 2\,\eta - 2\,\pi i\alpha \\ \textit{with} \qquad \eta &= 2 - \alpha \log\big(\frac{\alpha+1}{\alpha-1}\big) < 0 \end{aligned}$$

They correspond to the integration paths $\mathcal{I}, \mathcal{I}^*, \mathcal{I}^{**}$.



Example 8.10 $: F(x) = \frac{x^p - \alpha^p}{1-\alpha^p} = \frac{x^p+\beta^p}{1+\beta^p}$, $\epsilon = e^{\pi i/p}\beta$.
Here Ω is generated by the unit roots of order p. More precisely, due to the condition

$\sum n_j\, d_j = 0$ in (219) (with $d_j = 1$ here) we have

$$\Omega = 2\pi i\alpha\Big((\epsilon-1)\mathbb{Z} + (\epsilon^2-1)\mathbb{Z} + \ldots (\epsilon^{p-1}-1)\mathbb{Z}\Big) \qquad \textit{with} \quad \epsilon := e^{2\pi i/p}$$

Thus, except for $p \in \{2,3,4,6\}$ the point set Ω is never discrete, but this doesn't prevent there being, from any point of the ramified ν-plane, only a finite number of *visible* singularities.

Case 1: $0 < \beta$.

$$\begin{aligned}
\Omega_j &= (\eta+\Omega)\,\epsilon^j = \eta\,\epsilon^j + \Omega \qquad (1 \le j \le p \;\;,\;\; \epsilon = e^{\pi i/p}\beta) && (252)\\
\eta &= -p\sum_{1\le k}(-1)^k \frac{\beta^{-kp}}{kp+1} > 0 \qquad (\textit{if}\;\; 1 \le \beta) && (253)\\
\eta &= p - p\,b_p\,\beta - p\sum_{1\le k}(-1)^k \frac{\beta^{kp}}{kp-1} > 0 \qquad (\textit{if}\;\; 0 < \beta \le 1) && (254)\\
b_p &= \int_0^1 \frac{1+t^{p-2}}{1+t^p}dt \;=\; 1 - 2\sum_{1\le k}\frac{(-1)^k}{k^2p^2-1} && (255)
\end{aligned}$$

Case 2: $0 < \alpha < 1$.

$$\begin{aligned}
\Omega_j &= (\eta+\pi i\alpha+\Omega)\,\epsilon^j \;=\; (\eta-\pi i\alpha+\Omega)\,\epsilon^j && (256)\\
&= (\eta+\pi i\alpha)\,\epsilon^j + \Omega \;=\; (\eta-\pi i\alpha)\,\epsilon^j + \Omega \qquad (1 \le j \le p \;\;,\;\; \epsilon = e^{\pi i/p}\beta) &&\\
\eta &= p - p\,a_p\,\alpha - p\sum_{1\le k}\frac{\alpha^{kp}}{kp-1} > 0 && (257)\\
a_p &= \int_0^1 \frac{1-t^{p-2}}{1-t^p}dt \;=\; 1 - 2\sum_{1\le k}\frac{1}{k^2p^2-1} && (258)
\end{aligned}$$

Case 3: $1 < \alpha$.

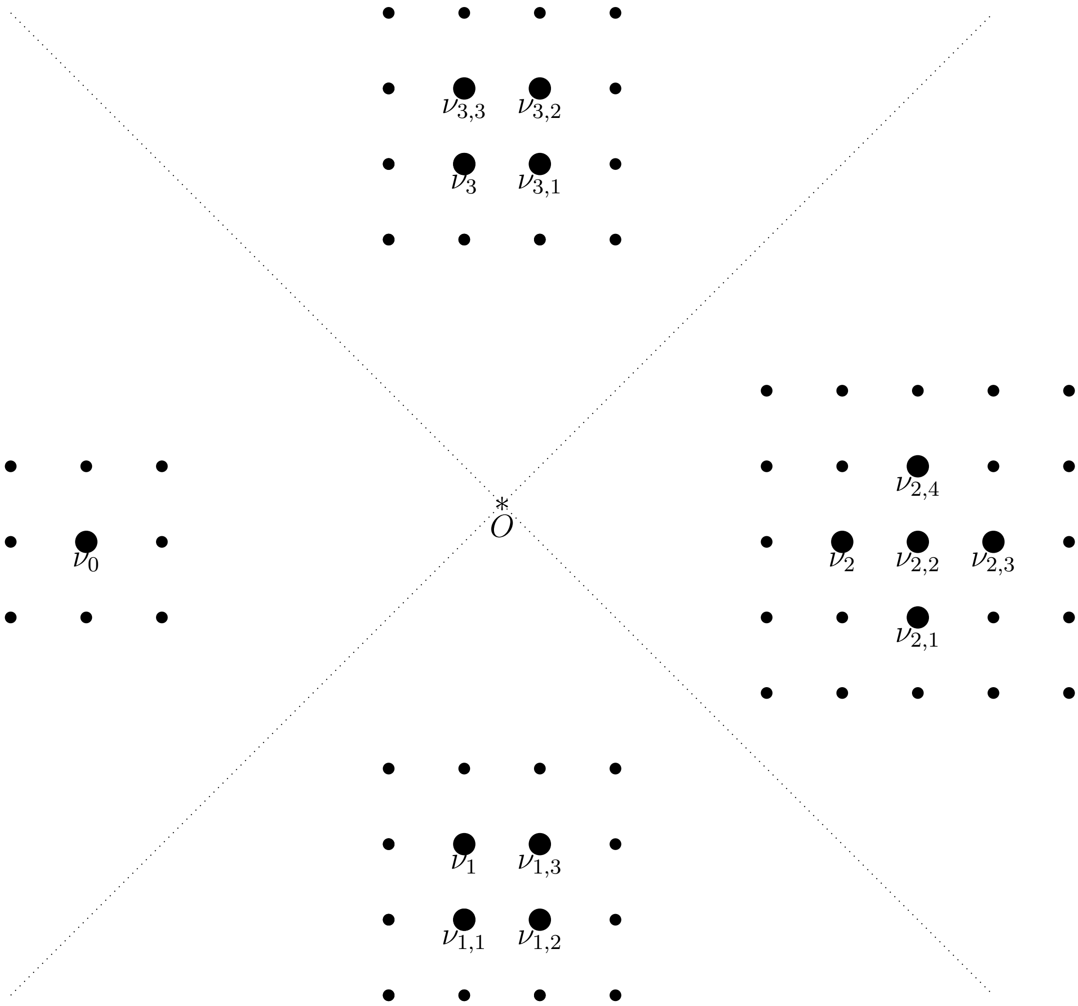
$$\begin{aligned}
\Omega_j &= (\eta+\Omega)\,\epsilon^j = \eta\,\epsilon^j + \Omega \qquad (1 \le j \le p \;\;,\;\; \epsilon = e^{\pi i/p}\beta) && (259)\\
\eta &= -p\sum_{1\le k}\frac{\alpha^{-kp}}{kp+1} < 0 && (260)
\end{aligned}$$

Remark: the expressions (255) for b_p are obtained by identifying the two distinct expressions (253),(254) for η which are are equally valid when $\beta = 1$. The expressions (258) for a_p are *formally* obtained in the same way, i.e. by equating the expressions (257),(260) when $\alpha = 1$, but since both diverge in that case, the derivation is illegitimate, and the proper way to proceed is by rotating α by $e^{\pi i/p}$ so as to fall back on the situation of case

1. Here are the $\mathbb{Z}$-irreducible equations verified by the first algebraic numbers $\alpha_p := \frac{p}{\pi}\, a_p$ and $\beta_p := \frac{p}{\pi}\, b_p$:

$$
\begin{array}{rclcrcl}
0 &=& \alpha_2 & \qquad & 0 &=& \beta_2 - 1 \\
0 &=& 3\,\alpha_3^2 - 1 & & 0 &=& 3\,\beta_3^2 - 4 \\
0 &=& \alpha_4 - 1 & & 0 &=& \beta_4^2 - 2 \\
0 &=& 5\,\alpha_5^4 - 10\,\alpha_5^2 + 1 & & 0 &=& 5\,\beta_5^4 - 20\,\beta_5^2 + 16 \\
0 &=& \alpha_6^2 - 3 & & 0 &=& \beta_6 - 2 \\
0 &=& \alpha_8^4 - 6\,\alpha_8^2 + 1 & & 0 &=& \beta_8^4 - 8\,\beta_8^2 + 8 \\
0 &=& \alpha_{10}^4 - 10\,\alpha_{10}^2 + 5 & & 0 &=& \beta_{10}^2 - 2\,\beta_{10} - 4 \\
0 &=& \alpha_{12}^2 - 4\,\alpha_{12} + 1 & & 0 &=& \beta_{12}^4 - 16\,\beta_{12}^2 + 16
\end{array}
$$

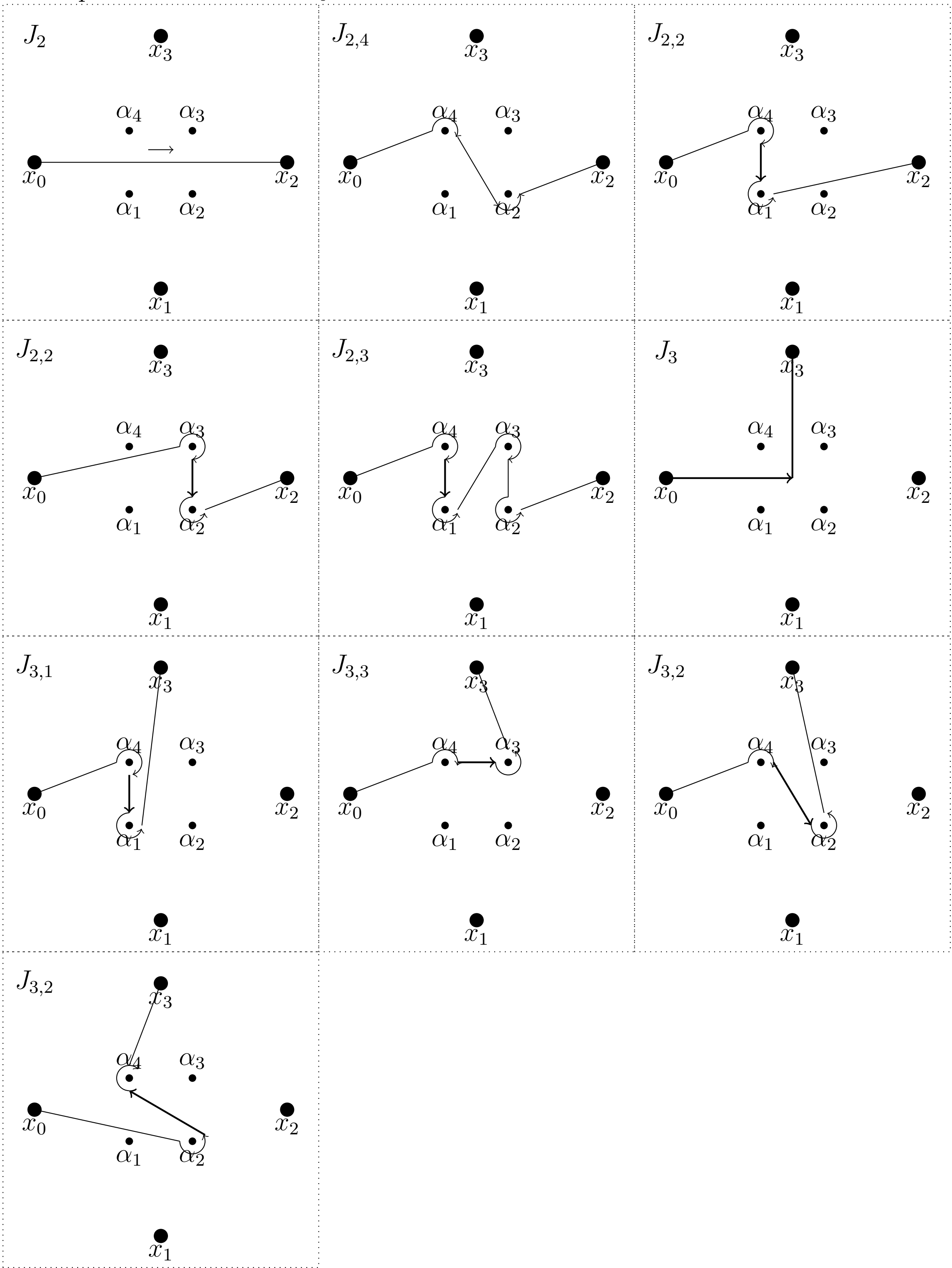
Let us illustrate the situation for $p = 4$ in all three 'real' cases. We choose one singularity ν_0, corresponding to $x_0 = -1$ (resp. $x_0 = 1$) in case 1 or 2 (resp. 3), as base point of the ν-plane, and plot as bold (resp. faint) points all singularities visible or semi-visible from ν_0 (resp. the closest invisible ones). For clarity, the scale (i.e. the relative values of α, η, π) has not been strictly respected – only the points' relatives position has.

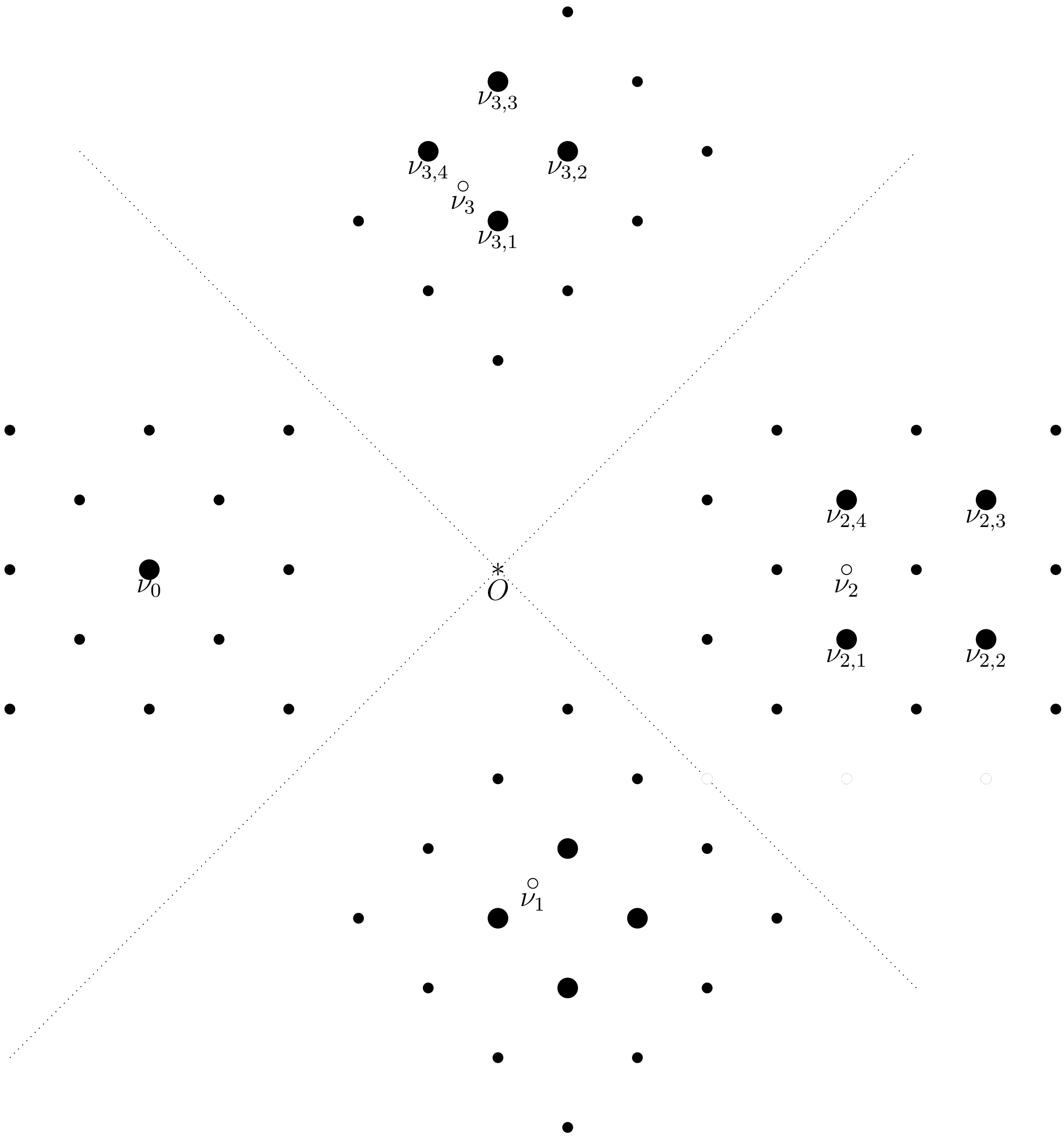


$$\boxed{F(x) = \frac{x^4+\beta^4}{1+\beta^4}; \beta > 0}$$

ν-plane viewed from ν_0

x -plane viewed from x_0

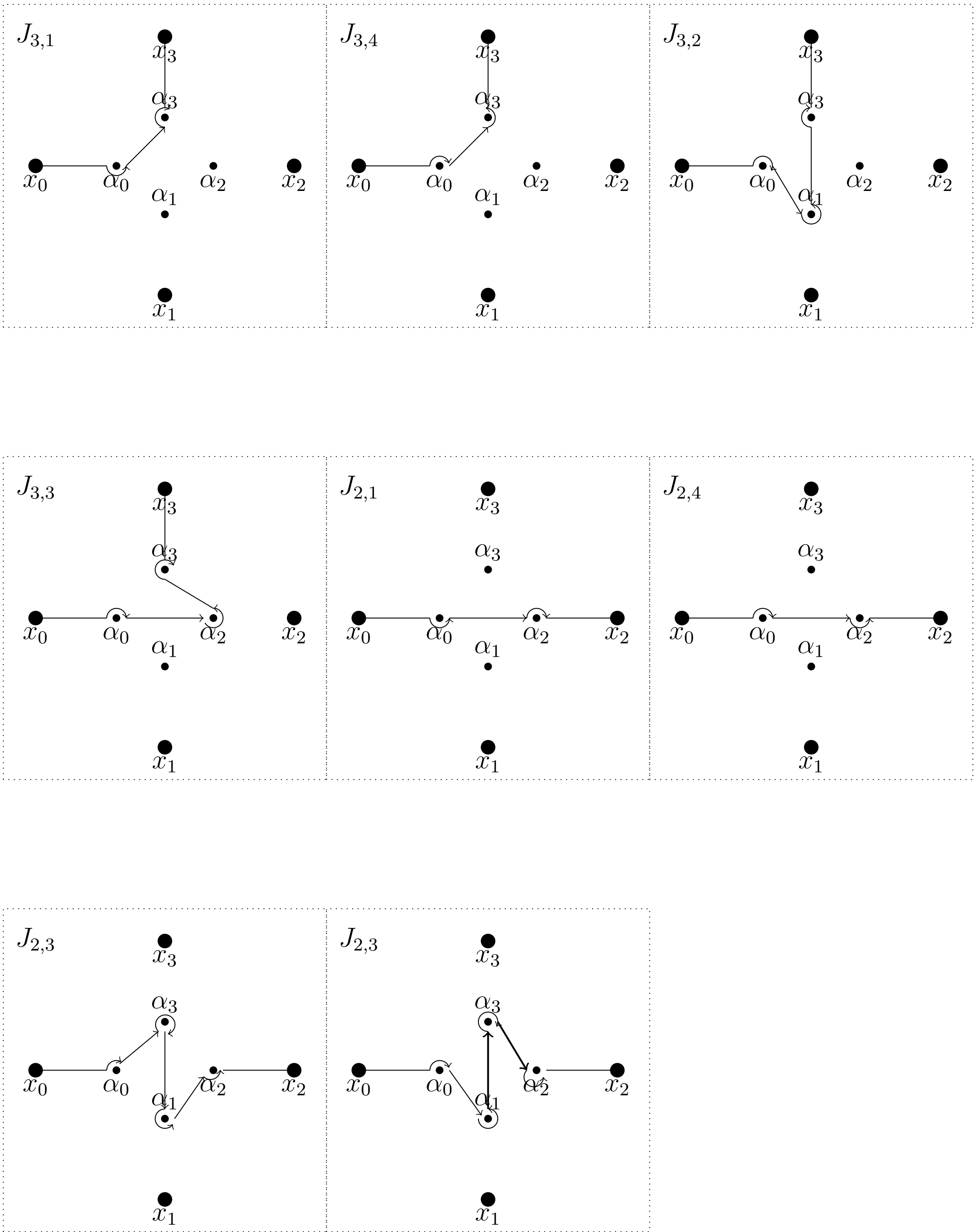


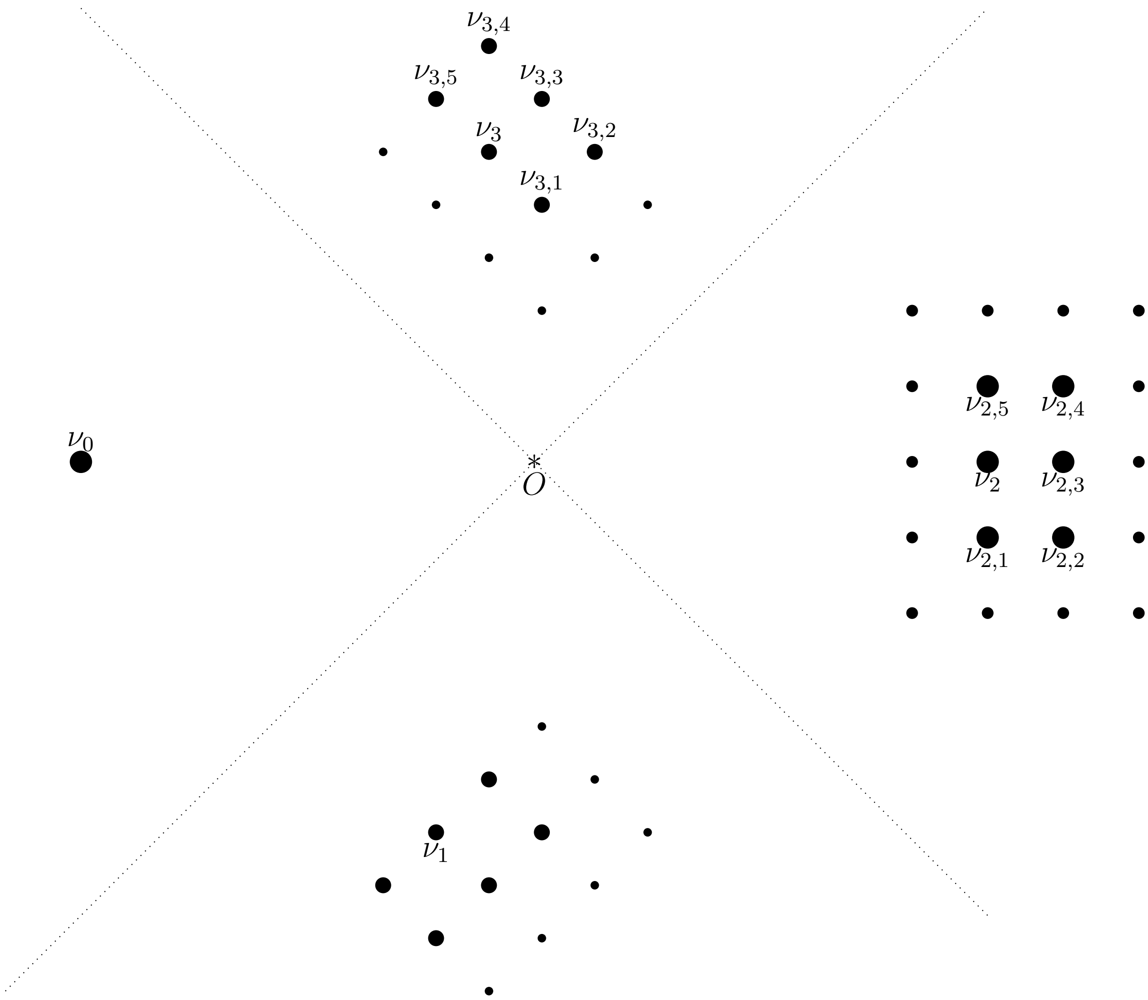


$\boxed{F(x) = \frac{x^4-\alpha^4}{1-\alpha^4}; 0 < \alpha < 1}$

ν-plane viewed from ν_0

x -plane viewed from x_0

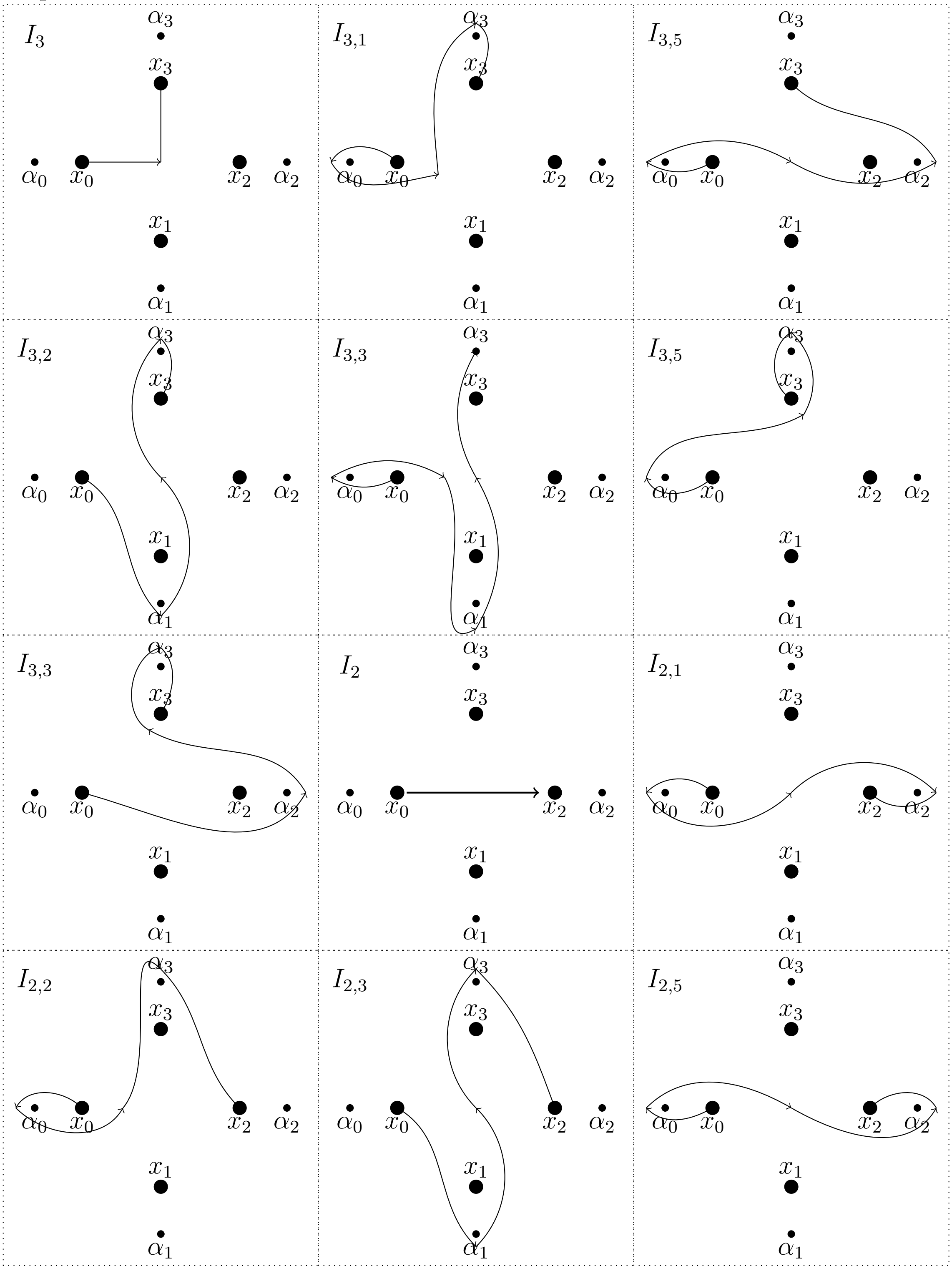




$$\boxed{F(x) = \frac{x^4-\alpha^4}{1-\alpha^4}; 1 < \alpha}$$

ν -plane viewed from ν_0.

x -plane viewed from x_0



Example 8.11 $: F(x) = \frac{1-x}{1+x}$ *or* $F(x) = \frac{1-x/\alpha}{1+x/\beta}$
This interesting case is the only one where, despite the equation $F(x) = 1$ having only one solution $x_0 = 0$, the function f has two logarithmic singularities, so that we get a non-trivial set $\Omega = 2\pi i\mathbb{Z}$ and infinitely many copies of one and the same inner generator. From any given singularity ν_0 there are two *visible* neighbouring singularities over $\dot{\nu}_0 \pm 2\pi i$

and infinitely many *semi-visible* ones over $\dot{\nu}_0 \pm 2\pi i\, k\, (k \geq 2)$.

