Twisted Resurgence Monomials and canonical-spherical synthesis of Local Objects.

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Abstract. Until very recently, received wisdom seemed to rule out the feasibility of a truly canonical and completely explicit synthesis for *local objects*, that is to say the possibility of constructing privileged representatives in each analytic conjugacy class of such objects. But in the mid 90s it emerged that there *does exist* a canonical synthesis after all. We call it *paralogarithmic*, because its building blocks are a new class of transcendental functions, the so-called paralogarithms, quite distinct from the classical but (in this context) unsuitable hyperlogarithms. We also call it *spherical*, because the most salient feature of the objects thus produced is a tendency to extend to the whole Riemann sphere (in the critical variable) and to go in pairs: a *direct object* and its *antipodal reflection.* Both objects – direct and antipodal – always exist; they are indisputably canonical upto the choice of one unremovable parameter (the "twist" c); and they connect under analytic continuation "whenever the invariants permit".

1 Introduction: Object Analysis and Object Synthesis.

Although the proper ambit of this paper is *canonical object synthesis* and the special functions – technically known as *well-behaved resurgence monomials* – on which synthesis relies, we have chosen to add a cursory treatment of three closely related topics: *well-behaved convolution averages, well-behaved alien derivations,* and *ramified-exponential growth.* Also, to make the whole thing tolerably self-contained, preliminary sections have been inserted, which recall the basic notions about moulds and resurgence. This apparatus is light, easy to master, and altogether a good investment. It makes possible a far-going *algebraisation* of the *analysis* difficulties. It also permits explicit constructions, and leads to proofs that are often quite concise.

The results about canonical synthesis go back to March 1996 (they were presented in a general survey I gave at the Orsay conference in honour of A. Douady in June 1996) but nothing in the way of a systematic written exposition has appeared so far [1]. This, I feel, is one more reason for attempting a fairly comprehensive treatment this time around.

1.1 The notion of Local Analytic Object.

By *local analytic object* we shall mean, **primarily :**
(1) germs of singular analytic vector fields at 0 on $\mathbb{C}^\nu$, often referred to as just *fields* for short
(2) germs of analytic diffeomorphisms of $\mathbb{C}^\nu$ into itself, with 0 as fixed point, or *diffeos* for short
and, **secondarily**, all those equations or systems (differential, difference, functional, etc) which may, in a standard manner, be rephrased in terms of fields or diffeos.

Fields will be noted

$$X = \sum_{1\le i\le \nu} X_i(x)\,\partial_{x_i} \qquad ; \qquad X_i(x) \in \mathbb{C}\{x_1,\ldots,x_\nu\} \tag{1}$$

but instead of diffeos proper:

$$f : x_i \mapsto f_i(x) \qquad i=1,\ldots,\nu\ ; \quad f_i(x) \in \mathbb{C}\{x_1,\ldots,x_\nu\} \tag{2}$$

it will often be more convenient to handle the corresponding substitution operators F (same symbols, but capitalised):

$$(F\varphi)(x) := \varphi(f(x)) \quad ; \qquad \forall\varphi(x) \in \mathbb{C}\{x_1,\ldots,x_\nu\} \tag{3}$$

The discussion hinges on the nature of the object's spectrum, ie the eigenvalues of its linear part: $\lambda = (\lambda_1,\ldots,\lambda_\nu)$ for a field; $l = (l_1,\ldots,l_\nu)$ for a diffeo. If the spectrum is "generic", then the object is analytically conjugate to its linear part – and that ends the matter, at least from the local point of view.

[1] though some sketchy indications found their way into [EM], 1996.

Difficulties arise only in the case of
– **resonance**: relations of type $0 = \lambda_i - \sum \lambda_j$ or $0 = l_i - \prod (l_j)^{m_j}$ with non-negative integers m_j.
– **quasiresonance**: when Bryuno's well-known diophantine condition, which minorises the above expressions in terms of $\|m\|$, is not fulfilled.
– **nihilence**: this condition, which presupposes resonance but bears on coefficients of all orders, occurs mostly in connection with symplectic or volume-preserving objects.

The more "complicated" an object, the larger its set of *invariants* tends to be. Alongside the *formal* and *analytic invariants* (ie relative to formal or analytic coordinate changes) we have the notion of *holomorphic invariants* – ie those invariants that depend holomorphically on the object **Ob** (ie in practical terms, on its Taylor coefficients), at least when **Ob** remains within a fixed formal conjugacy class. [2]

Resonance generates *formal invariants* (other that the spectrum itself), of which there may be an infinite number[3] if the resonance degree is ≥ 2.

Each of the above-mentioned complications – resonance, quasiresonance, nihilence – whether in isolation or in combination, gives rise to analytic invariants (strictly analytic, ie non formal). Moreover, when resonance alone is at work, there tend to exist [4] complete systems of analytic-cum-holomorphic invariants $\{\mathbb{A}_\omega\}$.

We shall henceforth deal with purely resonant objects. Resonance is more important, and less exceptional, than "Lebesgue measure" considerations would suggest, since it covers the cases:
(1) of identity-tangent diffeomorphisms
(2) of vector fields with (one or several) vanishing eigenvalues
and also since
(3) most singular differential equations or systems, when translated into time-autonomous systems (by the addition of one unknown) or, equivalently, into vector fields, tend to exhibit resonance.

"Object analysis" starts from some object **Ob** and is concerned with finding its invariants. For resonant objects, which alone matter to us here, there is a method of sweeping generality – the Bridge equation (see infra) – for conctructing complete systems $\{\mathbb{A}_\omega\}$ of analytic-cum-holomorphic invariants, in

[2] thus, when applied to *invariants*, the words *analytic* and *holomorphic* assume quite distinct meanings: analytic invariants are not necessarily holomorphic.

[3] more precisely, an infinite number of *independent* formal invariants

[4] with slight qualifications, see eg [E5]

the form of specific differential operators $\mathbb{A}_\omega$, with indices ω running through a countable set Ω generated by the object's spectrum. Moreover, relatively to any "suitable" basis $\{\mathbf{\Delta}_\omega^{\text{good}}\}$ in the algebra of alien derivations, the Bridge equation yields systems $\{\mathbb{A}_\omega^{\text{good}}\}$ that can be characterised by means of simple, transparent growth conditions on the invariants $\mathbb{A}_\omega^{\text{good}}$ as ω increases.

"Object synthesis" is the converse problem: starting from a prescribed system $\{\mathbb{A}_\omega\}$ with the proper growth pattern, find an object **Ob** whose invariants coincide with that system. There are actually four degrees there:
– *existence:* showing that such an object **Ob** does exist.[5]
– *constructiveness:* producing an effective procedure for constructing it.
– *expliciteness:* expanding the object **Ob**, in a manner both explicit and universal, by means of elementary special functions, the so-called resurgence monomials, that are not constructed *ad hoc*, but are given once and for all.
– *canonicity:* examining whether perchance there exists a "canonical" choice for **Ob** and also (since we don't want to forego *expliciteness*) a corresponding system of "canonical" resurgence monomials.

We shall recall (§5) the basic facts about *existential, constructive, explicit* synthesis, and sketch several methods, some going back to the late 70s, for achieving these. But our real concern here is with the more ambitious goal of *explicit-canonical* synthesis. We shall recall earlier attempts in this direction, based on the notion of *hyperlogarithmic monomials*, and show why these attempts, while interesting in their way and insightful, were doomed to partial failure. We shall then (§8) introduce a whole new class of *resurgence monomials*, based on "prodiffusions", and which on account of their nice growth properties, *do permit explicit synthesis* in all cases. Lastly, we shall (§§8,9,10) show that there exists a particular subclass, the so-called *paralogarithmic* or *spherical monomials*, which unquestionably stand out as "canonical" and which can be harnessed to synthesise objects **Ob** that will automatically inherit their 'canonicity'.

1.2 Object Analysis: the Bridge Equation.

Let **Ob** be some (purely) resonant object – field or diffeo – expressed in a particular analytic chart $x = \{x_1, \ldots, x_\nu\}$ that diagonalises the object's linear part. The object's complete linearisation is usually impossible, even formally, and what takes its place is either *formal normalisation*, which removes all but

[5]or, what amounts to the same, finding conditions for a system $\{\mathbb{A}_\omega\}$ to be "admissible", ie to be *someone's* invariants.

a few resonant monomials, or the more radical step of *formal trivialisation*, which forfeits *entireness* [6] but reduces the object to the simplest conceivable form, namely ∂_z for a field and $z \mapsto z+1$ for a diffeo.

Let $y = \{y_1, \ldots, y_\nu\}$ be a formal normal chart, and consider the formal-entire coordinate changes $y_i = \theta_i(x)$ and $x_i = \theta_i^{-1}(y)$ with the substitution operators Θ and Θ^{-1} that go with them : $\Theta^{\pm}\varphi := \varphi \circ \theta^{\pm}$

Consider also the "trivial chart" $\{z, u\} = \{z, u_1, \ldots, u_{\nu-1}\}$. Expressing the given coordinates $x = \{x_i\}$ in terms of the trivial coordinates $\{z, u_i\}$, we get the so-called *formal integral* $x(z,u) = \{x_1(z,u), \ldots, x_\nu(z,u)\}$, which verifies:

$$\partial_z\, x(z,u) \equiv X(x(z,u)) \qquad \text{for a field} \tag{4}$$

$$f(x(z,u)) \equiv x(z+1,u) \qquad \text{for a diffeo} \tag{5}$$

The Bridge Equation (B.E.) is an amazingly general and flexible tool for extracting the object's *invariants* from the divergence-resurgence of the *trivialising* or (direct/inverse) *normalising* transformations. Here are its three main forms :

$$\mathbf{\Delta}_\omega\, x(z,u) = \mathbb{A}_\omega\, x(z,u) \qquad \text{(B.E. for formal integral)} \tag{6}$$

$$[\mathbf{\Delta}_\omega, \Theta] = -\,\Theta\, \mathbb{A}_\omega \qquad \text{(B.E. for direct normaliser)} \tag{7}$$

$$[\mathbf{\Delta}_\omega, \Theta^{-1}] = +\,\mathbb{A}_\omega\, \Theta^{-1} \qquad \text{(B.E. for inverse normaliser)} \tag{8}$$

The indices ω on both sides of the Bridge Equation range through a countable set Ω essentially spanned by the object's multipliers, ie the λ_j in the case of a field, and the $\log(l_j)$ (to which one must add the universal multiplier $\lambda_0 := 2\pi i$) in the case of a diffeo.

The $\mathbf{\Delta}_\omega$ on the left-hand side denotes the alien derivation relative to the variable z and the index ω but with a built-in exponential factor $\mathbf{\Delta}_\omega := e^{-\omega z}\Delta_\omega$ that makes it commute with ∂_z and ensures the stability rule under analytic changes of equivalent variables $z \mapsto z_\star$ with $z \sim z_\star$:

$$\varphi(z) \equiv \varphi_*(z_*) \qquad \Longrightarrow \qquad \mathbf{\Delta}_\omega^{(z)}\, \varphi(z) \equiv \mathbf{\Delta}_\omega^{(z_*)}\, \varphi_*(z_*) \tag{9}$$

The alien-differentiation variable z, also known as *critical variable*[7], is always $\sim \infty$. In (6) it is simply the z inside the formal integral. In (7) (resp (8)) it is the inverse of some *resonant monomial*, ie $z := 1/x^m$ (resp $z := 1/y^m$). Due to the afore-mentioned invariance property of alien differentiation, the

[6] it no longer relies on entire power series

[7] or *critical time*.

critical variable is actually defined up to equivalence $\sim$ and so the proper, intrinsic notion is that of *critical class.*

The $\mathbb{A}_\omega$ on the right-hand side are ordinary (first-order) differential operators – in the variables (z, u_i) or (x_i) or (y_i) respectively. They are constructively determined, even overdetermined, by the requirement of equality in the Bridge Equation – whichever of its variants we choose to work with, and whichever critical variable we pick (within the critical class) for alien differentiation. Each single $\mathbb{A}_\omega$ is an invariant of the object **Ob**, and the total collection $\{\mathbb{A}_\omega, \omega \in \Omega\}$ constitutes a set, both complete and free, of analytic-cum-holomorphic invariants.

All these claims, as sketchy as they are sweeping, clearly cry for explanations, and qualifications too, which cannot be supplied here but are available in the literature ([E2],[E3],[E7]). We recalled these statements simply to serve as a general backdrop for the twin problems of object analysis and synthesis – but we shall illustrate the method on just six typical instances.

1.3 Object Synthesis: semi-formal candidates

Pick a purely resonant and, for simplicity, monocritical[8] object **Ob** with invariants $\{\mathbb{A}_\omega\}$ and consider the expansions:

$$\Theta \overset{always}{:=} 1 + \sum_{1 \leq r} \sum_{\omega_i \in \Omega} (-1)^r \mathcal{U}e^{\omega_1, \dots, \omega_r} \, \mathbb{A}_{\omega_r} \dots \mathbb{A}_{\omega_1} \tag{10}$$

$$\Theta^{-1} \overset{conditionally}{:=} 1 + \sum_{1 \leq r} \sum_{\omega_i \in \Omega} \mathcal{U}e^{\omega_r, \dots, \omega_1} \, \mathbb{A}_{\omega_r} \dots \mathbb{A}_{\omega_1} \tag{11}$$

Formally, ie ignoring all questions of convergence, they verify the Bridge equation: if we alien-differentiate them term by term, we can see that the operator Θ verifies (7) and its inverse Θ^{-1} inverse verifies (8)[9]. But (7) and (8) *do characterise* Θ and Θ^{-1} as direct and inverse normalisers. Thus, working our way back from the normal form $\mathbf{Ob}^{\mathrm{norm}}$, we get a new object $\mathbf{Ob} := \Theta \, \mathbf{Ob}^{\mathrm{nor}} \, \Theta^{-1}$ which not only has the prescribed invariants $\mathbb{A}_\omega$ but is also 'formally analytic', in the sense that all its alien derivatives in z – by assumption the only critical time – identically vanish.

[8] ie such that the associated objects – normalising transformations, fractional iterates, etc – have only one *critical time* z, or rather *critical time class* $\{z\}$.

[9] That much is always true. The cautionary word *conditionally* in the middle of (11) signals that this second expansion, unlike the first, holds only when the invariants $\mathbb{A}_\omega$ commute with ∂_z, which is always the case with vector fields. But even when there is no commutation, Θ^{-1} admits an analoguous expansion – and knowing Θ is sufficient anyway.

1.4 Object Synthesis: from semi-formal to effective.

But what is missing in this *formal*, or should we say *semi-formal*, construction, is the convergence of $\Theta^{\pm 1}$ in the relevant space, ie the space of resurgent functions with exponential growth in the Borel plane. Such convergence, which would automatically ensure the convergence of the synthesised object in its own space, namely that of multiplicative function-germs , has itself three sides to it:
(i) convergence on each compact subset of the ramified Borel-plane, ie on the Riemann function generated by singularities from Ω which puncture that plane.
(ii) proper growth at infinity, ie *exponential* growth *away from*, or *parallel to*, the singularity-carrying axes, and *"suitable"* growth (technically : *"ramified-exponential"* growth, see §9), *over* such axes.
(iii) for the synthesis construction to succeed, all this should apply, if not to the expansions (10), (11) themselves, at least to some re-arranged variant of these expansions.

Demand (i) calls for the introduction not only of well-behaved resurgence monomials, but also of a suitable *twist* parameter c.

Demand (ii) is then automatically fulfilled: it follows from the growth conditions which the invariants $\mathbb{A}_\omega$ verify as soon as they are expressed relative to a well-behaved basis of ALIEN.

That leaves (iii): it will turn out that the absolute convergence of (10) or (11) is too strong a demand, except in very simple – linear or affine – situations. In truly general cases, we must not only work with a well-behaved system of resurgence monomials (preferably the *canonical system*, since we are fortunate in having one) and adjust the *twist* parameter c, but we must also perform two further adaptations :
– we must work relative to a well-behaved basis of ALIEN (which means expressing, simultaneously, our *alien derivations*, *holomorphic invariants*, and *resurgence monomials* coherently in this basis)
– then we must *arborify* our expansions (10),(11), ie switch from an indexation by totally ordered sequences to one that relies on a partial order, of arborescent type.

2 Reminders about moulds, resurgent functions, alien derivations.

2.1 Moulds/comoulds.

Moulds are *functions of a variable number of variables*: they depend on sequences $\boldsymbol{\omega} := (\omega_1, \ldots, \omega_r)$ of arbitrary length $r = r(\boldsymbol{\omega})$. The sum $\|\boldsymbol{\omega}\|$ of a sequence is simply $\sum_1^r \omega_i$. Sequences are systematically written in boldface, with upper indexation when such is called for, and with the product denoting concatenation: e.g. $\boldsymbol{\omega} = \boldsymbol{\omega}^1.\boldsymbol{\omega}^2$. The elements ω_i which make up these sequences are written in normal print, with lower indexation. The sequences themselves are affixed to the moulds as upper indices $A^\bullet = \{A^{\boldsymbol{\omega}}\}$, since moulds are meant to be contracted

$$A^\bullet, B_\bullet \quad \mapsto \quad < A^\bullet, B_\bullet > := \sum A^{\boldsymbol{\omega}}\, B_{\boldsymbol{\omega}}$$

with dual objects (often differential operators or elements of an associative algebra), the so-called comoulds $B_\bullet = \{B_{\boldsymbol{\omega}}\}$, which carry their own indices in lower position.

2.1.1 Basic mould operations.

Moulds may be *added, multiplied, composed.*
Mould addition is what you expect: components of equal length get added.
Mould multiplication (*mu* or $\times$) is associative, but non-commutative:

$$C^\bullet = A^\bullet \times B^\bullet \Longleftrightarrow C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega} = \boldsymbol{\omega}^1.\boldsymbol{\omega}^2} A^{\boldsymbol{\omega}^1} B^{\boldsymbol{\omega}^2} \tag{12}$$

(This includes the trivial decompositions $\boldsymbol{\omega} = \boldsymbol{\omega}.\emptyset$ and $\boldsymbol{\omega} = \emptyset.\boldsymbol{\omega}$).

Mould composition ($\circ$) too is associative and non-commutative:

$$C^\bullet = (A^\bullet \circ B^\bullet) \quad \Longleftrightarrow \quad C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega} = \boldsymbol{\omega}^1 \ldots \boldsymbol{\omega}^s} A^{\|\boldsymbol{\omega}^1\|,\ldots,\|\boldsymbol{\omega}^s\|}\, B^{\boldsymbol{\omega}^1} \ldots B^{\boldsymbol{\omega}^s} \tag{13}$$

with a sum extending to all possible decompositions of $\boldsymbol{\omega}$ into $s \leq r(\boldsymbol{\omega})$ non-empty factor sequences $\boldsymbol{\omega}^{\boldsymbol{i}}$

The operations $(+, \times, \circ)$ on moulds interact in exactly the same way as their namesakes for power series. Thus $(A^\bullet \times B^\bullet) \circ C^\bullet \equiv (A^\bullet \circ C^\bullet) \times (B^\bullet \circ C^\bullet)$.