Example 8.12 $: F(x) = \frac{(1-x/\alpha_1)(1-x/\alpha_2)}{(1-x/\alpha_3)}$ *or* $F(x) = \frac{(1-x/\alpha_1)(1-x/\alpha_2)}{(1-x/\alpha_3)(1-x/\alpha_4)}$
Here the equation $F(x) = 0$ has two distinct solutions, so we have two distinct families of 'parallel' inner generators, and a set Ω which is generically discrete in the first sub-case (no α_4) and generically non-discrete in the second sub-case.[75]

8.4 Holomorphic/meromorphic inputs F. Examples.

Example 8.13 $: \quad F = \prod_{j=1}^{\infty} \left(1 - \frac{x}{\alpha_j}\right) e^{A_j(x)}$ *or* $F = \frac{\prod_{j=1}^{\infty}(1-x/\alpha_j)e^{A_j(x)}}{\prod_{j=1}^{\infty}(1-x/\beta_j)e^{B_j(x)}}$.
Predictably enough, we inherit here features from the case of polynomial inputs f and from that of rational inputs F, but three points need to be stressed :
(i) the presence of even a single zero α_j or of a single pole β_j is enough to weld all inner algebras $Inner_f$ into one (see §7.2.2 *supra*).
(ii) though, for a given x_0, the numbers $\eta_{0,j} := -\int_{x_0}^{x_j} f(x)dx$ may accumulate 0, the corresponding singularities ν_j never accumulate ν_0 in the ramified ν-plane.
(iii) the question of deciding which integration paths (in the x-plane) lead to visible singularities (in the ν-plane) is harder to decide than for purely polynomial inputs f or purely rational inputs F because, unlike in these two earlier situations, we don't always have the option of *deforming* a configuration with full radial symmetry. The precise criteria for visibility/invisibility shall be given in [SS 2].

Example 8.14 $: \ F =$ *trigonometric polynomial.*
The series associated with knots tend to fall into this class. Significant simplifications occur, especially when $f = -\log(F)$ is itself the derivative of a periodic function, because the number of singularities ν_j then becomes finite up to Ω-translations. The special case $F(x) = 4\sin^2(\pi x)$, which is relevant to the knot 4_1, is investigated at length in the next section.

9 Application to some knot-related power series.

9.1 The knot 4_1 and the attached power series G^{NP}, G^P.

Knot theory attaches to each knot $\mathcal{K}$ two types of power series : the so-called *non-perturbative* series $G^{NP}_{\mathcal{K}}$ and their *perturbative* companions $G^{P}_{\mathcal{K}}$. Both encode the bulk of the invariant information about $\mathcal{K}$ and both are largely equivalent, though non-trivially so : each one can be retieved from the other, either by non-trivial arithmetic manipulations (the Habiro approach) or under a non-trivial process of analytic continuation (the approach favoured in this section).

[75] As usual, this discrete/non-discrete dichotomy applies only to the projection on $\mathbb{C}$ of the ramified ν-plane, which is itself always a *discrete* Riemann surface, with only a *discrete* configuration of singular points visible from any given base point.

The main ingredient in the construction of $G^{NP}_{\mathcal{K}}$ and $G^{P}_{\mathcal{K}}$ is the so-called *quantum factorial*, classically denoted $(q)_m$:

$$(q)_m := \prod_{k=1}^{k=m} (1-q^k) \tag{261}$$

For the simplest knots, namely $\mathcal{K} = 3_1$ or 4_1 in standard notation, the general definitions yield :

$$\begin{array}{lcl lcl}
\Phi_{3_1}(q) & := & \sum_{m\geq 1}(q)_m & \Phi_{4_1}(q) & := & \sum_{m\geq 1}(q)_m(q^{-1})_m \\
\hat{G}{}^{NP}_{3_1}(\zeta) & := & \sum_{n\geq 0}\Phi_{3_1}(e^{2\pi i/n})\,\zeta^n & \hat{G}{}^{NP}_{4_1}(\zeta) & := & \sum_{n\geq 0}\Phi_{4_1}(e^{2\pi i/n})\,\zeta^n \\
\tilde{G}^{P}_{3_1}(n) & := & \Phi_{3_1}(e^{-1/n}) = \sum c_k\, n^{-k} & \tilde{G}^{P}_{4_1}(n) & := & \Phi_{4_1}(e^{-1/n}) = \sum c^*_k\, n^{-k} \\
\hat{G}{}^{P}_{3_1}(\nu) & := & \sum c_k\, \frac{\nu^{k-1}}{(k-1)!} & \hat{G}{}^{P}_{4_1}(\nu) & := & \sum c^*_k\, \frac{\nu^{k-1}}{(k-1)!}
\end{array}$$

A few words of explanation are in order here.

First, when we plug unit roots $q = e^{2\pi i/n}$ into the infinite series $\Phi_{3_1}(q)$ or $\Phi_{4_1}(q)$, these reduce to finite sums.

Second, the coefficients $\Phi_{3_1}(e^{2\pi i/n})$ or $\Phi_{4_1}(e^{2\pi i/n})$ thus defined are syntactically of *sum-product* type, relative to the driving functions :

$$F_{3_1}(x) := 1 - e^{2\pi i x} \;;\; F_{4_1}(x) := (1-e^{2\pi i x})(1-e^{-2\pi i x}) \;=\; 4\sin^2(\pi x) \tag{262}$$

Third, whereas the non-perturbative series $\hat{G}{}^{NP}$ clearly possess positive radii of convergence, their perturbative counterparts $\tilde{G}^P$ are divergent power series of $1/n$, of Gevrey type 1, i.e. with coefficients bounded by

$$|c_k| < Const\ k! \quad , \quad |c^*_k| < Const^*\, k!$$

Fourth, the perturbative series $\tilde{G}^P(n)$ being Gevrey-divergent, we have to take their Borel transforms $\hat{G}{}^{P}(\nu)$ to restore convergence.

Here, we won't discuss the series attached to knot 3_1, because that case has already been thoroughly investigated by Costin-Garoufalidis [CG1],[CG2] and also because it is rather atypical, with an uncharacteristically poor resurgence structure : indeed, $G^{NP}_{3_1}$ and $G^{P}_{3_1}$ give rise to only *one* inner generator Li, whereas it takes at least two of them for the phenomenon of ping-pong resurgence to manifest.

So we shall concentrate on the next knot, to wit 4_1, with its driving function $F(x) := 4\sin^2(\pi x)$. That case was/is also being investigated by Costin-Garoufalidis but with methods quite different from ours : see §12.2 below for a comparison. Here, we shall approach the problem as a special case of *sum-product* series, unravel the underlying resurgence structure, and highlight the typical interplay between the four types of generators : *original, exceptional, outer, inner.*

Our main *original* generator Lo and main *outer* generator Lu, both corresponding to the same base point $x = 0$, shall turn out to be essentially equivalent, respectively, to the *non-perturbative* and *perturbative* series of the classical theory, with only minor differences stemming from the *ingress factor* (see below) and a trivial $2\pi i$ rotation. The exact correspondence goes like this:

$$\hat{G}{}^{NP}_{4_1}(\zeta) \;\equiv\; \zeta\,\partial_\zeta\,\overset{\wedge}{Lo}(\zeta) \tag{263}$$

$$\hat{G}{}^{P}_{4_1}(\nu) \;\equiv\; \frac{1}{2\pi i}\,\partial_\zeta\,\overset{\wedge}{lu}(2\pi i\nu) \;=\; \frac{1}{2\pi i}\,\partial_\zeta\,\overset{\wedge}{Lu}(e^{2\pi i\nu}-1) \tag{264}$$

But we shall also introduce other generators, absent from the classical picture: namely an *exceptional* generator Le, relative to the base-point $x = 1/2$, as well as a new pair consisting of a secondary *original* generator Loo and a secondary *outer* generator Luu, also relative to the base-point $x = 1/2$.

We shall show that these generators don't self-reproduce under alien differentiation, but vanish without trace: they are mere *gates* for entering the true core of the resurgence algebra, namely the *inner algebra*, which in the present instance will be spanned by just two *inner* generators, Li and Lii.

9.2 Two contingent ingress factors.

Applying the rules of §3 we find that to the driving function Fo and its translate Foo:

$$Fo(x) = F(x) = 4\sin^2(\pi\, x) \;\; ; \;\; Foo(x) = F(x+\frac{1}{2}) = 4\cos^2(\pi\, x) \tag{265}$$

there correspond the following ingress factors:

$$Ig_{Fo}(n) := (4\,pi^2)^{-1/2}\,(2\pi\, n)^{2/2} = n \qquad ; \qquad Ig_{Foo}(n) := 4^{1/2} = 2 \tag{266}$$

Their elementary character stems from the fact the only contributing factors in $Fo(x)$ and $Foo(x)$ are $4\pi^2x^2$ and 4 respectively. All other *binomial* or *exponential* factors inside $Fo(x)$ and $Foo(x)$ contribute nothing, since they are *even* functions of x.

Leaving aside the totally trivial ingress factor $Ig_{Foo}(n) = 2$, we can predict what the effect will be of removing $Ig_{Fo}(n) = n$ from $\hat{G}{}^{NP}(\zeta)$ and all its alien derivatives: it will *smoothen* all singularities under what shall amount to one ζ-integration. In particular, it shall replace the leading terms $C_1(\zeta-\zeta_1)^{-5/2}$ and $C_3(\zeta-\zeta_3)^{-5/2}$ in the singularities of $\hat{G}{}^{NP}(\zeta)$ at ζ_1 and ζ_3 by the leading terms $C_1'(\zeta-\zeta_1)^{-3/2}$ and $C_3'(\zeta-\zeta_3)^{-3/2}$ typical of *inner generators* produced by driving functions $f(x)$ of tangency order $m = 1$ (see §4).

Remark: an alternative, more direct but less conceptual way of deriving the form of the ingress factor $Ig_{Fo}(n) = n$ would be to use the following trigonometric identities:

$$K_{n,n-1} \equiv n^2 \;,\; K_{2\,n,n-1} \equiv n \;,\; K_{2\,n,n} \equiv 4\,n \;,\; K_{2\,n+1,n} \equiv 2\,n+1 \tag{267}$$

with

$$K_{n,m} := \prod_{1\le k\le m} F(\frac{k}{m}) = 4^m \prod_{1\le k\le m} \sin^2(\pi\frac{k}{m}) \tag{268}$$

9.3 Two original generators Lo and Loo.

Here are the power series of *sum-product* type corresponding to the driving functions *Fo* and *Foo* (mark the lower summation bounds: first 1, then 0):

$$\overset{\wedge}{Jo}(\zeta) := \sum_{1\leq n} Jo_n\,\zeta^n \quad \textit{with} \quad Jo_n := \sum_{m=1}^{m=n}\prod_{k=1}^{k=m} Fo(\frac{k}{n}) \tag{269}$$

$$\overset{\wedge}{Joo}(\zeta) := \sum_{1\leq n} Joo_n\,\zeta^n \quad \textit{with} \quad Joo_n := \sum_{m=0}^{m=n}\prod_{k=0}^{k=m} Foo(\frac{k}{n}) \tag{270}$$

After removal of the respective ingress factors $Ig_{Fo}(n)$ and $Ig_{Foo}(n)$ these become our two *'original generators'*:

$$\overset{\wedge}{Lo}(\zeta) := \sum_{1\leq n} Lo_n\,\zeta^n = \sum_{1\leq n}\frac{1}{n}Jo_n\,\zeta^n \implies \overset{\wedge}{Lo}(\zeta) = \int_0^{\zeta}\overset{\wedge}{Jo}(\zeta')\frac{d\zeta'}{\zeta'} \tag{271}$$

$$\overset{\wedge}{Loo}(\zeta) := \sum_{1\leq n} Loo_n\,\zeta^n = \sum_{1\leq n}\frac{1}{2}Joo_n\,\zeta^n \implies \overset{\wedge}{Loo}(\zeta) = \frac{1}{2}\overset{\wedge}{Joo}(\zeta) \tag{272}$$

9.4 Two outer generators Lu and Luu.

The two outer generators $\overset{\wedge}{Lu}$ and $\overset{\wedge}{Luu}$ (resp. their variants $\overset{\wedge}{\ell u}$ and $\overset{\wedge}{\ell uu}$) are produced as outputs H (resp. $\hat{k}$) by inputting $F = Fo$ or $F = 1/Foo$ into the short chain §5.2 and duly removing the ingress factor Ig_{Fo} or Ig_{Foo}. Since both $Fo(x)$ and $Foo(x)$ are even functions of x, we find:

$$\begin{array}{llllll}
\tilde{\ell u}(n) & := & \frac{1}{n}\sum_{1\leq m}\prod_{1\leq k\leq m} Fo(\frac{k}{n}) & \tilde{\ell uu}(n) & := & \frac{1}{2}\sum_{1\leq m}\prod_{1\leq k\leq m}(1/Foo)(\frac{k}{n}) \\
\Downarrow & & & \Downarrow & & \\
\tilde{\ell u}(n) & := & \sum_{1\leq k} c_{2k+1}\, n^{-2k-1} & \tilde{\ell uu}(n) & := & \sum_{0\leq k} c^*_{2k}\, n^{-2k} \\
\Downarrow & & & \Downarrow & & \\
\overset{\wedge}{\ell u}(\nu) & := & \sum_{1\leq k} c_{2k+1}\,\frac{\nu^{2k}}{(2k)!} & \overset{\wedge}{\ell uu}(\nu) & := & \sum_{0\leq k} c^*_{2k}\,\frac{\nu^{2k-1}}{(2k-1)!} \\
\Downarrow & & & \Downarrow & & \\
\overset{\wedge}{Lu}(\zeta) & := & \overset{\wedge}{\ell u}(\log(1+\zeta)) & \overset{\wedge}{Luu}(\zeta) & := & \overset{\wedge}{\ell uu}(\log(1+\zeta))
\end{array}$$

There is a subtle difference between the two columns, though. Whereas in the left column, the sum product $\sum\prod$ truncated at order m yields the *exact* values of all coefficients c_{2k+1} up to order m, the same doesn't hold true for the right column: here, the truncation of $\sum\prod$ at order m yields only approximate values of the coefficients c_{2k} (of course, the larger m, the better the approximation). This is because $Fo(0) = 0$ but $1/Foo(0) \neq 0$. Therefore, whereas the short, four-link chain of §5.2 suffices to give the exact coefficients c_{2k+1}, one must resort to the more complex *nur*-transform, as articulated in the long, nine-link chain of §5.2, to get the exact value of any given coefficient c^*_{2k}.

9.5 Two inner generators Li and Lii.

The two outer generators $\overset{\wedge}{Li}$ and $\overset{\wedge}{Lii}$ (resp. their variants $\overset{\wedge}{\ell i}$ and $\overset{\wedge}{\ell ii}$) are produced as outputs h (resp. H) by inputting

$$f(x) = fi(x) = -\log\big(4\sin^2(\pi(x+\frac{5}{6}))\big) = +2\sqrt{3}\pi\,x + 4\pi^2\,x^2 + O(x^3) \tag{273}$$

$$f(x) = fii(x) = -\log\big(4\sin^2(\pi(x+\frac{1}{6}))\big) = -2\sqrt{3}\pi\,x + 4\pi^2\,x^2 + O(x^3) \tag{274}$$

into the long chain of §4.2 expressive of the *nir*-transform. However, due to an obvious symmetry, it is enough to calculate $\overset{\wedge}{\ell i}\,(\nu)$ and then deduce $\overset{\wedge}{\ell ii}\,(\nu)$ under (essentially) the chance $\nu \to -\nu$. Notice that the tangency order here is $m = 1$, leading to semi-integral powers of ν :

$$\overset{\wedge}{\ell i}\,(\nu) := \sum_{0\le n} d_{-\frac{3}{2}+n}\,\nu^{-\frac{3}{2}+n} \quad ; \quad \overset{\wedge}{\ell ii}\,(\nu) := \sum_{0\le n} (-1)^n\,d_{-\frac{3}{2}+n}\,\nu^{-\frac{3}{2}+n} \tag{275}$$

Notice, too, that there is no need to bother about the ingress factors here: the very definition of the *nir*-transform automatically provides for their removal.

9.6 One exceptional generator Le.

The exceptional generators $\overset{\wedge}{Le}$ (resp. their variant $\overset{\wedge}{\ell e}$) is produced as output h (resp. H) by inputting

$$f(x) = fo(x) = -\log\Big(4\sin^2\big(\pi(x+\frac{1}{2})\big)\Big) = -2\log 2 + \pi^2\,x^2 + O(x^4) \tag{276}$$

into the long chain of §4.2 expressive of the *nir*-transform. The tangency order here being $m = 0$ and $fo(x)$ being an even function of x, the series $\overset{\frown}{\ell e}$ (resp. $\overset{\wedge}{\ell e}$) carries only integral-even (resp. integral-odd) powers of ν :

$$\overset{\frown}{\ell e}\,(\nu) := \sum_{0\le n} c^{**}_{2\,n}\,\nu^{2\,n} \qquad \overset{\wedge}{\ell e}\,(\nu) := \sum_{0\le n} 2n\,c^{**}_{2\,n}\,\nu^{2\,n-1} \tag{277}$$

As with the *inner generators*, the *nir*-transform automatically takes care of removing the ingress factor.

9.7 A complete system of resurgence equations.

Before writing down the exact resurgence equations, let us depict them graphically, in the two pictures below, where each arrow connecting two generators signals that the *target generator* can be obtained as an alien derivative of the *source generator*.

$$\begin{array}{ccccccc}
 & & \longrightarrow & \longrightarrow & \textit{inner generator}\ \mathrm{Li} & & \\
 & \nearrow & & \nearrow & \uparrow\downarrow & \nwarrow & \\
\textit{original generator}\ \mathrm{Lo} & \longrightarrow & \textit{outer generator}\ \mathrm{Lu} & & \uparrow\downarrow & & \textit{exceptional generator}\ \mathrm{Le} \\
 & & & \searrow & \uparrow\downarrow & \swarrow & \\
 & & & & \textit{inner generator}\ \mathrm{Lii} & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & \longrightarrow & \longrightarrow & \textit{inner generator}\ \mathrm{Li} & & \\
 & \nearrow & & \nearrow & \uparrow\downarrow & \nwarrow & \\
\textit{original generator}\ \mathrm{Loo} & \longrightarrow & \textit{outer generator}\ \mathrm{Luu} & & \uparrow\downarrow & & \textit{exceptional generator}\ \mathrm{Le} \\
 & & & \searrow & \uparrow\downarrow & \swarrow & \\
 & & & & \textit{inner generator}\ \mathrm{Lii} & &
\end{array}$$

We observe that whereas each inner generator is both *source* and *target*, the other generators (– original, outer, exceptional –) are *sources* only. Moreover, although there is perfect symmetry between Li and Lii within the inner algebra, that symmetry breaks down when we adduce the original generators Lo or Loo: indeed, Li is a target for both Lo and Loo, but its counterpart Lii is a target for neither.[76] Altogether, we get the six resurgence algebras depicted below, with the inner algebra as their common core:

$$\begin{array}{lclcl}
\textit{inner algebra} & & & & \\
\{Li, Lii\} & \subset & \{Li, Lii, Lu\} & \subset & \{Li, Lii, Lu, Lo\} \\
\{Li, Lii\} & \subset & \{Li, Lii, Luu\} & \subset & \{Li, Lii, Luu, Loo, \} \\
\{Li, Lii\} & \subset & \{Li, Lii, Le\} & &
\end{array}$$

Next, we list the points ζ_i where the singularities occur in the *zeta*-plane, and their real logarithmic counterparts ν_i in the ν-plane.

$$\begin{aligned}
\nu_0 &:= -\infty \\
\nu_1 &:= \int_0^{1/6} f(x)dx = -\frac{Li_2(e^{2\pi i/6}) - Li_2(e^{-2\pi i/6})}{2\pi i} = -0.3230659470\ldots \\
\nu_2 &:= 0 \\
\nu_3 &:= \int_0^{5/6} f(x)dx = +\frac{Li_2(e^{2\pi i/6}) - Li_2(e^{-2\pi i/6})}{2\pi i} = +0.3230659470\ldots \\
\zeta_0 &:= 0 \\
\zeta_1 &:= \exp(\nu_1) = 0.723926112\cdots = 1/\zeta_3 \\
\zeta_2 &:= 1 \\
\zeta_3 &:= \exp(\nu_3) = 1.381356444\ldots
\end{aligned}$$

[76] at least, under *strict alien derivation*: this doesn't stand in contradiction to the fact that under *lateral* continuation (upper or lower) of Lo or Loo along the real axis, singularities $\pm 4\, iLii$ can be "seen" over the point ζ_3. See §9.8.3 below.

The assignment of *generators* to *singular points* goes like this: [77]

$$\begin{array}{llll} \overset{\wedge}{Lo} \;\; and \;\; \overset{\wedge}{Loo} & at\;\; \zeta_0 & & \\ \overset{\wedge}{Li} & at\;\; \zeta_1 & ; & \overset{\wedge}{\underline{Li}} \;\; at\;\; \underline{\zeta_1} \\ \overset{\wedge}{Lu} \;\; and \;\; \overset{\wedge}{Luu} \;\; and \;\; \overset{\wedge}{Le} & at\;\; \zeta_2 & ; & \overset{\wedge}{\underline{Luu}} \;\; at\;\; \underline{\zeta_2} \\ \overset{\wedge}{Lii} & at\;\; \zeta_3 & ; & \overset{\wedge}{\underline{Lii}} \;\; at\;\; \underline{\zeta_3} \end{array}$$

with

$$\underline{\zeta_1} := -\zeta_1 \;\; , \;\; \underline{\zeta_2} := -\zeta_2 \;\; , \;\; \underline{\zeta_3} := -\zeta_3$$

and

$$\overset{\wedge}{\underline{Li}}(\zeta) := \overset{\wedge}{Li}\,(-\zeta) \;\; , \;\; \overset{\wedge}{\underline{Lii}}(\zeta) := \overset{\wedge}{Lii}\,(-\zeta) \;\; , \;\; \overset{\wedge}{\underline{Luu}}(\zeta) := \overset{\wedge}{Luu}\,(-\zeta)$$

The correspondence between singularities in the ζ- and ν-planes is as follows:

$$\begin{array}{llllll} \textit{minors} & & \textit{minors} & \textit{majors} & & \textit{majors} \\ \zeta\ \textit{plane} & & \nu\ \textit{plane} & \zeta\ \textit{plane} & & \nu\ \textit{plane} \\ \overset{\wedge}{\mathrm{Li}}(\zeta) & = & \overset{\wedge}{\mathrm{li}}\big(\log(1+\zeta/\zeta_i)\big) & \overset{\vee}{\mathrm{Li}}(\zeta) & = & \overset{\vee}{\mathrm{li}}\big(-\log(1-\zeta/\zeta_i)\big) \\ \overset{\wedge}{\mathrm{Lii}}(\zeta) & = & \overset{\wedge}{\mathrm{lii}}\big(\log(1+\zeta/\zeta_{ii})\big) & \overset{\vee}{\mathrm{Lii}}(\zeta) & = & \overset{\vee}{\mathrm{lii}}\big(-\log(1-\zeta/\zeta_{ii})\big) \\ \overset{\wedge}{\mathrm{Lu}}(\zeta) & = & \overset{\wedge}{\mathrm{lu}}\big(\log(1+\zeta)\big) & \overset{\vee}{\mathrm{Lu}}(\zeta) & = & \overset{\vee}{\mathrm{lu}}\big(-\log(1-\zeta)\big) \\ \overset{\wedge}{\mathrm{Luu}}(\zeta) & = & \overset{\wedge}{\mathrm{luu}}\big(\log(1+\zeta)\big) & \overset{\vee}{\mathrm{Luu}}(\zeta) & = & \overset{\vee}{\mathrm{luu}}\big(-\log(1-\zeta)\big) \\ \overset{\wedge}{\mathrm{Le}}(\zeta) & = & \overset{\wedge}{\mathrm{le}}\big(\log(1+\zeta)\big) & & & \end{array}$$

With all these notations and definitions out of the way, we are now in a position to write down the resurgence equations connecting the various generators:

Resurgence algebra generated by *Lo*.

$$\begin{array}{lllllllll} \Delta_{\zeta_1}\overset{\Diamond}{Lo} & = & 2\,\overset{\Diamond}{Li} & \Delta_{\zeta_3-\zeta_2}\overset{\Diamond}{Lu} & = & \frac{2}{2\pi}\,\overset{\Diamond}{Lii} & \Delta_{\zeta_3-\zeta_1}\overset{\Diamond}{Li} & = & \frac{3}{2\pi}\,\overset{\Diamond}{Lii} \\ \Delta_{\zeta_2}\overset{\Diamond}{Lo} & = & 1\,\overset{\Diamond}{Lu} & \Delta_{\zeta_1-\zeta_2}\overset{\Diamond}{Lu} & = & \frac{2}{2\pi}\,\overset{\Diamond}{Li} & \Delta_{\zeta_1-\zeta_3}\overset{\Diamond}{Lii} & = & \frac{3}{2\pi}\,\overset{\Diamond}{Li} \\ \Delta_{\zeta_3}\overset{\Diamond}{Lo} & = & 0\,\overset{\Diamond}{Lii} & & & & & & \end{array}$$

[77] in the ζ-plane, for definiteness.

Resurgence algebra generated by *Loo.*

$$
\begin{array}{lll}
\Delta_{\zeta_1}\overset{\diamond}{Loo} = 2\,\overset{\diamond}{Li} & \Delta_{\zeta_3-\zeta_2}\overset{\diamond}{Luu} = \frac{2}{2\pi}\,\overset{\diamond}{Lii} & \Delta_{\zeta_3-\zeta_1}\overset{\diamond}{Li} = \frac{3}{2\pi}\,\overset{\diamond}{Lii} \\
\Delta_{\zeta_2}\overset{\diamond}{Loo} = 1\,\overset{\diamond}{Luu} & \Delta_{\zeta_1-\zeta_2}\overset{\diamond}{Luu} = -\frac{2}{2\pi}\,\overset{\diamond}{Li} & \Delta_{\zeta_1-\zeta_3}\overset{\diamond}{Lii} = \frac{3}{2\pi}\,\overset{\diamond}{Li} \\
\Delta_{\zeta_3}\overset{\diamond}{Loo} = 0\,\overset{\diamond}{Lii} & & \\
\Delta_{\underline{\zeta_1}}\overset{\diamond}{Loo} = 0\,\underline{\overset{\diamond}{Li}} & \Delta_{\underline{\zeta_3}-\underline{\zeta_2}}\underline{\overset{\diamond}{Luu}} = \frac{2}{2\pi}\,\underline{\overset{\diamond}{Lii}} & \Delta_{\underline{\zeta_3}-\underline{\zeta_1}}\underline{\overset{\diamond}{Li}} = \frac{3}{2\pi}\,\underline{\overset{\diamond}{Lii}} \\
\Delta_{\underline{\zeta_2}}\overset{\diamond}{Loo} = -2\,\underline{\overset{\diamond}{Luu}} & \Delta_{\underline{\zeta_1}-\underline{\zeta_2}}\underline{\overset{\diamond}{Luu}} = -\frac{2}{2\pi}\,\underline{\overset{\diamond}{Li}} & \Delta_{\underline{\zeta_1}-\underline{\zeta_3}}\underline{\overset{\diamond}{Lii}} = \frac{3}{2\pi}\,\underline{\overset{\diamond}{Li}} \\
\Delta_{\underline{\zeta_3}}\overset{\diamond}{Loo} = 0\,\underline{\overset{\diamond}{Lii}} & &
\end{array}
$$

Resurgence algebra generated by *Le.*

$$
\begin{array}{ll}
\Delta_{\zeta_3-\zeta_2}\overset{\diamond}{Le} = \frac{2}{2\pi}\,\overset{\diamond}{Lii} & \Delta_{\zeta_3-\zeta_1}\overset{\diamond}{Li} = \frac{3}{2\pi}\,\overset{\diamond}{Lii} \\
\Delta_{\zeta_1-\zeta_2}\overset{\diamond}{Le} = -\frac{2}{2\pi}\,\overset{\diamond}{Li} & \Delta_{\zeta_1-\zeta_3}\overset{\diamond}{Lii} = \frac{3}{2\pi}\,\overset{\diamond}{Li}
\end{array}
$$

9.8 Computational verifications.

In order to check numerically our dozen or so resurgence equations, we shall make systematic use of the method of §2.3 which describes *singularities* in terms of *Taylor coefficient asymptotics*. Three situations, however, may present themselves:

(i) The singularity under investigation is closest to zero. This is the most favourable situation, as it makes for a straightforward application of §2.3.

(ii) The singularity under investigation is not closest to zero, but becomes so after an *origin-preserving* conformal transform, after which we can once again resort to §2.3. This is no serious complication, because such conformal transforms don't diminish the accuracy with which Taylor coefficients of a given rank are computed.

(iii) The singularity under investigation is not closest to zero, nor can it be made so under a reasonably simple, origin-preserving conformal transform. We must then take recourse to *origin-changing* conformal transforms, the simplest instances of which are *shifts.* This is the least favourable case, because origin-changing conformal transforms – and be they simple shifts – entail a steep loss of numerical accuracy and demand great attention to the propriety of the truncations being performed.[78]

Fortunately, this third, least favourable situation shall occur but once (in §9.8.3 , when investigating the arrow $Lo \to Lii$) and even there we will manage the confirm the theoretical prediction with reasonable accuracy (up to 7 places). In all other instances, we shall achieve truly remarkable numerical accuracy, often with up to 50 or 60 exact digits.

[78] indeed, inept truncations can all too easily lead to meaningless results.

9.8.1 From Li to Lii and back (*inner* to *inner*).

Since the theory predicts that Li and Lii generate each other under alien differentiation, but that neither of them generates Lo nor Lu, we may directly solve the system $\mathbb{S}^{\mathbf{n},\mathbf{m}}_{i\,,ii}$:

$$\tilde{\ell i}_{-\frac{1}{2}+n} = 3\,\nu_{1,3}^{\frac{1}{2}-n} \sum_{0\leq m<\mathbf{m}} \tilde{riis}_{\frac{1}{2}+m}\,(-\frac{1}{2}+n)^{-\frac{1}{2}-m} \quad ; \quad \forall n \in]\mathbf{n}-\mathbf{m},\mathbf{n}]$$

with $\mathbf{n}$ equations and $\mathbf{m}$ unknowns $\tilde{riis}_{\frac{1}{2}+m}$. Then we may form:

$$\overset{\wedge}{riis}\,(\rho) \;:=\; \sum_{0\leq m<\mathbf{m}} \tilde{riis}_{\frac{1}{2}+m}\; \frac{\rho^{-\frac{1}{2}+m}}{(-\frac{1}{2}+m)!}$$

$$\overset{\frown}{liis}\,(\nu) \;:=\; \overset{\wedge}{riis}\,\Big(\log(1+\frac{\nu}{\nu_{1,3}})\Big) \;=\; \sum_{0\leq m<\mathbf{m}} \overset{\frown}{liis}_{-\frac{1}{2}+m}\;\; \nu^{-\frac{1}{2}+m}$$

and check that the ratios $rat_{-\frac{1}{2}+m} := \dfrac{\overset{\frown}{liis}_{-\frac{1}{2}+m}}{\overset{\frown}{lii}_{-\frac{1}{2}+m}}$ are indeed ~ 1. For instance, with the coefficients $\overset{\frown}{liis}_{-\frac{1}{2}+m}$ computed from $\mathbb{S}^{150,45}_{i\,,ii}$, we already get a high degree of accuracy:

$$|1 - rat_{-\frac{1}{2}}| < 10^{-58},\ldots,|1 - rat_{\frac{15}{2}}| < 10^{-40},\ldots,|1 - rat_{\frac{31}{2}}| < 10^{-24},\ldots$$

This confirms the (equivalent) pairs of resurgence equations

$$\Delta_{\nu_3-\nu_1}\overset{0}{\ell i} = \frac{3}{2\pi}\,\overset{0}{\ell ii} \quad ; \quad \Delta_{\nu_1-\nu_3}\overset{0}{\ell ii} = \frac{3}{2\pi}\,\overset{0}{\ell i}$$

$$\Delta_{\nu_3-\nu_1}\overset{\Diamond}{\ell i} = \frac{3}{2\pi}\,\overset{\Diamond}{\ell ii} \quad ; \quad \Delta_{\nu_1-\nu_3}\overset{\Diamond}{\ell ii} = \frac{3}{2\pi}\,\overset{\Diamond}{\ell i}$$

in the ν-plane, which in turn imply

$$\Delta_{\zeta_3-\zeta_1}\overset{\Diamond}{Li} = \frac{3}{2\pi}\,\overset{\Diamond}{Lii} \quad ; \quad \Delta_{\zeta_1-\zeta_3}\overset{\Diamond}{Lii} = \frac{3}{2\pi}\,\overset{\Diamond}{Li}$$

in the ζ-plane.

9.8.2 From Lo to Li (*original* to *close-inner*).

Since ζ_1 is closest to 0, we solve the system $\mathbb{S}^{\mathbf{n},\mathbf{m}}_{o\,,i}$:

$$\overset{\wedge}{Lo}_n = 2\,\zeta_1^{-n} \sum_{0\leq m<\mathbf{m}} \tilde{lis}_{-\frac{1}{2}+m}\;\, n^{\frac{1}{2}-m} \quad ; \quad \forall n \in]\mathbf{n}-\mathbf{m},\mathbf{n}]$$

with $\mathbf{n}$ equations and $\mathbf{m}$ unknowns $\tilde{lis}_{-\frac{1}{2}+m}$. Then we check that the ratios $rat_{-\frac{1}{2}+m} := \dfrac{\overset{\frown}{lis}_{-\frac{1}{2}+m}}{\overset{\frown}{li}_{-\frac{1}{2}+m}}$ are indeed ~ 1. For instance, with the coefficients $\overset{\frown}{lis}_{-\frac{1}{2}+m}$ computed from $\mathbb{S}^{700,50}_{o\,,i}$, we get this sort of accuracy:

$$|1 - rat_{-\frac{1}{2}}| < 10^{-54},\ldots,|1 - rat_{\frac{15}{2}}| < 10^{-29},\ldots,|1 - rat_{\frac{31}{2}}| < 10^{-6},\ldots$$

This confirms the resurgence equations $\Delta_{\zeta_1}\overset{\Diamond}{Lo} = 2\;\overset{\Diamond}{Li}$ in the ζ-plane.

9.8.3 From *Lo* to *Lii* (*original* to *distant-inner*).

The singular point ζ_3 being farthest from 0, we first resort to an origin-preserving conformal transform $\zeta \to \xi$:

$$
\begin{array}{lll}
h_{\zeta,\xi} & : \ \xi \mapsto \zeta := \zeta_1 - \left(\zeta_1^{1/4} - \xi\right)^4 & \forall \xi \\
h_{\xi,\zeta} & : \ \zeta \mapsto \xi := \zeta_1^{1/4} - \left(\zeta_1 - \zeta\right)^{1/4} & \forall \zeta \in [0, \zeta_1] \\
h^+_{\xi,\zeta} & : \ \zeta \mapsto \xi := \zeta_1^{1/4} - \left(\zeta - \zeta_1\right)^{1/4} e^{-i\pi/4} & \forall \zeta \in [\zeta_1, \infty] \\
h^-_{\xi,\zeta} & : \ \zeta \mapsto \xi := \zeta_1^{1/4} - \left(\zeta - \zeta_1\right)^{1/4} e^{+i\pi/4} & \forall \zeta \in [\zeta_1, \infty]
\end{array}
$$

$$
\begin{array}{llll}
h_{\xi,\zeta} : \zeta_1 \mapsto \xi_1 \ = \ 0.9224\ldots & ; & |\xi_1| \ = \ 0.9224\ldots & (\textit{farthest}) \\
h^{\pm}_{\xi,\zeta} : \zeta_2 \mapsto \xi_2^{\pm} \ = \ 0.4098 \pm 0.5126\, i \ldots & ; & |\xi_2^{\pm}| = 0.6563\ldots & (\textit{closest}) \\
h^{\pm}_{\xi,\zeta} : \zeta_3 \mapsto \xi_3^{\pm} \ = \ 0.2857 \pm 0.6367\, i \ldots & ; & |\xi_3^{\pm}| = 0.6979\ldots & (\textit{middling})
\end{array}
$$

Since the images $\xi_3^{\pm}$ are closer, but not closest, to 0, we must perform an additional shift $\xi \to \tau$:

$$
h_{\tau,\xi} : \xi \mapsto \tau \quad := \ \xi - \tfrac{i}{2} \qquad\qquad h_{\xi,\tau} : \tau \mapsto \xi := \tau + \tfrac{i}{2}
$$

$$
\begin{array}{llll}
h_{\tau,\xi} : \xi_1 \mapsto \tau_1 & = & 0.9224 - 0.5000\, i \ldots & |\tau_1| = 1.0492\ldots (\textit{farthest}) \\
h_{\tau,\xi} : \xi_2^+ \mapsto \tau_2^+ & = & 0.4098 + 0.0125\, i \ldots & |\tau_2^+| = 0.4100\ldots (\textit{middling}) \\
h_{\tau,\xi} : \xi_3^+ \mapsto \tau_3^+ & = & 0.2857 + 0.1367\, i \ldots & |\tau_3^+| = 0.3167\ldots (\textit{closest})
\end{array}
$$

The image τ_3^+ at last is closest, and we can now go through the usual motions. We form successively:

$$
\overset{\wedge}{Lo}_{\#}(\zeta) \ := \ \sum_{0<n<\mathbf{n}} \overset{\wedge}{Lo}_n \ \zeta^n \qquad\qquad (\textit{truncation})
$$

$$
\overset{\wedge}{Lo}_{\#\#}(\xi) \ := \ \overset{\wedge}{Lo}_{\#}(h_{\zeta,\xi}(\xi)) \qquad\qquad (\textit{conf. transf.})
$$

$$
\overset{\wedge}{Lo}_{\#\#\#}(\xi) \ := \ \overset{\wedge}{Lo}_{\#\#}(h_{\xi,\tau}(\tau)) = \sum_{0<n<\mathbf{n}} L_n \ \tau^n + (\ldots) \qquad\qquad (\textit{simple shift.})
$$

We then solve the system $\mathbb{S}^{\mathbf{n},\mathbf{m}}_{o,ii}$:

$$
L_n = 4\, i\, (\tau_3^+)^{-n} \sum_{0\leq m<\mathbf{m}} P_{-\frac{1}{2}+m} \ n^{\frac{1}{2}-m} \qquad\qquad \big(n \in]\mathbf{n}-\mathbf{m}, \mathbf{n}]\big)
$$

with $\mathbf{m}$ equations and $\mathbf{m}$ unknowns $P_{-\frac{1}{2}+m}$.

$$
\hat{P}(\nu) \ := \ \sum_{0\leq m<\mathbf{m}} P_{-\frac{1}{2}+m} \ \frac{\nu^{-\frac{3}{2}+m}}{(-\frac{3}{2}+m)!} + (\ldots)
$$

$$
\hat{R}(\tau) \ := \ \hat{P}(\log(1+\frac{\tau}{\tau_3^+})) \ = \ \sum_{0\leq m<\mathbf{m}} R_{-\frac{3}{2}+m} \ \tau^{-\frac{3}{2}+m} + (\ldots)
$$

Next, for comparison, we form series that carry the expected singularity Lii successively in the ν, ζ and τ-planes:

$$\begin{aligned}
\overset{\wedge}{\ell ii}\,(\nu) &:= \sum_{0\leq m<\mathbf{m}} \tilde{\ell ii}_{-\frac{1}{2}+m}\, \frac{\nu^{-\frac{3}{2}+m}}{(-\frac{3}{2}+m)!} + (\dots)\\
\overset{\wedge}{Lii}\,(\zeta) &:= \overset{\wedge}{\ell ii}\,\Big(\log(1+\frac{\zeta}{\zeta_3})\Big)\\
\hat{Q}\,(\tau) &:= \overset{\wedge}{Lii}\,(dh_{\zeta,\tau}(\tau)) \;=\; \sum_{0\leq m<\mathbf{m}} Q_{-\frac{3}{2}+m}\;\tau^{-\frac{3}{2}+m} + (\dots)
\end{aligned}$$

Lastly, we form the ratios $rat_{-\frac{3}{2}+m} := \frac{R_{-\frac{3}{2}+m}}{Q_{-\frac{3}{2}+m}}$ of homologous coefficients P, Q and check that these ratios are ~ 1. With the data derived from the linear system $\mathbb{S}^{800,4}_{o\,,\,ii}$ and with truncation at order $\mathbf{n}^* = \mathbf{20}$ in the computation of $\overset{\wedge}{Lo}_{\#\#\#}$, we get the following, admittedly poor degree[79] of accuracy:

$$|1 - rat_{-3/2}| < 10^{-7}, |1 - rat_{-1/2}| < 10^{-3}, |1 - rat_{+1/2}| < 10^{-2}, \dots$$

To compound the poor numerical accuracy, the theoretical interpretation is also rather roundabout in this case. By itself, the above results only show that:

$$\Delta^{\pm}_{\zeta_3}\, \overset{\diamond}{Lo} = \pm 4\, i\, \overset{\diamond}{Lii} \tag{278}$$

with the one-path lateral operators $\Delta^{\pm}_{\omega}$ of §2.3 which, unlike the multi-path averages Δ_ω, are *not* alien derivations. To infer from (278) the expected resurgence equation:

$$\Delta_{\zeta_3}\, \overset{\diamond}{Lo} = 0\, \overset{\diamond}{Lii} \tag{279}$$

we must apply the basic identity (5) of §2.3 to $\overset{\diamond}{Lo}$:

$$\Big(1 + \sum_{0<\omega} \Delta^{+}_{\omega}\Big)\, \overset{\diamond}{Lo} = \Big(\exp\big(2\pi i \sum_{0<\omega} \Delta_\omega\big)\Big)\, \overset{\diamond}{Lo} \tag{280}$$

and then equate the sole term coming from the left-hand side, namely $\Delta^{\pm}_{\zeta_3}\, \overset{\diamond}{Lo}$, with the 4 possible terms coming from the right-hand side, namely:

$$2\pi i\, \Delta_{\zeta_3}\, \overset{\diamond}{Lo} = unknown \tag{281}$$

$$\frac{(2\pi i)^2}{2} \Delta_{\zeta_3-\zeta_1}\Delta_{\zeta_1}\, \overset{\diamond}{Lo} = 1\;\; \overset{\diamond}{Lii} \tag{282}$$

$$\frac{(2\pi i)^2}{2} \Delta_{\zeta_3-\zeta_2}\Delta_{\zeta_2}\, \overset{\diamond}{Lo} = 3\;\; \overset{\diamond}{Lii} \tag{283}$$

$$\frac{(2\pi i)^3}{6} \Delta_{\zeta_3-\zeta_2}\Delta_{\zeta_2-\zeta_1}\Delta_{\zeta_1}\, \overset{\diamond}{Lo} = 0\;\; \overset{\diamond}{Lii} \tag{284}$$

Equating the terms in the left and right clusters, we find that the sole unknown term (281) does indeed vanish, as required by the theory.

[79]this is because of the recourse to the *shift* $\tau := \xi + \frac{i}{2}$ whereas in all the other computations we handled less disruptive *origin-preserving* conformal transforms $\zeta \to \xi$.

9.8.4 From *Lo* to *Lu* (*original* to *outer*).

A single, origin-preserving conformal transform $\zeta \to \xi$ takes the singular point ζ_2 to middling position $\xi_2^{\pm}$:

$$\begin{array}{llll} h_{\zeta,\xi} & : & \xi \mapsto \zeta := \zeta_1 - \left(\zeta_1^{1/2} - \xi\right)^2 & \forall \xi \\ h_{\xi,\zeta} & : & \zeta \mapsto \xi := \zeta_1^{1/2} - \left(\zeta_1 - \zeta\right)^{1/2} & \forall \zeta \in [0, \zeta_1] \\ h^+_{\xi,\zeta} & : & \zeta \mapsto \xi := \zeta_1^{1/2} + i\left(\zeta - \zeta_1\right)^{1/2} & \forall \zeta \in [\zeta_1, \infty] \\ h^-_{\xi,\zeta} & : & \zeta \mapsto \xi := \zeta_1^{1/2} - i\left(\zeta - \zeta_1\right)^{1/2} & \forall \zeta \in [\zeta_1, \infty] \end{array}$$

$$\begin{array}{lllllll} h_{\xi,\zeta} & : & \zeta_1 \mapsto \xi_1 & = 0.8508\ldots & ; & |\xi_1| = 0.8508\ldots & (\textit{closest}) \\ h^{\pm}_{\xi,\zeta} & : & \zeta_2 \mapsto \xi_2^{\pm} & = 0.8508 \pm 0.5254\, i \ldots & ; & |\xi_2^{\pm}| = 1.0000\ldots & (\textit{middling}) \\ h^{\pm}_{\xi,\zeta} & : & \zeta_3 \mapsto \xi_3^{\pm} & = 0.8508 \pm 0.8108\, i \ldots & ; & |\xi_3^{\pm}| = 1.7573\ldots & (\textit{farthest}) \end{array}$$

Then we form:

$$\overset{\wedge}{Lo}_{\#}(\zeta) := \sum_{0<n<\mathbf{n}} \overset{\wedge}{Lo}_n\ \zeta^n \qquad (\textit{truncation})$$

$$\overset{\wedge}{Lo}_{\#\#}(\xi) := \overset{\wedge}{Lo}_{\#}(h_{\zeta,\xi}(\xi)) \qquad (\textit{conf. transf.})$$

$$\overset{\wedge}{Lo}_{\#\#\#}(\xi) := \overset{\wedge}{Lo}_{\#\#}(\xi)\ (\xi_1 - \xi)^3 = \sum_{0<n<\mathbf{n}} L_n\ \xi^n + (\ldots) \qquad (\textit{sing. remov.})$$

Since $\zeta_1 - \zeta = (\xi_1 - \xi)^2$, all the semi-integral powers $(\zeta_1 - \zeta)^{n/2}$ present in $\overset{\wedge}{Lo}_{\#}(\zeta)$ at $\zeta \sim \zeta_2$ vanish from $\overset{\wedge}{Lo}_{\#\#}(\xi)$, except for the first two terms:

$$C_{-3}\ (\xi_1 - \xi)^{-3} + C_{-1}\ (\xi_1 - \xi)^{-1}$$

but even these two vanish from $\overset{\wedge}{Lo}_{\#\#\#}(\xi)$ due to multiplication by $(\xi_1 - \xi)^3$. So the points $\xi_2^{\pm}$ now carry the closest singularities of $\overset{\wedge}{Lo}_{\#\#\#}(\xi)$, and we can apply the usual Taylor coefficient asymptotics.

For comparison with the expected singularity $\overset{\wedge}{Lu}$, we construct a new triplet $\{\overset{\wedge}{Ro}_{\#}, \overset{\wedge}{Ro}_{\#\#}, \overset{\wedge}{Ro}_{\#\#\#}\}$, but with a more severe truncation ($\mathbf{n}^* \prec \mathbf{n}$) and with coefficients $\overset{\wedge}{Lo}_n$ replaced by the $\overset{\wedge}{Ro}_n$ defined as follows:

$$\overset{\wedge}{Ro}_n := \frac{1}{n} SP^F\left(\frac{1}{n}\right) \qquad \textit{with} \qquad SP^F(x) := \sum_{1 \le m \le \mathbf{n}^*} \prod_{1 \le k \le m} F(k\,x)$$

$$\overset{\wedge}{Ro}_{\#}(\zeta) := \sum_{0<n<\mathbf{n}^*} \overset{\wedge}{Ro}_n\ \zeta^n \qquad (\textit{truncation})$$

$$\overset{\wedge}{Ro}_{\#\#}(\xi) := \overset{\wedge}{Ro}_{\#}(h_{\zeta,\xi}(\xi)) \qquad (\textit{conf. transf.})$$

$$\overset{\wedge}{Ro}_{\#\#\#}(\xi) := \overset{\wedge}{Ro}_{\#\#}(\xi)\ (\xi_1 - \xi)^3 = \sum_{0<n<\mathbf{n}} R_n\ \xi^n + (\ldots) \qquad (\textit{sing. remov.})$$

Then, we solve the two parallel systems $\overline{\mathbb{S}}^{\mathbf{n},\mathbf{m}}_{o\,,\,u}$ and $\underline{\mathbb{S}}^{\mathbf{n},\mathbf{m}}_{o\,,\,u}$:

$$L_n = \sum_{\epsilon=\pm} \ (\xi_2^\epsilon)^{-n} \sum_{1\leq m\leq \mathbf{m}} L_m^\epsilon \, n^{-k} \qquad\qquad \big(n \ \in]\mathbf{n}-\mathbf{2}\,\mathbf{m},\mathbf{n}]\ \big)$$

$$R_n = \sum_{\epsilon=\pm} \ (\xi_2^\epsilon)^{-n} \sum_{1\leq m\leq \mathbf{m}} R_m^\epsilon \, n^{-k} \qquad\qquad \big(n \ \in]\mathbf{n}-\mathbf{2}\,\mathbf{m},\mathbf{n}]\ \big)$$

each with $2\,\mathbf{m}$ equations and $2\,\mathbf{m}$ unknowns, L_m^ϵ or R_m^ϵ respectively. We then check that the ratios $rat_n^\epsilon := \frac{L_m^\epsilon}{R_m^\epsilon}$ are ~ 1. With the data obtained from the systems $\overline{\mathbb{S}}^{495,7}_{o\,,\,u}$ and $\underline{\mathbb{S}}^{495,7}_{o\,,\,u}$ and with truncation at order $\mathbf{n}^* = \mathbf{30}$ in the $\overset{\wedge}{Ro}$ triplet, we get the following degree of accuracy :

$$|1 - rat_1^\pm| < 10^{-17}, \ldots, |1 - rat_3^\pm| < 10^{-13}, \ldots, |1 - rat_6^\pm| < 10^{-10}, \ldots,$$

The immediate implication is $\Delta^+_{\zeta_2} \overset{\diamond}{Lo} = 2\pi i \ \overset{\diamond}{Lu}$. To translate this into a statement about $\Delta_{\zeta_2} \overset{\diamond}{Lo}$, the argument is the same as in §9.8.3, only much simpler. Indeed, the only term coming from the left-hand side of (280) is now $\Delta^+_{\zeta_2} \overset{\diamond}{Lo}$ and the only two possible terms coming from the right-hand side are :

$$2\pi i \ \Delta_{\zeta_2} \overset{\diamond}{Lo} = \text{unknown} \quad \textit{and} \quad \frac{(2\pi i)^2}{2} \Delta_{\zeta_2-\zeta_1}\Delta_1 \overset{\diamond}{Lo} = 0 \tag{285}$$

Equating both sides, we find $\Delta_{\zeta_2} \overset{\diamond}{Lo} = \overset{\diamond}{Lu}$, as required by the theory.

9.8.5 From Lu to Li and Lii (*outer* to *inner*).

The singular points under investigation being closest, the investigation is straightforward. We form the linear system $\mathbb{S}^{\mathbf{n},\mathbf{m}}_{u,i/ii}$:

$$\overset{\wedge}{\ell u}_n = 2\,\big(\nu_3^{-n} + (-\nu_3)^{-n}\big) \sum_{0\leq m<\mathbf{2}\,\mathbf{m}} \tilde{riis}_{-\frac{1}{2}+m}\, n^{\frac{1}{2}-m} \qquad \Big(n \ \in]\mathbf{n}{-}\mathbf{2m},\mathbf{n}]\Big)$$

with $\mathbf{m}$ effective equations (for even values of n) and $\mathbf{m}$ unknowns $\tilde{riis}_{-\frac{1}{2}+m}$. We then form :

$$\overset{\wedge}{riis}\,(\xi) \ := \ \sum_{0\leq m<\mathbf{m}} \tilde{riis}_{-\frac{1}{2}+m} \ \frac{\xi^{-\frac{3}{2}+m}}{(-\frac{3}{2}+m)!}$$

$$\overset{\wedge}{liis}\,(\xi) \ := \ \overset{\wedge}{riis}\,\Big(\log(1+\frac{\nu}{\nu_3})\Big) \ = \ \sum_{0\leq m<\mathbf{m}} \overset{\wedge}{liis}_{-\frac{3}{2}+m} \ \ \nu^{-\frac{3}{2}+m}$$

and check that the ratios $rat_{-\frac{3}{2}+m} := \frac{\overset{\wedge}{liis}_{-\frac{3}{2}+m}}{\overset{\wedge}{lii}_{-\frac{3}{2}+m}}$ are indeed ~ 1. With the data obtained from the system $\mathbb{S}^{300,40}_{u\,,\,i/ii}$, we get this high degree of accuracy :

$$|1 - rat^\pm_{-3/2}| < 10^{-81}, \ \ldots \ , |1 - rat^\pm_{17/2}| < 10^{-39}, \ \ldots \ ,$$

$$|1 - rat^\pm_{37/2}| < 10^{-21}, \ \ldots \ , |1 - rat^\pm_{57/2}| < 10^{-10}, \ \ldots$$

This confirms, via the ν-plane, the expected resurgence equations in the ζ-plane, namely :

$$\Delta_{\zeta_3-\zeta_2}\overset{\diamond}{Lu} = \frac{2}{2\pi}\,\overset{\diamond}{Lii} \quad ; \quad \Delta_{\zeta_1-\zeta_2}\overset{\diamond}{Lu} = \frac{2}{2\pi}\,\overset{\diamond}{Li}$$

9.8.6 From *Loo* to *Li* (*original* to *close-inner*).

We proceed exactly as in §9.8.2. We form the linear system $\mathbb{S}^{\mathbf{n},\mathbf{m}}_{oo\,,\,i}$:

$$\overset{\wedge}{Loo}_n = \zeta_1^{-n} \sum_{0\le m<\mathbf{m}} \tilde{lis}_{-\frac{1}{2}+m}\, n^{\frac{1}{2}-m} \qquad \Big(n\ \in\,]\mathbf{n}-\mathbf{2m},\mathbf{n}]\Big)$$

with $\mathbf{m}$ equations and $\mathbf{m}$ unknowns $\tilde{lis}_{-\frac{1}{2}+m}$. We then check that the ratios $rat_{-\frac{3}{2}+m} := \frac{\overset{\wedge}{lis}_{-\frac{3}{2}+m}}{\overset{\wedge}{li}_{-\frac{3}{2}+m}}$ are indeed ~ 1. With the data obtained from the system $\mathbb{S}^{800,30}_{oo\,,\,i}$, we get this degree of accuracy :

$$|1-rat^{\pm}_{-1/2}| < 10^{-51},\ldots,|1-rat^{\pm}_{19/2}| < 10^{-22},\ldots,|1-rat^{\pm}_{39/2}| < 10^{-7},\ldots$$

This confirms the expected resurgence equations in the ζ-plane :

$$\Delta_{\zeta_1}\overset{\diamond}{Lo} = 2\ \overset{\diamond}{Li} \quad ; \quad \Delta_{\underline{\zeta_1}}\overset{\diamond}{Lo} = 0\ \underline{\overset{\diamond}{Li}}$$

An alternative method would to check that $\overset{\wedge}{Lo}(\zeta) - \overset{\wedge}{Loo}(\zeta)$ has radius of convergence 1, which means that $\overset{\wedge}{Lo}$ and $\overset{\wedge}{Loo}$ have the same singularity at ζ_1, namely $\overset{\diamond}{Li}$: see §9.8.2. With that method, too, the numerical confirmation is excellent.