2.1.2 Basic mould symmetries.

Nearly all useful moulds fall into a few basic *symmetry types.*
A mould $A^\bullet$ is said to be *symmetral* (resp. *alternal*) iff :

$$\sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega^1},\boldsymbol{\omega^2})} A^{\boldsymbol{\omega}} = A^{\boldsymbol{\omega^1}} A^{\boldsymbol{\omega^2}} \quad (resp.\ 0) \qquad \forall\, \boldsymbol{\omega^1} \neq \emptyset\,, \forall\, \boldsymbol{\omega^2} \neq \emptyset \tag{14}$$

A mould $A^\bullet$ is said to be *symmetrel* (resp. *alternel*) iff :

$$\sum_{\boldsymbol{\omega}\in \mathrm{she}(\boldsymbol{\omega^1},\boldsymbol{\omega^2})} A^{\boldsymbol{\omega}} = A^{\boldsymbol{\omega^1}} A^{\boldsymbol{\omega^2}} \quad (resp.\ 0) \qquad \forall\, \boldsymbol{\omega^1} \neq \emptyset\,, \forall\, \boldsymbol{\omega^2} \neq \emptyset \tag{15}$$

Here $sha(\boldsymbol{\omega^1},\boldsymbol{\omega^2})$ (resp. $she(\boldsymbol{\omega^1},\boldsymbol{\omega^2}\ )$) denotes the set of all sequences $\boldsymbol{\omega}$ obtained from $\boldsymbol{\omega^1}$ and $\boldsymbol{\omega^2}$ under ordinary (resp. contracting) shuffling. In a contracting shuffle, two adjacent indices ω_i and ω_j stemming from $\boldsymbol{\omega^1}$ and $\boldsymbol{\omega^2}$ respectively may coalesce to $\omega_{ij} := \omega_i + \omega_j$.

Thus, for a sequence $\boldsymbol{\omega^1} := (\omega_1)$ of length 1 and a sequence $\boldsymbol{\omega^2} := (\omega_2, \omega_3)$ of length 2, the symmetrality (resp alternality) condition reads:

$$\begin{aligned} A^{\omega_1,\omega_2,\omega_3} + A^{\omega_2,\omega_1,\omega_3} + A^{\omega_2,\omega_3,\omega_1} \ &\equiv\ A^{\omega_1}\ A^{\omega_2,\omega_3} \\ (resp \ &\equiv\ 0) \end{aligned}$$

and the symmetrelity (resp alternelity) condition reads:

$$\begin{aligned} A^{\omega_1,\omega_2,\omega_3} + A^{\omega_2,\omega_1,\omega_3} + A^{\omega_2,\omega_3,\omega_1} + A^{\omega_1+\omega_2,\omega_3} + A^{\omega_2,\omega_1+\omega_3} \ &\equiv\ A^{\omega_1}\ A^{\omega_2,\omega_3} \\ (resp \ &\equiv\ 0) \end{aligned}$$

2.1.3 Mould-comould contractions.

Let B_ω be the homogeneous components of some local-analytic, ν-dimensional vector field X (resp of the postcomposition operator F associated to some local-analytic ν-dimensional diffeomorphism f) and let

$$B_{\boldsymbol{\omega}} = B_{\omega_1,\dots,\omega_r} := B_{\omega_r} \dots B_{\omega_1} \qquad\qquad (reversion!) \tag{16}$$

The comould $B_\bullet$ so defined is said to be co-symmetral (resp co-symmetrel) if its action on a product $\varphi_1\varphi_2$ obeys the Leibniz rule:

$$B_{\boldsymbol{\omega}}\,(\varphi_1\,\varphi_2) = \sum\,(B_{\boldsymbol{\omega^1}}\,\varphi_1)\,(B_{\boldsymbol{\omega^2}}\,\varphi_2) \tag{17}$$

with a sum extending to all pairs $(\boldsymbol{\omega^1},\boldsymbol{\omega^2})$ such that $\boldsymbol{\omega} \in sha(\boldsymbol{\omega^1},\boldsymbol{\omega^2})$ (resp $\boldsymbol{\omega} \in she(\boldsymbol{\omega^1},\boldsymbol{\omega^2})$).

The four main symmetry types admit a simple characterisation in terms of mould-comould contractions :

$$A^\bullet \ : \quad B_\bullet \mapsto C_\bullet \quad with \quad C_{\omega_0} := \sum_{\|\boldsymbol{\omega}\|=\omega_0} A^{\boldsymbol{\omega}}\, B_{\boldsymbol{\omega}} \tag{18}$$

Indeed :

$$\begin{array}{llll} A^\bullet & : B_\bullet & \to & C_\bullet \\ \\ \textit{alternal} & : \textit{field} & \to & \textit{field} \\ \textit{symmetral} & : \textit{field} & \to & \textit{diffeo} \\ \textit{alternel} & : \textit{diffeo} & \to & \textit{field} \\ \textit{symmetrel} & : \textit{diffeo} & \to & \textit{diffeo} \end{array}$$

Most stability properties follow from this interpretation. Thus :

$$\begin{array}{ccccc} \textit{symmetral}^\bullet & \times & \textit{symmetral}^\bullet & = & \textit{symmetral}^\bullet \\ \textit{symmetrel}^\bullet & \times & \textit{symmetrel}^\bullet & = & \textit{symmetrel}^\bullet \\ \textit{alternal}^\bullet & \circ & \textit{alternal}^\bullet & = & \textit{alternal}^\bullet \\ \textit{symmetrel}^\bullet & \circ & \textit{symmetrel}^\bullet & = & \textit{symmetrel}^\bullet \end{array}$$

2.1.4 Arborification and co-arborification.

Straightforward mould expansions $\sum_{\boldsymbol{\omega}} A^{\boldsymbol{\omega}}\, B_{\boldsymbol{\omega}}$, which typically pair a symmetral or alternal mould $A^\bullet$ with a cosymmetral comould $B_\bullet$ (or a symmetrel or alternel mould $A^\bullet$ with a cosymmetrel comould $B_\bullet$) often fail to converge absolutely [10] , ie we often have $\sum_{\boldsymbol{\omega}} \|A^{\boldsymbol{\omega}}\ B_{\boldsymbol{\omega}}\| = +\infty$, although the underlying power series may well be convergent. Fortunately, there is an extremely general method for restoring convergence. In essence, it replaces expansions indexed by *totally ordered* sequences $\boldsymbol{\omega}$ by others whose indices are *arborescent sequences* $\boldsymbol{\omega}^{\prec}$ or $\boldsymbol{\omega}^{\nprec}$, like this :

$$\sum_{\boldsymbol{\omega}} A^{\boldsymbol{\omega}}\, B_{\boldsymbol{\omega}} \ \mapsto \ \sum_{\boldsymbol{\omega}^{\prec}} A^{\boldsymbol{\omega}^{\prec}}\ B_{\boldsymbol{\omega}^{\prec}} \qquad (\textit{ordinary} \quad \textit{arborification}) \tag{19}$$

$$\sum_{\boldsymbol{\omega}} A^{\boldsymbol{\omega}}\, B_{\boldsymbol{\omega}} \ \mapsto \ \sum_{\boldsymbol{\omega}^{\nprec}} A^{\boldsymbol{\omega}^{\nprec}}\ B_{\boldsymbol{\omega}^{\nprec}} \qquad (\textit{contracting} \quad \textit{arborification}) \tag{20}$$

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[10]for any reasonable choice of operator norm $\|.\|$, or rather, since we are dealing with germs all along, for any *system* of such norms. For definiteness, think of the system of uniform norms $\|\varphi\|_{\mathcal{D}_1} := \sup_{x\in\mathcal{D}_1} |\varphi(x)|$ on functions germs, and of the corresponding system of norms $\|B\|_{\mathcal{D}_1,\mathcal{D}_2} := \sup_{\|\varphi\|_{\mathcal{D}_1}\le 1} |B\varphi|_{\mathcal{D}_2}$ on operators, relatively to small enough neighbourhoods $\mathcal{D}_2 \subset \mathcal{D}_1$ of the origin.

The dual arborification/coarborification transforms verify :

arborification $\Longrightarrow$

ordinary :	$A^{\boldsymbol{\omega}^{\prec}}$	$:=$	$\sum_{\boldsymbol{\omega}^{\prec}<\boldsymbol{\omega}} \quad A^{\boldsymbol{\omega}}$	(*complete definition*)
contracting :	$A^{\boldsymbol{\omega}^{\prec\!\!\prec}}$	$:=$	$\sum_{\boldsymbol{\omega}^{\prec\!\!\prec} \ll \boldsymbol{\omega}} A^{\boldsymbol{\omega}}$	(*complete definition*)

coarborification $\Longrightarrow$

ordinary :	$B_{\boldsymbol{\omega}}$	$:=$	$\sum_{\boldsymbol{\omega}^{\prec}<\boldsymbol{\omega}} \quad B_{\boldsymbol{\omega}^{\prec}}$	(*mere constraint*)
contracting :	$B_{\boldsymbol{\omega}}$	$:=$	$\sum_{\boldsymbol{\omega}^{\prec\!\!\prec} \ll \boldsymbol{\omega}} B_{\boldsymbol{\omega}^{\prec\!\!\prec}}$	(*mere constraint*)

and are devised in such a way as to :

(1) leave the expansions formally unchanged : they amount to a simple redistribution of terms.

(2) drastically reduce the size of the comould part : comoulds typically get divided by a factor of order $r! := r(\boldsymbol{\omega})!$

(3) prevent a concomitant increase of the mould part : moulds typically retain the same order of magnitude, despite being changed into a sum of almost $r!$ similar terms !

But whereas the reduction (2) is automatic and universal, the non-increase (3) relies on specific identities, of an algebraic or combinatorial nature, which can never be taken for granted, and yet tend to take place, with providential regularity, whenever we require them !

The ordinary (resp contracting) *arborification* rule boils down to summing all the terms $A^{\boldsymbol{\omega}}$ with totally ordered sequences $\boldsymbol{\omega}$ whose order is compatible with the arborescent order of $\boldsymbol{\omega}^{\prec}$ (resp with that of $\boldsymbol{\omega}^{\prec\!\!\prec}$, but allowing contractions of consecutive elements ω_i). The following example should make this amply clear. Assume :

$$\begin{array}{ccccccc} & & & & \omega_4 & & \\ & & & \nearrow & & & \\ \boldsymbol{\omega}^{\prec}\ (\,or\ \boldsymbol{\omega}^{\prec\!\!\prec}\,) & := & \omega_1 & \rightarrow & \omega_2 & \rightarrow & \omega_3 \end{array}$$

Then the arborification rules means :

$$\begin{aligned} A^{\boldsymbol{\omega}^{\prec}} &:= A^{\omega_1,\omega_2,\omega_3,\omega_4} + A^{\omega_1,\omega_2,\omega_4,\omega_3} + A^{\omega_1,\omega_4,\omega_2,\omega_3} \\ A^{\boldsymbol{\omega}^{\prec\!\!\prec}} &:= A^{\omega_1,\omega_2,\omega_3,\omega_4} + A^{\omega_1,\omega_2,\omega_4,\omega_3} + A^{\omega_1,\omega_4,\omega_2,\omega_3} \\ &\quad + A^{\omega_1,\omega_2,\omega_3+\omega_4} + A^{\omega_1,\omega_2+\omega_4,\omega_3} \\ &:= A^{\boldsymbol{\omega}^{\prec}} + A^{\omega_1,\omega_2,\omega_3+\omega_4} + A^{\omega_1,\omega_2+\omega_4,\omega_3} \end{aligned}$$

Unlike the *arborification identities*, the *coaborification constraints* leave a lot of latitude. However, for any given choice of variables, there exists a 'canonical' choice, the so-called 'homogeneous' coarborification, which shall be described, and made use of, in §11.2.

Lastly, we may note that no difference of meaning attaches to the notations $\boldsymbol{\omega}^{\prec}$ and $\boldsymbol{\omega}^{\not\prec}$: both stand for sequences with an arborescent order on them. The distinction is simply for clarity, the former notation being used in the context of symmetral-alternal moulds and ordinary arborification, and the latter in the the context of symmetrel-alternel moulds and contracting arborification.

2.1.5 Armoulds and ormoulds. Operations and symmetries.

Depending on the nature of the order on the indexing sequence, we have three types of objects :

total order	$\boldsymbol{\omega}$	:	*moulds*	$A^{\bullet}$
arborescent order	$\boldsymbol{\omega}^{\prec}$	:	*armoulds*	$A^{\bullet^{\prec}}$
arbitrary partial order	$\boldsymbol{\omega}^{\sharp}$	:	*ormoulds*	$A^{\bullet^{\sharp}}$

An *arborescent order* is one under which each ω_i has either one antecedent $\omega_{i\_}$ or none at all, in which case ω_i is declared a *root.* Under the arborification transform (§2.1.4), moulds induce armoulds. Similarly, moulds (as well as armoulds) induce ormoulds. It turns out that all basic mould *operations* (unary or binary) and *symmetries* uniquely and unambiguously extend to both *armoulds* and *ormoulds*: that extension is natural not only in the sense of commuting with the induction *stronger order* $\rightarrow$ *weaker order* , but also of preserving all the main properties of the operations or symmetries in question (such as, say, the associativity of $\times$ or $\circ$, or the simple rule for calculating the *multiplicative inverse* of a symmetral/el mould, namely *multiplication* by $(-1)^r$ and *order reversal* without/with possible *contraction* of order-adjacent indices ω_i) even when applied to *'primitive'* armoulds or ormoulds, ie to ar/ormoulds that are *not* induced by any mould. Although most ar/ormoulds encountered in practice are of the induced sort, and therefore carry no more information than the underlying moulds, the change of perspective nonetheless often proves very convenient.

We shall now proceed to list a number of rather elementary yet very useful moulds and armoulds which are going to occur and recur throughout this paper, and to mention their main properties. These properties are of a combinatorial or algebraic nature but, as usual in our 'formalisation approach', they 'absorb' and resolve most of the 'analysis' or 'divergence' difficulties which stand in the way – here, in the way of Object Synthesis.

Let us first settle some nomenclature:
– *constant-type* moulds depend only on the length r of the sequence $\boldsymbol{\omega}$. [11]
– *sum-type (* resp *difference-type)* moulds depend on partial sums (resp differences) of their indices ω_i
– *flat* (resp *polar*) moulds are expressible as simple superpositions of the *sign function* (resp the *polar function* $t \mapsto 1/t$).

2.1.6 Constant-type moulds.

mould	value	symmetry type	associated series
$\mathbf{1}^\bullet$	$1 \;\; if \; r=0 \; (0 \; otherwise)$	*symmetral*	1
$\mathrm{I}^\bullet$	$1 \;\; if \; r=1 \; (0 \; otherwise)$	*alternal*	x
$\log^\bullet$	$\frac{(-1)^{r-1}}{r}$	*alternel*	$\log(1+x)$
$\exp_a^\bullet$	$\frac{a^r}{r!}$	*symmetral*	$e^{a\,x}$
$\mathrm{tu}_a^\bullet$	$\frac{(-1)^r}{r!}\frac{\Gamma(r-a)}{\Gamma(-a)}$	*symmetrel*	$(1+x)^a$

2.1.7 Difference-type flat moulds.

$$
\begin{aligned}
\mathrm{sad}^{\emptyset} &:= 1 \\
\mathrm{sad}^{t_1,\ldots,t_r} &:= 1 \;\; if \;\; t_1 < t_2 < \cdots < t_r \\
\mathrm{sad}^{t_1,\ldots,t_r} &:= 0 \;\; otherwise
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{lad}^{\emptyset} &:= 0 \\
\mathrm{lad}^{t_1,\ldots,t_r} &:= (-1)^q \, \frac{p!\,q!}{(p+q+1)!} \; = \; (-1)^q \, \frac{p!\,q!}{r!} \\
with \quad & p := \sum_{t_i<t_{i+1}} 1 \quad and \quad q := \sum_{t_i>t_{i+1}} 1
\end{aligned}
$$

[11] they correspond one-to-one to power series $M^\bullet \mapsto M^\emptyset + \sum_{1\geq r} M^{\overbrace{1,\ldots,1}^{r\ times}} x^r$ and this correspondence commutes with all three operations $+, \times, \circ$.