9.8.7 From *Loo* to *Lii* (*original* to *distant-inner*).

The verication hasn't been done yet. The theory, however, predicts a vanishing alien derivative $\Delta_{\zeta_3}(\overset{\diamond}{Loo}) = 0$ just as with $\overset{\diamond}{Lo}$. Therefore, the upper/lower lateral singularity seen at ζ_3 when continuing $\overset{\diamond}{Loo}\,(\zeta)$ should be $\pm 4\,i\ \overset{\diamond}{Lii}$, just as was the case with the lateral continuations of $\overset{\diamond}{Lo}\,(\zeta)$.

9.8.8 From *Loo* to *Luu* (*original* to *outer*).

We form the linear system $\mathbb{S}^{\mathbf{n},\mathbf{m}}_{oo\,,\,uu}$:

$$\begin{aligned} \overset{\wedge}{Loo}_n - \overset{\wedge}{Lo}_n \quad = \quad & -\zeta_2^{-n} \sum_{1\le m\le \mathbf{m}} \tilde{lu}_{1+2m}\, n^{-1-2\,m} \\ & +\zeta_2^{-n} \sum_{1\le m\le \mathbf{m}} \tilde{luus}_{\,2m}\ \ n^{-2\,m} \\ -2\quad & (-\zeta_2)^{-n} \sum_{1\le m\le \mathbf{m}} \tilde{luus}_{\,2m}\ \ n^{-2\,m} \end{aligned}$$

with $\mathbf{m}$ equations[80] and $\mathbf{m}$ unknowns $\tilde{luus}_{2m}$.[81] We then check that the ratios $rat_{2m} := \frac{\overset{\wedge}{luus}_{2m}}{\overset{\wedge}{luu}_{2m}}$ are indeed ~ 1. With the data obtained from the system $\mathbb{S}^{600,30}_{oo\,,\,uu}$, we get this level of accuracy :

$$|1 - rat_2| < 10^{-48}, \ldots, |1 - rat_{12}| < 10^{-28}, \ldots, |1 - rat_{24}| < 10^{-15}, \ldots$$

This directly confirms the expected resurgence equations :

$$\Delta_{\zeta_2} \overset{\diamond}{Loo} = \overset{\diamond}{Luu} \qquad ; \qquad \Delta_{\underline{\zeta_2}} \overset{\diamond}{Loo} = -2\, \underline{\overset{\diamond}{Luu}}$$

9.8.9 From *Luu* to *Li* and *Lii* (*outer* to *inner*).

We proceed exactly as in §9.8.8. We solve the linear system $\mathbb{S}^{\mathbf{n},\mathbf{m}}_{uu\,,\,i/ii}$:

$$\overset{\wedge}{luu}_n = \frac{2}{2\pi}\,(\nu_3^{-n} - (-\nu_3)^{-n}) \sum_{0 \le m < \mathbf{m}} \tilde{riis}_{-\frac{1}{2}+m}\; n^{\frac{1}{2}-m} \qquad \big(n \in]\mathbf{n}{-}\mathbf{2m}\,,\,\mathbf{n}]\big)$$

with $\mathbf{m}$ effective equations (for n odd) and $\mathbf{m}$ unknowns $\tilde{riis}_{-\frac{1}{2}+m}$. Then we form :

$$\begin{aligned} \overset{\wedge}{riis}\,(\xi) \;&:=\; \sum_{0 \le m < \mathbf{m}} \tilde{riis}_{-\frac{1}{2}+m}\, \frac{\xi^{-\frac{3}{2}+m}}{(-\frac{3}{2}+m)!} + (\ldots) \\ \overset{\wedge}{liis}\,(\xi) \;&:=\; \overset{\wedge}{riis}\,\Big(\log(1+\frac{\nu}{\nu_3})\Big) \;=:\; \sum_{0 \le m < \mathbf{m}} \overset{\wedge}{liis}_{-\frac{3}{2}+m}\;\; \nu^{-\frac{3}{2}+m} + (\ldots) \end{aligned}$$

and check that the ratios $rat_{-\frac{3}{2}+m} := \frac{\overset{\wedge}{liis}_{-\frac{3}{2}+m}}{\overset{\wedge}{lii}_{-\frac{3}{2}+m}}$ of homologous coefficients are indeed ~ 1. For the data corresponding to $\mathbb{S}^{300,40}_{uu\,,\,i/ii}$, we find this excellent level of accuracy :

$$|1 - rat_{-\frac{3}{2}}| < 10^{-74}, \ldots, |1 - rat_{\frac{21}{2}}| < 10^{-37}, \ldots, |1 - rat_{\frac{45}{2}}| < 10^{-18}, \ldots$$

This confirms, via the ν-plane, the expected resurgence equations in the ζ-plane, namely :

$$\Delta_{\zeta_3-\zeta_2} \overset{\diamond}{Luu} = \frac{2}{2\pi}\,\overset{\diamond}{Lii} \quad ; \quad \Delta_{\zeta_1-\zeta_2} \overset{\diamond}{Luu} = -\frac{2}{2\pi}\,\overset{\diamond}{Li}$$

and also, by mirror symmetry :

$$\Delta_{\underline{\zeta_3}-\underline{\zeta_2}} \underline{\overset{\diamond}{Luu}} = \frac{2}{2\pi}\,\underline{\overset{\diamond}{Lii}} \quad ; \quad \Delta_{\underline{\zeta_1}-\underline{\zeta_2}} \underline{\overset{\diamond}{Luu}} = -\frac{2}{2\pi}\,\underline{\overset{\diamond}{Li}}$$

[80] with n ranging through the interval $]\mathbf{n}{-}\mathbf{2m}, \mathbf{n}]$.

[81] the coefficients $\tilde{lu}_{1+2m}$ are already known, from §9.8.4.

9.8.10 From *Le* to *Li* and *Lii* (*exceptional* to *inner*).

As in the preceding subsection, we solve the linear system $\mathbb{S}^{\mathbf{n},\mathbf{m}}_{e\,,i/ii}$:

$$\overset{\wedge}{le}_n = \frac{2}{2\pi}\,(\nu_3^{-n} - (-\nu_3)^{-n}) \sum_{0\leq m<\mathbf{m}} \tilde{riis}_{-\frac{1}{2}+m}\, n^{\frac{1}{2}-m} \qquad \big(n\in]\mathbf{n}-\mathbf{2m}\,,\,\mathbf{n}]\big)$$

with $\mathbf{m}$ effective equations (for n odd) and $\mathbf{m}$ unknowns $\tilde{riis}_{-\frac{1}{2}+m}$. Then we form:

$$\overset{\wedge}{riis}\,(\xi) \;:=\; \sum_{0\leq m<\mathbf{m}} \tilde{riis}_{-\frac{1}{2}+m}\,\frac{\xi^{-\frac{3}{2}+m}}{(-\frac{3}{2}+m)!} + (\dots)$$

$$\overset{\wedge}{liis}\,(\xi) \;:=\; \overset{\wedge}{riis}\,\big(\log(1+\frac{\nu}{\nu_3})\big) \;=\; \sum_{0\leq m<\mathbf{m}} \overset{\wedge}{liis}_{-\frac{3}{2}+m}\;\nu^{-\frac{3}{2}+m} + (\dots)$$

and check that the ratios $rat_{-\frac{3}{2}+m} := \dfrac{\overset{\wedge}{liis}_{-\frac{3}{2}+m}}{\overset{\wedge}{lii}_{-\frac{3}{2}+m}}$ of homologous coefficients are indeed ~ 1.

For the data corresponding to $\mathbb{S}^{300,40}_{e\,,i/ii}$, we find this excellent level of accuracy:

$$|1-rat_{-\frac{3}{2}}| < 10^{-73},\dots,|1-rat_{\frac{21}{2}}| < 10^{-38},\dots,|1-rat_{\frac{45}{2}}| < 10^{-17},\dots$$

This confirms, via the ν-plane, the expected resurgence equations in the ζ-plane, to wit:

$$\Delta_{\zeta_3-\zeta_2}\overset{\diamond}{Le} = \frac{2}{2\pi}\,\overset{\diamond}{Lii} \quad ; \quad \Delta_{\zeta_1-\zeta_2}\overset{\diamond}{Le} = -\frac{2}{2\pi}\,\overset{\diamond}{Li}$$

An alternative method is to check that $\overset{\wedge}{\ell e}(\nu) - \overset{\wedge}{\ell uu}(\nu)$ has a radius of convergence larger than $|\nu_1| = |\nu_3|$, which implies that $\overset{\wedge}{\ell e}$ and $\overset{\wedge}{\ell uu}$ have the same singularity at ν_1 and ν_3, namely $\frac{2}{2\pi}\,\overset{\diamond}{Li}$ and $\frac{2}{2\pi}\,\overset{\diamond}{Lii}$: see §9.8.9. Here too, the numerical accuracy is excellent.

10 General tables.

10.1 Main formulas.

10.1.1 Functional transforms.

standard case	$\beta(\tau) := \frac{1}{e^{\tau/2)}-e^{-\tau/2}}$	$\beta^{\dagger}(\tau) := \frac{1}{e^{\tau/2)}-e^{-\tau/2}} - \frac{1}{\tau}$
free β case	$\beta(\tau) := \tau^{-1} + \sum_{1\leq k}\beta_k\,\tau^k$	$\beta^{\dagger}(\tau) := \sum_{1\leq k}\beta_k\,\tau^k$

....

mir-transform: $\not{g} := 1/g \mapsto \hbar := 1/h$ with

$$\frac{1}{\hbar(\nu)} = \Big[\frac{1}{\not{g}(\nu)}\exp\Big(-\beta^{\dagger}\big(I\,\not{g}(\nu)\partial_\nu\big)\,\not{g}(\nu)\Big)\Big]_{I=\partial_\nu^{-1}} \tag{286}$$

....

nir-transform: $f \mapsto h$ with

$$h(\nu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n\nu}\frac{dn}{n}\int_0^{+\infty}\exp^{\#}\Big(-\beta(\partial_\tau)f(\frac{\tau}{n})\Big)\,d\tau \tag{287}$$

. . . .
nir-translocation : $\quad f \mapsto \nabla h := (nir - e^{-\eta\partial_\nu} nir\ e^{\epsilon\partial_x})(h)$ with

$$\nabla h(\epsilon,\nu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n\nu}\frac{dn}{n}\int_0^{\epsilon\, n} \exp_{\#}\Big(-\beta(\partial_\tau) f(\frac{\tau}{n})\Big)\, d\tau \tag{288}$$

. . . .
nur-transform : $\quad f \mapsto h$ with

$$h(\nu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{n\nu}\frac{dn}{n}\sum_{\tau\in\frac{1}{2}+\mathbb{N}} \exp^{\#}\Big(-\beta(\partial_\tau) f(\frac{\tau}{n})\Big)\, d\tau \tag{289}$$

. . . .
nur in terms of *nir* :

$$\mathrm{nur}(f) = \sum_{k\in\mathbb{Z}} (-1)^k\, \mathrm{nir}(k\, 2\pi i + f) \tag{290}$$

. . . .
For the interpretation of $exp^{\#}, exp_{\#}$ see §4.3.

10.1.2 SP coefficients and SP series.

Basic data : $\quad F = \exp(-f) \quad , \quad \eta_F := \int_0^1 f(x)dx \quad , \quad \omega_F = e^{-\eta_F}$

asymptotic series	*funct. germs*	
$\tilde{Ig}_{_F}(n) = \exp\Big(-\frac{1}{2}f(0) + \sum_{1\le s\, odd}\frac{\mathfrak{b}_s}{n^s} f^{(s)}(0)\Big)$	$Ig_{_F}(n)$	*ingress factor*
$\tilde{Eg}_{_F}(n) = \exp\Big(-\frac{1}{2}f(1) - \sum_{1\le s\, odd}\frac{\mathfrak{b}_s}{n^s} f^{(s)}(1)\Big)$	$Eg_{_F}(n)$	*egress factor*

. . . .

"*raw*"	"*cleansed*"
$P_F(n) := \prod_{0\le k\le n} F(\frac{k}{n})$	$P_F^{\#}(n) := (\omega_F)^n = \frac{P_F(n)}{Ig_{_F}(n) Eg_{_F}(n)}$
$J_F(n) := \sum_{0\le m<n}\prod_{0\le k\le m} F(\frac{k}{n})$	$J_F^{\#}(n) := J_F(n)/Ig_{_F}(n)$
$j_F(\zeta) := \sum_{0\le n} J_F(n)\,\zeta^n$	$j_F^{\#}(\zeta) := \sum_{0\le n} J_F^{\#}(n)\,\zeta^n$

10.1.3 Parity relations.

$$F^{\vDash}(x) := 1/F(1-x) \qquad \Longrightarrow$$

$$1 = \tilde{Ig}_{_F}(n)\,\tilde{Eg}_{_{F^{\vDash}}}(n) = \tilde{Ig}_{_{F^{\vDash}}}(n)\,\tilde{Eg}_{_F}(n)$$

$$J_{F^{\vDash}}(n) = J_F(n)/P_F(n) \qquad \textit{and} \qquad J^{\#}_{F^{\vDash}}(n) = J^{\#}_F(n)/P^{\#}_F(n)$$

$$j_{F^{\vDash}}(\zeta) \neq j_F(\zeta/\omega_F) \qquad \textit{but} \qquad j^{\#}_{F^{\vDash}}(\zeta) = j^{\#}_F(\zeta/\omega_F)$$

. . . .

$$
\begin{array}{lll}
 & F^{\vdash}(x) := 1/F(-x) \quad , \quad f^{\vdash}(x) := -f(-x) & \Longrightarrow \\
 & \mathrm{nur}(f^{\vdash})(\nu) = -\mathrm{nur}(f)(\nu) & (\textit{tangency } \kappa = 0) \\
 & \mathrm{nir}(f^{\vdash})(\nu) = -\mathrm{nir}(f)(\nu) & (\textit{tangency } \kappa = 0) \\
 & \mathrm{nir}(f^{\vdash}) \ \textit{and} \ \mathrm{nir}(f) \ \textit{unrelated} & (\textit{tangency } \kappa \textit{ even} \geq 2) \\
 & \mathrm{nir}(f^{\vdash})(\nu) = -\mathrm{nur}(f)(\epsilon_\kappa \nu) \quad \textit{with} \quad \epsilon_\kappa^{\frac{1}{\kappa+1}} = -1 & (\textit{tangency } \kappa \textit{ odd} \geq 1) \\
\Rightarrow & h^{\vdash}_{\frac{k}{\kappa+1}} = (-1)^{k-1} h_{\frac{k}{\kappa+1}} \quad \textit{with} \quad : (f, f^{\vdash}) \overset{\mathrm{nir}}{\mapsto} (h, h^{\vdash}) & (\textit{tangency } \kappa \textit{ odd} \geq 1)
\end{array}
$$

. . . .

10.2 The *Mir* mould.

10.2.1 Layered form.

length : $r = 1$, **order** : $d = 1$, **factor** : $c_{1,1} = 1/24$

$$\mathrm{Mir}[0,1] \;=\; 1\, c_{3,2} \;=\; 1/24$$

length : $r = 3$, **order** : $d = 2$, **factor** : $c_{3,2} = 1/1152$

$$\mathrm{Mir}[1,2,0,0] \;=\; 1\, c_{3,2} \;=\; 1/1152$$

length : $r = 3$, **order** : $d = 3$, **factor** : $c_{3,3} = 7/5760$

$$
\begin{array}{lclcl}
\mathrm{Mir}[0,3,0,0] & = & -1\, c_{3,3} & = & -7/5760 \\
\mathrm{Mir}[1,1,1,0] & = & -4\, c_{3,3} & = & -7/1440 \\
\mathrm{Mir}[2,0,0,1] & = & -1\, c_{3,3} & = & -7/5760
\end{array}
$$

length : $r = 5$, **order** : $d = 3$, **factor** : $c_{5,3} = 1/82944$

$$\mathrm{Mir}[2,3,0,0,0,0] \;=\; 1\, c_{5,3} \;=\; 1/82944$$

length : $r = 5$, **order** : $d = 4$, **factor** : $c_{5,4} = 7/138240$

$$
\begin{array}{lcrcr}
\mathrm{Mir}[1,4,0,0,0,0] & = & -1\, c_{5,4} & = & -7/138240 \\
\mathrm{Mir}[2,2,1,0,0,0] & = & -4\, c_{5,4} & = & -7/34560 \\
\mathrm{Mir}[3,1,0,1,0,0] & = & -1\, c_{5,4} & = & -7/138240
\end{array}
$$

length : $r = 5$, **order** : $d = 5$, **factor** : $c_{5,5} = 31/967680$

$$
\begin{array}{lcrcr}
\mathrm{Mir}[0,5,0,0,0,0] & = & 1\, c_{5,5} & = & 31/967680 \\
\mathrm{Mir}[1,3,1,0,0,0] & = & 26\, c_{5,5} & = & 403/483840 \\
\mathrm{Mir}[2,1,2,0,0,0] & = & 34\, c_{5,5} & = & 527/483840 \\
\mathrm{Mir}[2,2,0,1,0,0] & = & 32\, c_{5,5} & = & 31/30240 \\
\mathrm{Mir}[3,0,1,1,0,0] & = & 15\, c_{5,5} & = & 31/64512 \\
\mathrm{Mir}[3,1,0,0,1,0] & = & 11\, c_{5,5} & = & 341/967680 \\
\mathrm{Mir}[4,0,0,0,0,1] & = & 1\, c_{5,5} & = & 31/967680
\end{array}
$$

10.2.2 Compact form.

length : $r = 1$, **gcd** : $d_3 = 24$

$$^\star\text{Mir}[1] = 1/d_1 = 1/24$$

length : $r = 3$, **gcd** : $d_3 = 5760$

$$\begin{array}{rclcl} ^\star\text{Mir}[1,2,0] &=& -2/d_3 &=& -1/2880 \\ ^\star\text{Mir}[2,0,1] &=& -7/d_3 &=& -7/5760 \end{array}$$

length : $r = 5$, **gcd** : $d_5 = 2903040$

$$\begin{array}{rcrcr} ^\star\text{Mir}[1,4,0,0,0] &=& 16/d_5 &=& 1/181440 \\ ^\star\text{Mir}[2,2,1,0,0] &=& 540/d_5 &=& 1/5376 \\ ^\star\text{Mir}[3,0,2,0,0] &=& 372/d_5 &=& 31/241920 \\ ^\star\text{Mir}[3,1,0,1,0] &=& 504/d_5 &=& 1/5760 \\ ^\star\text{Mir}[4,0,0,0,1] &=& 93/d_5 &=& 31/967680 \end{array}$$

length : $r = 7$, **gcd** : $d_7 = 1393459200$

$$\begin{array}{rcrcr} ^\star\text{Mir}[1,6,0,0,0,0,0] &=& -144/d_7 &=& -1/9676800 \\ ^\star\text{Mir}[2,4,1,0,0,0,0] &=& -28824/d_7 &=& -1201/58060800 \\ ^\star\text{Mir}[3,2,2,0,0,0,0] &=& -141576/d_7 &=& -5899/58060800 \\ ^\star\text{Mir}[4,0,3,0,0,0,0] &=& -38862/d_7 &=& -2159/77414400 \\ ^\star\text{Mir}[3,3,0,1,0,0,0] &=& -88928/d_7 &=& -397/6220800 \\ ^\star\text{Mir}[4,1,1,1,0,0,0] &=& -186264/d_7 &=& -2587/19353600 \\ ^\star\text{Mir}[5,0,0,2,0,0,0] &=& -16116/d_7 &=& -1343/116121600 \\ ^\star\text{Mir}[4,2,0,0,1,0,0] &=& -67878/d_7 &=& -419/8601600 \\ ^\star\text{Mir}[5,0,1,0,1,0,0] &=& -29718/d_7 &=& -1651/77414400 \\ ^\star\text{Mir}[5,1,0,0,0,1,0] &=& -16428/d_7 &=& -1369/116121600 \\ ^\star\text{Mir}[6,0,0,0,0,0,1] &=& -1143/d_7 &=& -127/154828800 \end{array}$$

length : $r = 9$, **gcd** : $d_9 = 367873228800$

$$
\begin{array}{lclcr}
{}^\star\mathrm{Mir}[1,8,0,0,0,0,0,0,0] & = & 768/d_9 & = & 1/479001600 \\
{}^\star\mathrm{Mir}[2,6,1,0,0,0,0,0,0] & = & 789504/d_9 & = & 257/119750400 \\
{}^\star\mathrm{Mir}[3,4,2,0,0,0,0,0,0] & = & 13702656/d_9 & = & 811/21772800 \\
{}^\star\mathrm{Mir}[4,2,3,0,0,0,0,0,0] & = & 26034672/d_9 & = & 542389/7664025600 \\
{}^\star\mathrm{Mir}[5,0,4,0,0,0,0,0,0] & = & 3801840/d_9 & = & 2263/218972160 \\
{}^\star\mathrm{Mir}[3,5,0,1,0,0,0,0,0] & = & 6324224/d_9 & = & 193/11226600 \\
{}^\star\mathrm{Mir}[4,3,1,1,0,0,0,0,0] & = & 52597760/d_9 & = & 10273/71850240 \\
{}^\star\mathrm{Mir}[5,1,2,1,0,0,0,0,0] & = & 40989024/d_9 & = & 47441/425779200 \\
{}^\star\mathrm{Mir}[5,2,0,2,0,0,0,0,0] & = & 18164736/d_9 & = & 73/1478400 \\
{}^\star\mathrm{Mir}[6,0,1,2,0,0,0,0,0] & = & 6350064/d_9 & = & 18899/1094860800 \\
{}^\star\mathrm{Mir}[4,4,0,0,1,0,0,0,0] & = & 11628928/d_9 & = & 90851/2874009600 \\
{}^\star\mathrm{Mir}[5,2,1,0,1,0,0,0,0] & = & 33372912/d_9 & = & 695269/7664025600 \\
{}^\star\mathrm{Mir}[6,0,2,0,1,0,0,0,0] & = & 5886720/d_9 & = & 73/4561920 \\
{}^\star\mathrm{Mir}[6,1,0,1,1,0,0,0,0] & = & 9462768/d_9 & = & 28163/1094860800 \\
{}^\star\mathrm{Mir}[7,0,0,0,2,0,0,0,0] & = & 429240/d_9 & = & 511/437944320 \\
{}^\star\mathrm{Mir}[5,3,0,0,0,1,0,0,0] & = & 7436800/d_9 & = & 83/4105728 \\
{}^\star\mathrm{Mir}[6,1,1,0,0,1,0,0,0] & = & 7391376/d_9 & = & 51329/2554675200 \\
{}^\star\mathrm{Mir}[7,0,0,1,0,1,0,0,0] & = & 736848/d_9 & = & 731/364953600 \\
{}^\star\mathrm{Mir}[6,2,0,0,0,0,1,0,0] & = & 1941144/d_9 & = & 80881/15328051200 \\
{}^\star\mathrm{Mir}[7,0,1,0,0,0,1,0,0] & = & 490560/d_9 & = & 73/54743040 \\
{}^\star\mathrm{Mir}[7,1,0,0,0,0,0,1,0] & = & 209712/d_9 & = & 4369/7664025600 \\
{}^\star\mathrm{Mir}[8,0,0,0,0,0,0,0,1] & = & 7665/d_9 & = & 73/3503554560
\end{array}
$$

10.3 The *mir* transform: from $\not{g}$ to $\hbar$.

10.3.1 Tangency 0, ramification 1.

Recall that $\not{g} = 1/g$ and $\hbar = 1/h$.