2.1.8 Difference-type polar moulds.

$$\begin{aligned}
\mathrm{tas}_{a,b}^{\emptyset} &:= 1 \\
\mathrm{tas}_{a,b}^{t_1} &:= \frac{a-b}{(a-t_1)(t_1-b)} \\
\mathrm{tas}_{a,b}^{t_1,\ldots,t_r} &:= \frac{a-b}{(a-t_1)(t_1-t_2)\ldots(t_{r-1}-t_r)(t_r-b)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{tas}_{\star}^{\emptyset} &:= 0 \\
\mathrm{tas}_{\star}^{t_1} &:= \frac{1}{(-t_1)(t_1)} \\
\mathrm{tas}_{\star}^{t_1,\ldots,t_r} &:= \frac{1}{(-t_1)(t_1-t_2)\ldots(t_{r-1}-t_r)(t_r)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{tas}_{\star\star}^{\emptyset} &:= 0 \\
\mathrm{tas}_{\star\star}^{t_1} &:= 1 \\
\mathrm{tas}_{\star\star}^{t_1,\ldots,t_r} &:= \frac{1}{(t_1-t_2)\ldots(t_{r-1}-t_r)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{tas}_{a,b}^{\bullet} \times \mathrm{tas}_{b,c}^{\bullet} &= \mathrm{tas}_{a,c}^{\bullet} \\
\mathrm{tas}_{a,b}^{\bullet} \times \mathrm{tas}_{b,a}^{\bullet} &= \mathbf{1}^{\bullet}
\end{aligned}$$

2.1.9 Sum-type flat moulds.

Some abbreviations first :

$$\boldsymbol{x} := (x_1,\ldots,x_r) \tag{21}$$
$$\check{x}_i := x_1+\cdots+x_i \tag{22}$$
$$\hat{x}_i := x_i+\cdots+x_r \tag{23}$$
$$\|\boldsymbol{x}\| := x_1+\cdots+x_r = \hat{x}_1 = \check{x}_r \tag{24}$$
$$\sigma_+(x) := 1 \ \ if \ \ x>0 \ \ (resp \ \ :=0 \ \ if \ \ x<0) \tag{25}$$
$$\sigma_-(x) := 1 \ \ if \ \ x<0 \ \ (resp \ \ :=0 \ \ if \ \ x>0) \tag{26}$$
$$\delta(x) := 1 \ \ if \ \ x=0 \ \ (resp \ \ :=0 \ \ if \ \ x\neq 0) \tag{27}$$

$$
\begin{aligned}
\mathrm{sofo}^{\boldsymbol{x}}_{\pm} &:= (-1)^r\ \sigma_{\pm}(\check{x}_1)\ldots\sigma_{\pm}(\check{x}_r)\\
\mathrm{antisofo}^{\boldsymbol{x}}_{a} &:= (-1)^r\ \sigma_{\pm}(\hat{x}_1)\ldots\sigma_{\pm}(\hat{x}_r)\\
\mathrm{sefo}^{\boldsymbol{x}}_{\pm} &:= (-1)^{r-1}\sigma_{\pm}(\check{x}_1)\ldots\sigma_{\pm}(\check{x}_{r-1})\ \sigma_{\mp}(\check{x}_r)\\
\mathrm{antisefo}^{\boldsymbol{x}}_{\pm} &:= (-1)^{r-1}\sigma_{\mp}(\hat{x}_1)\ \sigma_{\pm}(\hat{x}_{r-1})\ldots\sigma_{\pm}(\hat{x}_r)\\
\mathrm{lefo}^{\boldsymbol{x}}_{\pm} &:= (-1)^r\sigma_{\pm}(\check{x}_1)\ldots\sigma_{\pm}(\check{x}_{r-1})\ \delta(\check{x}_r)\\
\mathrm{antilefo}^{\boldsymbol{x}}_{\pm} &:= (-1)^r\delta(\hat{x}_1)\ \sigma_{\pm}(\hat{x}_{r-1})\ldots\sigma_{\pm}(\hat{x}_r)
\end{aligned}
$$

2.1.10 Sum-type polar moulds.

$$
\begin{aligned}
\mathrm{sa}^{\boldsymbol{\omega}}_a &:= \textstyle\prod_{i=1}^{i=r}\frac{\omega_i}{\check{\omega}_i} & \mathrm{musa}^{\boldsymbol{\omega}}_a &:= (-1)^r\textstyle\prod_{i=1}^{i=r}\frac{\omega_i}{\hat{\omega}_i}\\
\mathrm{romo}^{\boldsymbol{\omega}}_a &:= \textstyle\prod_{i=1}^{i=r}(a\,\frac{\omega_i}{\check{\omega}_i}-1) & \mathrm{antiromo}^{\boldsymbol{\omega}}_a &:= \textstyle\prod_{i=1}^{i=r}(a\,\frac{\omega_i}{\hat{\omega}_i}-1)\\
\mathrm{remo}^{\boldsymbol{\omega}}_a &:= a\,\frac{\omega_r}{\check{\omega}_r}\textstyle\prod_{i=1}^{i=r-1}(a\,\frac{\omega_i}{\check{\omega}_i}-1) & \mathrm{antiremo}^{\boldsymbol{\omega}}_a &:= a\,\frac{\omega_1}{\hat{\omega}_1}\textstyle\prod_{i=2}^{i=r}(a\,\frac{\omega_i}{\hat{\omega}_i}-1)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{somo}^{\bullet}_{a,b} &:= \mathrm{remo}^{\bullet}_a \times \mathrm{antiromo}^{\bullet}_{1-b} && (28)\\
&:= \mathrm{romo}^{\bullet}_a \times \mathrm{antiremo}^{\bullet}_{1-b} && (29)\\
&= \mathrm{romo}^{\bullet}_{a/b} \times \mathrm{remo}^{\bullet}_b && (30)\\
\mathrm{somo}^{\bullet}_{\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]} &:= \mathrm{somo}^{\bullet}_{\frac{c-b}{d-b},\,\frac{a-b}{d-b}} && (31)\\
\mathrm{somo}^{\bullet}_{\left[\begin{smallmatrix} b & 0\\ a & 1\end{smallmatrix}\right]} &:= \mathrm{somo}^{\bullet}_{a,b} && (32)
\end{aligned}
$$

2.1.11 Some mould properties for future use.

Symmetry types.[12]

All the above moulds fall into one or the other of the main symmetry types.

Alternal: $\mathrm{lad}^{\bullet}$, $\mathrm{tas}^{\bullet}_{\star}$, $\mathrm{tas}^{\bullet}_{\star\star}$

Symmetral: $\mathrm{exp}^{\bullet}_a$, $\mathrm{sad}^{\bullet}$, $\mathrm{tas}^{\bullet}_{a,b}$, $\mathrm{sa}^{\bullet}$, $\mathrm{musa}^{\bullet}$

Alternel: $\mathrm{log}^{\bullet}$, $\mathrm{lefo}^{\bullet}_{\pm}$, $\mathrm{redo}^{\bullet}_{\pm}$, $\mathrm{redom}^{\bullet}$

Symmetrel: $\mathrm{tu}^{\bullet}_a$, $\mathrm{sofo}^{\bullet}_{\pm}$, $\mathrm{sefo}^{\bullet}_{\pm}$, $\mathrm{romo}^{\bullet}_a$, $\mathrm{remo}^{\bullet}_a$, $\mathrm{somo}^{\bullet}_{a,b}$

[12] *flat* moulds should be regarded as distribution-valued: for them the *symmetries* hold *almost everywhere*, not necessarily *everywhere*.

Pairs of the form $mould^\bullet, antimould^\bullet$ have the same symmetry type.

Useful identities and closure properties:

$$\mathrm{sofo}^\bullet_+ \times \mathrm{sefo}^\bullet_- = \mathbf{1}^\bullet \tag{33}$$

$$\mathrm{antisofo}^\bullet_+ \times \mathrm{antisefo}^\bullet_- = \mathbf{1}^\bullet \tag{34}$$

$$\mathrm{remo}^\bullet_a \times \mathrm{antiromo}^\bullet_{1-a} = \mathbf{1}^\bullet \tag{35}$$

$$\mathrm{romo}^\bullet_a \times \mathrm{antiremo}^\bullet_{1-a} = \mathbf{1}^\bullet \tag{36}$$

$$\begin{array}{llcll}
\textit{multplicative inverse}: & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]} & \leftrightarrow & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} c & b \\ a & d\end{smallmatrix}\right]} & (\ a\ ,\ c\ \textit{exchanged}\ ) \\
\textit{composition inverse}: & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]} & \leftrightarrow & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} b & a \\ d & c\end{smallmatrix}\right]} & (\ \textit{columns exchanged}\ ) \\
\textit{sequence reversal}: & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]} & \overset{\textbf{anti}}{\leftrightarrow} & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} c & d \\ a & b\end{smallmatrix}\right]} & (\ \textit{rows exchanged}\ )
\end{array}$$

$$\begin{array}{llcl}
\textit{multplication}: & \mathrm{somo}^\bullet_{a_1,a_2} \times \mathrm{somo}^\bullet_{a_2,a_3} & = & \mathrm{somo}^\bullet_{a_1,a_3} \\
\textit{composition}: & \mathrm{somo}^\bullet_{a_1,b_1} \circ \mathrm{somo}^\bullet_{a_2,b_2} & = & \mathrm{somo}^\bullet_{(a_2-b_2)a_1+b_2\,,\,(a_2-b_2)b_1+b_2} \\
\textit{multplication}: & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_0 & b_1 \\ a_2 & b_2\end{smallmatrix}\right]} \times \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_1 & b_1 \\ a_0 & b_2\end{smallmatrix}\right]} & = & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2\end{smallmatrix}\right]} \\
\textit{composition}: & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2\end{smallmatrix}\right]} \circ \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} b_1 & c_1 \\ b_2 & c_2\end{smallmatrix}\right]} & = & \mathrm{somo}^\bullet_{\left[\begin{smallmatrix} a_1 & c_1 \\ a_2 & c_2\end{smallmatrix}\right]}
\end{array}$$

Smooth or size-preserving arborification.

All the above moulds possess the property of smooth arborification (meaning that their arborified variants admit essentially the same type of bounds) the only exception being the moulds $log^\bullet$ and $tu^\bullet_a$ for $a \notin \mathbb{Z}$ and in particular for $a = 1/2$. This is in relation with the fact that the *standard* alien derivations (which admit $log^\bullet$ as their left-lateral mould) and the *standard* or *median* convolution average (which admits $tu^\bullet_{1/2}$ as its right- *and* left-lateral mould) are not *well-behaved.*

Of course, for alternal or symmetral (resp alternel or symmetrel) moulds, one should take the ordinary (resp contracting) form of arborification.

Form-preserving arborification.

All the sum-type moulds listed above, ie all those moulds whose definition involves forward sums $\hat{x}_i$ or $\hat{\omega}_i$ (resp backward sums $\check{x}_i$ or $\breve{\omega}_i$) have the stronger and very useful property of form-preserving arborification. This means that they retain their *outward analytical expression*, except that the sums $\hat{x}_i$ or $\hat{\omega}_i$ (resp $\check{x}_i$ or $\breve{\omega}_i$) are now relative to the arborescent (resp antiarborescent) order. The same holds for the difference-type moulds $tas^\bullet_{a,\infty}$ and $tas^\bullet_{\infty,b}$.

Thus, it is an easy matter to check that for any arborescent sequence $\boldsymbol{\omega}^{\prec}$ (resp antiarborescent sequence $\boldsymbol{\omega}^{\succ}$) we still have :

$$\mathrm{sa}_a^{\boldsymbol{\omega}^{\succ}} := \prod_{i=1}^{i=r} \frac{\omega_i}{\breve{\omega}_i} \qquad \mathrm{musa}_a^{\boldsymbol{\omega}^{\prec}} := (-1)^r \prod_{i=1}^{i=r} \frac{\omega_i}{\hat{\omega}_i}$$

exactly as in §2.1.10, except that $\hat{\omega}_i$ (resp $\breve{\omega}_i$) now denotes the sum of all indices ω_j that follow (resp precede) ω_i *inside* $\boldsymbol{\omega}^{\prec}$ (resp $\boldsymbol{\omega}^{\succ}$). Of course, as in the case of totally ordered sequences, ω_i itself should be included in that sum.

2.1.12 From alternel/symmetrel to alternal/symmetral.

Let us also mention two mould transforms which turn alternel (resp symmetrel) moulds $A^\bullet$ into alternal (resp symmetral) moulds $B^\bullet$. The first transform is quite elementary and applies to all cases. The second transform is more subtle, but also more relevant to the present investigation. It applies only to moulds $A^\bullet$ with indices n_i in $\mathbb{N}$ and turns them into 'flat' or 'piecewise-constant' moulds $B^\bullet$ with indices t_i in $\mathbb{R}$. Both transforms respect multiplication in the sense that $transf(A_1^\bullet \times A_2^\bullet) \equiv transf(A_1^\bullet) \times transf(A_2^\bullet)$. Here is how they are defined :

First mould transform:

$$\begin{aligned} \textit{direct} \ : \qquad & A^\bullet \ \mapsto \ B^\bullet := A^\bullet \circ \exp_1^\bullet && (37) \\ \textit{inverse} \ : \qquad & B^\bullet \ \mapsto \ A^\bullet := B^\bullet \circ \log^\bullet && (38) \end{aligned}$$

Second mould transform:

$$
\begin{array}{rcl}
A^\bullet \leftrightarrow B^\bullet & \textit{with} & B^{\,t_1,\dots,t_r} := SA^{\,\epsilon_1,\dots,\epsilon_{r-1},+} \quad \textit{and} \\
\epsilon_1 := \mathrm{sign}(t_1-t_2)\ , & \dots & ,\ \epsilon_{r-1} := \mathrm{sign}(t_{r-1}-t_r) \\
SA^{+} & := & -A^{\,1} \\
SA^{+,+} & := & +A^{\,1,1} \\
SA^{-,+} & := & +A^{\,1,1}+A^{\,2} \\
SA^{+,+,+} & := & -A^{\,1,1,1} \\
SA^{+,-,+} & := & -A^{\,1,1,1}-A^{\,1,2} \\
SA^{-,+,+} & := & -A^{\,1,1,1}-A^{\,2,1} \\
SA^{-,-,+} & := & -A^{\,1,1,1}-A^{\,1,2}-A^{\,2,1}-A^{\,3} \\
etc\dots & & \\
Or\ \ generally & : & \\
direct\ : \qquad SA^{\boldsymbol{\epsilon^1},\dots,\boldsymbol{\epsilon^s}} & := & (-1)^r \sum\nolimits^{\star} \ A^{\mathbf{n^1},\dots,\mathbf{n^s}} \\
inverse\ : \qquad A^{\,r_1,\dots,\mathbf{r_s}} & := & (-1)^s \sum\nolimits^{\star\star} \ \epsilon_1\dots\epsilon_r \ SA^{\,\epsilon_1,\dots,\epsilon_r}
\end{array}
$$

In the last but one identity, all sign subsequences $\boldsymbol{\epsilon^i}$ consist of (r_i-1) initial $-$ signs and one final $+$ sign (r_i may be 1) and $\sum^\star$ extends to all integer sequences $\mathbf{n^i}$ of sum r_i , whereas in the last (reverse) identity the sum $\sum^{\star\star}$ extends to all $\epsilon_j \in \{+,-\}$ except when $j \in \{r_1, r_1+r_2, \dots, r_1+\dots+r_s\}$, in which case ϵ_j has to be $+$.

2.2 Resurgent functions.

Roughly, the algebra of *real* resurgent fonctions [13] consists of all (analytic or cohesive) function germs at $+0$ that possess an *endless* (analytic or cohesive, [14] but usually ramified) *forward* continuation *over* the whole of $\mathbb{R}^+$.

Similarly, the general algebra of *complex* resurgent fonctions consists of all analytic function germs at $0_\bullet$ that possess an *endless* (analytic but usually

[13]This algebra underlies for ex. the construction of the trigebra of analysable germs.

[14]in the present applications, all our resurgent functions are going to be analytic over $\mathbb{R}^+$, with at most a discrete set of singularities there, but the most general notion of resurgent function (ie the broadest setting in which that hall-mark of resurgence, *alien derivations*, may still be defined) involves the more comprehensive notion of *cohesive functions*. Cohesiveness is a *regular* and *stable* form of *quasi-analyticity*. Like analyticity, it implies infinite smoothness and unique continuation, but the notion of cohesive singularity slightly differs from that of analytic singularity. Cohesiveness seldom occurs in problems involving only pure power series, but becomes rather generic when we move on to truly general *transseries* and *analysable functions*, as in the so-called Dulac problem. Detailed expositions are available in [E6],[E7],[E10] but here we can forget about cohesiveness.

ramified) continuation *over* the whole of $\mathbb{C}_\bullet$. Here, $\mathbb{C}_\bullet$ denotes the Riemann surface of the logarithm function with its 'ramified' origin $0_\bullet$ and a 'privileged direction' $\arg\zeta = 0$. Points *on* or *over* $\mathbb{C}_\bullet$ will be denoted by ζ. *Endless continuability* for a (ramified) analytic germ $\hat{\varphi}$ at $0_\bullet$ means *lateral analytic continuability* along any broken line Γ on $\mathbb{C}_\bullet$ starting from $0_\bullet$ under right *and* left circumvention of a discrete point set $Sing_{\hat{\varphi},\Gamma} \subset \Gamma$.

Resurgent functions are subject to the convolution product, and there act upon them an incredibly rich array of alien derivations, which measure their singular behaviour away from $0_\bullet$. The label *resurgent* owes its origin to the fact that, at least in natural situations, alien derivatives tend to 'resemble', or be simply related to, the original function which thus 'resurfaces' at its singular points.

Local aspect: minors/majors :

More precisely, *locally* at $0_\bullet$, resurgent functions may be thought of as microfunctions: they are pairs $\overset{\diamond}{\varphi}(\zeta) = \{\check{\varphi}(\zeta), \hat{\varphi}(\zeta)\}$ consisting of a major $\check{\varphi}$ defined upto a regular germ, and of a minor $\hat{\varphi}$, which is the "variation" of the major. Depending on the situation, one may find it easier to work with real-majors and natural-majors

$$\hat{\varphi}(\zeta) = -\frac{1}{2\pi i}\left(\check{\varphi}_{\mathrm{real}}(e^{\pi i}\zeta) - \check{\varphi}_{\mathrm{real}}(e^{-\pi i}\zeta)\right) \qquad (\zeta \text{ close to } 0_\bullet) \quad (39)$$

$$\hat{\varphi}(\zeta) = \check{\varphi}_{\mathrm{nat}}(\zeta) - \check{\varphi}_{\mathrm{nat}}(e^{-2\pi i}\zeta) \qquad (\zeta \text{ close to } 0_\bullet) \quad (40)$$

$$\check{\varphi}_{\mathrm{real}}(\zeta) \equiv 2\pi i\,\check{\varphi}_{\mathrm{nat}}(e^{-\pi i}\zeta) \qquad (\zeta \text{ close to } 0_\bullet) \quad (41)$$

Real-majors are particularly recommended when dealing with acceleration theory and, of course, with real resurgent functions : they make it possible to associate real majors $\check{\varphi}$ to any real $\overset{\diamond}{\varphi}$. Unless otherwise specified, all formulae below use real-majors.

The minor is exactly defined, but does not encapsulate the whole information about $\overset{\diamond}{\varphi}$, except in the important case of "integrable resurgent functions". For these the convolution product is defined by

$$(\hat{\varphi}_1 \star \hat{\varphi}_2)(\zeta) := \int_0^\zeta \hat{\varphi}_1(\zeta_1)\,\hat{\varphi}_2(\zeta - \zeta_1)\,d\zeta_1 \qquad (42)$$

for ζ close to $+0$ and by analytic or cohesive continuation in the large. The convolution for general germs $\overset{\diamond}{\varphi}$ is defined by :

$$\overset{\diamond}{\varphi}_3 = \overset{\diamond}{\varphi}_1 \star \overset{\diamond}{\varphi}_2 \quad \Longleftrightarrow \tag{43}$$

$$\check{\varphi}_{3,u}(\zeta) = \frac{1}{2\pi i}\int_{I_{\zeta,u}} \check{\varphi}_1(\zeta_1)\, \check{\varphi}_2(\zeta-\zeta_1)\, d\zeta_1 \qquad (0<\zeta<u\ll 1) \tag{44}$$

with integration on the interval

$$I(\zeta,u) := [\frac{1}{2}\zeta + e^{-\frac{\pi i}{2}}\,u, \frac{1}{2}\zeta + e^{+\frac{\pi i}{2}}\,u] \tag{45}$$

The above definition makes sense since, upto addition of a regular germ, the major $\check{\varphi}_{3,u}$ doesn't depend on the choice of u. Moreover, for integrable functions $\overset{\diamond}{\varphi}$, the two definitions (42) and (44) coincide.

Global aspect: alien derivations :

What really matters, however, is not the local or microfunction aspect, but the *global* properties of resurgent functions $\overset{\diamond}{\varphi}$, which come from their having endlessly continuable, but usually highly ramified minors $\hat{\varphi}$. This a source of many fascinating developments, chief amongst which is the existence of a system $\{\Delta_\omega\,,\ \omega \in \mathbb{R}^+\}$ of so-called alien derivations[15], which measure the singularities of the minor $\hat{\varphi}$ *over* the points ω and which, together, freely generate an infinite dimensional Lie algebra with a non-countable basis – despite their acting on functions of one single variable !

The three models: formal, geometric, convolutive :

Actually, resurgent functions exist simultaneously in three different models :
– the *formal model*, with decreasing power series $\tilde{\varphi}(z)$ (or suitable generalisations, like the so-called transseries) as elements, and with formal multiplication as product
– the *convolutive model*, defined a moment ago, with endlessly continuable microfunctions $\overset{\diamond}{\varphi}(\zeta) = \{\hat{\varphi}(\zeta), \check{\varphi}(\zeta)\}$ as elements, and with local-global convolution as product
– the *geometric model*, with sectorial analytic germs $\varphi(z)$ as elements and with point-wise multiplication as product.

Roughly speaking, the way things work out in practice is like this : when formally solving singular equations or systems, we tend to get divergent solutions $\tilde{\varphi}(z)$ of resurgent type, which do not converge directly to $\varphi(z)$, but have to be summed in a round-about way, via the convolutive model $\overset{\diamond}{\varphi}(\zeta)$. The

[15] because they are indeed derivations, relative to the convolution product.

detour, however, is rewarding, since it yields a treasure-trove of information about the equation, in particular its analytic invariants.

$$
\begin{array}{lcccl}
\text{geometric model} & \varphi(z) & \text{Laplace} & & \\
 & \uparrow & \searrow & & \\
 & \vdots & & \overset{\diamond}{\varphi}(\zeta) & \text{convolutive model} \\
 & \uparrow & \nearrow & & \\
\text{formal model} & \tilde{\varphi}(z) & \text{Borel} & &
\end{array}
$$

Standard Borel transform: $\quad \varphi(z) \to \overset{\diamond}{\varphi}(\zeta) = \big(\hat{\varphi}(\zeta), \check{\varphi}(\zeta)\big)$

$$\hat{\varphi}(\zeta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(z\zeta)\,\varphi(z)\,dz \qquad (1 \ll c\ ;\ \arg\zeta = 0) \tag{46}$$

$$\check{\varphi}(\zeta) = \int_{+u}^{+\infty} \exp(-z\zeta)\,\varphi(z)\,dz \qquad (1 \ll u\ ;\ |\arg\zeta| \leq \pi) \tag{47}$$

Standard Laplace transform: $\quad \overset{\diamond}{\varphi}(\zeta) = \big(\hat{\varphi}(\zeta), \check{\varphi}(\zeta)\big) \to \ \varphi(z)$

$$\varphi(z) = \int_{+0}^{+\infty} \exp(-z\zeta)\,\hat{\varphi}(\zeta)\,d\zeta \qquad (\textit{for } \overset{\diamond}{\varphi} \textit{ integrable at } 0_\bullet) \tag{48}$$

$$\varphi(z) = \frac{1}{2\pi i}\int_{e^{-\pi i}\infty}^{e^{\pi i}\infty} \exp(z\zeta)\,\check{\varphi}(\zeta)\,d\zeta \qquad (\textit{for any } \overset{\diamond}{\varphi}\ ;\ \arg z = 0) \tag{49}$$

Elementary (standard) Borel/Laplace transforms :

$$
\begin{aligned}
\varphi(z) &= z^{-\sigma} \qquad\qquad (\textit{for } \sigma \in \mathbb{C} - \mathbb{N}^\star) \\
\hat{\varphi}(\zeta) &= \zeta^{\sigma-1}/\Gamma(\sigma) \\
\check{\varphi}(\zeta) &= \zeta^{\sigma-1}\Gamma(1-\sigma)
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
\varphi(z) &= z^{-n} \qquad\qquad (\textit{for } n \in \mathbb{N}^\star) \\
\hat{\varphi}(\zeta) &= \zeta^{n-1}/\Gamma(n) \\
\check{\varphi}(\zeta) &= (-1)^n\,\zeta^{n-1}\log\zeta\,/\Gamma(n)
\end{aligned} \tag{51}
$$

2.3 Alien derivations or automorphisms. Their weights.

Alongside the *natural derivation* $\partial := \partial_z$ in the *multiplicative* (ie formal or geometric) models and its counterpart $\hat{\partial} : \hat{\varphi}(\zeta) \mapsto -\,\zeta\,\hat{\varphi}(\zeta)$ in the *convolutive* model, resurgent functions are acted upon by so-called *alien derivations* Δ_ω. These are primarily defined in the convolutive model:

$$\Delta_\omega \quad : \quad \overset{\diamond}{\varphi} := (\hat{\varphi}, \check{\varphi}) \quad \mapsto \quad \overset{\diamond}{\varphi}_\omega = (\hat{\varphi}_\omega, \check{\varphi}_\omega) \tag{52}$$

where they measure the singularities of minors *over*[16] a given singular point ω in $\mathbb{R}^+$ or $\mathbb{C}_\bullet$ [17] but in such a way as to verify the Leibniz rule:

$$\Delta_\omega (\overset{\diamond}{\varphi}_1 \star \overset{\diamond}{\varphi}_2) \equiv \Delta_\omega (\overset{\diamond}{\varphi}_1) \star \overset{\diamond}{\varphi}_2 + \overset{\diamond}{\varphi}_1 \star \Delta_\omega (\overset{\diamond}{\varphi}_2) \tag{53}$$

As a microfunction, $\overset{\diamond}{\varphi}_\omega$ is defined by:

$$\hat{\varphi}_\omega (\zeta) \ := \ + \sum_{\epsilon_i = \pm} \mathbf{d}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_r \\ \omega_1 & ,\dots, & \omega_r \end{smallmatrix}\right)} \ \hat{\varphi}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_r \\ \omega_1 & ,\dots, & \omega_r \end{smallmatrix}\right)}(\omega + \zeta) \tag{54}$$

$$\check{\varphi}_\omega (\zeta) \ := \ - \sum_{\epsilon_i = \pm} \mathbf{d}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_{r-1}, & * \\ \omega_1 & ,\dots, & \omega_{r-1}, & \omega_r \end{smallmatrix}\right)} \ \hat{\varphi}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_{r-1} \\ \omega_1 & ,\dots, & \omega_{r-1} \end{smallmatrix}\right)}(\omega - \zeta) \tag{55}$$

$$\mathbf{d}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_{r-1}, & * \\ \omega_1 & ,\dots, & \omega_{r-1}, & \omega_r \end{smallmatrix}\right)} \ := \ +2\pi i \, \mathbf{d}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_{r-1}, & + \\ \omega_1 & ,\dots, & \omega_{r-1}, & \omega_r \end{smallmatrix}\right)} = -2\pi i \, \mathbf{d}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_{r-1}, & - \\ \omega_1 & ,\dots, & \omega_{r-1}, & \omega_r \end{smallmatrix}\right)}$$

for $\arg\zeta = \arg\omega_i = \arg\omega$ and ζ small[18]. Here, of course, $\hat{\varphi}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_r \\ \omega_1 & ,\dots, & \omega_r \end{smallmatrix}\right)}(\omega + \zeta)$ denotes the determination of $\hat{\varphi}(\zeta)$ on the open interval $]\omega_r, \omega_{r+1}[$ that answers to the right (resp left) circumvention of $\hat{\omega}_i$ [19] to the right (resp left) if $\epsilon_i = +$ (resp $-$). Due to (56), $\overset{\diamond}{\varphi}_\omega = 0$ if $\hat{\varphi}$ has no singularities over ω. Moreover, since the right-hand side of (54) or (55) should not change when additional, ficticious singular points $\hat{\omega}_i$ are added, the *weights* $\mathbf{d}^\bullet$ are subject to obvious *self-consistency* relations (164): see §6.1.

The simplest choice for "singularity measuring" operators would be the right-lateral operators Δ^+_ω, with weights:

$$\begin{aligned} \mathbf{d}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_r \\ \omega_1 & ,\dots, & \omega_r \end{smallmatrix}\right)} \ &:= \ \epsilon_r \, 1 \qquad \textit{if} \ \ (\epsilon_1, \dots, \epsilon_{r-1}) = (+, \dots, +) \\ &:= \ 0 \qquad \textit{otherwise} \end{aligned}$$

and the left-lateral operators Δ^-_ω, with weights:

$$\begin{aligned} \mathbf{d}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_r \\ \omega_1 & ,\dots, & \omega_r \end{smallmatrix}\right)} \ &:= \ -\epsilon_r \, 1 \qquad \textit{if} \ \ (\epsilon_1, \dots, \epsilon_{r-1}) = (-, \dots, -) \\ &:= \ 0 \qquad \textit{otherwise} \end{aligned}$$

[16] ie *at* various points lying over ω on various Riemann leaves.

[17] depending on whether we are dealing with *real* or *general* resurgent functions.

[18] analytic continuation extends these definitions for large ζ.

[19] $\hat{\omega}_i$ being of the i-th singular point $\hat{\omega}_i := \sum_1^i \omega_k$ of $\hat{\varphi}$ lying on the interval $]0_\bullet, \omega[$.

But the "atomicity" of these operators (ie the fact that they involve one path only) isn't much of an asset, because they are not *first-order* derivations. Instead of verifying the Leibniz rule (53), they verify messier relations, which for simplicity we write down in the multiplicative models:

$$\Delta^{\pm}_{\omega}(\varphi_1\,\varphi_2) \equiv (\Delta^{\pm}_{\omega}\varphi_1)\,\varphi_2 + \varphi_1\,(\Delta^{\pm}_{\omega}\varphi_2) + \sum_{\substack{\omega_1+\omega_2=\omega \\ \arg\omega_i=\arg\omega}} (\Delta^{\pm}_{\omega_1}\varphi_1)\,(\Delta^{\pm}_{\omega_2}\varphi_2) \tag{56}$$

with a sum extending to all ω_1, ω_2 co-axial with ω. [20]

The so-called *standard* alien derivations Δ_ω *do* verify (53), but they involve all the 2^r determinations which forward analytic continuation may create over $[\omega_r, \omega_{r+1}]$. Still, their weights are very simple, as they do not depend on the gaps ω_i, but only on the number p, q of $+, -$ signs in the sequence $\{\epsilon_1, \ldots, \epsilon_{r-1}\}$:

$$\mathbf{d}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_r}} := \frac{\epsilon_r}{2\pi i}\;\frac{p!\,q!}{(p+q+1)!} \qquad \begin{array}{l} p := \#\{1 \le i < r\,;\ \epsilon_i = +\} \\ q := \#\{1 \le i < r\,;\ \epsilon_i = -\} \end{array} \tag{57}$$

Together, these *standard*[21] derivations Δ_ω freely generate the algebra ALIEN of alien derivations but, as we shall see later on, their simplicity appeal is deceptive, and in many applications, preference should be given to another system, the so-called *organic* alien derivations.

There being no risk of confusion, the same symbols Δ_ω serve to denote the alien derivations in all three models – *convolutive* (where they are defined and where only they can be directly calculated), *formal*, and *geometric*.

In the two later models (collectively referred to as *multiplicative* models) it is also useful to consider the exponential-carrying variants $\blacktriangle_\omega := e^{-\omega z}\Delta_\omega$ which have the advantage of commuting with the *natural* derivation $\partial := \partial_z$ and of enjoying the invariance property (9), but at the price of introducing an exponential factor external to the algebra of resurgent functions.

2.4 Resurgence monomials.

They are basically systems of resurgent fuctions that
(1) are as elementary as possible and, ideally, canonical

[20] For any given pair of test functions φ_1, φ_2 the above sum makes sense, since it can never involve more than a finite number of non-zero terms.

[21] usually, the standard alien derivations are defined without the factor $\frac{1}{2\pi i}$, so as to give them real weights. But in this investigation, more important than having real weights is the property for an operator of turning a *real* resurgent function into another *real* resurgent function. In view of the self-consistency relation (165), this makes the introduction of the factor $\frac{1}{2\pi i}$ necessary.

(2) are easy to multiply or convolute[22]
(3) behave simply under natural differentiation
(3′) behave simply under alien differentiation
(4) are numerous enough to generate the resurgent solutions of singular equations – be these of differential, difference, functional, etc, type
(4′) are numerous enough to permit the reconstitution of any resurgent function with a prescribed resurgence pattern, ie verifying a prescribed system of resurgence equations.

Such systems of monomials do exist, but there is a tension between the demands (3),(4) and (3′),(4′).

If we give precedence to (3),(4), we get the so-called ∂-friendly resurgence monomials $\mathcal{V}^{\bullet}(z)$, which are particularly useful in all problems of "Object Analysis" (object $\mapsto$ invariants)

If we give precedence to (3′),(4′), we get the so-called Δ-friendly resurgence monomials $\mathcal{U}^{\bullet}(z)$, which are particularly useful in all problems of "Object Synthesis" (invariants $\mapsto$ object)

The alien derivatives $\Delta_\omega \mathcal{V}^{\bullet}(z)$ of the ∂-friendly monomials involve simpler ∂-friendly monomials, but also transcendental constants $V^{\bullet}$, the so-called ∂-friendly monics, which measure as it were the Δ-unfriendliness of the $\mathcal{V}^{\bullet}(z)$.

Similarly, the natural derivatives $\partial_z \mathcal{U}^{\bullet}(z)$ of the Δ-friendly monomials involve simpler Δ-friendly monomials, but also transcendental constants $U^{\bullet}$, the so-called Δ-friendly monics, which so to speak measure the ∂-unfriendliness of the $\mathcal{U}^{\bullet}(z)$.

The ∂-friendly monics make it possible to express the analytic invariants $\mathbb{A}_\omega$ of a local object **Ob** in terms of its Taylor coefficients.

The Δ-friendly monics, or at any rate particularly well-chosen systems of such monics, make the reverse passage possible: expressing the Taylor coefficients of a (canonical) object $\mathbf{Ob}^{\mathrm{can}}$ in terms of a prescribed system of invariants $\mathbb{A}_\omega$.

The monics also serve as a bridge between the two types of monomials: see (65) below.

Lastly, just as there exist two useful variants of alien derivations, the exponential-free Δ_ω and the exponential-carrying $\boldsymbol{\Delta}_\omega$, we have to consider, alongside the exponential-free resurgence monomials, the exponential-carrying

[22] depending on the model we happen to be working in. Here, for notational convenience, we shall assume it to be the formal model

variants, but with the opposite sign conventions (for 'orthogonality') :

$$
\begin{aligned}
\mathbb{\Delta}_\omega &:= e^{-\omega\, z}\, \Delta_\omega && (58)\\
\mathcal{V}\!e^{\boldsymbol{\omega}}(z) &:= e^{+\|\boldsymbol{\omega}\|\, z}\, \mathcal{V}^{\boldsymbol{\omega}}(z) && (59)\\
\mathcal{U}\!e^{\boldsymbol{\omega}}(z) &:= e^{+\|\boldsymbol{\omega}\|\, z}\, \mathcal{U}^{\boldsymbol{\omega}}(z) && (60)
\end{aligned}
$$

To make the above statements sound a little less abstract, let us construct four very elementary systems of resurgence monomials and monics. They are often referred to as *hyperlogarithmic*, for such indeed is their dependence on the indices ω_i and (in the convolutive model) on the variable ζ. The construction involves four steps :

$$
\mathcal{V}\!e^{\bullet}(z) \longrightarrow V^{\bullet} \longrightarrow U^{\bullet} \longrightarrow \mathcal{U}\!e^{\bullet}(z) \qquad (61)
$$

and the definitions, relative to the *formal model*, go as follows :

$$
\begin{aligned}
\partial_z \tilde{\mathcal{V}}\!e^{\omega_1,\dots,\omega_r}(z) &:= -\,\tilde{\mathcal{V}}\!e^{\omega_1,\dots,\omega_{r-1}}(z)\; e^{\omega_r\, z}\, z^{-1} && (62)\\
\Delta_{\omega_0}\, \tilde{\mathcal{V}}\!e^{\omega_1,\dots,\omega_r}(z) &=: \sum_{\omega_1+\dots+\omega_i=\omega_0} V^{\,\omega_1,\dots,\omega_i}\; \tilde{\mathcal{V}}\!e^{\omega_{i+1},\dots,\omega_r}(z) && (63)\\
U^{\bullet} &:= \text{inverse of } V^{\bullet} \text{ under mould-composition } \circ && (64)\\
\tilde{\mathcal{U}}\!e^{\bullet}(z) &:= \tilde{\mathcal{V}}\!e^{\bullet}(z) \circ U^{\bullet} && (65)
\end{aligned}
$$

Clearly[23] :

$$
\begin{aligned}
\tilde{\mathcal{V}}\!e^{\omega_1,\dots,\omega_r}(z) &\in e^{(\omega_1+\dots+\omega_r)\, z}\; \mathbb{C}[[z^{-1}]] && (66)\\
\tilde{\mathcal{U}}\!e^{\omega_1,\dots,\omega_r}(z) &\in e^{(\omega_1+\dots+\omega_r)\, z}\; \mathbb{C}[[z^{-1}]] && (67)
\end{aligned}
$$

The above systems of ∂-friendly monomials and monics suffice for object analysis in the *monocritical* examples 1, 2, 3, 4 of the next section. In fact, larger systems, with the one-index factor $e^{\omega_r\, z}\, z^{-1}$ in (62) replaced by the two-index factor $e^{\omega_r\, z}\, z^{-1-\sigma_r}$, suffice for most *monocritical* examples. Lastly, by considering factors of type $e^{P_r(z)}\, z^{-1}$ (resp $e^{P_r(z)}\, z^{-1-\sigma_r}$), with a general polynomial $P_r(z)$ inside the exponential, one can construct systems large enough to deal with object analysis in our *polycritical* examples 5 and 6 (resp in most conceivable *polycritical* examples). In none of these instances, however, are the corresponding Δ-frienly systems adequate to the job of object synthesis : as we shall see, two extra ingredients – a *twist parameter* along with an *antipodal symmetry* – are called for to make things click.

[23] in theory the indices ω_i are in $\mathbb{C}_\bullet$ but here they can be assumed to be in $\mathbb{C}$ since the minors of our resurgent functions (in all six examples) are not ramified at $0_\bullet \in \mathbb{C}_\bullet$.

3 Object Analysis: six basic examples.

In the following six examples, we consider analytic local objects **Ob** (diffeos, vector fields, differential equations or systems) which we assume to be *formally* conjugate to a normal form $\mathbf{Ob}^{\mathrm{nor}}$. In each instance, we choose the simplest formal class that is apt to illustrate the particular point of analysis we wish to make.