$$
\begin{aligned}
h_1 - g_1 &= \tfrac{1}{24}\not{g}_1 \\
h_2 - g_2 &= \tfrac{1}{24}\not{g}_2 + \tfrac{1}{2304}\not{g}_0\not{g}_1^2 \\
h_3 - g_3 &= \tfrac{1}{24}\not{g}_3 - \tfrac{1}{17280}\not{g}_1{}^3 - \tfrac{1}{960}\not{g}_0\not{g}_1\not{g}_2 - \tfrac{7}{5760}\not{g}_0{}^2\not{g}_3 + \tfrac{1}{497664}\not{g}_0{}^2\not{g}_1{}^3 \\
h_4 - g_4 &= \tfrac{1}{24}\not{g}_4 - \tfrac{1}{1920}\not{g}_0\not{g}_2{}^2 - \tfrac{1}{2880}\not{g}_1{}^2\not{g}_2 - \tfrac{1}{720}\not{g}_0\not{g}_1\not{g}_3 - \tfrac{7}{5760}\not{g}_0{}^2\not{g}_4 \\
&\quad - \tfrac{23}{1658880}\not{g}_0{}^2\not{g}_1{}^2\not{g}_2 - \tfrac{11}{9953280}\not{g}_0\not{g}_1{}^4 - \tfrac{7}{552960}\not{g}_0{}^3\not{g}_1\not{g}_3 + \tfrac{1}{191102976}\not{g}_0{}^3\not{g}_1{}^4 \\
h_5 - g_5 &= \tfrac{1}{24}\not{g}_5 - \tfrac{11}{28800}\not{g}_1\not{g}_2{}^2 - \tfrac{23}{14400}\not{g}_0\not{g}_1\not{g}_4 - \tfrac{7}{14400}\not{g}_1{}^2\not{g}_3 - \tfrac{17}{14400}\not{g}_0\not{g}_2\not{g}_3 \\
&\quad - \tfrac{7}{5760}\not{g}_0{}^2\not{g}_5 + \tfrac{143}{21772800}\not{g}_0\not{g}_1{}^3\not{g}_2 + \tfrac{61}{2419200}\not{g}_0{}^2\not{g}_1\not{g}_2{}^2 + \tfrac{73}{1209600}\not{g}_0{}^3\not{g}_1\not{g}_4 \\
&\quad + \tfrac{19}{537600}\not{g}_0{}^2\not{g}_1{}^2\not{g}_3 + \tfrac{13}{302400}\not{g}_0{}^3\not{g}_2\not{g}_3 + \tfrac{31}{967680}\not{g}_0{}^4\not{g}_5 + \tfrac{1}{21772800}\not{g}_1{}^5 \\
&\quad - \tfrac{37}{597196800}\not{g}_0{}^3\not{g}_1{}^3\not{g}_2 - \tfrac{1}{176947200}\not{g}_0{}^2\not{g}_1{}^5 - \tfrac{7}{132710400}\not{g}_0{}^4\not{g}_1{}^2\not{g}_3 \\
&\quad + \tfrac{1}{114661785600}\not{g}_0{}^4\not{g}_1{}^5 \\
h_6 - g_6 &= +\tfrac{1}{24}\not{g}_6 - \tfrac{13}{14400}\not{g}_1\not{g}_2\not{g}_3 - \tfrac{17}{28800}\not{g}_1{}^2\not{g}_4 - \tfrac{19}{14400}\not{g}_0\not{g}_2\not{g}_4 - \tfrac{1}{576}\not{g}_0\not{g}_1\not{g}_5 \\
&\quad - \tfrac{7}{5760}\not{g}_0{}^2\not{g}_6 - \tfrac{11}{86400}\not{g}_2{}^3 - \tfrac{17}{28800}\not{g}_0\not{g}_3{}^2 + \tfrac{17}{241920}\not{g}_0{}^2\not{g}_1\not{g}_2\not{g}_3 \\
&\quad + \tfrac{163}{10886400}\not{g}_0\not{g}_1{}^2\not{g}_2{}^2 + \tfrac{41}{2721600}\not{g}_0\not{g}_1{}^3\not{g}_3 + \tfrac{13}{241920}\not{g}_0{}^2\not{g}_1{}^2\not{g}_4 \\
&\quad + \tfrac{59}{1209600}\not{g}_0{}^3\not{g}_2\not{g}_4 + \tfrac{17}{14515200}\not{g}_2\not{g}_1{}^4 + \tfrac{13}{181440}\not{g}_0{}^3\not{g}_1\not{g}_5 + \tfrac{31}{967680}\not{g}_0{}^4\not{g}_6 \\
&\quad + \tfrac{61}{7257600}\not{g}_0{}^2\not{g}_2{}^3 + \tfrac{13}{604800}\not{g}_0{}^3\not{g}_3{}^2 + \tfrac{461}{1161216000}\not{g}_0{}^4\not{g}_1\not{g}_2\not{g}_3 \\
&\quad + \tfrac{5347}{20901888000}\not{g}_0{}^3\not{g}_1{}^2\not{g}_2{}^2 + \tfrac{25259}{83607552000}\not{g}_0{}^3\not{g}_1{}^3\not{g}_3 + \tfrac{211}{464486400}\not{g}_0{}^4\not{g}_1{}^2\not{g}_4 \\
&\quad + \tfrac{12109}{167215104000}\not{g}_0{}^2\not{g}_1{}^4\not{g}_2 + \tfrac{31}{139345920}\not{g}_0{}^5\not{g}_1\not{g}_5 + \tfrac{37}{37158912000}\not{g}_0\not{g}_1{}^6 \\
&\quad + \tfrac{49}{1327104000}\not{g}_0{}^5\not{g}_3{}^2 - \tfrac{17}{114661785600}\not{g}_0{}^4\not{g}_1{}^4\not{g}_2 - \tfrac{1}{68797071360}\not{g}_0{}^3\not{g}_1{}^6 \\
&\quad - \tfrac{7}{57330892800}\not{g}_0{}^5\not{g}_1{}^3\not{g}_3 + \tfrac{1}{99067782758400}\not{g}_0{}^5\not{g}_1{}^6
\end{aligned}
$$

10.3.2 Tangency 1, ramification 2.

$$
\begin{aligned}
h_{1/2} - g_{1/2} &= \tfrac{1}{24}\, \mathscr{g}_{1/2} \\
h_1 - g_1 &= \tfrac{1}{24}\, \mathscr{g}_1 \\
h_{3/2} - g_{3/2} &= \tfrac{1}{24}\, \mathscr{g}_{3/2} + \tfrac{1}{3456}\, \mathscr{g}_{1/2}{}^3 \\
\ell_2 - g_2 &= \tfrac{1}{24}\, \mathscr{g}_2 + \tfrac{5}{9216}\, \mathscr{g}_{1/2}{}^2 \mathscr{g}_1 \\
h_{5/2} - g_{5/2} &= \tfrac{1}{24}\, \mathscr{g}_{5/2} - \tfrac{1}{9600}\, \mathscr{g}_{1/2} \mathscr{g}_1{}^2 - \tfrac{7}{28800}\, \mathscr{g}_{1/2}{}^2 \mathscr{g}_{3/2} + \tfrac{1}{1244160}\, \mathscr{g}_{1/2}{}^5 \\
h_3 - g_3 &= \tfrac{1}{24}\, \mathscr{g}_3 - \tfrac{1}{17280}\, \mathscr{g}_1{}^3 - \tfrac{1}{1920}\, \mathscr{g}_1 \mathscr{g}_{3/2} \mathscr{g}_{1/2} - \tfrac{1}{2304}\, \mathscr{g}_{1/2}{}^2 \mathscr{g}_2 + \tfrac{1}{497664}\, \mathscr{g}_{1/2}{}^4 \mathscr{g}_1 \\
h_{7/2} - g_{7/2} &= \tfrac{1}{24}\, \mathscr{g}_{7/2} - \tfrac{53}{201600}\, \mathscr{g}_1{}^2 \mathscr{g}_{3/2} - \tfrac{13}{40320}\, \mathscr{g}_{1/2} \mathscr{g}_{3/2}{}^2 - \tfrac{11}{14400}\, \mathscr{g}_{1/2} \mathscr{g}_1 \mathscr{g}_2 \\
&\quad - \tfrac{113}{201600}\, \mathscr{g}_{1/2}{}^2 \mathscr{g}_{5/2} - \tfrac{11}{21772800}\, \mathscr{g}_{1/2}{}^3 \mathscr{g}_1{}^2 - \tfrac{113}{43545600}\, \mathscr{g}_{1/2}{}^4 \mathscr{g}_{3/2} \\
&\quad + \tfrac{1}{836075520}\, \mathscr{g}_{1/2}{}^7 \\
h_4 - g_4 &= \tfrac{1}{24}\, \mathscr{g}_4 - \tfrac{1}{1280}\, \mathscr{g}_{1/2} \mathscr{g}_{3/2} \mathscr{g}_2 - \tfrac{7}{23040}\, \mathscr{g}_1 \mathscr{g}_{3/2}{}^2 - \tfrac{11}{11520}\, \mathscr{g}_{1/2} \mathscr{g}_1 \mathscr{g}_{5/2} \\
&\quad - \tfrac{1}{2880}\, \mathscr{g}_1{}^2 \mathscr{g}_2 - \tfrac{1}{1536}\, \mathscr{g}_{1/2}{}^2 \mathscr{g}_3 - \tfrac{229}{159252480}\, \mathscr{g}_{1/2}{}^2 \mathscr{g}_1{}^3 \\
&\quad - \tfrac{229}{39813120}\, \mathscr{g}_{1/2}{}^3 \mathscr{g}_1 \mathscr{g}_{3/2} - \tfrac{49}{15925248}\, \mathscr{g}_{1/2}{}^4 \mathscr{g}_2 + \tfrac{11}{3057647616}\, \mathscr{g}_{1/2}{}^6 \mathscr{g}_1 \\
h_{9/2} - g_{9/2} &= \tfrac{1}{24}\, \mathscr{g}_{9/2} - \tfrac{43}{60480}\, \mathscr{g}_1 \mathscr{g}_{3/2} \mathscr{g}_2 - \tfrac{67}{60480}\, \mathscr{g}_{1/2} \mathscr{g}_1 \mathscr{g}_3 - \tfrac{17}{40320}\, \mathscr{g}_{1/2} \mathscr{g}_2{}^2 \\
&\quad - \tfrac{13}{120960}\, \mathscr{g}_{3/2}{}^3 - \tfrac{29}{40320}\, \mathscr{g}_{1/2}{}^2 \mathscr{g}_{7/2} - \tfrac{17}{40320}\, \mathscr{g}_1{}^2 \mathscr{g}_{5/2} - \tfrac{11}{12096}\, \mathscr{g}_{1/2} \mathscr{g}_{3/2} \mathscr{g}_{5/2} \\
&\quad + \tfrac{907}{457228800}\, \mathscr{g}_{1/2}{}^2 \mathscr{g}_1{}^2 \mathscr{g}_{3/2} + \tfrac{757}{228614400}\, \mathscr{g}_{1/2}{}^3 \mathscr{g}_1 \mathscr{g}_2 + \tfrac{79}{914457600}\, \mathscr{g}_{1/2} \mathscr{g}_1{}^4 \\
&\quad + \tfrac{659}{457228800}\, \mathscr{g}_{1/2}{}^3 \mathscr{g}_{3/2}{}^2 + \tfrac{1531}{914457600}\, \mathscr{g}_{1/2}{}^4 \mathscr{g}_{5/2} - \tfrac{13}{12541132800}\, \mathscr{g}_{1/2}{}^5 \mathscr{g}_1{}^2 \\
&\quad - \tfrac{29}{4180377600}\, \mathscr{g}_{1/2}{}^6 \mathscr{g}_{3/2} + \tfrac{1}{902961561600}\, \mathscr{g}_{1/2}{}^9 \\
h_5 - g_5 &= \tfrac{1}{24}\, \mathscr{g}_5 - \tfrac{11}{28800}\, \mathscr{g}_1 \mathscr{g}_2{}^2 - \tfrac{7}{14400}\, \mathscr{g}_1{}^2 \mathscr{g}_3 - \tfrac{47}{57600}\, \mathscr{g}_1 \mathscr{g}_{3/2} \mathscr{g}_{5/2} \\
&\quad - \tfrac{71}{57600}\, \mathscr{g}_{1/2} \mathscr{g}_1 \mathscr{g}_{7/2} - \tfrac{53}{57600}\, \mathscr{g}_2 \mathscr{g}_{1/2} \mathscr{g}_{5/2} - \tfrac{59}{57600}\, \mathscr{g}_{1/2} \mathscr{g}_{3/2} \mathscr{g}_3 \\
&\quad - \tfrac{41}{115200}\, \mathscr{g}_{3/2}{}^2 \mathscr{g}_2 - \tfrac{89}{115200}\, \mathscr{g}_{1/2}{}^2 \mathscr{g}_4 + \tfrac{1}{21772800}\, \mathscr{g}_1{}^5 \\
&\quad + \tfrac{383}{43545600}\, \mathscr{g}_{1/2}{}^3 \mathscr{g}_1 \mathscr{g}_{5/2} + \tfrac{797}{116121600}\, \mathscr{g}_{1/2}{}^2 \mathscr{g}_1{}^2 \mathscr{g}_2 \\
&\quad + \tfrac{361}{174182400}\, \mathscr{g}_{1/2} \mathscr{g}_1{}^3 \mathscr{g}_{3/2} + \tfrac{41}{6220800}\, \mathscr{g}_{1/2}{}^3 \mathscr{g}_{3/2} \mathscr{g}_2 \\
&\quad + \tfrac{17}{3225600}\, \mathscr{g}_{1/2}{}^2 \mathscr{g}_1 \mathscr{g}_{3/2}{}^2 + \tfrac{19}{4976640}\, \mathscr{g}_{1/2}{}^4 \mathscr{g}_3 - \tfrac{1}{188743680}\, \mathscr{g}_{1/2}{}^4 \mathscr{g}_1{}^3 \\
&\quad - \tfrac{47}{2548039680}\, \mathscr{g}_{1/2}{}^5 \mathscr{g}_1 \mathscr{g}_{3/2} - \tfrac{107}{15288238080}\, \mathscr{g}_{1/2}{}^6 \mathscr{g}_2 + \tfrac{7}{1834588569600}\, \mathscr{g}_{1/2}{}^8 \mathscr{g}_1
\end{aligned}
$$

10.3.3 Tangency 2, ramification 3.

$$
\begin{aligned}
h_{2/3} - g_{2/3} &= \tfrac{1}{24}\, \not{g}_{2/3} \\
h_1 - g_1 &= \tfrac{1}{24}\, \not{g}_1 \\
h_{4/3} - g_{4/3} &= \tfrac{1}{24}\, \not{g}_{4/3} \\
\ell_{5/3} - g_{5/3} &= \tfrac{1}{24}\, \not{g}_{5/3} \\
h_2 - g_2 &= \tfrac{1}{24}\, \not{g}_2 + \tfrac{1}{5184}\, {\not{g}_{2/3}}^3 \\
h_{7/3} - g_{7/3} &= \tfrac{1}{24}\, \not{g}_{7/3} - \tfrac{1}{40320}\, {\not{g}_{2/3}}^2 \not{g}_1 \\
h_{8/3} - g_{8/3} &= \tfrac{1}{24}\, \not{g}_{8/3} - \tfrac{7}{57600}\, \not{g}_{2/3} {\not{g}_1}^2 - \tfrac{1}{5760}\, {\not{g}_{2/3}}^2 \not{g}_{4/3} \\
h_3 - g_3 &= \tfrac{1}{24}\, \not{g}_3 - \tfrac{1}{17280}\, {\not{g}_1}^3 - \tfrac{11}{25920}\, \not{g}_{2/3} \not{g}_1 \not{g}_{4/3} - \tfrac{1}{3456}\, {\not{g}_{2/3}}^2 \not{g}_{5/3} \\
h_{10/3} - g_{10/3} &= \tfrac{1}{24}\, \not{g}_{10/3} - \tfrac{53}{201600}\, \not{g}_{2/3} {\not{g}_{4/3}}^2 - \tfrac{47}{201600}\, {\not{g}_1}^2 \not{g}_{4/3} \\
&\quad - \tfrac{59}{100800}\, \not{g}_{2/3} \not{g}_1 \not{g}_{5/3} - \tfrac{11}{28800}\, {\not{g}_{2/3}}^2 \not{g}_2 + \tfrac{1}{2903040}\, {\not{g}_{2/3}}^5 \\
h_{11/3} - g_{11/3} &= \tfrac{1}{24}\, \not{g}_{11/3} - \tfrac{23}{31680}\, \not{g}_{2/3} \not{g}_1 \not{g}_2 - \tfrac{29}{63360}\, {\not{g}_{2/3}}^2 \not{g}_{7/3} - \tfrac{17}{63360}\, \not{g}_1 {\not{g}_{4/3}}^2 \\
&\quad - \tfrac{37}{126720}\, {\not{g}_1}^2 \not{g}_{5/3} - \tfrac{1}{1584}\, \not{g}_{2/3} \not{g}_{4/3} \not{g}_{5/3} + \tfrac{1}{22809600}\, {\not{g}_{2/3}}^4 \not{g}_1 \\
h_4 - g_4 &= \tfrac{1}{24}\, \not{g}_4 - \tfrac{1}{2880}\, {\not{g}_1}^2 \not{g}_2 - \tfrac{1}{1620}\, \not{g}_1 \not{g}_{4/3} \not{g}_{5/3} - \tfrac{11}{12960}\, \not{g}_{2/3} \not{g}_1 \not{g}_{7/3} \\
&\quad - \tfrac{1}{2880}\, \not{g}_{2/3} {\not{g}_{5/3}}^2 - \tfrac{1}{10368}\, {\not{g}_{4/3}}^3 - \tfrac{19}{25920}\, \not{g}_{2/3} \not{g}_{4/3} \not{g}_2 - \tfrac{1}{1920}\, {\not{g}_{2/3}}^2 \not{g}_{8/3} \\
&\quad - \tfrac{47}{44789760}\, {\not{g}_{2/3}}^3 {\not{g}_1}^2 - \tfrac{1}{1119744}\, {\not{g}_{2/3}}^4 \not{g}_{4/3} \\
h_{13/3} - g_{13/3} &= \tfrac{1}{24}\, \not{g}_{13/3} - \tfrac{131}{187200}\, \not{g}_1 \not{g}_{4/3} \not{g}_2 - \tfrac{179}{187200}\, \not{g}_{2/3} \not{g}_1 \not{g}_{8/3} - \tfrac{43}{74880}\, {\not{g}_{2/3}}^2 \not{g}_3 \\
&\quad - \tfrac{119}{374400}\, {\not{g}_{4/3}}^2 \not{g}_{5/3} - \tfrac{5}{14976}\, \not{g}_1 {\not{g}_{5/3}}^2 - \tfrac{149}{374400}\, {\not{g}_1}^2 \not{g}_{7/3} \\
&\quad - \tfrac{31}{37440}\, \not{g}_{2/3} \not{g}_{4/3} \not{g}_{7/3} - \tfrac{11}{14400}\, \not{g}_{2/3} \not{g}_{5/3} \not{g}_2 - \tfrac{1}{22014720}\, {\not{g}_{2/3}}^2 {\not{g}_1}^3 \\
&\quad + \tfrac{37}{330220800}\, {\not{g}_{2/3}}^3 \not{g}_1 \not{g}_{4/3} + \tfrac{23}{264176640}\, {\not{g}_{2/3}}^4 \not{g}_{5/3} \\
h_{14/3} - g_{14/3} &= \tfrac{1}{24}\, \not{g}_{14/3} - \tfrac{173}{221760}\, \not{g}_1 \not{g}_{4/3} \not{g}_{7/3} - \tfrac{23}{31680}\, \not{g}_1 \not{g}_{5/3} \not{g}_2 - \tfrac{233}{221760}\, \not{g}_{2/3} \not{g}_1 \not{g}_3 \\
&\quad - \tfrac{179}{443520}\, \not{g}_{2/3} {\not{g}_2}^2 - \tfrac{197}{443520}\, {\not{g}_1}^2 \not{g}_{8/3} - \tfrac{29}{31680}\, \not{g}_{2/3} \not{g}_{4/3} \not{g}_{8/3} \\
&\quad - \tfrac{31}{88704}\, {\not{g}_{4/3}}^2 \not{g}_2 - \tfrac{5}{8064}\, {\not{g}_{2/3}}^2 \not{g}_{10/3} - \tfrac{149}{443520}\, \not{g}_{4/3} {\not{g}_{5/3}}^2 \\
&\quad - \tfrac{37}{44352}\, \not{g}_{2/3} \not{g}_{5/3} \not{g}_{7/3} + \tfrac{1}{9313920}\, \not{g}_{2/3} {\not{g}_1}^4 + \tfrac{1}{776160}\, {\not{g}_{2/3}}^2 {\not{g}_1}^2 \not{g}_{4/3} \\
&\quad + \tfrac{1}{1451520}\, {\not{g}_{2/3}}^3 {\not{g}_{4/3}}^2 + \tfrac{17}{10160640}\, {\not{g}_{2/3}}^3 \not{g}_1 \not{g}_{5/3} + \tfrac{23}{31933440}\, {\not{g}_{2/3}}^4 \not{g}_2 \\
&\quad + \tfrac{1}{3065610240}\, {\not{g}_{2/3}}^7
\end{aligned}
$$

10.3.4 Tangency 3, ramification 4.

$$
\begin{aligned}
h_{3/4} - g_{3/4} &= \tfrac{1}{24}\, \not{g}_{3/4} \\
h_1 - g_1 &= \tfrac{1}{24}\, \not{g}_1 \\
h_{5/4} - g_{5/4} &= \tfrac{1}{24}\, \not{g}_{5/4} \\
h_{3/2} - g_{3/2} &= \tfrac{1}{24}\, \not{g}_{3/2} \\
h_{7/4} - g_{7/4} &= \tfrac{1}{24}\, \not{g}_{7/4} \\
h_2 - g_2 &= \tfrac{1}{24}\, \not{g}_2 \\
h_{9/4} - g_{9/4} &= \tfrac{1}{24}\, \not{g}_{9/4} + \tfrac{1}{86400}\, {\not{g}_{3/4}}^3 \\
h_{5/2} - g_{5/2} &= \tfrac{1}{24}\, \not{g}_{5/2} - \tfrac{1}{14400}\, {\not{g}_{3/4}}^2 \not{g}_1 \\
h_{11/4} - g_{11/4} &= \tfrac{1}{24}\, \not{g}_{11/4} - \tfrac{59}{443520}\, \not{g}_{3/4} {\not{g}_1}^2 - \tfrac{71}{443520}\, {\not{g}_{3/4}}^2 \not{g}_{5/4} \\
h_3 - g_3 &= \tfrac{1}{24}\, \not{g}_3 - \tfrac{1}{17280}\, {\not{g}_1}^3 - \tfrac{1}{2560}\, \not{g}_{3/4} \not{g}_1 \not{g}_{5/4} - \tfrac{11}{46080}\, {\not{g}_{3/4}}^2 \not{g}_{3/2} \\
h_{13/4} - g_{13/4} &= \tfrac{1}{24}\, \not{g}_{13/4} - \tfrac{49}{224640}\, {\not{g}_1}^2 \not{g}_{5/4} - \tfrac{19}{37440}\, \not{g}_{3/4} \not{g}_1 \not{g}_{3/2} \\
&\quad - \tfrac{53}{224640}\, \not{g}_{3/4} {\not{g}_{5/4}}^2 - \tfrac{23}{74880}\, {\not{g}_{3/4}}^2 \not{g}_{7/4} \\
h_{7/2} - g_{7/2} &= \tfrac{1}{24}\, \not{g}_{7/2} - \tfrac{53}{201600}\, {\not{g}_1}^2 \not{g}_{3/2} - \tfrac{31}{50400}\, \not{g}_{3/4} \not{g}_1 \not{g}_{7/4} \\
&\quad - \tfrac{1}{1800}\, \not{g}_{3/4} \not{g}_{5/4} \not{g}_{3/2} - \tfrac{37}{100800}\, {\not{g}_{3/4}}^2 \not{g}_2 - \tfrac{1}{4032}\, \not{g}_1 {\not{g}_{5/4}}^2 \\
h_{15/4} - g_{15/4} &= \tfrac{1}{24}\, \not{g}_{15/4} - \tfrac{113}{158400}\, \not{g}_{3/4} \not{g}_1 \not{g}_2 - \tfrac{89}{158400}\, \not{g}_{3/2} \not{g}_1 \not{g}_{5/4} - \tfrac{17}{190080}\, {\not{g}_{5/4}}^3 \\
&\quad - \tfrac{97}{316800}\, \not{g}_{3/4} {\not{g}_{3/2}}^2 - \tfrac{97}{316800}\, {\not{g}_1}^2 \not{g}_{7/4} - \tfrac{101}{158400}\, \not{g}_{3/4} \not{g}_{5/4} \not{g}_{7/4} \\
&\quad - \tfrac{133}{316800}\, {\not{g}_{3/4}}^2 \not{g}_{9/4} + \tfrac{1}{53222400}\, {\not{g}_{3/4}}^5 \\
h_4 - g_4 &= \tfrac{1}{24}\, \not{g}_4 - \tfrac{1}{2880}\, {\not{g}_1}^2 \not{g}_2 - \tfrac{7}{23040}\, \not{g}_1 {\not{g}_{3/2}}^2 - \tfrac{37}{46080}\, \not{g}_{3/4} \not{g}_1 \not{g}_{9/4} \\
&\quad - \tfrac{29}{46080}\, \not{g}_1 \not{g}_{5/4} \not{g}_{7/4} - \tfrac{43}{92160}\, {\not{g}_{3/4}}^2 \not{g}_{5/2} - \tfrac{11}{15360}\, \not{g}_{3/4} \not{g}_{5/4} \not{g}_2 \\
&\quad - \tfrac{3}{10240}\, {\not{g}_{5/4}}^2 \not{g}_{3/2} - \tfrac{31}{46080}\, \not{g}_{3/4} \not{g}_{3/2} \not{g}_{7/4} - \tfrac{29}{70778880}\, {\not{g}_{3/4}}^4 \not{g}_1 \\
h_{17/4} - g_{17/4} &= \tfrac{1}{24}\, \not{g}_{17/4} - \tfrac{563}{636480}\, \not{g}_{3/4} \not{g}_1 \not{g}_{5/2} - \tfrac{419}{636480}\, \not{g}_{3/2} \not{g}_1 \not{g}_{7/4} \\
&\quad - \tfrac{443}{636480}\, \not{g}_1 \not{g}_{5/4} \not{g}_2 - \tfrac{7}{19584}\, \not{g}_{3/4} {\not{g}_{7/4}}^2 - \tfrac{407}{1272960}\, {\not{g}_{5/4}}^2 \not{g}_{7/4} \\
&\quad - \tfrac{467}{636480}\, \not{g}_{3/4} \not{g}_{3/2} \not{g}_2 - \tfrac{79}{254592}\, \not{g}_{5/4} {\not{g}_{3/2}}^2 - \tfrac{647}{1272960}\, {\not{g}_{3/4}}^2 \not{g}_{11/4} \\
&\quad - \tfrac{491}{1272960}\, {\not{g}_1}^2 \not{g}_{9/4} - \tfrac{503}{636480}\, \not{g}_{3/4} \not{g}_{5/4} \not{g}_{9/4} - \tfrac{1}{7128576}\, {\not{g}_{3/4}}^3 {\not{g}_1}^2 \\
&\quad - \tfrac{43}{641571840}\, {\not{g}_{3/4}}^4 \not{g}_{5/4}
\end{aligned}
$$

10.4 The *nir* transform : from f to h.

In all the tables that follow, the vertical bars ||||| separate clusters of terms with different homogeneous degree in f.