3.1 Example 1: shift-like diffeomorphism.

$$f^{\mathrm{nor}} \quad : \quad z \longrightarrow z + 2\pi i \tag{68}$$

$$f \quad : \quad z \longrightarrow z + 2\pi i + \sum_{2 \leq n} a_n \, z^{-n} \tag{69}$$

Remark: we might of course have chosen the *unit shift* as our normal form, but choosing the $2\pi i$-shift has the advantage of placing the singularities over $\mathbb{Z}$ in the Borel-plane, and of rendering the parallel with Example 2 (*infra*) more obvious.

We may work with the formal normalising map $f^\star$ or its inverse ${}^\star f$:

$$f^\star \circ f \equiv f^{\mathrm{nor}} \circ f^\star \qquad ie \qquad f^\star(f(z)) \equiv 2\pi i + f^\star(z) \tag{70}$$

$$f \circ {}^\star f \equiv {}^\star f \circ f^{\mathrm{nor}} \qquad ie \qquad f({}^\star f(z)) \equiv {}^\star f(z + 2\pi i) \tag{71}$$

Both are *generically divergent but always resurgent. They verify the resurgence equations:*

$$\Delta_n \, f^\star(z) \;\; \equiv \;\; -A_n \, \exp(-n \, f^\star(z)) \qquad (\forall n \in \mathbb{Z}^\star) \tag{72}$$

$$\Delta_n \, {}^\star f(z) \;\; \equiv \;\; +A_n \, e^{-nz} \partial_z \, {}^\star f(z) \;\; =: \;\; \mathbb{A}_n \, {}^\star f(z) \qquad (\forall n \in \mathbb{Z}^\star) \tag{73}$$

which in turn yield the complete and 'free' [24] *system of analytic invariants:*

$$\mathbb{A} = \{ \mathbb{A}_n := A_n \, e^{-nz} \partial_z \; ; \quad n \in \mathbb{Z}^\star \;\; , \;\; A_n \in \mathbb{C} \} \tag{74}$$

For details see [E2],[E3].

[24] see the remark at the end of §3.

3.2 Example 2: Euler-like differential equation.

Example 2: singular, non-linear differential equation.[25]

$$d_z\, y^{\mathrm{nor}} \;=\; y^{\mathrm{nor}} \tag{75}$$

$$d_z\, y \;=\; y + \sum_{1+n\geq 0} b_n(z)\, y^{1+n} \qquad \in\; y + \mathbb{C}\{y, z^{-1}\} \tag{76}$$

We may work with the formal integral:

$$y(z,u) \in \mathbb{C}[[z^{-1},\, u\, e^z]] \qquad (u \;=\; \textit{integration parameter}) \tag{77}$$

which is *generically divergent* (in z) *but always resurgent* (again, in z) *and verifies the Bridge equation:*

$$\boldsymbol{\Delta}_n\, y(z,u) \equiv A_n\, u^{n+1}\, \partial_u\, y(z,u) =: \mathbb{A}_n\, y(z,u) \qquad (n = -1, 1, 2, 3, \dots) \tag{78}$$

yielding the complete and 'free' system of analytic invariants:

$$\mathbb{A} = \{\mathbb{A}_n := A_n\, u^{n+1}\, \partial_u \quad ; \quad n \in \{-1\} \cup \mathbb{N}^\star \;\;,\;\; A_n \in \mathbb{C}\} \tag{79}$$

For details, see [E3].

3.3 Example 3: monocritical linear differential system.

$$d_z\, y_i^{\mathrm{nor}} \;=\; \lambda_i\, y_i^{\mathrm{nor}} \qquad (1 \leq i \leq \nu\; ;\; \lambda_i \neq \lambda_j \;\; if \;\; i \neq j) \tag{80}$$

$$d_z\, y_i \;=\; \lambda_i\, y_i + \sum_{1\leq j\leq \nu} b_{i,j}(z)\, y_j \qquad b_{i,j}(z) \in \mathbb{C}\{z^{-1}\} \tag{81}$$

Here the formal integral reduces to

$$y(z,u) = \sum_{1\leq i\leq \nu} b_i(z)\, e^{\lambda_i\, z}\, u_i \qquad \textit{with} \quad b_i(z) \in \mathbb{C}[[z^{-1}]] \tag{82}$$

The Bridge equation reads:

$$\boldsymbol{\Delta}_{\lambda_i-\lambda_j}\, y(z,u) \;\equiv\; A_{\lambda_i-\lambda_j}\, u_i\, \partial_{u_j}\, y(z,u) =: \mathbb{A}_{\lambda_i-\lambda_j}\, y(z,u) \quad (i \neq j) \tag{83}$$

and once again yields a complete and free, but this time finite, system of analytic invariants:

$$\{\mathbb{A}_{\lambda_i-\lambda_j} := A_{\lambda_i-\lambda_j}\, u_i\, \partial_{u_j} \quad ; \quad 1 \leq i \neq j \leq \nu\} \tag{84}$$

For details, see [E3].

[25] under addition of a variable $x := z^{-1}$ it translates into a two-dimensional vector field, local (at $0 \in \mathbb{C}^2$), singular, and resonant (with one vanishing and one non-vanishing multiplier)

3.4 Example 4: monocritical non-linear differential system.

$$d_z\, y_i^{\mathrm{nor}} \;=\; \lambda_i\, y_i^{\mathrm{nor}} \qquad (1 \le i \le \nu\ ;\ \lambda\ \textit{ not res. nor quasi.res.}) \tag{85}$$

$$d_z\, y_i \;=\; \lambda_i\, y_i + \sum_{\substack{1+n_i \ge 0 \\ n_j \ge 0\ if\ j \ne i}} b_{i,n}(z)\, y_i\, y^n \;\in \lambda_i\, y_i + \mathbb{C}\{z^{-1}, y_1, \ldots, y_\nu\} \tag{86}$$

The formal integral involves ν integration parameters u_i, each with its accompanying exponential factor :

$$y(z,u) \in \mathbb{C}[[z^{-1},\, u_1\, e^{\lambda_1 z}, \ldots, u_\nu\, e^{\lambda_\nu z}]] \qquad (\text{with } \mathbb{Q}\text{-independent } \lambda_i\text{'s}) \tag{87}$$

The Bridge equation reads :

$$\mathbb{\Delta}_\omega\, y(z,u) \equiv \mathbb{A}_\omega\, y(z,u) \qquad (\forall \omega = \mathbf{\Omega}) \tag{88}$$

with indices ω running through a set :

$$\mathbf{\Omega} = \{\omega\ ;\ \omega = \sum_{1 \le i \le \nu} m_i\, \lambda_i\ ,\ m_i \ge -1\ ,\ \sum_{m_i = -1} 1 = 0 \textit{ or } 1\} \tag{89}$$

and with differential operators of the form :

$$\mathbb{A}_\omega := u_1^{m_1} \ldots u_\nu^{m_\nu} \sum_{1 \le i \le \nu} A_\omega^i\, u_i\, \partial_{u_i} \qquad if\ \ \omega = \sum m_i\, \lambda_i \qquad (A_\omega^i \in \mathbb{C}) \tag{90}$$

which, together, constitute a complete and free system $\{\mathbb{A}_\omega\ ;\ \omega = \mathbf{\Omega}\}$ of analytic invariants. For details, see [E3].

3.5 Example 5: polycritical linear differential system.

$$p_i^{-1}\, t^{1+p_i}\ \ d_t\, y_i^{\mathrm{nor}} \ + \lambda_i\, y_i^{\mathrm{nor}} \;=\; 0 \qquad (1 \le i \le \nu) \tag{91}$$

$$p_i^{-1}\, t^{1+p_i}\ \ d_t\, y_i \ + \lambda_i\, y_i \;=\; \sum_{1 \le j \le \nu} b_i(t)\, y_j \quad (b_i(t) \in \mathbb{C}\{t\}) \tag{92}$$

The Bridge equation reads :

$$\mathbb{\Delta}_\varpi\, y(z,u) \equiv \mathbb{A}_\varpi\, y(z,u) \qquad (\forall \omega = \mathbf{\Omega}) \tag{93}$$

Here the alien derivations carry indices $\varpi = \binom{\omega}{q}$, with a lower $q \in \{p_1, \ldots, p_s\}$ signalling the *critical variable* $z_q := t^{-q}$ respective to which the alien derivation operates, and with an upper ω of the form $\lambda_i - \lambda_j$ or λ_i or λ_j for $p_i = p_j = q$. The corresponding operators $\mathbb{A}_\varpi$ read:

$$
\begin{array}{lclll}
\mathbb{A}_\varpi & := & A_\varpi\, u_i\, \partial_{u_j} & if & \varpi=(\begin{smallmatrix}\lambda_i-\lambda_j\\ q\end{smallmatrix}) \\
\mathbb{A}_\varpi & := & u_i\, (\sum_{p_k<q} A^k_\varpi\, \partial_{u_k}) & if & \varpi=(\begin{smallmatrix}\lambda_i\\ q\end{smallmatrix}) \\
\mathbb{A}_\varpi & := & (\sum_{p_k<q} A^k_\varpi\, u_k\,)\partial_{u_j} & if & \varpi=(\begin{smallmatrix}-\lambda_j\\ q\end{smallmatrix})
\end{array}
$$

As usual, *the Bridge Equation yields a complete and free but (due to linearity) finite system of analytic invariants :*

$$
\{A_{\lambda_i-\lambda_j}\ ;\ p_i=p_j\}\ \cup\ \{A^k_{\lambda_i}\ ;\ p_i>p_k\}\ \cup\ \{A^{\,k}_{-\lambda_j}\ ;\ p_j>p_k\} \tag{94}
$$

For details, see [E3],[E4],[E7].

3.6 Example 6: polycritical non-linear differential system.

$$
\begin{array}{llcll}
p_i^{-1}\, t^{1+p_i}\ \ d_t\, y_i^{\mathrm{nor}} & +\lambda_i\, y_i^{\mathrm{nor}} & = & 0 & (1\le i\le \nu) \qquad (95) \\
p_i^{-1}\, t^{1+p_i}\ \ d_t\, y_i & +\lambda_i\, y_i & = & b_i(t,y_1,\ldots,y_r) & \in \mathbb{C}\{t,y\} \qquad (96)
\end{array}
$$

The Bridge equation reads :

$$
\blacktriangle_\varpi\, y(z,u) \equiv \mathbb{A}_\varpi\, y(z,u) \qquad\qquad (\forall \omega = \mathbf{\Omega}) \tag{97}
$$

Here the alien derivations carry indices $\varpi=(\begin{smallmatrix}\omega\\ q\end{smallmatrix})$, with ω running through a set :

$$
\mathbf{\Omega}_q = \{\omega\ ;\ \omega=\sum_{p_i=q} m_i\,\lambda_i\ ,\ m_i\ge -1\ ,\ \sum_{m_i=-1} 1 = 0 \textit{ or } 1\} \tag{98}
$$

and with differential operators of the form :

$$
\begin{array}{lcl}
\mathbb{A}_\varpi & := & u^{n(\varpi)}\, \{\sum_{p_j\ge q} A^j_\varpi(u)\, u_j\, \partial_{u_j} + \sum_{p_j<q} A^j_\varpi(u)\, \partial_{u_j}\} \quad \textit{with} \qquad (99) \\
u^{n(\varpi)} & := & \prod_{p_i=q} u_i^{m_i} \quad \textit{if} \quad \omega = \sum m_i\,\lambda_i \qquad (100) \\
A^j_\varpi(u) & \in & \mathbb{C}[[u_j; p_j<q]]) \qquad (101)
\end{array}
$$

which, together, constitute a complete and 'free' system $\{\mathbb{A}_\omega\ ;\ \omega=\mathbf{\Omega}\}$ of analytic invariants. For details, see [E3],[E4],[E7],[Braa] .

Remark : Of course, in our four non-linear examples, the countably infinite systems of invariants $\{\mathbb{A}_\omega\}$ are *"free"* only in the sense of being subject to no *finite constraints*, ie constraints involving finite subsets of them. But they are clearly subject to *'infinite' constraints* which, relative to a nice ("well-behaved", see *infra*) basis of ALIEN, reduce to the existence of *exponential bounds in* ω.

4 The reverse problem: Object Synthesis.

In this section, we examine what happens when we try to perform canonical Object Synthesis with the elementary, hyperlogarithmic resurgence monomials (constructed at the end of §2.4) as our sole tool: *in some very simple cases the method works; in most others it fails.* Yet these failures are instructive, for they help identify the two main difficulties and already suggest the correct remedy, namely the introduction of a well-placed 'twist parameter'.

4.1 Standard or hyperlogarithmic resurgence monomials and monics.

Let us at the oustet collect a few formulas about the hyperlogarithmic system of monomials and monics:

$$
\begin{array}{lccccc}
 & \partial\!-\!friendly & & \Delta\!-\!friendly & & type \\
monomials & \mathcal{V}^\bullet(z) & \longleftrightarrow & \mathcal{U}^\bullet(z) & & symmetral \\
 & \downarrow & \nwarrow\!\!\!\!\searrow\!\!\!\swarrow & \uparrow & & \\
monics & V^\bullet & \longleftrightarrow & U^\bullet & & alternal
\end{array}
$$

$$
\begin{aligned}
U^\bullet \circ V^\bullet &= V^\bullet \circ U^\bullet = I^\bullet = \textit{identity mould} && (*)\\
\mathcal{U}^\bullet &= \mathcal{V}^\bullet \circ U^\bullet && (**)\\
\mathcal{V}^\bullet &= \mathcal{U}^\bullet \circ V^\bullet && (***)
\end{aligned}
$$

We are mainly interested in the Δ-friendly monomials. So let us first mention the rules for multiplying and differentiating them :

$$
\mathcal{U}^{\boldsymbol{\omega}'}(z)\,\mathcal{U}^{\boldsymbol{\omega}''}(z) = \sum_{\boldsymbol{\omega}\in\mathbf{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} \mathcal{U}^{\boldsymbol{\omega}}(z) \tag{102}
$$

$$
(\partial_z + ||\boldsymbol{\omega}||)\,\mathcal{U}^{\omega_1,\dots,\omega_r}(z) = -\sum_{1\le i\le r} \mathcal{U}^{\omega_1,\dots,\omega_i}(z)\,U^{\omega_{i+1},\dots,\omega_r}\,z^{-1} \tag{103}
$$

$$
\begin{aligned}
\Delta_{\omega_0}\,\mathcal{U}^{\omega_1,\dots,\omega_r}(z) &= \mathcal{U}^{\omega_2,\dots,\omega_r}(z) \quad if \quad \omega_0=\omega_1 && (104)\\
&= 0 \qquad\qquad\quad\;\, if \quad \omega_0\neq\omega_1 && (105)
\end{aligned}
$$

Then, for the sake of symmetry, let us mention the rules for multiplying and differentiating the ∂-friendly monomials :

$$\begin{aligned}
\mathcal{V}^{\boldsymbol{\omega}'}(z)\,\mathcal{V}^{\boldsymbol{\omega}''}(z) &= \sum_{\boldsymbol{\omega}\in\mathbf{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} \mathcal{V}^{\boldsymbol{\omega}}(z) && (102\,bis)\\
(\partial_z+||\boldsymbol{\omega}||)\,\mathcal{V}^{\omega_1,\dots,\omega_r}(z) &= -\,\mathcal{V}^{\omega_1,\dots,\omega_{r-1}}\,z^{-1} && (103\,bis)\\
\Delta_{\omega_0}\,\mathcal{V}^{\omega_1,\dots,\omega_r}(z) &= \sum_{\omega_1+\dots\omega_i=\omega_0} V^{\omega_1,\dots,\omega_i}\,\mathcal{V}^{\omega_{i+1},\dots,\omega_r}(z) && (104\,bis)\\
&= 0 \quad if\ \ \omega_1+\dots\omega_i\neq\omega_0 \qquad \forall i && (105\,bis)
\end{aligned}$$

The ∂-friendly monomials are actually simpler objects than their Δ-friendly counterpart. For one thing, when viewed in the formal model, ie as power series of z^{-1}, the $\mathcal{V}^\bullet(z)$ have Taylor coefficients that are rational fonctions (with rational coefficients) of the indices ω_i, whereas the $\mathcal{U}^\bullet(z)$ always involve transcendental ingredients. More importantly perhaps, the $\mathcal{V}^\bullet(z)$ provide the best starting point for constructing the entire hyperlogarithmic system, according to the following scheme :

$$\mathcal{V}^\bullet(z) \overset{1}{\Longrightarrow} V^\bullet \overset{2}{\Longrightarrow} U^\bullet \overset{3}{\Longrightarrow} \mathcal{U}^\bullet(z)$$

First, we calculate (step 0) the monomials $\mathcal{V}^\bullet$ inductively, under successive integrations, by rephrasing (103 bis) in the convolutive model :

$$\begin{aligned}
(-\zeta+||\boldsymbol{\omega}||)\,\hat{\mathcal{V}}^{\omega_1,\dots,\omega_r}(\zeta) &= -\,(\hat{\mathcal{V}}^{\omega_1,\dots,\omega_{r-1}} * 1)(\zeta) && (103\,ter)\\
&= -\int_0^\zeta \hat{\mathcal{V}}^{\omega_1,\dots,\omega_{r-1}}(\zeta_0)\,d\zeta_0
\end{aligned}$$

Then (step 1) we get the monics $V^\bullet$, again inductively, by interpreting the alien derivation rule (104 bis), again in the convolutive model. This yields the $V^\bullet$ as finite integrals.

Then (step 2) we get the monics $U^\bullet$ by taking the composition inverse of the mould $V^\bullet$: see relation (*) above and also (13).

Lastly (step 3) we get the momomials $\mathcal{U}^\bullet$ by postcomposing the mould $\mathcal{V}^\bullet$ by $U^\bullet$: see relation (**).

Such is the natural sequence, but there also exist direct formulas for the Δ-friendly monomials, like this one, which is capable of a treble interpretation – in the *formal, geometric*, and *convolutive* models [26] :

$$\mathcal{U}^{\omega_1,\dots,\omega_r}(z) := \mathit{SPA}\int_0^\infty \frac{\exp(-\omega_1 y_1\cdots-\omega_r y_r)}{(y_r-y_{r-1})\dots(y_2-y_1)(y_1-z)}\,dy_1\dots dy_r \qquad (106)$$

and has its counterpart for the Δ-friendly monics.

[26]for the interpretation in the various models and the meaning of *SPA* – *special path averaging* – see also §10.3 *infra.*

4.2 Semi-formal synthesis in Example 1.

Recall that in this example each invariant reduces to a single scalar:

$$\mathbb{A}_n \quad := \quad A_n \, e^{-nz} \, \partial_z \tag{107}$$

So we may start from some admissible system $\{\dots A_{-2}, A_{-1}, A_1, A_2 \dots\}$ and set:

$$F^\star \quad := \quad 1 + \sum_r \sum_{n_i} (-1)^r \, \mathcal{U}e^{n_1, n_2, \dots, n_r}(z) \, \mathbb{A}_{n_r} \dots \mathbb{A}_{n_2} \mathbb{A}_{n_1} \tag{108}$$

Simple algebra[27] shows that $F^\star$ is a formal[28] automorphism of $\mathbb{C}[[z^{-1}]]$:

$$F^\star \, (\varphi_1 * \varphi_2) \equiv (F^\star \, \varphi_1) * (F^\star \, \varphi_2) \qquad\qquad \forall \, \varphi_1, \varphi_2 \tag{109}$$

and therefore a substitution operator of the form $F^\star \, \varphi \equiv \varphi \circ f^\star$ with[29] :

$$f^\star(z) \quad := \quad z - \sum_r \sum_{n_i} \mathcal{U}^{n_1, \dots, n_r}(z) \, \Gamma_{n_1, \dots, n_r} \, A_{n_1} \dots A_{n_r} \tag{110}$$

$$\Gamma_{n_1} \quad := \quad 1 \tag{111}$$

$$\Gamma_{n_1, \dots, n_r} \quad := \quad n_1 (n_1 + n_2) \dots (n_1 + \dots n_{r-1}) \quad (r \geq 1) \tag{112}$$

Then $f^\star$ formally verifies the system of resurgence equations:

$$\mathbb{\Delta}_n f^\star(z) = -A_n \; \exp(-n f^\star(z)) \qquad (\forall n) \tag{113}$$

$$\Delta_n f^\star(z) = -A_n \; \exp(-n \, (f^\star(z) - z)) \qquad (\forall n) \tag{114}$$

Therefore, if we set

$$f(z) := f^\star(2\pi i + {}^{\star}\!f(z)) \tag{115}$$

and apply the rule for alien-differentiating a composition product:

$$\mathbb{\Delta}_\omega (g \circ h) \equiv (\mathbb{\Delta}_\omega g) \circ h + (\partial_z \, g) \circ h \, . \, \mathbb{\Delta}_\omega h \qquad (if \;\; h(z) \sim z) \tag{116}$$

we find that all alien derivatives of f are identically zero. So f is "formally" an identity-tangent analytic diffeo with the prescribed invariants.

[27] using the multiplication rule (102) for the $\mathcal{U}^\bullet$.

[28] but, to get lighter notations, we drop the tildas

[29] mark the replacement of $\mathcal{U}e^\bullet$ by $\mathcal{U}^\bullet$ and the non-dependence of $\Gamma_{n_1, \dots, n_r}$ on n_r .

4.3 Semi-formal synthesis in Example 2.

Here again, the invariants $\mathbb{A}_n := A_n\, u^{n+1}\, \partial_u$ reduce to a single scalar A_n , but they no longer depend on z and their index n can no longer assume negative values other than the exceptional -1. So let us start from some admissible system $\{A_{-1}, A_1, A_2, A_3, \dots\}$ and set:

$$\Theta \quad := \quad 1 + \sum_r \sum_{n_i} (-1)^r\, \mathcal{U}e^{n_1,n_2,\dots,n_r}(z)\ \ \mathbb{A}_{n_r} \dots \mathbb{A}_{n_2}\mathbb{A}_{n_1} \tag{117}$$

$$\Theta^{-1} \quad := \quad 1 + \sum_r \sum_{n_i} \mathcal{U}e^{n_r,\dots,n_2,n_1}(z)\ \ \mathbb{A}_{n_r} \dots \mathbb{A}_{n_2}\mathbb{A}_{n_1} \tag{118}$$

Due once again to (102), the operatorrs $\Theta^{\pm 1}$ are formal algebra automorphisms or, more precisely, substitution operators :

$$\Theta^{\pm 1}(\varphi_1\ \varphi_2) \equiv \Theta^{\pm 1}(\varphi_1)\ \ \Theta^{\pm 1}(\varphi_2) \tag{119}$$

This standard procedure, when applied with respect to the *hyperlogarithmic* resurgence monomials mentioned at the end of §2.4, leads from an admissible system of prescribed invariants:

$$\{A_{-1}, A_1, A_2, A_3, \dots\} \tag{120}$$

to an singular differential equation of the form:

$$d_z y = y - z^{-1}\{B_{-1} + B_1 y^2 + B_2 y^3 + B_3 y^4 + \dots\} \tag{121}$$

with a perfect symmetry of form between the invariants $\{A_n\}$ of our "object" and its Taylor coefficients $\{B_n\}$. Indeed, if we introduce operators $\mathbb{B}_n := B_n\, u^{n+1} \partial_u$ parallel to the operators $\mathbb{A}_n := A_n\, u^{n+1} \partial_u$ and then plug into (117),(118) the formula (103) that express the natural derivatives of the Δ-friendly monomials $\mathcal{U}^\bullet(z)$, *setting aside for the time being all questions of convergence*, we find the following, engagingly symmetric formulae for the correspondance $\{A_n\} \leftrightarrow \{B_n\}$:

$$\mathbb{B}_{n_0} \quad = \quad \sum_{r\geq 1} \ \sum_{n_i} U^{n_1,\dots,n_r}\, \mathbb{A}_{n_r} \dots\, \mathbb{A}_{n_2}\, \mathbb{A}_{n_1} \tag{122}$$

$$(\textit{due to alternality}) \quad = \quad \sum_{r\geq 1} \frac{1}{r} \sum_{n_i} U^{n_1,\dots,n_r} [\mathbb{A}_{n_r} \dots [\mathbb{A}_{n_2}, \mathbb{A}_{n_1}]] \tag{123}$$

$$\mathbb{A}_{n_0} \quad = \quad \sum_{r\geq 1} \ \sum_{n_i} V^{n_1,\dots,n_r}\, \mathbb{B}_{n_r} \dots\, \mathbb{B}_{n_2}\, \mathbb{B}_{n_1} \tag{124}$$

$$(\textit{due to alternality}) \quad = \quad \sum_{r\geq 1} \frac{1}{r} \sum_{n_i} V^{n_1,\dots,n_r} [\mathbb{B}_{n_r} \dots [\mathbb{B}_{n_2}, \mathbb{B}_{n_1}]] \tag{125}$$

4.4 Semi-formal synthesis in the remaining examples.

For Example 3 and 4, the procedure is exactly the same as above, in §4.2, for Example 2 :

$$\mathbb{A}_\omega := u^{n+1} \sum A^j_\omega \, u_j \partial_{u_j} \tag{126}$$

$$\Theta := 1 + \sum_r \sum_{\omega_i} (-1)^r \mathcal{U}e^{\omega_1,\omega_2,\ldots,\omega_r}(z) \, \mathbb{A}_{\omega_r} \ldots \mathbb{A}_{\omega_2} \mathbb{A}_{\omega_1} \tag{127}$$

$$\Theta^{-1} := 1 + \sum_r \sum_{\omega_i} \mathcal{U}e^{\omega_r,\ldots,\omega_2,\omega_1}(z) \; \mathbb{A}_{\omega_r} \ldots \mathbb{A}_{\omega_2} \mathbb{A}_{\omega_1} \tag{128}$$

except for the required modifications in the size of the index reservoir Ω (see (83),(89)) and the shape of the invariants $\mathbb{A}_\omega$ (see (84),(90)).

For the polycritical examples 5 (linear) and 6 (non-linear), the changes are more thorough-going :
– due to the need to apply *acceleration transforms*[30] for jumping from one critical time to the next,
– due to the need to choose a specific integration axis θ_i in each critical Borel plane, which results in a general *multipolarisation* that affects not only the *geometric model*, but also the resurgence monomials and the very invariants $\mathbb{A}_\omega$, except of course those attached to the 'first' or 'slowest' critical time,
– due to the presence in the invariants $\mathbb{A}_\omega$ of parameters u_j attached to the 'earlier' or 'slower' critical times, with the result that each $\mathbb{A}_\omega$ may now carry (in the non-linear Example 6) *infinitely* many scalar coefficients.

But in all these cases, hyperlogarithmic object synthesis displays one striking feature, already apparent in the formulas (122),(123) and (124),(125), namely a rigorous (at least at the formal level) duality between the *invariants* and the *Taylor coefficients* of the canonical object $\mathbf{Ob}^{\text{can}}$ produced by synthesis. The natural indexation, in particular, is exactly the same – whereas for generic objects **Ob** in a given conjugacy class the Taylor coefficients tend to be "infinitely more numerous"[31] than the invariants. This extremely nice feature of *semi-formal, hyperlogarithmic synthesis* will get lost, alas, when we go over to *effective, paralogarithmic synthesis* but it won't disappear altogether : the Taylor coefficients of the new $\mathbf{Ob}^{\text{can}}$ will be exactly *twice as numerous* (no more !) than its invariants.

[30] see [E4],[E7].

[31] not in terms of cardinality, of course, which is always countable, but in terms of natural indexability by sets of type $\mathbb{N}^m \times \mathbb{Z}^n$.

4.5 Inadequacy of the standard resurgence monomials for the purpose of synthesis.

In Example 1 and 2 alike, the analytic invariants $\mathbb{A}_n$ and the operators $\mathbb{B}_n$ that encode the Taylor-coefficients[32] of $\mathbf{Ob}^{\mathrm{can}}$ essentially reduce to scalars A_n , B_n. Looking at the (formally inverse) relations (122-125) which exchange them, we are led to distinguish exactly five cases:

4.5.1 Unary case:

All $\{A_n\, , B_n\}$ vanish, save for one value of the index, $n = n_0$, which, due to scale-invariance[33] , we may take to be 1. Then the relations (122-125) immediately give:

$$\{A_1\} \iff \{B_1\} \qquad \textit{with} \qquad A_1 = B_1 \tag{129}$$

4.5.2 Binary case:

All $\{A_n\, , B_n\}$ vanish, save for two opposite values of the index, $n = \pm n_0$, which, due to scale-invariance, we may take to be ± 1. After some calculations (see [E2]), the relations (122-125) yield:

$$\{A_{-1}, A_1\} \longleftarrow \{B_{-1}, B_1\} \tag{130}$$

$$\textit{with} \qquad \frac{1}{2}\sqrt{A_{-1}\, A_1} = \sin\,(\frac{1}{2}\sqrt{B_{-1}\, B_1}) \tag{131}$$

$$\textit{and} \qquad A_1/A_{-1} = B_1/B_{-1} \tag{132}$$

So, going from semi-formal to effective synthesis, we register the existence of countably many hyperlogarithmic $\mathbf{Ob}^{\mathrm{can}}$, with no clear privileged choice except for small values of the product $A_{-1}\, A_1$.

4.5.3 Unilateral case:

All $\{A_n\, , B_n\}$ vanish, save for indices n that are positive multiples of some n_0 , which we may take to be 1.
Actually, to get true unilaterality[34] we must assume not only that $\ldots A_{-3} = A_{-2} = A_{-1} = 0$ but also that A_1 and at least *one* other A_n with positive index n are $\neq 0$.

[32] directly so in Example 2; indirectly so, via equation (115), in Example 1.

[33] $U^{l\,\omega_1,\ldots,l\,\omega_r} \equiv U^{\omega_1,\ldots,\omega_r} \quad ; \quad V^{l\,\omega_1,\ldots,l\,\omega_r} \equiv V^{\omega_1,\ldots,\omega_r}$

[34] ie to exclude the unary and binary cases.

This is the most promising case because (unlike the unary case) it is non-trivial and (unlike with the binary case) there is at the semi-formal level a clear one-to-one correspondance:

$$\{A_1, A_2, A_3, \dots\} \longleftrightarrow \{B_1, B_2, B_3, \dots\} \tag{133}$$

since for any given n_0 the right-hand side of the relations(122-125) reduces to a finite number of summands. Unfortunately, going from semi-formal to effective synthesis, it can be shown that for *nearly all* [35] admissible unilateral systems $\{A_n\}$ of invariants, the corresponding sequence $\{B_n\}$ of Taylor coefficients has Gevrey-1 growth [36] in n, which disqualifies the hyperlogarithmic $\mathbf{Ob}^{\mathrm{can}}$ as an effective synthesis.

4.5.4 Sesquilateral case:

All $\{A_n\,, B_n\}$ vanish, save for n a positive multiple of some n_0 and also for $n = -n_0$. Here again, we may take n_0 to be 1. In fact, it is enough to assume that $A_n \neq 0$ for $n = \pm 1$ and for at least *one* other positive n. In terms of effective hyperlogarithmic synthesis, the picture is much the same as in the binary case: there exists one priviledged solution $\mathbf{Ob}^{\mathrm{can}}$ if $A_{-1}\, A_1$ is small enough. Otherwise, there are several (countably many) solutions on an equal footing.

4.5.5 Bilateral case:

All $\{A_n\,, B_n\}$ vanish, save for n a positive or negative multiple of n_0, which we may take to be 1. In fact, it is enough to assume that $A_n \neq 0$ for at least two distinct positive values of n and two distinct negative values. For definiteness let us assume that $A_{\pm 1} \neq 0, A_{\pm 2} \neq 0$.

Then there exists one priviledged $\mathbf{Ob}^{\mathrm{can}}$ if $A_{-1}\, A_1$ is small enough *and* if the two sequences $\{A_n\,,\, n > 0\}$ and $\{A_n\,,\, n < 0\}$, or rather the Fourier series $\pi^{\pm}$ built from the related sequences $\{A'_n\,,\, n > 0\}$ and $\{A'_n\,,\, n < 0\}$ (see (140),(141) *infra*) in some right and left half-planes, verify a suitable "overlapping condition", like the one mentioned in §5.1 in the context of non-canonical synthesis. Failing that condition, the picture becomes quite murky, with either a countably infinity of solutions, or none at all.

[35] the exact criterion is mentioned in §4.6 below

[36] ie $0 < \limsup \frac{1}{n}\, |B_n|^{1/n} < \infty$

4.6 First intimations of "antipodality":

Let us take a closer look at the unilateral case. Assume for simplicty that $A_1 \neq 0, A_2 = 0$ [37] , which under (122-125) also implies that $B_1 \neq 0, B_2 = 0$. Assume further that the invariants A_n are expressed relative to the system of *standard alien derivations*: assume, in other words, that the A_n are the constants occuring in (72-73) with Δ_n defined as in (54),(57).

To describe the precise obstructions to effective *hyperlogarithmic synthesis* in *complete, closed form*, we require a fair number of auxiliary objects. First, we successively define six power series A_* , A^* , A and B_* , B^* , B :

$$
\begin{array}{lcllcl}
A_*(y) & := & \sum_{n\geq 1} A_n\, y^{n+1} & \qquad B_*(y) & := & \sum_{n\geq 1} B_n\, y^{n+1} \\
A^\star(y) & := & \int \frac{dy}{A(y)} & \qquad B^\star(y) & := & \int \frac{dy}{B(y)} \\
A^* \circ A & \equiv & 2\pi i + A^* & \qquad B^* \circ B & \equiv & 2\pi i + B^*
\end{array}
$$

Further, we form the partial differential operator :

$$D := \partial_z + \{y - z^{-1} B_*(y)\}\, \partial_y \tag{134}$$

We have to solve equation $DK = 0$ formally in decreasing powers of z.
We also have to solve equation $DQ = 0$ formally in increasing powers of z.
In both cases, we get increasing (but not always positive) powers of y and it is convenient to expand the solutions as follows :

$$
\begin{array}{lcll}
K(z,y) & := & e^{-z}\, y + e^{-z} \sum\limits_{n\geq 1} K_n(z)\, y^{n+1} & \quad \textit{with} \;\; K_n(z) \in \mathbb{C}[[z^{-1}]] \\
Q(z,y) & := & \log z + B^\star(y) + \sum\limits_{m\geq 1} Q_m(y)\, z^m & \quad \textit{with} \;\; B^*(y), Q_n(y) \in y^{-1}\, \mathbb{C}[[y]]
\end{array}
$$

Then we require three projectors P^0, P^+, P^- acting on series $\sum_{-\infty}^{+\infty} c_n\, z^{-n}$:
P^0 retains only the z-constants
P^+ retains only the positive powers of z
P^- retains only the negative powers of z

Since we have expressed our $\{A_n\}$ relative to the *standard* alien derivations, their quality of being an admissible system of invariants is not directly recognisable on $A_*(y)$ or $A^*(y)$, but rather on $A(y)$. More precisely, $\{A_n\}$

[37] No restrictions are imposed on $A_3, A_4 \ldots$. The condition $A_2 = 0$ is simply for convenience (to remove the logarithms in A^* and B^*). Dropping it would make no difference to the analysis.

constitutes an admissible system of invariants **iff** $A(y)$ *is local-analytic* [38]. As for $A_*(y)$ and $A^*(y)$), being respectively the *infinitesimal generator* and *normaliser* of the local-analytic diffeo A, they are merely resurgent, with critical time $s := -(2\pi i/A_1)\, y^{-1}$ and with the resurgence equations :

$$\Delta_n A^\star \;=\; -\,C_n\, e^{-nA^\star} \qquad (n \in \mathbb{Z}^\star\ ,\ \Delta_n := \Delta_n^{\{s\}}) \quad (135)$$

The constants C_n have a rather strange status : they are *'invariants of invariants'* since the Taylor coefficients of the diffeo A are themselves rational functions of the invariants A_n. Yet they hold the key to our problem. Indeed, we are now in a position to state the main results of this section : *as power series of* y *or Laurent series of* s := const.y^{-1}*, the series* B$_*$, B* , K , Q *are generically divergent, but always* resurgent*, with critical time* s*, and they verify the system of resurgence equations :*

$$\begin{aligned}
\Delta_n B^\star/\partial_s B^\star &= -\,C_n\, P^0\, (e^{-nQ}/\partial_s Q) && (n \in \mathbb{N}^\star) \quad (136)\\
\Delta_n K/\partial_s K &= +\,C_n\, P^-\, (e^{-nQ}/\partial_s Q) && (n \in \mathbb{N}^\star) \quad (137)\\
\Delta_n Q/\partial_s Q &= -\,C_n\, (P^0 + P^+)\, (e^{-nQ}/\partial_s Q) && (n \in \mathbb{N}^\star) \quad (138)\\
\textit{with} \quad C_n \in \mathbb{C} \quad &, \quad P^- + P^0 + P^+ = id && (139)
\end{aligned}$$

That system is *complete and closed*, in the sense that it enables us to calculate all the iterated alien derivatives $\Delta_{n_r} \dots \Delta_{n_1} B^*$ and $\Delta_{n_r} \dots \Delta_{n_1} B_*$, whereas *no smaller system could do that.* But if we take a closer look at the above system, especially at (136), we notice that only the alien derivations Δ_n with positive n are capable of acting non-trivially on B^* and B_*. Those with negative n yield 0, because for them the right-hand side of (136) vanishes. The upshot is that *unilateral hyperlogarithmic synthesis is possible if and only if the diffeo A is* semi-iterable *in the sense of having half its analytic invariants C_n, namely those with negative n, identically zero.* [39]

For details and complements, see [E3]. The above results demand some very careful analysis. Strangely enough, they are harder to prove that the forthcoming, more complete, and far more satisfactory, results about *paralogarithmic synthesis.*

The above resurgence is of a highly unusual type. In [E3] it is dubbed *"synthesis resurgence"* and contrasted with two other, far more common,

[38] see [E3].