10.4.1 Tangency 0, ramification 1.

$$
\begin{aligned}
h_0 &= f_0{}^{-1} \\
h_1 &= -f_0{}^{-3}f_1 \ \ \|\|\| + \tfrac{1}{24}\,f_0{}^{-1}f_1 \\
h_2 &= +\tfrac{3}{2}\,f_0{}^{-5}f_1{}^2 - f_0{}^{-4}f_2 \ \ \|\|\| - \tfrac{1}{48}\,f_0{}^{-3}f_1{}^2 + \tfrac{1}{24}\,f_0{}^{-2}f_2 \ \ \|\|\| + \tfrac{1}{2304}\,f_0{}^{-1}f_1{}^2 \\
h_3 &= -\tfrac{5}{2}\,f_0{}^{-7}f_1{}^3 + \tfrac{10}{3}\,f_0{}^{-6}f_1f_2 - f_0{}^{-5}f_3 \ \ \|\|\| + \tfrac{1}{48}\,f_0{}^{-5}f_1{}^3 - \tfrac{1}{18}\,f_0{}^{-4}f_1f_2 + \tfrac{1}{24}\,f_0{}^{-3}f_3 \\
&\qquad \|\|\| - \tfrac{1}{6912}\,f_0{}^{-3}f_1{}^3 + \tfrac{1}{1728}\,f_0{}^{-2}f_1f_2 - \tfrac{7}{5760}\,f_0{}^{-1}f_3 \ \ \|\|\| + \tfrac{1}{497664}\,f_0{}^{-1}f_1{}^3 \\
h_4 &= \tfrac{35}{8}\,f_0{}^{-9}f_1{}^4 - \tfrac{35}{4}\,f_0{}^{-8}f_1{}^2f_2 + \tfrac{5}{3}\,f_0{}^{-7}f_2{}^2 + \tfrac{15}{4}\,f_0{}^{-7}f_1f_3 - f_0{}^{-6}f_4 \ \ \|\|\| \\
&\quad -\tfrac{5}{192}\,f_0{}^{-7}f_1{}^4 + \tfrac{25}{288}\,f_0{}^{-6}f_1{}^2f_2 - \tfrac{7}{96}\,f_0{}^{-5}f_1f_3 - \tfrac{1}{36}\,f_0{}^{-5}f_2{}^2 + \tfrac{1}{24}\,f_0{}^{-4}f_4 \ \ \|\|\| \\
&\quad +\tfrac{1}{9216}\,f_0{}^{-5}f_1{}^4 - \tfrac{7}{13824}\,f_0{}^{-4}f_1{}^2f_2 + \tfrac{17}{23040}\,f_0{}^{-3}f_1f_3 + \tfrac{1}{3456}\,f_0{}^{-3}f_2{}^2 - \tfrac{7}{5760}\,f_0{}^{-2}f_4 \\
&\quad \|\|\| - \tfrac{1}{1990656}\,f_0{}^{-3}f_1{}^4 + \tfrac{1}{331776}\,f_0{}^{-2}f_1{}^2f_2 - \tfrac{7}{552960}\,f_0{}^{-1}f_1f_3 \|\|\| + \tfrac{1}{191102976}\,f_0{}^{-1}f_1{}^4 \\
h_5 &= -\tfrac{63}{8}\,f_0{}^{-11}f_1{}^5 + 21\,f_0{}^{-10}f_1{}^3f_2 - \tfrac{28}{3}\,f_0{}^{-9}f_1f_2{}^2 - \tfrac{21}{2}\,f_0{}^{-9}f_1{}^2f_3 + \tfrac{7}{2}\,f_0{}^{-8}f_2f_3 \\
&\quad +\tfrac{21}{5}\,f_0{}^{-8}f_1f_4 - f_0{}^{-7}f_5 \ \ \|\|\| + \tfrac{7}{192}\,f_0{}^{-9}f_1{}^5 - \tfrac{7}{48}\,f_0{}^{-8}f_1{}^3f_2 + \tfrac{1}{8}\,f_0{}^{-7}f_1{}^2f_3 \\
&\quad +\tfrac{7}{72}\,f_0{}^{-7}f_1f_2{}^2 - \tfrac{1}{16}\,f_0{}^{-6}f_2f_3 - \tfrac{11}{120}\,f_0{}^{-6}f_1f_4 + \tfrac{1}{24}\,f_0{}^{-5}f_5 \ \ \|\|\| - \tfrac{1}{9216}\,f_0{}^{-7}f_1{}^5 \\
&\quad +\tfrac{1}{1728}\,f_0{}^{-6}f_1{}^3f_2 - \tfrac{43}{57600}\,f_0{}^{-5}f_1{}^2f_3 - \tfrac{1}{1728}\,f_0{}^{-5}f_1f_2{}^2 + \tfrac{37}{57600}\,f_0{}^{-4}f_2f_3 \\
&\quad +\tfrac{31}{28800}\,f_0{}^{-4}f_1f_4 - \tfrac{7}{5760}\,f_0{}^{-3}f_5 \ \ \|\|\| + \tfrac{1}{3317760}\,f_0{}^{-5}f_1{}^5 - \tfrac{1}{497664}\,f_0{}^{-4}f_1{}^3f_2 \\
&\quad +\tfrac{1}{230400}\,f_0{}^{-3}f_1{}^2f_3 + \tfrac{1}{414720}\,f_0{}^{-3}f_1f_2{}^2 - \tfrac{7}{1382400}\,f_0{}^{-2}f_2f_3 - \tfrac{7}{691200}\,f_0{}^{-2}f_1f_4 \\
&\quad +\tfrac{31}{967680}\,f_0{}^{-1}f_5 \ \ \|\|\| - \tfrac{1}{955514880}\,f_0{}^{-3}f_1{}^5 + \tfrac{1}{119439360}\,f_0{}^{-2}f_1{}^3f_2 \\
&\quad -\tfrac{7}{132710400}\,f_0{}^{-1}f_1{}^2f_3 \ \ \|\|\| + \tfrac{1}{114661785600}\,f_0{}^{-1}f_1{}^5
\end{aligned}
$$

10.4.2 Tangency 1, ramification 2.

$$
\begin{aligned}
h_{-1/2} &= 2^{-1/2}\{{f_1}^{-1/2}\} \\
h_0 &= \quad \{-\tfrac{2}{3}{f_1}^{-2}f_2\} \\
h_{1/2} &= 2^{1/2}\{\tfrac{5}{6}\,{f_1}^{-7/2}{f_2}^2 - \tfrac{3}{4}\,{f_1}^{-5/2}f_3 \;\;\|\|\| + \tfrac{1}{24}\,{f_1}^{1/2}\} \\
h_1 &= 2 \;\; \{-\tfrac{4}{5}\,{f_1}^{-3}f_4 - \tfrac{32}{27}\,{f_1}^{-5}{f_2}^3 + 2\,{f_1}^{-4}f_2f_3 \;\;\|\|\| + \tfrac{1}{36}\,{f_1}^{-1}f_2\} \\
h_{3/2} &= 2^{3/2}\{-\tfrac{5}{6}\,{f_1}^{-7/2}f_5 + \tfrac{7}{3}\,{f_1}^{-9/2}f_2f_4 - \tfrac{35}{8}\,{f_1}^{-11/2}{f_2}^2f_3 + \tfrac{385}{216}\,{f_1}^{-13/2}{f_2}^4 \\
&\qquad + \tfrac{35}{32}\,{f_1}^{-9/2}{f_3}^2 \;\;\|\|\| - \tfrac{7}{432}\,{f_1}^{-5/2}{f_2}^2 + \tfrac{1}{32}\,{f_1}^{-3/2}f_3 \;\;\|\|\| + \tfrac{1}{3456}\,{f_1}^{3/2}\} \\
h_2 &= 2^2 \;\; \{-\tfrac{16}{3}\,{f_1}^{-6}{f_2}^2f_4 - 5\,{f_1}^{-6}f_2{f_3}^2 + \tfrac{80}{9}\,{f_1}^{-7}{f_2}^3f_3 + \tfrac{8}{3}\,{f_1}^{-5}f_2f_5 \\
&\qquad + \tfrac{12}{5}\,{f_1}^{-5}f_3f_4 - \tfrac{6}{7}\,{f_1}^{-4}f_6 - \tfrac{224}{81}\,{f_1}^{-8}{f_2}^5 \;\;\|\|\| + \tfrac{1}{30}\,{f_1}^{-2}f_4 \\
&\qquad + \tfrac{5}{324}\,{f_1}^{-4}{f_2}^3 - \tfrac{1}{24}\,{f_1}^{-3}f_2f_3 \;\;\|\|\| + \tfrac{5}{13824}\,f_2\} \\
h_{5/2} &= 2^{5/2}\{-\tfrac{231}{20}\,{f_1}^{-13/2}f_2f_3f_4 - \tfrac{7}{8}\,{f_1}^{-9/2}f_7 - \tfrac{5005}{288}\,{f_1}^{-17/2}{f_2}^4f_3 \\
&\qquad + \tfrac{1001}{90}\,{f_1}^{-15/2}{f_2}^3f_4 - \tfrac{231}{128}\,{f_1}^{-13/2}{f_3}^3 + \tfrac{17017}{3888}\,{f_1}^{-19/2}{f_2}^6 \\
&\qquad - \tfrac{77}{12}\,{f_1}^{-13/2}f_5{f_2}^2 + \tfrac{21}{8}\,{f_1}^{-11/2}f_3f_5 + \tfrac{63}{50}\,{f_1}^{-11/2}{f_4}^2 \\
&\qquad + \tfrac{1001}{64}\,{f_1}^{-15/2}{f_2}^2{f_3}^2 + 3\,{f_1}^{-11/2}f_2f_6 \;\;\|\|\| - \tfrac{91}{5184}\,{f_1}^{-11/2}{f_2}^4 \\
&\qquad + \tfrac{5}{144}\,{f_1}^{-5/2}f_5 - \tfrac{17}{768}\,{f_1}^{-7/2}{f_3}^2 + \tfrac{35}{576}\,{f_1}^{-9/2}{f_2}^2f_3 - \tfrac{19}{360}\,{f_1}^{-7/2}f_2f_4 \\
&\qquad\;\; \|\|\| - \tfrac{7}{38400}\,{f_1}^{-1/2}f_3 + \tfrac{1}{20736}\,{f_1}^{-3/2}{f_2}^2 \;\;\|\|\|\,\tfrac{1}{1244160}\,{f_1}^{5/2}\} \\
h_3 &= 2^3 \;\; \{\tfrac{112}{3}\,{f_1}^{-8}{f_2}^2f_3f_4 - \tfrac{40}{3}\,{f_1}^{-7}f_2f_3f_5 + \tfrac{10}{3}\,{f_1}^{-6}f_2f_7 + \tfrac{20}{7}\,{f_1}^{-6}f_3f_6 \\
&\qquad - \tfrac{160}{21}\,{f_1}^{-7}{f_2}^2f_6 + \tfrac{8}{3}\,{f_1}^{-6}f_4f_5 + \tfrac{1120}{81}\,{f_1}^{-8}{f_2}^3f_5 - \tfrac{32}{5}\,{f_1}^{-7}f_2{f_4}^2 \\
&\qquad - \tfrac{8}{9}\,{f_1}^{-5}f_8 - 6\,{f_1}^{-7}{f_3}^2f_4 - \tfrac{1792}{81}\,{f_1}^{-9}{f_2}^4f_4 - \tfrac{1120}{27}\,{f_1}^{-9}{f_2}^3{f_3}^2 \\
&\qquad + \tfrac{35}{3}\,{f_1}^{-8}f_2{f_3}^3 - \tfrac{5120}{729}\,{f_1}^{-11}{f_2}^7 + \tfrac{896}{27}\,{f_1}^{-10}{f_2}^5f_3 \;\;\|\|\| + \tfrac{1}{28}\,{f_1}^{-3}f_6 \\
&\qquad - \tfrac{7}{108}\,{f_1}^{-4}f_2f_5 - \tfrac{5}{54}\,{f_1}^{-6}{f_2}^3f_3 + \tfrac{16}{729}\,{f_1}^{-7}{f_2}^5 + \tfrac{11}{135}\,{f_1}^{-5}{f_2}^2f_4 \\
&\qquad + \tfrac{5}{72}\,{f_1}^{-5}f_2{f_3}^2 - \tfrac{1}{20}\,{f_1}^{-4}f_3f_4 \;\;\|\|\| - \tfrac{1}{23328}\,{f_1}^{-3}{f_2}^3 - \tfrac{1}{2880}\,{f_1}^{-1}f_4 \\
&\qquad + \tfrac{1}{5760}\,{f_1}^{-2}f_2f_3 \;\;\|\|\| + \tfrac{1}{746496}\,f_1f_2\}
\end{aligned}
$$

10.4.3 Tangency 2, ramification 3.

$$
\begin{aligned}
h_{-2/3} &= 3^{-2/3}\,\{f_2{}^{-1/3}\}\\
h_{-1/3} &= 3^{-1/3}\{-\tfrac{1}{2}\,f_2{}^{-5/3}f_3\}\\
h_0 &= \quad \{\tfrac{9}{16}f_2{}^{-3}f_3^2-\tfrac{3}{5}f_2{}^{-2}f_4\}\\
h_{1/3} &= 3^{1/3}\,\{-\tfrac{2}{3}\,f_2{}^{-7/3}f_5-\tfrac{35}{48}\,f_2{}^{-13/3}f_3{}^3+\tfrac{7}{5}\,f_2{}^{-10/3}f_3f_4\}\\
h_{2/3} &= 3^{2/3}\,\{\tfrac{385}{384}\,f_2{}^{-17/3}f_3{}^4+\tfrac{5}{3}\,f_2{}^{-11/3}f_3f_5+\tfrac{4}{5}\,f_2{}^{-11/3}f_4{}^2-\tfrac{5}{7}\,f_2{}^{-8/3}f_6\\
&\qquad -\tfrac{11}{4}\,f_2{}^{-14/3}f_3{}^2f_4\quad ||||||+\tfrac{1}{24}\,f_2{}^{1/3}\}\\
h_1 &= 3\,\{\tfrac{9}{5}\,f_2{}^{-4}f_4f_5+\tfrac{81}{16}\,f_2{}^{-6}f_4f_3{}^3-\tfrac{3}{4}\,f_2{}^{-3}f_7+\tfrac{27}{14}\,f_2{}^{-4}f_3f_6-\tfrac{27}{8}\,f_2{}^{-5}f_3{}^2f_5\\
&\qquad -\tfrac{81}{25}\,f_2{}^{-5}f_3f_4{}^2-\tfrac{729}{512}\,f_2{}^{-7}f_3{}^5||||||+\tfrac{1}{48}\,f_3f_2{}^{-1}\}\\
h_{4/3} &= 3^{4/3}\,\{-\tfrac{91}{75}\,f_2{}^{-16/3}f_4{}^3-\tfrac{91}{12}\,f_2{}^{-16/3}f_3f_4f_5-\tfrac{7}{9}\,f_2{}^{-10/3}f_8-\tfrac{65}{16}\,f_2{}^{-16/3}f_3{}^2f_6\\
&\qquad +\tfrac{35}{16}\,f_2{}^{-13/3}f_3f_7+2\,f_2{}^{-13/3}f_4f_6-\tfrac{1729}{192}\,f_2{}^{-22/3}f_3{}^4f_4+\tfrac{19019}{9216}\,f_2{}^{-25/3}f_3{}^6\\
&\qquad +\tfrac{455}{72}\,f_2{}^{-19/3}f_3{}^3f_5+\tfrac{35}{36}\,f_2{}^{-13/3}f_5{}^2+\tfrac{91}{10}\,f_2{}^{-19/3}f_3{}^2f_4{}^2\\
&\qquad ||||||+\tfrac{1}{40}\,f_2{}^{-4/3}f_4-\tfrac{5}{384}\,f_2{}^{-7/3}f_3{}^2\}\\
h_{5/3} &= 3^{5/3}\,\{-\tfrac{4}{5}\,f_2{}^{-11/3}f_9-\tfrac{44}{5}\,f_2{}^{-17/3}f_4f_6f_3-\tfrac{77}{16}\,f_2{}^{-17/3}f_3{}^2f_7-\tfrac{1309}{60}\,f_2{}^{-23/3}f_3{}^3f_4{}^2\\
&\quad -\tfrac{6545}{576}\,f_2{}^{-23/3}f_3{}^4f_5+\tfrac{187}{24}\,f_2{}^{-20/3}f_3{}^3f_6+\tfrac{30107}{1920}\,f_2{}^{-26/3}f_3{}^5f_4-\tfrac{55913}{18432}\,f_2{}^{-29/3}f_3{}^7\\
&\quad +\tfrac{1309}{60}\,f_2{}^{-20/3}f_3{}^2f_4f_5+\tfrac{44}{21}\,f_2{}^{-14/3}f_6f_5+\tfrac{2618}{375}\,f_2{}^{-20/3}f_3f_4{}^3+\tfrac{22}{9}\,f_2{}^{-14/3}f_3f_8\\
&\quad +\tfrac{11}{5}\,f_2{}^{-14/3}f_7f_4-\tfrac{308}{75}\,f_2{}^{-17/3}f_4{}^2f_5-\tfrac{77}{18}\,f_2{}^{-17/3}f_5{}^2f_3||||||-\tfrac{1}{30}\,f_2{}^{-8/3}f_3f_4\\
&\quad +\tfrac{7}{576}\,f_2{}^{-11/3}f_3{}^3+\ 1/36\,f_2{}^{-5/3}f_5\}\\
h_2 &= 3^2\,\{\tfrac{729}{28}\,f_2{}^{-7}f_3{}^2f_4f_6-\tfrac{135}{14}\,f_2{}^{-6}f_3f_5f_6-\tfrac{1701}{32}\,f_2{}^{-8}f_3{}^3f_4f_5+\tfrac{243}{10}\,f_2{}^{-7}f_3f_4{}^2f_5\\
&\quad -\tfrac{9}{2}\,f_2{}^{-6}f_5{}^2f_4-\tfrac{81}{8}\,f_2{}^{-6}f_3f_4f_7+\tfrac{295245}{65536}\,f_2{}^{-11}f_3{}^8+\tfrac{27}{10}\,f_2{}^{-5}f_3f_9\\
&\quad -\tfrac{3645}{256}\,f_2{}^{-8}f_3{}^4f_6+\tfrac{243}{125}\,f_2{}^{-7}f_4{}^4-\tfrac{9}{11}\,f_2{}^{-4}f_{10}-\tfrac{137781}{5120}\,f_2{}^{-10}f_3{}^6f_4\\
&\quad +\tfrac{54}{49}\,f_2{}^{-5}f_6{}^2-\tfrac{45}{8}\,f_2{}^{-6}f_3{}^2f_8-\tfrac{5103}{200}\,f_2{}^{-8}f_3{}^2f_4{}^3+\tfrac{5103}{256}\,f_2{}^{-9}f_3{}^5f_5\\
&\quad +\tfrac{15309}{320}\,f_2{}^{-9}f_3{}^4f_4{}^2+\tfrac{12}{5}\,f_2{}^{-5}f_8f_4+\tfrac{405}{32}\,f_2{}^{-7}f_3{}^2f_5{}^2-\tfrac{162}{35}\,f_2{}^{-6}f_4{}^2f_6\\
&\quad +\tfrac{1215}{128}\,f_2{}^{-7}f_3{}^3f_7+\tfrac{9}{4}\,f_2{}^{-5}f_5f_7\quad ||||||-\tfrac{27}{2048}\,f_2{}^{-5}f_3{}^4-\tfrac{1}{24}\,f_2{}^{-3}f_3f_5\\
&\quad +\tfrac{5}{168}\,f_2{}^{-2}f_6-\tfrac{11}{600}\,f_2{}^{-3}f_4{}^2+\tfrac{3}{64}\,f_2{}^{-4}f_3{}^2f_4\quad ||||||+\tfrac{1}{5184}\,f_2{}^{-1}\}
\end{aligned}
$$

10.4.4 Tangency 3, ramification 4.

$$h_{-3/4} = 4^{-3/4}\{f_3{}^{-3/4}\}$$

$$h_{-1/2} = 4^{-1/2}\{-\tfrac{2}{5}\, f_3{}^{-3/2} f_4\}$$

$$h_{-1/4} = 4^{-1/4}\{\tfrac{21}{50}\, f_3{}^{-11/4} f_4{}^2 - \tfrac{1}{2}\, f_3{}^{-7/4} f_5\}$$

$$h_0 = \{-\tfrac{64}{125} f_3{}^{-4} f_4^3 + \tfrac{16}{15} f_3{}^{-3} f_4 f_5 - \tfrac{4}{7} f_3{}^{-2} f_6\}$$

$$h_{1/4} = 4^{1/4}\{-\tfrac{39}{20}\, f_3{}^{-17/4} f_4{}^2 f_5 + \tfrac{9}{7}\, f_3{}^{-13/4} f_4 f_6 + \tfrac{5}{8}\, f_3{}^{-13/4} f_5{}^2 + \tfrac{663}{1000}\, f_3{}^{-21/4} f_4{}^4 - \tfrac{5}{8}\, f_3{}^{-9/4} f_7\}$$

$$h_{1/2} = 4^{1/2}\{\tfrac{84}{25}\, f_3{}^{-11/2} f_4{}^3 f_5 - \tfrac{2772}{3125}\, f_3{}^{-13/2} f_4{}^5 - \tfrac{12}{5}\, f_3{}^{-9/2} f_4{}^2 f_6 - \tfrac{7}{3}\, f_3{}^{-9/2} f_4 f_5{}^2 + \tfrac{3}{2}\, f_3{}^{-7/2} f_4 f_7 + \tfrac{10}{7}\, f_3{}^{-7/2} f_5 f_6 - \tfrac{2}{3}\, f_3{}^{-5/2} f_8\}$$

$$h_{3/4} = 4^{3/4}\{-\tfrac{231}{80}\, f_3{}^{-19/4} f_4{}^2 f_7 - \tfrac{11}{2}\, f_3{}^{-19/4} f_4 f_5 f_6 - \tfrac{385}{432}\, f_3{}^{-19/4} f_5{}^3 + \tfrac{209}{50}\, f_3{}^{-23/4} f_4{}^3 f_6 + \tfrac{1463}{240}\, f_3{}^{-23/4} f_4{}^2 f_5{}^2 - \tfrac{7}{10}\, f_3{}^{-11/4} f_9 + \tfrac{11}{14}\, f_3{}^{-15/4} f_6{}^2 + \tfrac{302841}{250000}\, f_3{}^{-31/4} f_4{}^6 + \tfrac{77}{45}\, f_3{}^{-15/4} f_4 f_8 + \tfrac{77}{48}\, f_3{}^{-15/4} f_7 f_5 - \tfrac{33649}{6000}\, f_3{}^{-27/4} f_5 f_4{}^4 \;\;|||||| + \tfrac{1}{24}\, f_3{}^{1/4}\}$$

$$h_1 = 4\;\{+\tfrac{28672}{3125}\, f_3{}^{-8} f_4{}^5 f_5 - \tfrac{131072}{78125}\, f_3{}^{-9} f_4{}^7 - \tfrac{8}{11}\, f_3{}^{-3} f_{10} - \tfrac{32}{5}\, f_3{}^{-5} f_4 f_5 f_7 - \tfrac{256}{75}\, f_3{}^{-5} f_4{}^2 f_8 - \tfrac{768}{245}\, f_3{}^{-5} f_4 f_6{}^2 - \tfrac{64}{21}\, f_3{}^{-5} f_5{}^2 f_6 + \tfrac{48}{25}\, f_3{}^{-4} f_4 f_9 + \tfrac{16}{9}\, f_3{}^{-4} f_5 f_8 + \tfrac{12}{7}\, f_3{}^{-4} f_6 f_7 - \tfrac{6144}{875}\, f_3{}^{-7} f_4{}^4 f_6 - \tfrac{1024}{75}\, f_3{}^{-7} f_4{}^3 f_5{}^2 + \tfrac{512}{35}\, f_3{}^{-6} f_4{}^2 f_5 f_6 + \tfrac{128}{25}\, f_3{}^{-6} f_4{}^3 f_7 + \tfrac{128}{27}\, f_3{}^{-6} f_4 f_5{}^3 \;\;|||||| + \tfrac{1}{60}\, f_3{}^{-1} f_4\}$$

$$h_{5/4} = 4^{5/4}\{\tfrac{23476167}{10000000}\, f_3{}^{-41/4} f_4{}^8 + \tfrac{13}{7}\, f_3{}^{-17/4} f_6 f_8 + \tfrac{57681}{5000}\, f_3{}^{-33/4} f_4{}^5 f_6 + \tfrac{1547}{1152}\, f_3{}^{-25/4} f_5{}^4 - \tfrac{1989}{500}\, f_3{}^{-21/4} f_4{}^2 f_9 - \tfrac{221}{64}\, f_3{}^{-21/4} f_5{}^2 f_7 - \tfrac{663}{196}\, f_3{}^{-21/4} f_5 f_6{}^2 - \tfrac{1480479}{100000}\, f_3{}^{-37/4} f_4{}^6 f_5 + \tfrac{5967}{700}\, f_3{}^{-25/4} f_4{}^2 f_6{}^2 - \tfrac{13923}{1600}\, f_3{}^{-29/4} f_4{}^4 f_7 - \tfrac{1547}{96}\, f_3{}^{-29/4} f_4{}^2 f_5{}^3 + \tfrac{117}{55}\, f_3{}^{-17/4} f_4 f_{10} + \tfrac{39}{20}\, f_3{}^{-17/4} f_9 f_5 + \tfrac{1547}{250}\, f_3{}^{-25/4} f_4{}^3 f_8 + \tfrac{117}{128}\, f_3{}^{-17/4} f_7{}^2 - \tfrac{3}{4}\, f_3{}^{-13/4} f_{11} - \tfrac{663}{20}\, f_3{}^{-29/4} f_4{}^3 f_5 f_6 + \tfrac{663}{40}\, f_3{}^{-25/4} f_5{}^2 f_6 f_4 + \tfrac{44863}{1600}\, f_3{}^{-33/4} f_4{}^4 f_5{}^2 - \tfrac{1989}{280}\, f_3{}^{-21/4} f_4 f_6 f_7 + \tfrac{13923}{800}\, f_3{}^{-25/4} f_4{}^2 f_5 f_7 - \tfrac{221}{30}\, f_3{}^{-21/4} f_4 f_5 f_8 \;\;|||||| + \tfrac{1}{48}\, f_3{}^{-5/4} f_5 - \tfrac{13}{1200}\, f_3{}^{-9/4} f_4{}^2\}$$

10.5 The *nur* transform: from f to h.

10.5.1 Tangency 0.

$$h_0 = f_0{}^{<-1>}$$

$$h_1 = -f_0{}^{<-3>} f_1 \;\;|||||| + \tfrac{1}{24}\, f_0{}^{<-1>} f_1$$

$$h_2 = +\tfrac{3}{2}\, f_0{}^{<-5>} f_1{}^2 - f_0{}^{<-4>} f_2 \;\;|||||| - \tfrac{1}{48}\, f_0{}^{<-3>} f_1{}^2 + \tfrac{1}{24}\, f_0{}^{<-2>} f_2 \;\;|||||| + \tfrac{1}{2304}\, f_0{}^{<-1>} f_1{}^2$$

$$\alpha^{<-k>} := \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{(\alpha + 2\pi\, i\, n)^k} = \frac{1}{(k-1)!}\Big[\partial_\sigma^{k-1} \frac{2}{\sinh(\frac{\alpha-\sigma}{2})}\Big]_{\sigma=0} \equiv (-1)^k\,(-\alpha)^{<-k>}$$

$$\begin{aligned}
\alpha^{<-1>} &= \tfrac{\sqrt{a}}{a-1} \qquad \textit{with} \quad a := e^\alpha \\
\alpha^{<-2>} &= \tfrac{\sqrt{a}(a+1)}{2\,(a-1)^2} \\
\alpha^{<-3>} &= \tfrac{\sqrt{a}\left(a^2+6\,a+1\right)}{8\,(a-1)^3} \\
\alpha^{<-4>} &= \tfrac{\sqrt{a}\left(a^3+23\,a^2+23\,a+1\right)}{48\,(a-1)^4} \\
\alpha^{<-5>} &= \tfrac{\sqrt{a}\left(a^4+76\,a^3+230\,a^2+76\,a+1\right)}{384\,(a-1)^5} \\
\alpha^{<-6>} &= \tfrac{\sqrt{a}\left(a^5+237\,a^4+1682\,a^3+1682\,a^2+237\,a+1\right)}{3840\,(a-1)^6} \\
\alpha^{<-7>} &= \tfrac{\sqrt{a}\left(a^6+722\,a^5+10543\,a^4+23548\,a^3+10543\,a^2+722\,a+1\right)}{46080\,(a-1)^7} \\
\alpha^{<-8>} &= \tfrac{\sqrt{a}\left(a^7+2179\,a^6+60657\,a^5+259723\,a^4+259723\,a^3+60657\,a^2+2179\,a+1\right)}{645120\,(a-1)^8} \\
\alpha^{<-9>} &= \tfrac{\sqrt{a}\left(a^8+6552\,a^7+331612\,a^6+2485288\,a^5+4675014\,a^4+2485288\,a^3+331612\,a^2+6552\,a+1\right)}{10321920\,(a-1)^9}
\end{aligned}$$

10.5.2 Tangency > 0.

We set $Dh := nur(f) - nir(f)$ and to calculate the general Dh_n, we take h_n in the *nir*-table §10.4 and perform the substitution:

$$f_0^{-k} \longrightarrow \sum_{n\in\mathbb{Z}^*} \frac{1}{(2\pi\, i)^k} = -\beta_{k-1} = \frac{1}{(k-1)!}\Big[\partial_\sigma^{k-1} \frac{2}{\sinh(-\frac{\sigma}{2})}\Big]_{\sigma=0}$$

$$
\begin{aligned}
Dh_0 &= Dh_1 = 0 \\
Dh_2 &= +\tfrac{17}{5760}\, f_2 \qquad\qquad \equiv \big(\beta_3 - \tfrac{1}{24}\,\beta_1\big)\, f_2 \\
Dh_3 &= +\tfrac{1}{5040}\, f_2 f_1 \qquad\qquad \equiv \big(-\tfrac{10}{3}\,\beta_5 + \tfrac{1}{18}\,\beta_3 - \tfrac{1}{1728}\,\beta_1\big) f_1 f_2 \\
Dh_4 &= -\tfrac{43}{322560}\, f_4 + \tfrac{11927}{1114767360}\, f_2 f_1^2 \\
Dh_5 &= -\tfrac{18839}{2322432000}\, f_4 f_1 - \tfrac{27241}{4644864000}\, f_3 f_2 + \tfrac{28283}{200658124800}\, f_2 f_1^3 \\
Dh_6 &= +\tfrac{769}{154828800}\, f_6 - \tfrac{143}{746496000}\, f_4 f_1^2 - \tfrac{28709}{111476736000}\, f_3 f_2 f_1 - \tfrac{677}{20065812480}\, f_2^3 \\
&\quad + \tfrac{1657}{2407897497600}\, f_2 f_1^4 \\
Dh_7 &= +\tfrac{319}{1560674304}\, f_6 f_1 + \tfrac{13277}{117050572800}\, f_5 f_2 + \tfrac{373}{4335206400}\, f_4 f_3 - \tfrac{41539}{28092137472000}\, f_4 f_1^3 \\
&\quad - \tfrac{156581}{56184274944000}\, f_3 f_2 f_1^2 - \tfrac{667}{936404582400}\, f_2^3 f_1 + \tfrac{6133}{3467372396544000}\, f_2 f_1^5 \\
Dh_8 &= -\tfrac{163}{1114767360}\, f_8 + \tfrac{32843}{10487731322880}\, f_6 f_1^2 + \tfrac{27151}{8739776102400}\, f_5 f_2 f_1 \\
&\quad + \tfrac{207481}{93640458240000}\, f_4 f_3 f_1 + \tfrac{2693}{3121348608000}\, f_4 f_2^2 + \tfrac{269489}{374561832960000}\, f_3^2 f_2 \\
&\quad - \tfrac{32447}{6472428473548800}\, f_4 f_1^4 - \tfrac{6953}{577895399424000}\, f_3 f_2 f_1^3 \\
&\quad - \tfrac{7157}{1618107118387200}\, f_2^3 f_1^2 + \tfrac{633697}{2236871280458465280000}\, f_2 f_1^6
\end{aligned}
$$

. . . .