[39] at least in the Examples 1 and 2 under discussion here. Incidentally, the above construction relates more directly to Example 2, but can be easily transposed to Example 1.

types: *"equational resurgence"*[40] and *"co-equational resurgence"*[41]. Yet, remarkably enough, all three types give rise to the same sort of *acting alien algebras* $ALIEN^{act} := ALIEN/ALIEN^{nil}$, which is to say that in all three cases, the ideal $ALIEN^{nil}$ of $ALIEN$ which annihilates the relevant resurgence algebra [42] is generated by derivations of the form

$$c_{n_1+n_2}\,\mathbb{\Delta}_{n_1+n_2} - (n_1 - n_2)\, c_{n_1}\, c_{n_1}\, [\mathbb{\Delta}_{n_1}, \mathbb{\Delta}_{n_2}]$$

for a well-defined system of (usually transcendental) structure constants $\{c_n\}$.

4.7 The need for one free parameter at least.

Perhaps the main lesson from this excursion into the *unilateral case* is this: in order to describe the divergence-resurgence pattern of the series B_* , B^* as elements of $\mathbb{C}[[s^{-1}]]$ we are compelled to adduce elements of $\mathbb{C}[[s]]$. In other words, what goes on at one pole $s = \infty$ (and $z = \infty$) of the Riemann sphere is inextricably linked to what goes on at the other pole $s = 0$ (and $z = 0$). [43] This "antipodal pairing" is going to be an outstanding feature of "canonical" or "sperical" synthesis.

But another lesson emerges, this time from the binary and sesquilateral cases: no one-to-one correspondance $\{\mathbb{A}_\omega\} \leftrightarrow \mathbf{Ob}^{\mathrm{can}}$ that holds for all systems $\{\mathbb{A}_\omega\}$ – and not just 'small' ones – can be achieved without introducing at least one free parameter capable of restoring 'smallness' where it is missing. Fortunately, as we shall see, one parameter is enough.

5 Methods for non-canonical Object Synthesis.

Having seen the limitations of *canonical hyperlogarithmic* synthesis and before turning to the completely satisfactory answer of *canonical paralogarithmic* synthesis, let us recall a few basic results about *non-canonical* synthesis, if only to obtain a characterisation of the admissible systems of invariants.

[40] because there the resurgence-bearing variable is the (or one of the) variables(s) of the equation under investigation.

[41] because it is roughly dual to the preceding. It is also known as *parametric resurgence* because there the resurgence-bearing variable is, typically, a singular perturbation parameter.

[42] ie the algebra generated by the resurgent functions under consideration and *all* their alien derivatives, of *all* orders.

[43] despite the absence, generally speaking, of an analytic continuation, from one pole to the next, of the functions involved.

5.1 Main and earliest method.

We go back to Example 1. For any shift-like diffeo f, let f^*_+ (resp f^*_-) denote *the* Borel-Laplace sum of the normaliser f^* that is defined and regular in a U-shaped domain containing an upper (lower) half-plane $\Im z > const$ (resp $\Im z < const$) and two vertical half-planes $|\Re z| > const$. For our present purpose, it is convenient to express the invariants $\{\mathbb{A}_n\}$ in the guise of two $2\pi i$-periodic functions π^+ and π^- defined each in some vertical half-plane, respectively right and left:

$$\pi^+(z) = z - \sum_{n\in\mathbb{N}^+} A'_n\, e^{-nz} \quad ; \quad \Re z \gg +1 \tag{140}$$

$$\pi^-(z) = z - \sum_{n\in\mathbb{N}^-} A'_n\, e^{-nz} \quad ; \quad \Re z \ll -1 \tag{141}$$

and linking the normalisers $f^*_\pm$ on their common half-planar domains of definitions:

$$\pi^+ \circ f^*_-(z) = f^*_+(z) \qquad for \quad \Re z \gg +1 \tag{142}$$

$$\pi^- \circ f^*_+(z) = f^*_-(z) \qquad for \quad \Re z \ll -1 \tag{143}$$

The Fourier coefficients A'_n of $\pi_\pm$ are not exactly the constants A'_n occuring in the resurgence equations (72),(73), but rather those that we would get if we were to write down (72),(73) with the *lateral* operator $\mathbb{\Delta}^\pm_\omega$ instead of $\mathbb{\Delta}_\omega$ [44].

Each conjugacy class of shift-like diffeos f is characterised by an analytic pair (π^+,π^-) and conversely to each analytic pair (π^+,π^-) there answers an analytic conjugacy class of shift-like diffeos f.

The direct part of the statement is very easy, from (142),(143). As for the converse statement, the original proof ([E2],pp 450-456) splits into an infinitesimal and a global step.

The *infinitesimal* step starts from an arbitrary shift-like diffeo f with analytic invariants (π^+,π^-) as in (140),(141). Then, given any infinitesimal analytic perturbation $(\pi^+ + \delta\pi^+, \pi^- + \delta\pi^-)$, one proceeds to construct a corresponding infinitesimal perturbation $f + \delta f$ via the normalisers $f^*_\pm + \delta f^*_\pm$. The construction goes like this:

[44] Up to a $2\pi i$ factor, these A'_n are "polar residues" (in the Borel plane) at the points n as approached from below or from above.

$$
\begin{aligned}
2\pi i\,\frac{\delta f_+^\star(z)}{\partial f_+^\star(z)} &= -\int_{t^+}^{+\infty}\frac{\left(\frac{\delta\,{}^+\pi}{\partial\,{}^+\pi}\right)\circ f_+^\star(y)}{\partial f_+^\star(y)}\frac{dy}{y-z}\\
&\quad +\int_{t^-}^{-\infty}\frac{\left(\frac{\delta\,\pi^-}{\partial\,\pi^-}\right)\circ f_+^\star(y)}{\partial f_+^\star(y)}\frac{dy}{y-z} \qquad (144)\\
2\pi i\,\frac{\delta f_-^\star(z)}{\partial f_-^\star(z)} &= -\int_{t^-}^{+\infty}\frac{\left(\frac{\delta\,{}^-\pi}{\partial\,{}^-\pi}\right)\circ f_-^\star(y)}{\partial f_-^\star(y)}\frac{dy}{y-z}\\
&\quad +\int_{t^+}^{-\infty}\frac{\left(\frac{\delta\,\pi^+}{\partial\,\pi^+}\right)\circ f_-^\star(y)}{\partial f_-^\star(y)}\frac{dy}{y-z} \qquad (145)
\end{aligned}
$$

with ${}^\pm\pi$ standing for the reciprocal functions of $\pi^\pm$ and with $\pm\Re(t^\pm)\gg 1$. The construction clearly implies:

$$
\begin{aligned}
\frac{\delta\,f_+^\star}{\partial\,f_+^\star}-\frac{\delta\,f_-^\star}{\partial\,f_-^\star} &= +\frac{\left(\frac{\delta\,\pi^+}{\partial\,\pi^+}\right)\circ f_-^\star}{\partial\,f_-^\star} \qquad for \quad \Re(z)\gg +1\\
\frac{\delta\,f_+^\star}{\partial\,f_+^\star}-\frac{\delta\,f_-^\star}{\partial\,f_-^\star} &= -\frac{\left(\frac{\delta\,\pi^-}{\partial\,\pi^-}\right)\circ f_+^\star}{\partial\,f_+^\star} \qquad for \quad \Re(z)\ll -1
\end{aligned}
$$

which means that $f+\delta f$ and $f_\pm^*+\delta f_\pm^*$. verify the perturbed form of the normaliser's characteristic equation (142),(143).

The *global* step consists in joining some elementary but non-trivial pair $\{\pi_0^-,\pi_0^+\}$ for which f_0 is *known*[45] to a given pair $\{\pi_1^-,\pi_1^+\}$ for which f_1 is *sought*, by an λ-analytic path $\{\pi_\lambda^-,\pi_\lambda^+,\ \ f_\lambda\,,f_{\lambda,\pm}^*\,,\ t_\lambda^\pm\}_{\lambda\in[0,1]}$. This is done by turning (144),(145) into a *differential system* in λ, with the integration end points $t_\lambda^\pm$ suitably chosen, ie far enough to the right or left for the compositions $\pi_\lambda^\pm\circ f_{\lambda,\mp}^*$ and ${}^\pm\pi_\lambda\circ f_{\lambda,\pm}^*$ to make sense. For details, see [E2] [46].

5.2 Malgrange's method.

Using polar coordinates $x=e^{i\theta}\,r$ on $\mathbb{C}$ with $(\theta,r)\in(\mathbb{T},\mathbb{R}^+)$ and considering the blow-up $\mathbb{S}:=(\mathbb{T},0)$, Malgrange defines sheaves Γ_p on $\mathbb{S}$ by

[45] if $\{\pi_1^-,\pi_1^+\}$ corresponds to the *unilateral* (resp sesqui- or bilateral) case, then the obvious choice for $\{\pi_0^-,\pi_0^+\}$ is to take an instance of *unary* (resp *binary*) hyperlogarithmic synthesis, as in §4.5.1 (resp §4.5.2).

[46] in [E] we took $f^{\mathrm{nor}}(z)=z+1$ instead of $f^{\mathrm{nor}}(z)=z+2\,\pi i$ here, which explains the slight differences in the two presentations.

letting Γ_{p,θ_0} contain all functions that are holomorphic in some small sectorial neigbourhood of 0 of bisectrix θ_0 with an asymptotic expansion of the form $x + \sum_{n \geq p+1} a_n\, x^n$. Then he sets $\Gamma_\infty := \cap\, \Gamma_p$ and shows that solving (142),(143) reduces to a special case of the following statement: *the image of the natural mapping $H^1(\mathbb{S}, \Gamma_\infty) \to H^1(\mathbb{S}, \Gamma_p)$ reduces to the trivial element* $O_{H^1(\mathbb{S},\Gamma_p)}$. Malgrange then derives that statement, without calculations, from the classical Newlander-Nirenberg theorem.

5.3 The quasiconformal method.

Another early proof of the (non-canonical) synthesis theorem for identity-tangent, one-dimensional diffeos was given by S. M. Voronin. Though independent of Malgrangre's proof, it resembles it in that it solves the problem first in a smooth setting ($\mathcal{C}^\infty$ with M., $\mathcal{C}^1$ with V.), then in a complex-analytic one. But the main tool here is the beautiful Ahlfors-Bers theory (see [A]) of quasiconformal mappings.[47] Starting from any given pair $\{\pi^+, \pi^-\}$, Voronin constructs $\{f^*_+, f^*_-, f\}$, with all the required relations, on an abstract Riemann surface S. Then he shows that S is *quasiconformally*, and therefore *conformally*, equivalent to a punctured neighbourhood of 0 in $\mathbb{C}$. These quasiconformal methods were later extensively used in holomophic dynamics by authors like Douady, Hubbard, Lavaurs, etc.

5.4 Comparison.

Anteriority aside, the method outlined in §5.1 would seem to be more explicit, elementary, and direct. Above all, it has the merit of extending with very little modification not only to most other problems of analytic synthesis (like Malgrange and Voronin's methods), but also to *non-analytic* ones (*unlike* Malgrange's or Voronin's methods, which are too dependent on "geometry", ie on the *multiplicative plane*, to tackle situation when *geometry disappears*).

Consider for example shift-like diffeos, but formal ones, of Gevrey rather than analytic type. In E2 it was shown that for all Gevrey classes $\mathcal{G}_t$ between $\mathcal{G}_0$ (analytic) and $\mathcal{G}_1$, there exist non-trivial Gevrey moduli (ie Gevrey conjugacy classes), with invariants $\mathbb{A}_n := e^{-n\,z}\, A_n\, \partial_z$ which, relative to any well-behaved basis of ALIEN, are characterised by growth bounds

$$|A_n| < C_0\ \exp\left(C_1\, n^{\frac{1}{1-t}}\right) \qquad (0 < C_i < \infty)$$

[47] ie mappings $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ with locally integrable, first-order derivatives such that $|\partial_{\bar{z}}\, f| \leq k\, |\partial_z\, f|$ almost everywhere for some $k < 1$.

Here, the geometric model no longer exists and geometric methods become useless, whereas in the convolutive model, things hardly change. Indeed, the integrals (144),(145) can easily be rephrased in terms of their Borel transforms, and we find ourselves dealing with normalisers $\hat{f}^*(\zeta)$ and $\delta\hat{f}^*(\zeta)$ which have exactly the same singularities and verify the same resurgence equations as in the analytic case, but differ only by their growth pattern at ∞ (away from the singularities) :

$$|\hat{f}^*(\zeta)| < C_0 \ \exp\left(C_1 \ |\zeta|^{\frac{1}{1-t}}\right) \qquad (0 < C_i < \infty)$$

6 Four closely related challenges.

We shall discuss in this section four closely related challenges, pertaining to four different notions central to resummation theory – *uniformising averages, alien derivations, resurgenge monomials, ramified growth* – and arising from the need to simultaneously satisfy a series of conditions :
C1: compatibility with the multiplicative structure
C2: respecting realness
C3: respecting lateral (exponential) growth
C4: scale invariance
which are much the same in all four cases, yet call for slight reformulations.
For uniformising averages these conditions shall become $\mathbf{A1}, \ldots, \mathbf{A4}$.
For alien derivations : $\mathbf{D1}, \ldots, \mathbf{D4}$.
For resurgence monomials : $\mathbf{M1}, \ldots, \mathbf{M4}$.
For ramified growth : $\mathbf{R1}, \ldots, \mathbf{R4}$.

Each demand is easy enough to meet in isolation, but combining them is more difficult, for reasons we shall try to make clear in a moment. Fortunately, there *are ways* of reconciling these demands, and we then speak of *well-behaved* averages, derivations, monomials, etc.

But let us first devote a section to the central difficulty in all these problem : overcoming *faster-than-lateral growth.*

6.1 The main obstacle: faster-than-lateral growth.

As pointed out, summation or accelero-summation yields at every step i a function $\hat{\varphi}_i(\zeta_i)$ which has precisely the right rate of growth, ie the one that makes the next acceleration (or Laplace) possible. But this applies only to singularity-free axes or, on singularity-carrying axes, to the two lateral determinations.[48] Most other determinations of $\hat{\varphi}_i(\zeta_i)$, especially the ones that correspond to oft-crossing paths, tend to display slightly faster-than-lateral growth.

For instance, if the lateral growth of $\hat{\varphi}(\zeta)$ is exponential (this is the growth required for Laplace integration), the other determinations generally admit no better uniform bounds than $|\hat{\varphi}(\zeta)| \leq \gamma_0 \, e^{\gamma_1 |\zeta| |\log \zeta|}$ (so that they cannot be subjected to Laplace). Therefore, unless we resort to carefully honed averages $\mathbf{m}$, the averaged function $\mathbf{m}\,\hat{\varphi}(\zeta)$ itself is going to display this slightly super-exponential growth.

This nuisance of faster-than-lateral growth is extremely common, generic almost. In the Dulac problem for instance, it affects nearly all transit maps G_i attached to summits of semi-hyperbolic type in a given polycycle.

In order to show just how prevalent and inescapable the phenomenon of faster-than-lateral growth is, let us adduce the simplest conceivable illustration, linked to as harmless an operation as *inversion*:

$$\tilde{A}(z) \;:=\; \sum_{n\geq 1} n!\, z^{-n} \tag{146}$$

$$\hat{A}(\zeta) \;:=\; \sum_{n\geq 0} \zeta^n \;=\; 1/(1-\zeta) \tag{147}$$

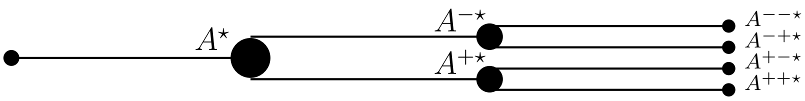
$$\tilde{B}(z) \;:=\; \sum_{n\geq 1} A^n(z) \;=\; A(z)/(1-A(z)) \tag{148}$$

$$\hat{B}(\zeta) \;:=\; \sum_{n\geq 1} \hat{A}^{\star n}(\zeta) \tag{149}$$

The divergent series $\tilde{A}(z)$ verifies the Euler equation $(1+\partial_z)\tilde{A}(z) = z^{-1}$ and its Borel transform $\hat{A}(\zeta)$, with its single pole at $\zeta = 1$, is the simplest instance of a non-trivial resurgent function. Yet a simple Möbius transform turns $\tilde{A}(z)$ into a series $\tilde{B}(z)$, also solution of a first-order differential equation, but with a Borel transform $\hat{B}(\zeta)$ that has singularities at every point $\zeta = n \in \mathbb{N}$, with simple poles as leading terms and logarithmic singularities as corrections:

$$\begin{aligned}
\hat{B}(\zeta) \;=\; & +R_B^{\epsilon_1,\ldots,\epsilon_{n-1},\star}.(\zeta-n)^{-1} && (\textit{simple pole})\\
& +Reg_1^{\epsilon_1,\ldots,\epsilon_{n-1},\star}(\zeta-n).\log(\zeta-n) && (\textit{logarithmic singularity})\\
& +Reg_0^{\epsilon_1,\ldots,\epsilon_{n-1},\star}(\zeta-n) && (\textit{regular part})
\end{aligned}$$

[48] and that too even if we take care of choosing in the critical time class $\{z_i\}$ a suitably slow time z_i , which precaution has the effect of smoothing the singularities of $\hat{\varphi}_i(\zeta_i)$.



This even provides us with a discretised model of the phenomenon of faster-than-lateral growth. Indeed, the residues $R_B^{\epsilon_1,\dots,\epsilon_{n-1},\star}$ of address $\{\epsilon_i\}$ are calculable by a simple induction (see [E8]) which readily shows that they verify no better bounds than

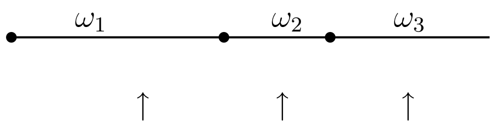
$$|R_B^{\epsilon_1,\dots,\epsilon_{n-1},\star}| \leq c_0\, c_1^n\, n! \tag{150}$$

Therefore, although $\hat{B}$ has exponential growth (at most) on each singularity-free axis $\arg\zeta = \theta \neq 0$, on paths that criss-cross [49] the axis $\arg\zeta = 0$ (especially for constantly alternating ϵ_i's) it admits no better bounds than $|\hat{B}(\zeta)| \leq \gamma_0\, e^{\gamma_1|\zeta||\log\zeta|}$.