10.6 Translocation of *nir*.

10.6.1 Standard case.

$$
\begin{aligned}
(\delta_1\,h)_0 &= +\tfrac{1}{24}f_1\\
(\delta_1\,h)_1 &= +\tfrac{1}{1152}f_1^2\\
(\delta_1\,h)_2 &= -\tfrac{7}{1920}f_3+\tfrac{1}{165888}f_1^3\\
(\delta_1\,h)_3 &= -\tfrac{7}{138240}f_1f_3+\tfrac{1}{47775744}f_1^4\\
(\delta_1\,h)_4 &= +\tfrac{31}{193536}f_5-\tfrac{7}{26542080}f_3f_1^2+\tfrac{1}{22932357120}f_1^5\\
(\delta_1\,h)_5 &= +\tfrac{49}{221184000}f_3^2+\tfrac{31}{23224320}f_1f_5-\tfrac{7}{9555148800}f_1^3f_3+\tfrac{1}{16511297126400}f_1^6\\
(\delta_1\,h)_6 &= -\tfrac{127}{22118400}f_7+\tfrac{31}{6688604160}f_1^2f_5+\tfrac{49}{31850496000}f_1f_3^2-\tfrac{7}{5503765708800}f_1^4f_3\\
&\quad+\tfrac{1}{16643387503411200}f_1^7\\
(\delta_2\,h)_0 &= +\tfrac{1}{24}f_2-\tfrac{1}{2304}f_0f_1^2\\
(\delta_2\,h)_1 &= +\tfrac{1}{576}f_1f_2+\tfrac{7}{1920}f_0f_3-\tfrac{1}{165888}f_0f_1^3\\
(\delta_2\,h)_2 &= -\tfrac{7}{960}f_4+\tfrac{7}{92160}f_0f_1f_3+\tfrac{1}{55296}f_1^2f_2-\tfrac{1}{31850496}f_0f_1^4\\
(\delta_2\,h)_3 &= -\tfrac{7}{138240}f_3f_2-\tfrac{7}{69120}f_4f_1-\tfrac{31}{96768}f_0f_5+\tfrac{7}{13271040}f_0f_3f_1^2\\
&\quad+\tfrac{1}{11943936}f_1^3f_2-\tfrac{1}{11466178560}f_0f_1^5\\
(\delta_2\,h)_4 &= +\tfrac{31}{64512}f_6-\tfrac{7}{13271040}f_3f_1f_2-\tfrac{31}{9289728}f_0f_1f_5-\tfrac{49}{88473600}f_0f_3^2\\
&\quad-\tfrac{7}{13271040}f_1^2f_4+\tfrac{7}{3822059520}f_0f_1^3f_3+\tfrac{1}{4586471424}f_1^4f_2-\tfrac{1}{6604518850560}f_0f_1^6\\
(\delta_2\,h)_5 &= +\tfrac{31}{7741440}f_6f_1+\tfrac{31}{23224320}f_2f_5+\tfrac{127}{7372800}f_0f_7+\tfrac{49}{55296000}f_3f_4\\
&\quad-\tfrac{7}{3185049600}f_1^2f_2f_3-\tfrac{49}{10616832000}f_0f_3^2f_1-\tfrac{31}{2229534720}f_0f_1^2f_5\\
&\quad-\tfrac{7}{4777574400}f_1^3f_4+\tfrac{7}{1834588569600}f_0f_1^4f_3+\tfrac{1}{2751882854400}f_1^5f_2\\
&\quad-\tfrac{1}{5547795834470400}f_0f_1^7\\
(\delta_3\,h)_0 &= +\tfrac{1}{24}f_3-\tfrac{1}{864}f_0f_1f_2-\tfrac{7}{5760}f_3f_0^2-\tfrac{1}{6912}f_1^3+\tfrac{1}{497664}f_0^2f_1^3\\
(\delta_3\,h)_1 &= +\tfrac{7}{720}f_0f_4+\tfrac{17}{5760}f_3f_1+\tfrac{1}{864}f_2^2-\tfrac{7}{138240}f_3f_0^2f_1-\tfrac{1}{41472}f_0f_1^2f_2\\
&\quad-\tfrac{1}{497664}f_1^4+\tfrac{1}{47775744}f_0^2f_1^4\\
(\delta_3\,h)_2 &= -\tfrac{7}{576}f_5+\tfrac{7}{34560}f_0f_4f_1+\tfrac{7}{69120}f_0f_3f_2+\tfrac{1}{41472}f_1f_2^2+\tfrac{1}{23040}f_3f_1^2\\
&\quad+\tfrac{31}{96768}f_0^2f_5-\tfrac{7}{13271040}f_3f_0^2f_1^2-\tfrac{1}{5971968}f_0f_1^3f_2-\tfrac{1}{95551488}f_1^5\\
&\quad+\tfrac{1}{11466178560}f_0^2f_1^5\\
(\delta_3\,h)_3 &= -\tfrac{31}{24192}f_0f_6-\tfrac{7}{51840}f_4f_2-\tfrac{5}{18144}f_1f_5-\tfrac{7}{138240}f_3^2+\tfrac{31}{6967296}f_1f_0^2f_5\\
&\quad+\tfrac{7}{4976640}f_0f_1^2f_4+\tfrac{1}{5971968}f_1^2f_2^2+\tfrac{49}{66355200}f_0^2f_3^2+\tfrac{31}{119439360}f_1^3f_3\\
&\quad+\tfrac{7}{4976640}f_0f_3f_1f_2-\tfrac{7}{2866544640}f_0^2f_1^3f_3-\tfrac{1}{1719926784}f_0f_1^4f_2\\
&\quad-\tfrac{1}{34398535680}f_1^6+\tfrac{1}{4953389137920}f_0^2f_1^6
\end{aligned}
$$

10.6.2 Free-β case.

$$
\begin{aligned}
(\delta_1\, h)_0 &= -f_1\beta_1 \\
(\delta_1\, h)_1 &= -2f_2\beta_2 + \tfrac{1}{2}f_1^2\beta_1^2 \\
(\delta_1\, h)_2 &= -3f_3\beta_3 + f_1f_2\beta_1\beta_2 - \tfrac{1}{12}f_1^3\beta_1^3 \\
(\delta_1\, h)_3 &= -4f_4\beta_4 - 4f_4\beta_4 + f_1f_3\beta_1\beta_3 + \tfrac{1}{3}f_2^2\beta_2^2 - \tfrac{1}{6}f_1^2f_2\beta_1^2\beta_2 + \tfrac{1}{144}f_1^4\beta_1^4 \\
(\delta_1\, h)_4 &= -5f_5\beta_5 + f_1f_4\beta_1\beta_4 + \tfrac{1}{2}f_2f_3\beta_2\beta_3 - \tfrac{1}{8}f_1^2f_3\beta_1^2\beta_3 - \tfrac{1}{12}f_1f_2^2\beta_1\beta_2^2 \\
&\quad + \tfrac{1}{72}f_1^3f_2\beta_1^3\beta_2 - \tfrac{1}{2880}f_1^5\beta_1^5 \\
(\delta_1\, h)_5 &= -6f_6\beta_6 + f_1f_5\beta_1\beta_5 + \tfrac{2}{5}f_2f_4\beta_2\beta_4 + \tfrac{3}{20}f_3^2\beta_3^2 - \tfrac{1}{10}f_1^2f_4\beta_1^2\beta_4 \\
&\quad - \tfrac{1}{90}f_2^3\beta_2^3 - \tfrac{1}{10}f_1f_2f_3\beta_1\beta_2\beta_3 + \tfrac{1}{120}f_1^3f_3\beta_1^3\beta_3 + \tfrac{1}{120}f_1^2f_2^2\beta_1^2\beta_2^2 \\
&\quad - \tfrac{1}{1440}f_1^4f_2\beta_1^4\beta_2 + \tfrac{1}{86400}f_1^6\beta_1^6 \\
(\delta_2\, h)_0 &= -f_2\beta_1 + f_0f_2\beta_2 - \tfrac{1}{4}f_0f_1^2\beta_1^2 \\
(\delta_2\, h)_1 &= -3f_3\beta_2 + 3f_0f_3\beta_3 + f_1f_2\beta_1^2 - f_0f_1f_2\beta_1\beta_2 + \tfrac{1}{12}f_0f_1^3\beta_1^3 \\
(\delta_2\, h)_2 &= -6f_4\beta_3 + \tfrac{3}{2}f_1f_3\beta_1\beta_2 + 6f_0f_4\beta_4 + f_2^2\beta_1\beta_2 - \tfrac{1}{2}f_0f_2^2\beta_2^2 - \tfrac{1}{4}f_1^2f_2\beta_1^3 \\
&\quad - \tfrac{3}{2}f_0f_1f_3\beta_1\beta_3 + \tfrac{1}{4}f_0f_1^2f_2\beta_1^2\beta_2 - \tfrac{1}{96}f_0f_1^4\beta_1^4 \\
(\delta_2\, h)_3 &= -10f_5\beta_4 + 2f_1f_4\beta_1\beta_3 + f_2f_3\beta_1\beta_3 + 10f_0f_5\beta_5 + f_2f_3\beta_2^2 \\
&\quad - 2f_0f_1f_4\beta_1\beta_4 - \tfrac{1}{4}f_1^2f_3\beta_1^2\beta_2 - \tfrac{1}{3}f_1f_2^2\beta_1^2\beta_2 - f_0f_2f_3\beta_2\beta_3 + \tfrac{1}{36}f_1^3f_2\beta_1^4 \\
&\quad + \tfrac{1}{4}f_0f_1^2f_3\beta_1^2\beta_3 + \tfrac{1}{6}f_0f_1f_2^2\beta_1\beta_2^2 - \tfrac{1}{36}f_0f_1^3f_2\beta_1^3\beta_2 + \tfrac{1}{1440}f_0f_1^5\beta_1^5 \\
(\delta_3\, h)_0 &= -f_3\beta_1 + \tfrac{1}{3}f_1f_2\beta_2 + 2f_0f_3\beta_2 - \tfrac{1}{12}f_1^3\beta_1^2 - f_0^2f_3\beta_3 - \tfrac{2}{3}f_0f_1f_2\beta_1^2 \\
&\quad + \tfrac{1}{3}f_0^2f_1f_2\beta_1\beta_2 - \tfrac{1}{36}f_0^2f_1^3\beta_1^3 \\
(\delta_3\, h)_1 &= -4f_4\beta_2 + f_1f_3\beta_1^2 + f_1f_3\beta_3 + \tfrac{2}{3}f_2^2\beta_1^2 + 8f_0f_4\beta_3 - 4f_0^2f_4\beta_4 \\
&\quad - \tfrac{1}{3}f_1^2f_2\beta_1\beta_2 - \tfrac{4}{3}f_0f_2^2\beta_1\beta_2 - 2f_0f_1f_3\beta_1\beta_2 + \tfrac{1}{36}f_1^4\beta_1^3 + \tfrac{1}{3}f_0^2f_2^2\beta_2^2 \\
&\quad + \tfrac{1}{3}f_0f_1^2f_2\beta_1^3 + f_0^2f_1f_3\beta_1\beta_3 - \tfrac{1}{6}f_0^2f_1^2f_2\beta_1^2\beta_2 + \tfrac{1}{144}f_0^2f_1^4\beta_1^4 \\
(\delta_3\, h)_2 &= -10f_5\beta_3 + 2f_1f_4\beta_1\beta_2 + 20f_0f_5\beta_4 + 2f_1f_4\beta_4 + 3f_2f_3\beta_1\beta_2 - 10f_0^2f_5\beta_5 \\
&\quad - \tfrac{1}{4}f_1^2f_3\beta_1^3 - \tfrac{1}{3}f_1f_2^2\beta_1^3 - \tfrac{1}{6}f_1f_2^2\beta_2^2 - \tfrac{1}{2}f_1^2f_3\beta_1\beta_3 - 2f_0f_2f_3\beta_2^2 \\
&\quad - 2f_0f_2f_3\beta_1\beta_3 - 4f_0f_1f_4\beta_1\beta_3 + \tfrac{1}{12}f_1^3f_2\beta_1^2\beta_2 + \tfrac{2}{3}f_0f_1f_2^2\beta_1^2\beta_2 \\
&\quad + \tfrac{1}{2}f_0f_1^2f_3\beta_1^2\beta_2 + f_0^2f_2f_3\beta_2\beta_3 + 2f_0^2f_1f_4\beta_1\beta_4 - \tfrac{1}{288}f_1^5\beta_1^4 - \tfrac{1}{18}f_0f_1^3f_2\beta_1^4 \\
&\quad - \tfrac{1}{6}f_0^2f_1f_2^2\beta_1\beta_2^2 - \tfrac{1}{4}f_0^2f_1^2f_3\beta_1^2\beta_3 + \tfrac{1}{36}f_0^2f_1^3f_2\beta_1^3\beta_2 - \tfrac{1}{1440}f_0^2f_1^5\beta_1^5
\end{aligned}
$$

11 Tables relative to the 4_1 knot.

11.1 The original generators Lo and Loo .

$$
\begin{aligned}
[\tfrac{p}{q}] &:= \cos(\pi\tfrac{p}{q}) \\
\hat{Jo}_2 &= 4\,[\tfrac{0}{2}] \quad ; \quad \hat{Jo}_4 = 26\,[\tfrac{0}{4}] \quad ; \quad \hat{Jo}_6 = 60\,[\tfrac{0}{6}] + 56\,[\tfrac{2}{6}] \\
\hat{Jo}_8 &= 186\,[\tfrac{0}{8}] + 90\,[\tfrac{2}{8}] \quad ; \quad \hat{Jo}_{10} = 348\,[\tfrac{0}{10}] + 366\,[\tfrac{2}{10}] + 22\,[\tfrac{4}{10}] \\
\hat{Jo}_{12} &= 650\,[\tfrac{0}{12}] + 748\,[\tfrac{2}{12}] + 624\,[\tfrac{4}{12}] \\
\hat{Jo}_{14} &= 1396\,[\tfrac{0}{14}] + 1854\,[\tfrac{2}{14}] + 1030\,[\tfrac{4}{14}] + 568\,[\tfrac{6}{14}] \\
\hat{Jo}_{16} &= 2776\,[\tfrac{0}{16}] + 3804\,[\tfrac{2}{16}] + 2816\,[\tfrac{4}{16}] + 1570\,[\tfrac{6}{16}] \\
\hat{Jo}_{18} &= 4862\,[\tfrac{0}{18}] + 8078\,[\tfrac{2}{18}] + 6550\,[\tfrac{4}{18}] + 4802\,[\tfrac{6}{18}] + 1472\,[\tfrac{8}{18}] \\
\hat{Jo}_{20} &= 9864\,[\tfrac{0}{20}] + 16588\,[\tfrac{2}{20}] + 14484\,[\tfrac{4}{20}] + 10242\,[\tfrac{6}{20}] + 4646\,[\tfrac{8}{20}] \\
\hat{Jo}_{22} &= 19238\,[\tfrac{0}{22}] + 34168\,[\tfrac{2}{22}] + 29144\,[\tfrac{4}{22}] + 23360\,[\tfrac{6}{22}] + 14032\,[\tfrac{8}{22}] + 4882\,[\tfrac{10}{22}] \\
\hat{Jo}_{24} &= 36622\,[\tfrac{0}{24}] + 68070\,[\tfrac{2}{24}] + 61092\,[\tfrac{4}{24}] + 49618\,[\tfrac{6}{24}] + 36436\,[\tfrac{8}{24}] + 18362\,[\tfrac{10}{24}] \\
\hat{Jo}_{26} &= 72910\,[\tfrac{0}{26}] + 136798\,[\tfrac{2}{26}] + 123574\,[\tfrac{4}{26}] + 105408\,[\tfrac{6}{26}] + 78140\,[\tfrac{8}{26}] \\
&+ 49554\,[\tfrac{10}{26}] + 17140\,[\tfrac{12}{26}] \\
\hat{Jo}_{28} &= 142414\,[\tfrac{0}{28}] + 270968\,[\tfrac{2}{28}] + 250954\,[\tfrac{4}{28}] + 217464\,[\tfrac{6}{28}] + 171476\,[\tfrac{8}{28}] \\
&+ 118824\,[\tfrac{10}{28}] + 62910\,[\tfrac{12}{28}] \\
\hat{Jo}_{30} &= 276046\,[\tfrac{0}{30}] + 536500\,[\tfrac{2}{30}] + 501662\,[\tfrac{4}{30}] + 444608\,[\tfrac{6}{30}] + 367512\,[\tfrac{8}{30}] \\
&+ 275698\,[\tfrac{10}{30}] + 168622\,[\tfrac{12}{30}] + 57032\,[\tfrac{14}{30}] \\
\hat{Jo}_{32} &= 546414\,[\tfrac{0}{32}] + 1059780\,[\tfrac{2}{32}] + 998970\,[\tfrac{4}{32}] + 899322\,[\tfrac{6}{32}] + 761478\,[\tfrac{8}{32}] \\
&+ 597972\,[\tfrac{10}{32}] + 413774\,[\tfrac{12}{32}] + 208304\,[\tfrac{14}{32}] \\
\hat{Jo}_{34} &= 1069006\,[\tfrac{0}{34}] + 2090050\,[\tfrac{2}{34}] + 1978918\,[\tfrac{4}{34}] + 1807392\,[\tfrac{6}{34}] + 1566682\,[\tfrac{8}{34}] \\
&+ 1276434\,[\tfrac{10}{34}] + 946554\,[\tfrac{12}{34}] + 579596\,[\tfrac{14}{34}] + 192316\,[\tfrac{16}{34}] \\
\hat{Jo}_{36} &= 2088162\,[\tfrac{0}{36}] + 4103632\,[\tfrac{2}{36}] + 3916262\,[\tfrac{4}{36}] + 3606154\,[\tfrac{6}{36}] + 3192168\,[\tfrac{8}{36}] \\
&+ 2678024\,[\tfrac{10}{36}] + 2087512\,[\tfrac{12}{36}] + 1424860\,[\tfrac{14}{36}] + 723470\,[\tfrac{16}{36}] \\
\hat{Jo}_{38} &= 4092062\,[\tfrac{0}{38}] + 8053558\,[\tfrac{2}{38}] + 7720542\,[\tfrac{4}{38}] + 7184188\,[\tfrac{6}{38}] + 6434868\,[\tfrac{8}{38}] \\
&+ 5524442\,[\tfrac{10}{38}] + 4463534\,[\tfrac{12}{38}] + 3272004\,[\tfrac{14}{38}] + 2000440\,[\tfrac{16}{38}] + 677250\,[\tfrac{18}{38}] \\
\hat{Jo}_{40} &= 7996624\,[\tfrac{0}{40}] + 15773130\,[\tfrac{2}{40}] + 15189306\,[\tfrac{4}{40}] + 14233202\,[\tfrac{6}{40}] \\
&+ 12919072\,[\tfrac{8}{40}] + 11281960\,[\tfrac{10}{40}] + 9387198\,[\tfrac{12}{40}] + 7241732\,[\tfrac{14}{40}] \\
&+ 4922634\,[\tfrac{16}{40}] + 2500390\,[\tfrac{18}{40}]
\end{aligned}
$$

$$
\begin{aligned}
\hat{J}oo_2 &= 4\left[\tfrac{0}{2}\right] \\
\hat{J}oo_4 &= 12\left[\tfrac{0}{4}\right] \\
\hat{J}oo_6 &= 20\left[\tfrac{0}{6}\right] + 16\left[\tfrac{2}{6}\right] \\
\hat{J}oo_8 &= 44\left[\tfrac{0}{8}\right] + 24\left[\tfrac{2}{8}\right] \\
\hat{J}oo_{10} &= 68\left[\tfrac{0}{10}\right] + 72\left[\tfrac{2}{10}\right] + 8\left[\tfrac{4}{10}\right] \\
\hat{J}oo_{12} &= 108\left[\tfrac{0}{12}\right] + 128\left[\tfrac{2}{12}\right] + 96\left[\tfrac{4}{12}\right] \\
\hat{J}oo_{14} &= 196\left[\tfrac{0}{14}\right] + 264\left[\tfrac{2}{14}\right] + 152\left[\tfrac{4}{14}\right] + 80\left[\tfrac{6}{14}\right] \\
\hat{J}oo_{16} &= 340\left[\tfrac{0}{16}\right] + 480\left[\tfrac{2}{16}\right] + 352\left[\tfrac{4}{16}\right] + 200\left[\tfrac{6}{16}\right] \\
\hat{J}oo_{18} &= 540\left[\tfrac{0}{18}\right] + 904\left[\tfrac{2}{18}\right] + 728\left[\tfrac{4}{18}\right] + 520\left[\tfrac{6}{18}\right] + 160\left[\tfrac{8}{18}\right] \\
\hat{J}oo_{20} &= 980\left[\tfrac{0}{20}\right] + 1664\left[\tfrac{2}{20}\right] + 1440\left[\tfrac{4}{20}\right] + 1032\left[\tfrac{6}{20}\right] + 472\left[\tfrac{8}{20}\right] \\
\hat{J}oo_{22} &= 1740\left[\tfrac{0}{22}\right] + 3104\left[\tfrac{2}{22}\right] + 2656\left[\tfrac{4}{22}\right] + 2128\left[\tfrac{6}{22}\right] + 1280\left[\tfrac{8}{22}\right] + 440\left[\tfrac{10}{22}\right] \\
\hat{J}oo_{24} &= 3052\left[\tfrac{0}{24}\right] + 5688\left[\tfrac{2}{24}\right] + 5088\left[\tfrac{4}{24}\right] + 4136\left[\tfrac{6}{24}\right] + 3008\left[\tfrac{8}{24}\right] + 1528\left[\tfrac{10}{24}\right] \\
\hat{J}oo_{26} &= 5596\left[\tfrac{0}{26}\right] + 10520\left[\tfrac{2}{26}\right] + 9512\left[\tfrac{4}{26}\right] + 8112\left[\tfrac{6}{26}\right] + 6016\left[\tfrac{8}{26}\right] + 3816\left[\tfrac{10}{26}\right] \\
&+ 1328\left[\tfrac{12}{26}\right] \\
\hat{J}oo_{28} &= 10156\left[\tfrac{0}{28}\right] + 19360\left[\tfrac{2}{28}\right] + 17912\left[\tfrac{4}{28}\right] + 15552\left[\tfrac{6}{28}\right] + 12256\left[\tfrac{8}{28}\right] \\
& \quad +8496\left[\tfrac{10}{28}\right] + 4488\left[\tfrac{12}{28}\right] \\
\hat{J}oo_{30} &= 18412\left[\tfrac{0}{30}\right] + 35792\left[\tfrac{2}{30}\right] + 33448\left[\tfrac{4}{30}\right] + 29632\left[\tfrac{6}{30}\right] + 24480\left[\tfrac{8}{30}\right] \\
&+ 18344\left[\tfrac{10}{30}\right] + 11240\left[\tfrac{12}{30}\right] + 3808\left[\tfrac{14}{30}\right] \\
\hat{J}oo_{32} &= 34124\left[\tfrac{0}{32}\right] + 66240\left[\tfrac{2}{32}\right] + 62424\left[\tfrac{4}{32}\right] + 56232\left[\tfrac{6}{32}\right] + 47592\left[\tfrac{8}{32}\right] \\
&+ 37392\left[\tfrac{10}{32}\right] + 25864\left[\tfrac{12}{32}\right] + 13024\left[\tfrac{14}{32}\right] \\
\hat{J}oo_{34} &= 62860\left[\tfrac{0}{34}\right] + 122936\left[\tfrac{2}{34}\right] + 116408\left[\tfrac{4}{34}\right] + 106320\left[\tfrac{6}{34}\right] + 92168\left[\tfrac{8}{34}\right] \\
&+ 75096\left[\tfrac{10}{34}\right] + 55704\left[\tfrac{12}{34}\right] + 34096\left[\tfrac{14}{34}\right] + 11312\left[\tfrac{16}{34}\right] \\
\hat{J}oo_{36} &= 116012\left[\tfrac{0}{36}\right] + 228032\left[\tfrac{2}{36}\right] + 217576\left[\tfrac{4}{36}\right] + 200360\left[\tfrac{6}{36}\right] + 177312\left[\tfrac{8}{36}\right] \\
&+ 148768\left[\tfrac{10}{36}\right] + 115904\left[\tfrac{12}{36}\right] + 79136\left[\tfrac{14}{36}\right] + 40168\left[\tfrac{16}{36}\right] \\
\hat{J}oo_{38} &= 215340\left[\tfrac{0}{38}\right] + 423848\left[\tfrac{2}{38}\right] + 406344\left[\tfrac{4}{38}\right] + 378128\left[\tfrac{6}{38}\right] + 338688\left[\tfrac{8}{38}\right] \\
&+ 290776\left[\tfrac{10}{38}\right] + 234952\left[\tfrac{12}{38}\right] + 172224\left[\tfrac{14}{38}\right] + 105296\left[\tfrac{16}{38}\right] + 35640\left[\tfrac{18}{38}\right]
\end{aligned}
$$

$$
\begin{aligned}
\hat{D}Jo_n &:= \hat{J}o_n - \tfrac{1}{2}\ \hat{J}oo_n = n\,(\hat{L}o_n - \hat{L}oo_n) \\
\ldots\ & \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\hat{D}Jo_2 &= 0 \\
\hat{D}Jo_4 &= 2\,[\tfrac{0}{4}] \\
\hat{D}Jo_6 &= 8\,[\tfrac{2}{6}] \\
\hat{D}Jo_8 &= 10\,[\tfrac{0}{8}] - 6\,[\tfrac{2}{8}] \\
\hat{D}Jo_{10} &= 8\,[\tfrac{0}{10}] + 6\,[\tfrac{2}{10}] - 18\,[\tfrac{4}{10}] \\
\hat{D}Jo_{12} &= 2\,[\tfrac{0}{12}] - 20\,[\tfrac{2}{12}] + 48\,[\tfrac{4}{12}] \\
\hat{D}Jo_{14} &= 24\,[\tfrac{0}{14}] + 6\,[\tfrac{2}{14}] - 34\,[\tfrac{4}{14}] + 8\,[\tfrac{6}{14}] \\
\hat{D}Jo_{16} &= 56\,[\tfrac{0}{16}] - 36\,[\tfrac{2}{16}] - 30\,[\tfrac{6}{16}] \\
\hat{D}Jo_{18} &= 2\,[\tfrac{0}{18}] - 58\,[\tfrac{2}{18}] - 2\,[\tfrac{4}{18}] + 122\,[\tfrac{6}{18}] + 32\,[\tfrac{8}{18}] \\
\hat{D}Jo_{20} &= 64\,[\tfrac{0}{20}] - 52\,[\tfrac{2}{20}] + 84\,[\tfrac{4}{20}] - 78\,[\tfrac{6}{20}] - 74\,[\tfrac{8}{20}] \\
\hat{D}Jo_{22} &= 98\,[\tfrac{0}{22}] + 24\,[\tfrac{2}{22}] - 72\,[\tfrac{4}{22}] - 48\,[\tfrac{6}{22}] - 48\,[\tfrac{8}{22}] + 42\,[\tfrac{10}{22}] \\
\hat{D}Jo_{24} &= -2\,[\tfrac{0}{24}] - 186\,[\tfrac{2}{24}] + 36\,[\tfrac{4}{24}] - 14\,[\tfrac{6}{24}] + 340\,[\tfrac{8}{24}] + 26\,[\tfrac{10}{24}] \\
\hat{D}Jo_{26} &= 162\,[\tfrac{0}{26}] + 38\,[\tfrac{2}{26}] - 82\,[\tfrac{4}{26}] - 48\,[\tfrac{6}{26}] - 68\,[\tfrac{8}{26}] - 54\,[\tfrac{10}{26}] \\
&- 124\,[\tfrac{12}{26}] \\
\hat{D}Jo_{28} &= 230\,[\tfrac{0}{28}] - 72\,[\tfrac{2}{28}] + 186\,[\tfrac{4}{28}] - 264\,[\tfrac{6}{28}] - 108\,[\tfrac{8}{28}] - 120\,[\tfrac{10}{28}] \\
&+ 78\,[\tfrac{12}{28}] \\
\hat{D}Jo_{30} &= -134\,[\tfrac{0}{30}] - 380\,[\tfrac{2}{30}] - 58\,[\tfrac{4}{30}] + 128\,[\tfrac{6}{30}] + 312\,[\tfrac{8}{30}] + 538\,[\tfrac{10}{30}] \\
&+ 22\,[\tfrac{12}{30}] - 88\,[\tfrac{14}{30}] \\
\hat{D}Jo_{32} &= 430\,[\tfrac{0}{32}] - 60\,[\tfrac{2}{32}] + 186\,[\tfrac{4}{32}] - 390\,[\tfrac{6}{32}] + 6\,[\tfrac{8}{32}] - 300\,[\tfrac{10}{32}] \\
&- 50\,[\tfrac{12}{32}] - 80\,[\tfrac{14}{32}] \\
\hat{D}Jo_{34} &= 386\,[\tfrac{0}{34}] + 138\,[\tfrac{2}{34}] - 18\,[\tfrac{4}{34}] - 48\,[\tfrac{6}{34}] - 174\,[\tfrac{8}{34}] - 198\,[\tfrac{10}{34}] \\
&- 414\,[\tfrac{12}{34}] - 36\,[\tfrac{14}{34}] + 12\,[\tfrac{16}{34}] \\
\hat{D}Jo_{36} &= -54\,[\tfrac{0}{36}] - 944\,[\tfrac{2}{36}] - 106\,[\tfrac{4}{36}] - 326\,[\tfrac{6}{36}] + 552\,[\tfrac{8}{36}] + 200\,[\tfrac{10}{36}] \\
&+ 1240\,[\tfrac{12}{36}] + 412\,[\tfrac{14}{36}] + 446\,[\tfrac{16}{36}] \\
\hat{D}Jo_{38} &= 602\,[\tfrac{0}{38}] + 446\,[\tfrac{2}{38}] + 6\,[\tfrac{4}{38}] - 244\,[\tfrac{6}{38}] - 204\,[\tfrac{8}{38}] - 302\,[\tfrac{10}{38}] \\
&- 554\,[\tfrac{12}{38}] - 252\,[\tfrac{14}{38}] - 184\,[\tfrac{16}{38}] + 90\,[\tfrac{18}{38}] \\
\hat{D}Jo_{40} &= 704\,[\tfrac{0}{40}] - 150\,[\tfrac{2}{40}] + 666\,[\tfrac{4}{40}] - 558\,[\tfrac{6}{40}] + 672\,[\tfrac{8}{40}] - 600\,[\tfrac{10}{40}] \\
&- 162\,[\tfrac{12}{40}] - 828\,[\tfrac{14}{40}] - 726\,[\tfrac{16}{40}] - 570\,[\tfrac{18}{40}]
\end{aligned}
$$

11.2 The outer generators Lu and Luu.

$$\overset{\frown}{\ell u}_3 = \tfrac{2}{3}\,\pi^2$$

$$\overset{\frown}{\ell u}_5 = \tfrac{47}{90}\,\pi^4$$

$$\overset{\frown}{\ell u}_7 = \tfrac{12361}{28350}\,\pi^6$$

$$\overset{\frown}{\ell u}_9 = \tfrac{10771487}{28576800}\,\pi^8$$

$$\overset{\frown}{\ell u}_{11} = \tfrac{23554574521}{70727580000}\,\pi^{10}$$

$$\overset{\frown}{\ell u}_{13} = \tfrac{108745677770927}{364105581840000}\,\pi^{12}$$

$$\overset{\frown}{\ell u}_{15} = \tfrac{941991271620333481}{3479028834481200000}\,\pi^{14}$$

$$\overset{\frown}{\ell u}_{17} = \tfrac{1405225117570189347436 7}{56777750578733184000000}\,\pi^{16}$$

$$\overset{\frown}{\ell u}_{19} = \tfrac{338235853283681239466745241}{1485476288391396292992000000}\,\pi^{18}$$

$$\overset{\frown}{\ell u}_{21} = \tfrac{1248078007992564127241918581780 7}{59270503906816712090380800000000}\,\pi^{20}$$

$$\overset{\frown}{\ell u}_{23} = \tfrac{677464224865543063964283025725745801}{3463946059826089104698125094400000000}\,\pi^{22}$$

$$\overset{\frown}{\ell u}_{25} = \tfrac{52286450764941086752360153711155317617247}{286814733753600177869004757816320000000000}\,\pi^{24}$$

$$\overset{\frown}{\ell u}_{27} = \tfrac{5576709094295748822539682325837453212563351161}{32718390752941940290406717747896704000000000000}\,\pi^{26}$$

$$\overset{\frown}{\ell u}_{29} = \tfrac{802296252573476631792613234163716692455738580968687}{5021225992072493692488138159334211369472000000000000}\,\pi^{28}$$

$$\overset{\frown}{\ell u}_{31} = \tfrac{152478111551613219380016146074649058491512195804169257321}{1015668487546463661648038146179327604759948800000000000000}\,\pi^{30}$$

$$\overset{\frown}{\ell u}_{33} = \tfrac{37594440841319141317941608961252368209065786417175227931648127}{265991388866568275421681414026611747755373471334400000000000000}\,\pi^{32}$$

$$\overset{\frown}{\ell u}_{35} = \tfrac{11835141477327660321229365086671544475799466913046047829189854440281}{88786595646716157494380147595012868342004887864066048000000000000000}\,\pi^{34}$$

$$\overset{\frown}{\ell u}_{37} = \tfrac{4690764674086518863265619885194480115513650223236872317083239413123511567}{37253079801449165361492022327915499298938410850004832419840000000000000000}\,\pi^{36}$$

$$
\begin{aligned}
\overset{\frown}{\ell u}u_2 &= \frac{20}{81}\ \pi^{2} \\
\overset{\frown}{\ell u}u_4 &= \frac{1219}{6561}\ \pi^{4} \\
\overset{\frown}{\ell u}u_6 &= \frac{401353}{2657205}\ \pi^{6} \\
\overset{\frown}{\ell u}u_8 &= \frac{36170973257}{281238577200}\ \pi^{8} \\
\overset{\frown}{\ell u}u_{10} &= \frac{7690394022421}{68340974259600}\ \pi^{10} \\
\overset{\frown}{\ell u}u_{12} &= \frac{6713640059454013219}{66980988871833960000}\ \pi^{12} \\
\overset{\frown}{\ell u}u_{14} &= \frac{27080598984361034138281 1}{2995215671777347922904000 * Pi^{1}4} \\
\overset{\frown}{\ell u}u_{16} &= \frac{15978614160083802139741 1486857}{19408997553117214540417920 00000}\ \pi^{16} \\
\overset{\frown}{\ell u}u_{18} &= \frac{2058426515481430718046750683489449}{2726071342325525251059858535 6800000}\ \pi^{18} \\
\overset{\frown}{\ell u}u_{20} &= \frac{841315703072694930111846430381192408987}{1207773516983949755549110961239680000000 0}\ \pi^{20} \\
\overset{\frown}{\ell u}u_{22} &= \frac{484644134135218172634419869853025470122 90219}{750629677804502793844677045913923045120000000}\ \pi^{22} \\
\overset{\frown}{\ell u}u_{24} &= \frac{262801525467520688318567177931229981418306153441965 7}{437426742473733906902927707668973364420717568000000000}\ \pi^{24} \\
\overset{\frown}{\ell u}u_{26} &= \frac{60844567261073718471236142418467451722253518312277829879}{1084770774949808596303293005485611664049855577600000000000}\ \pi^{26} \\
\overset{\frown}{\ell u}u_{28} &= \frac{936035972176127532431386156553924195730342423518027758335584419}{178245845519117719195329675465177057557014499002268160000000000000}\ \pi^{28} \\
\overset{\frown}{\ell u}u_{30} &= \frac{544041892757758967281183226421541460467968286401711736533940563 3983}{1103844112964347104482277529305756939803762244230319029760000000000000}\ \pi^{30} \\
\overset{\frown}{\ell u}u_{32} &= \frac{934698272550163813181716127854037356437874429913136800167921387091723 1163657}{2016357601215648368128116515723911862323010614153503785708863488000000000000000}\ \pi^{32} \\
\overset{\frown}{\ell u}u_{34} &= \frac{2626109148916837130178692639592153052122950393809810251987365466545464729588 00889}{60119060268523269795693059173750604073678336962874145099775926655385600000000000000}\ \pi^{34}
\end{aligned}
$$

11.3 The inner generators Li and Lii.

$$\widehat{\ell i}_n \;:=\; (\sqrt{3}\,\pi)^n \;\; \widehat{\ell i}^*_n \qquad with \qquad \widehat{\ell i}^*_n \in \mathbb{Q}^+ \qquad \forall n \in -\tfrac{1}{2} + \mathbb{N}$$

$$\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\widehat{\ell i}^*_{-1/2} = \frac{1}{2} \qquad \widehat{\ell i}^*_{1/2} = \frac{11}{108} \qquad \widehat{\ell i}^*_{3/2} = \frac{697}{34992} \qquad \widehat{\ell i}^*_{5/2} = \frac{724351}{141717600}$$

$$\widehat{\ell i}^*_{7/2} = \frac{278392949}{214277011200}$$

$$\widehat{\ell i}^*_{9/2} = \frac{244284791741}{728970392102400}$$

$$\widehat{\ell i}^*_{11/2} = \frac{1140363907117019}{12990252387264768000}$$

$$\widehat{\ell i}^*_{13/2} = \frac{212114205337147471}{9119157175859867136000}$$

$$\widehat{\ell i}^*_{15/2} = \frac{367362844229968131557}{59092138499571939041280000}$$

$$\widehat{\ell i}^*_{17/2} = \frac{44921192873529779078383921}{26852058655590484819748044800000}$$

$$\widehat{\ell i}^*_{19/2} = \frac{3174342130562495575602143407}{7012781282343667708197471191040000}$$

$$\widehat{\ell i}^*_{21/2} = \frac{699550295824437662808791404905733}{5686033191537069214483591616407142400000}$$

$$\widehat{\ell i}^*_{23/2} = \frac{14222388631469863165732695954913158931}{423723193433342397863317247254660251648000000}$$

$$\widehat{\ell i}^*_{25/2} = \frac{5255000379400316520126835457783180207189}{572026311135012237115478283793791339724800000000}$$

$$\widehat{\ell i}^*_{27/2} = \frac{382316740742735395243571101350366537155843}{151638974842699607584430425049335958785228800000000}$$

$$\widehat{\ell i}^*_{29/2} = \frac{16169753990012178960071991589211345955648397560689}{23313286852215065204225881629456261442356585431040000000000}$$

$$\widehat{\ell i}^*_{31/2} = \frac{119390469635156067915857712883546381438702433035719259}{6244230750497283064299860135633565064720787841849753600000000000}$$

$$\widehat{\ell i}^*_{33/2} = \frac{1116398659629170045249141261665722279335124967712466031771}{211417164750337009991064664472281245961316434749348957388800000000000}$$

$$\widehat{\ell i}^*_{35/2} = \frac{18640268802093894932827883966494225124248789786595330256830993}{12760730869817921915944551409486627074781263905145381747425280000000000000}$$

$$\widehat{\ell i}^*_{37/2} = \frac{24603580980642759676977649849254935255106862146820276902982636291}{607980114319063421253672019385217467506001479812995888209251532800000000000000}$$

$$\widehat{\ell i}^*_{39/2} = \frac{1346006922185440944067965138463911390263357783573661737568466592369}{119904252110924855600289403475275931852487944018771710822485432729600000000000000}$$

11.4 The exceptional generator Le.

$$
\begin{aligned}
\overset{\frown}{\ell e}_{2n+1} &\equiv 0 \\
\overset{\frown}{\ell e}_{2n} &\equiv \left(\tfrac{\pi}{4}\right)^{2n} \lambda^{-3n-2} \; \overset{\frown}{\ell e}^{*}{}_{2n} \quad \textit{with} \quad \lambda := \log 2 \quad \textit{and} \\
\overset{\frown}{\ell e}^{*}{}_{2n} &\equiv \textstyle\sum_{k=1}^{k=3n} \overset{\frown}{\ell e}^{*}{}_{2n,k} \;\; \lambda^{k} \;\; , \;\; \overset{\frown}{\ell e}^{*}{}_{2n,k} \in \mathbb{Q} \\
\cdots\cdots \;\; \cdots & \;\; \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\overset{\frown}{\ell e}^{*}{}_{0} &= \tfrac{1}{2}\lambda \\
\overset{\frown}{\ell e}^{*}{}_{2} &= +\lambda - \tfrac{1}{6}\lambda^3 \\
\overset{\frown}{\ell e}^{*}{}_{4} &= +\tfrac{10}{3}\lambda + \tfrac{2}{3}\lambda^2 - \tfrac{2}{9}\lambda^3 - \tfrac{1}{9}\lambda^4 + \tfrac{1}{108}\lambda^5 + \tfrac{7}{540}\lambda^6 \\
\overset{\frown}{\ell e}^{*}{}_{6} &= +\tfrac{112}{9}\lambda + \tfrac{224}{45}\lambda^2 + \tfrac{26}{135}\lambda^3 - \tfrac{52}{135}\lambda^4 - \tfrac{43}{405}\lambda^5 + \tfrac{34}{2025}\lambda^6 + \tfrac{331}{24300}\lambda^7 \\
&\quad -\tfrac{7}{12150}\lambda^8 - \tfrac{62}{42525}\lambda^9 \\
\overset{\frown}{\ell e}^{*}{}_{8} &= +\tfrac{440}{9}\lambda + \tfrac{88}{3}\lambda^2 + \tfrac{5872}{945}\lambda^3 - \tfrac{176}{315}\lambda^4 - \tfrac{1802}{2835}\lambda^5 - \tfrac{101}{945}\lambda^6 + \tfrac{2801}{85050}\lambda^7 + \tfrac{76}{4725}\lambda^8 \\
&\quad -\tfrac{17677}{14288400}\lambda^9 - \tfrac{8383}{4762800}\lambda^{10} + \tfrac{941}{23814000}\lambda^{11} + \tfrac{2159}{11907000}\lambda^{12} \\
\overset{\frown}{\ell e}^{*}{}_{10} &= +\tfrac{16016}{81}\lambda + \tfrac{64064}{405}\lambda^2 + \tfrac{2312024}{42525}\lambda^3 + \tfrac{277024}{42525}\lambda^4 - \tfrac{262084}{127575}\lambda^5 - \tfrac{18416}{18225}\lambda^6 \\
&\quad -\tfrac{8186}{91125}\lambda^7 + \tfrac{7864}{127575}\lambda^8 + \tfrac{904423}{45927000}\lambda^9 - \tfrac{220411}{80372250}\lambda^{10} - \tfrac{4359167}{1928934000}\lambda^{11} \\
&\quad +\tfrac{235303}{2411167500}\lambda^{12} + \tfrac{754321}{3214890000}\lambda^{13} - \tfrac{1469}{482233500}\lambda^{14} - \tfrac{4526}{189448875}\lambda^{15} \\
\overset{\frown}{\ell e}^{*}{}_{12} &= +\tfrac{198016}{243}\lambda + \tfrac{198016}{243}\lambda^2 + \tfrac{1369472}{3645}\lambda^3 + \tfrac{1637584}{18225}\lambda^4 + \tfrac{15535676}{4209975}\lambda^5 - \tfrac{20733676}{4209975}\lambda^6 \\
&\quad -\tfrac{2758192}{1804275}\lambda^7 - \tfrac{1401608}{63149625}\lambda^8 + \tfrac{13879099}{126299250}\lambda^9 + \tfrac{198337}{8419950}\lambda^{10} - \tfrac{45253739}{7956852750}\lambda^{11} \\
&\quad -\tfrac{474994117}{159137055000}\lambda^{12} + \tfrac{462208231}{1909644660000}\lambda^{13} + \tfrac{602881969}{1909644660000}\lambda^{14} - \tfrac{89414441}{10503045630000}\lambda^{15} \\
&\quad -\tfrac{3387032899}{105030456300000}\lambda^{16} + \tfrac{48691583}{183803298525000}\lambda^{17} + \tfrac{977403607}{298680360103125}\lambda^{18} \\
\overset{\frown}{\ell e}^{*}{}_{14} &= +\tfrac{826880}{243}\lambda + \tfrac{330752}{81}\lambda^2 + \tfrac{59576704}{25515}\lambda^3 + \tfrac{33079552}{42525}\lambda^4 + \tfrac{3920611264}{29469825}\lambda^5 \\
&\quad -\tfrac{79270784}{11609325}\lambda^6 - \tfrac{11637846268}{1149323175}\lambda^7 - \tfrac{1380040216}{638512875}\lambda^8 + \tfrac{2690948354}{17239847625}\lambda^9 \\
&\quad +\tfrac{153941804}{820945125}\lambda^{10} + \tfrac{6173415169}{241357866750}\lambda^{11} - \tfrac{1334139179}{120678933375}\lambda^{12} - \tfrac{28535702003}{7240736002500}\lambda^{13} \\
&\quad +\tfrac{1985726927}{3620368001250}\lambda^{14} + \tfrac{2489824430263}{5734662913980000}\lambda^{15} - \tfrac{21102007}{907670610000}\lambda^{16} \\
&\quad -\tfrac{11083211947}{247794076530000}\lambda^{17} + \tfrac{534608160781}{652317906465225000}\lambda^{18} + \tfrac{988383013399}{217439302155075000}\lambda^{19} \\
&\quad -\tfrac{567191}{22187683893375}\lambda^{20} - \tfrac{357848408}{776568936268125}\lambda^{21}
\end{aligned}
$$

12 Acknowledgments and references.

12.1 Acknowledgments.

Since our interest in knot-connected power series (i.e. the series $G_{\mathcal{K}}^{NP}$ and $G_{\mathcal{K}}^{P}$ associated with a knot $\mathcal{K}$: cf §9.1) and the closely related notion of SP-series (a natural and conceptually more appealing generalisation, in terms of which we chose to reframe the problem) was first awakened after the 2006 visit to Orsay of Stavros Garoufalidis and his pioneering joint work with Ovidiu Costin and since, despite tackling the problem from very different angles, we have been keeping in touch for about one year, comparing methods and results, we feel we owe it to the reader to outline the main differences between our two approaches – to justify, as it were, their parallel existence.

The very first step is the same in both cases: we all rely on a quite natural method [82] for deducing the shape of a function's closest singularity, or singularities, from the exact asymptotics of its Taylor coefficients at 0. [83]

But then comes the question of handling the other singularities – those farther afield – and this is where our approaches start diverging. In [C.G1.]-[C.G4], the idea is to re-write the functions under investigation in the form of multiple integrals amenable to the Riemann-Hilbert theory and then use the well-oiled machinery that goes with that theory. In this approach, the global picture (exact location of the singularities on the various Riemann leaves, rough nature of these singularities etc) emerges first, and the exact description of each singularity, while also achievable at the cost of some extra work, comes second.

Our own approach reverses this sequence: the local aspect takes precedence, and we then piece the global picture together from the local data. To that end, we distinguish three types of "resurgence generators" (i.e. local singularities that generate the resurgence algebra under alien derivation) : the actually occuring *inner* and *outer* generators[84], and the auxiliary *exceptional* or *movable* generators. The basic object here is the *inner resurgence algebra*, spanned by the *inner* generators, which recur indefinitely under alien derivation. The *outer* generators, on the other hand, produce only *inner* ones under alien derivation.[85] We give exact descriptions of both the *inner* and *outer* generators by means of special integro-differential functionals of infinite order: *nir*, *mir* and *nur*, *mur*.

So much for the local aspect. To arrive at the global picture, we resort to an auxiliary construct, the so-called *exceptional* or *movable* generators, which are very useful on account of three features:
(i) they depend on a arbitrary *base point*, which can be taken as close as we wish to any particular singularity we want to zoom in onto [86]

[82] see [O.C.] and §2.3 of the present paper.

[83] this convergence is hardly surprising: the functions on hand (knot-related or SP) tend to verify no useable equations, whether differential or functional, that might give us a handle on their analytic properties, and so the Taylor coefficients are all we have to go by. Two of us (O.C. in [C2] and J.E. in a 1993 letter to prof. G.K. Immink) hit independently on the same method – which must also have occurred, time and again, more or less explicitely, to many an analyst grappling with singularities.

[84] while there are only two *outer* generators (which may coalesce into one), there can be any number of *inner* generators.

[85] which is only natural, since the *outer* generators can be interpreted as infinite sums of (self-reproducing) *inner* generators.

[86] thus bringing it within the purview of the method of Taylor coefficient asymptotics (see above).

(ii) their own set of singularities include all the *inner* generators of the SP function
(iii) they may also possess *parasitical* singularities[87] (i.e. singularities other than the above), but these always lie farther away from the *base point* than the closest *inner* generators.
Thus, by moving the base point around, we can reduce the global investigation to a local, or should we say, semi-local one, and derive the full picture, beginning with the crucial *inner algebra.*

A further difference between our approaches is this: while O. Costin and S. Garoufalidis are more directly concerned with the knot-related series $G_{\mathcal{K}}^{NP}$ and $G_{\mathcal{K}}^{P}$ and the so-called *volume conjecture* which looms ominously over the whole field, the framework we have chosen for our investigation is that of SP-series, i.e. general Taylor series with coefficients that are syntactically of sum-product type. But this latter difference might well be less than appears, since each of the two methods would seem, in principle, to be capable of extension in both directions.

12.2 References.

[**C1**] O. Costin, *Asymptotics and Borel summability,* 2008, CRC Press.
[**C2**] O. Costin, *Global reconstruction of analytic functions from local expansions,* Preprint 2007, Ohio State University.
[**CG1**] O. Costin and S. Garoufalidis, *Resurgence of the Kontsevich-Zagier power series,* preprint 2006 math. GT/0609619.
[**CG2**] O. Costin and S. Garoufalidis, *Resurgence of the fractional polylogarithms,* preprint 2007, math. CA/0701743.
[**CG3**] O. Costin and S. Garoufalidis, *Resurgence of the Euler-MacLaurin summation formula,* preprint 2007, math. CA/0703641.
[**CG4**] O. Costin and S. Garoufalidis,(forthcoming paper on general knot-related series)
[**E1**] J. Ecalle, *Les fonctions resurgentes: Vol.1*, Publ. Math. Orsay, 1981
[**E2**] J. Ecalle, *Les fonctions resurgentes: Vol.2*, Publ. Math. Orsay, 1981
[**E3**] J. Ecalle, *Les fonctions resurgentes: Vol.3*, Publ. Math. Orsay, 1985
[**E4**] J. Ecalle., *Six lectures on transseries etc...,* in: Bifurcations and Periodic Orbits etc, D. Schlomiuk ed., 1993, Kluwer, p 75-184.
[**E5**] J. Ecalle, *Recent advances in the analysis of divergence and singularities*, in: Normal Forms etc, Y.Ilyashenko and C.Rousseau eds., 2004, Kluwer,p 87-186.
[**ES**] J. Ecalle and Sh. Sharma, *On a remarkable resurgence algebra linked to sum-product Taylor series.*, (forthcoming).
[**LR**] J.-L. Loday and M.O. Ronco, *Hopf algebra of a permutation and of planar binary trees.* J. Alg. Combinatorics, **15**, 2002, pp 253-270.
[**HNT**] F. Hivert, J.-C. Novelli, J.-Y. Thibon, *The algebra of binary search trees.* Theoret. Computer. Sc.. **339**, 2005, pp 129-165.
[**SS1**] Sh. Sharma, *Some examples of SP-series : a theoritical and numerical investigation*, (forthcoming).
[**SS2**] Sh. Sharma, *SP-series and the interplay between their resurgence and differential properties*, (forthcoming).

[87] this is the case *iff* the driving function F has at least one zero or one pole.