In [E8], §A4, pp195-197, details may be found about the asymptotics of the "load" $ld(\epsilon_1,\dots,\epsilon_{r-1},*)$, which up to a multiplicative constant is the same as that of the residue $R_B^{\epsilon_1,\dots,\epsilon_{n-1},\star}$. At a distance r from the origin, the *larger* values occur on "oft-crossing paths", ie for sequences $(\epsilon_1,\dots,\epsilon_{n-1},\star)$ with frequent sign changes, and the *largest* value occur on ever-crossing paths, ie for strictly alternating sequences $(\pm,\mp,\pm,\mp\dots,*)$. The corresponding loads are roughly equal to $(\pi/8)^r\, r!$ and that is also the order of magnitude for values $|\hat{B}(\zeta)|$ on the branches with the same $\boldsymbol{\epsilon}$-address .

6.2 Challenge 1 : Searching for well-behaved averages.

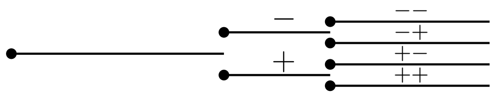
In the process of mono- or polycritical resummation, it is not the – usually highly ramified – Borel transform $\hat{\varphi}(\zeta)$ as such that gets subjected to the Laplace or acceleration transform, but some suitable uniformisation $\mathbf{m}\,\hat{\varphi}(\zeta)$ of $\hat{\varphi}(\zeta)$, evaluated on the relevant integration axis $\{\arg z = \theta_0\}$, which for simplicity we shall take to be $\mathbb{R}^+$ (ie $\theta_0 = 0$).[50]



uniform function $\mathbf{m}\,\hat{\varphi}(\zeta)$

[49]while avoiding small disks $|\zeta - n| < \rho < 1/2$ of fixed radius around each singular points. But we may remove this restriction by reasoning on the function $\iint \hat{B}(\zeta)$ which remains continous at each n.

[50]this applies in particular to the theory of transseries and analysable functions, where integration always takes place on $\mathbb{R}^+$.



multiform function $\hat{\varphi}(\zeta)$

A uniformising average $\mathbf{m} : \hat{\varphi} \mapsto \mathbf{m}\,\hat{\varphi}$ is defined by a system of weights $\mathbf{m}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_r}}$ subject to the self-consistency relations:

$$\sum_{\epsilon_i \in\{+,-\}} \mathbf{m}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_i\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_i\ ,\ldots,\ \omega_r}} = \mathbf{m}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_{i+1}\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_i+\omega_{i+1}\ ,\ldots,\ \omega_r}} \qquad \forall i < r \tag{151}$$

$$\sum_{\epsilon_r \in\{+,-\}} \mathbf{m}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_i\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_i\ ,\ldots,\ \omega_r}} = \mathbf{m}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_i\ ,\ldots,\ \epsilon_{r-1}}{\omega_1\ ,\ldots,\ \omega_i\ ,\ldots,\ \omega_{r-1}}} \tag{152}$$

and its action is as follows:

$$\mathbf{m}\,\hat{\varphi}(\zeta) := \sum_{\epsilon_i=\pm} \mathbf{m}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_r}}\ \hat{\varphi}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_r}}(\zeta) \tag{153}$$

with $\hat{\varphi}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_r}}$ denoting the determination of $\hat{\varphi}$ on the branch of address $(\epsilon_1, \ldots, \epsilon_r)$ over the interval $\zeta \in\,]\omega_1 + \ldots \omega_r\ ,\ \omega_1 + \ldots \omega_{r+1}[$ between two consecutive singularities [51]

To be really useful in the present context of Borel-Laplace resummation, or in the more general one of accelero-summation, a uniformising average must fulfill four main conditions:

A1: It must respect convolution[52], ie $\mathbf{m}\,(\hat{\varphi}_1 \star \hat{\varphi}_2) = (\mathbf{m}\,\hat{\varphi}_1) \star (\mathbf{m}\,\hat{\varphi}_2)$

A2: It must respect realness, ie $\mathbf{m}\,\hat{\varphi}(\zeta)$, as a global function, must be real whenever $\hat{\varphi}(\zeta)$, as a germ at $+0$, is real.

A3: It must respect lateral growth, that is to say $\mathbf{m}\,\hat{\varphi}(\zeta)$ must not grow significantly faster than the two lateral determinations (right or left) of $\hat{\varphi}(\zeta)$ along the positive real axis.

A4: It should be scale invariant, ie commute with real dilatations $\zeta \mapsto l\zeta$

Such averages will be declared "well-behaved".

[51] Of course, when the singularities of the minor $\hat{\varphi}$ are not of integrable type, we have to supplement the present formula with an anologous one for the major $\check{\varphi}$.

[52] observe that we are dealing here with two slightly different interpretations of the convolution product: in $\mathbf{m}\,(\hat{\varphi}_1 \star \hat{\varphi}_2)$ we convolute two function germs near the origin, then use analytic (or cohesive) forward continuation to get a global ramified function, and lastly we uniformise it by means of $\mathbf{m}$, whereas in $(\mathbf{m}\,\hat{\varphi}_1) \star (\mathbf{m}\,\hat{\varphi}_2)$ we directly convolute two global, uniform functions.

A1 is essential to get an algebra morphism[53].
A2 is natural and, in many instances[54], indispensible.
A3 ensures the convergence of the acceleration (or Laplace) integrals.
A4 is less essential, but very natural.

A2's translation in terms of weights is straightforward: the weights should change into their complex conjugates when all signs ϵ_i are changed. If real, the weights should remain unchanged. The same with **A4**: keeping all signs unchanged while multiplying all gaps ω_i by the same positive factor l should leave the weights unchanged. As for **A1** and **A3** we shall see in a moment what they imply in terms of weights.

But first we must introduce convenient, *non-redundant* ways of defining averages. To any uniformising average $\mathbf{m}$ we attach the moulds[55]:

$$\mathbf{rem}^{\omega_1,\dots,\omega_r} := (-1)^r\, \mathbf{m}^{\binom{+\ ,\dots,\ +}{\omega_1,\dots,\omega_r}} \qquad (\text{ “right-lateral mould” }) \tag{154}$$

$$\mathbf{lem}^{\omega_1,\dots,\omega_r} := (-1)^r\, \mathbf{m}^{\binom{-\ ,\dots,\ -}{\omega_1,\dots,\omega_r}} \qquad (\text{ “left-lateral mould” }) \tag{155}$$

$$\mathbf{nam}^{t_1,\dots,t_r}_{\omega_*,t_*} := \epsilon_1\dots\epsilon_r\, \mathbf{m}^{\binom{\epsilon_1,\dots,\epsilon_r}{\omega_*,\dots,\omega_*}} \qquad (\text{ “neutral mould” }) \tag{156}$$

$$\textit{with} \quad \epsilon_i := \mathit{sign}(t_i - t_{i-1}) \quad (\forall i < r) \quad \textit{and} \quad \epsilon_r := \mathit{sign}(t_r - t_*)$$

Due to the self-consistency relations, both the right- and left-lateral moulds encapsulate all the information about the entire weight system $\{\mathbf{m}^{\binom{\epsilon_1,\dots,\epsilon_r}{\omega_1,\dots,\omega_r}}\}$, and each one can be deduced from the other in a simple manner. As for the “neutral mould”, its upper indices t_i are real numbers, and so too are its lower index t_*, while the ‘common gap’ ω_* must of course be positive.

The initial consonants **r/l/n** in the names of our moulds stand of course for *right/left/neutral*; the inner vowels **e/a** stand respectively for *symmetrel/al* (for that's what our moulds will have to be); and the final **m** stands for *mean value.*

We are now in a position to state the criteria for well-behavedness.

The four following conditions are equivalent:
A1 : the uniformising average $\mathbf{m}$ *respects convolution*

[53]Operating exclusively with algebra morphisms is a must in all non-linear problems. Indeed, if $\tilde{\varphi}$ be the formal solution of a non-linear differential equation E, then its sum φ under Borel-Laplace (or, in polycritical instances, under accelero-summation) is going to verify the original equation E if and only if every single step in the resummation process is an algebra morphism.

[54]eg physics, real geometry, analysability theory, the Dulac problem, etc.

[55]see §2.1

$\mathbf{A1_1}$: the right-lateral mould $\mathbf{rem}^\bullet$ is *symmetrel* [56]
$\mathbf{A1_2}$: the left-lateral mould $\mathbf{lem}^\bullet$ is *symmetrel*
$\mathbf{A1_3}$: the neutral mould $\mathbf{nam}^\bullet_{\omega_*,t_*}$ is *symmetral* for all ω_*, t_*

The two following conditions are equivalent:
$\mathbf{A2}$: the uniformising average $\mathbf{m}$ *respects realness*
$\mathbf{A2_1}$: the moulds $\mathbf{rem}^\bullet$ and $\mathbf{lem}^\bullet$ are complex conjugate

The seven following conditions are equivalent:
$\mathbf{A3}$: the uniformising average $\mathbf{m}$ *respects lateral growth*
$\mathbf{A3_1}$: we have universal bounds $|\mathbf{rem}^{\boldsymbol{\omega}^{\prec\!\!\!<}}| \leq C_1\,(D_1)^r$
$\mathbf{A3_2}$: we have universal bounds $|\mathbf{rem}^{\boldsymbol{\omega}^{\succ\!\!\!>}}| \leq C_2\,(D_2)^r$
$\mathbf{A3_3}$: we have universal bounds $|\mathbf{lem}^{\boldsymbol{\omega}^{\prec\!\!\!<}}| \leq C_3\,(D_3)^r$
$\mathbf{A3_4}$: we have universal bounds $|\mathbf{lem}^{\boldsymbol{\omega}^{\succ\!\!\!>}}| \leq C_4\,(D_4)^r$
$\mathbf{A3_5}$: we have universal bounds $|\mathbf{nam}^{\mathbf{t}^{\prec}}_{\omega_*,t_*}| \leq C_5\,(D_5)^{\omega_*\,r}$
$\mathbf{A3_6}$: we have universal bounds $|\mathbf{nam}^{\mathbf{t}^{\succ}}_{\omega_*,t_*}| \leq C_6\,(D_6)^{\omega_*\,r}$
for some finite constants C_i, D_i.

Here, r always denotes the cardinal of the sequences involved. We recall that $M^{\bullet^{\prec}}$ (resp $M^{\bullet^{\succ}}$) denotes the forward (resp backward) *arborification* of the mould $M^\bullet$. In symmetric fashion, $M^{\bullet^{\prec\!\!\!<}}$ (resp $M^{\bullet^{\succ\!\!\!>}}$) denotes the forward (resp backward) *contracting arborification* of the mould $M^\bullet$. The defining relations read:

$$M^{\boldsymbol{\omega}^{\prec\!\!\!<}} := \sum_{\boldsymbol{\omega}^1 \gg \boldsymbol{\omega}^{\prec\!\!\!<}} M^{\boldsymbol{\omega}^1} \quad ; \quad M^{\boldsymbol{\omega}^{\succ\!\!\!>}} := \sum_{\boldsymbol{\omega}^2 \gg \boldsymbol{\omega}^{\succ\!\!\!>}} M^{\boldsymbol{\omega}^2} \tag{157}$$

The above symbol $\boldsymbol{\omega}^{\prec\!\!\!<}$ (resp $\boldsymbol{\omega}^{\succ\!\!\!>}$) denotes any sequence $\{\omega_i\}$ with an *arborescent* (resp *anti-arborescent*) order on it, ie an order such that each element ω_i has at most one predecessor ω_{i_-} (resp one successor ω_{i_+}), whereas the sums on the right-hand side extend to all *totally ordered* sequences $\boldsymbol{\omega}^1$ (resp $\boldsymbol{\omega}^2$) that can be obtained from $\boldsymbol{\omega}^{\prec\!\!\!<}$ (resp $\boldsymbol{\omega}^{\succ\!\!\!>}$) with the possible *contraction* $\omega_i, \ldots, \omega_j \mapsto \omega_i + \cdots + \omega_j$ of several consecutive elements. For some details see §2.1 and for more details go to [E5],[EV2],[EV3].

The four following conditions are equivalent:
$\mathbf{A4}$: the uniformising average $\mathbf{m}$ *is scale-invariant.*
$\mathbf{A4_1}$: the mould $\mathbf{rem}^\bullet$ is a homogeneous function of $\boldsymbol{\omega}$.
$\mathbf{A4_2}$: the mould $\mathbf{lem}^\bullet$ is a homogeneous function of $\boldsymbol{\omega}$.
$\mathbf{A4_3}$: the mould $\mathbf{nam}^\bullet_{\omega_*,t_*}$ is independent of its first lower index ω_*

Tempting answer: the standard (or uniform, or median) average.

[56] see §2.1

Let **mur** and **mul** [57] be the right- and left-lateral averages, with weights :

$$\mathbf{mur}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} \;:=\; 1 \quad if \quad \epsilon_1 = \cdots = \epsilon_r = + \quad (res \;\; 0 \;\; otherwise) \tag{158}$$

$$\mathbf{mul}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} \;:=\; 1 \quad if \quad \epsilon_1 = \cdots = \epsilon_r = - \quad (res \;\; 0 \;\; otherwise) \tag{159}$$

Both **mur** and **mul** verify **A1**, **A3** and even **A4** [58], but clearly not $\mathbf{A4_2}$ (preservation of realness). The half-sum $1/2\,(\mathbf{mur} + \mathbf{mul})$ does verify **A2**, but ceases to verify **A1** : it is not a convolution average and so of no use at all, except in linear problems. However, since **mur** and **mul** connect under post-composition by alien automorphisms :

$$\mathbf{mul} \;:=\; \mathbf{mur}\,.\,\mathbf{rul} \qquad with \qquad \mathbf{rul} := 1 + \sum_{\omega>0} \mathbb{\Delta}^-_\omega \tag{160}$$

$$\mathbf{mur} \;:=\; \mathbf{mul}\,.\,\mathbf{lur} \qquad with \qquad \mathbf{lur} := 1 + \sum_{\omega>0} \mathbb{\Delta}^+_\omega \tag{161}$$

it is very tempting to restore the right-left symmetry without destroying multiplicativity, by setting :

$$\mathbf{mun} \;:=\; \mathbf{mul}\,.\,(\mathbf{lur})^{\frac{1}{2}} \;\equiv\; \mathbf{mur}\,.\,(\mathbf{rul})^{\frac{1}{2}} \tag{162}$$

A simple calculation shows that the new average **mun** has weights that do not depend on the gaps ω_i, but only on the total number (p, q) of $(+, -)$ signs in the sequence $\boldsymbol{\epsilon}$: [59]

$$\mathbf{mun}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} := \frac{\Gamma(p+\frac{1}{2})\Gamma(q+\frac{1}{2})}{\Gamma(r+1)\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} = \frac{(2p)!\,(2q)!}{4^{p+q}\,p!\,q!\,(p+q)!} \tag{163}$$

This **mun** is known as *median* or *standard* or *uniform* average. Its very construction ensures at once that it verifies **A1**, **A2** and **A4**. But it can be shown [60] that it fails in regard to **A3** . So, for certain applications at least (see §7.6), we will have to look for better averages.

6.3 Challenge 2 : Searching for well-behaved alien derivations.

The singularities of minors $\hat{\varphi}(\zeta)$ in the Borel plane deserve close attention because :

---

[57] **m** for *mean value* , **u** for *uniform*, **r/l/n** for *right/left/neutral.*

[58] since **mur** and **mul** *load* only the lateral paths

[59] These properties, in fact, completely caracterise **mun** among all convolution and realness preserving averages.

[60] see §7.6

– they are (mainly) responsible for the divergence of $\tilde{\varphi}(z)$ in the formal model
– they command the asymptotic behaviour of $\varphi(z)$ in the geometric model, along the borders of its regularity sectors
– they carry, in the shape of residue-like coefficients in front of their leading terms, the analytic invariants of φ [61].

Hence the need for operators capable of measuring these singularities precisely and conveniently. Such operators do exist : they are the so-called alien derivations Δ_ω. These are determined by systems of weights $\mathbf{d}^{\binom{\epsilon_1,\dots,\epsilon_i,\dots,\epsilon_r}{\omega_1,\dots,\omega_i,\dots,\omega_r}}$ subject to the self-consistency relations :

$$\sum_{\epsilon_i\in\{+,-\}} \mathbf{d}^{\binom{\epsilon_1,\dots,\epsilon_i,\dots,\epsilon_r}{\omega_1,\dots,\omega_i,\dots,\omega_r}} \;=\; \mathbf{d}^{\binom{\epsilon_1,\dots,\quad\epsilon_{i+1}\quad,\dots,\epsilon_r}{\omega_1,\dots,\omega_i+\omega_{i+1},\dots,\omega_r}} \qquad \forall i<r \tag{164}$$

$$\sum_{\epsilon_r\in\{+,-\}} \mathbf{d}^{\binom{\epsilon_1,\dots,\epsilon_i,\dots,\epsilon_r}{\omega_1,\dots,\omega_i,\dots,\omega_r}} \;=\; 0 \tag{165}$$

and their action in the *convolutive model* is given by :

$$\Delta_\omega\, \hat{\varphi}(\zeta) := \sum_{\epsilon_i=\pm} \mathbf{d}^{\binom{\epsilon_1,\dots,\epsilon_r}{\omega_1,\dots,\omega_r}}\;\; \hat{\varphi}^{\binom{\epsilon_1,\dots,\epsilon_r}{\omega_1,\dots,\omega_r}}(\zeta+\omega) \tag{166}$$

for ζ close to $+0$ and by analytic continuation in the large. There being no scope for confusion, we also use the same symbols to denote the alien derivations acting in the *multiplicative models (formal or geometric)*, ie the pull-backs by Borel-Laplace of the operators Δ_ω as defined by (166).

To each system of alien derivations we may associate the moulds :

$$\begin{aligned}
\mathbf{red}^{\omega_1,\dots,\omega_r} \;&:=\; (-1)^r\, \mathbf{d}^{\binom{+,\dots,+}{\omega_1,\dots,\omega_r}} && (\text{“right-lateral mould”}) \quad (167)\\
\mathbf{led}^{\omega_1,\dots,\omega_r} \;&:=\; (-1)^r\, \mathbf{d}^{\binom{-,\dots,-}{\omega_1,\dots,\omega_r}} && (\text{“left-lateral mould”}) \quad (168)\\
\mathbf{nad}^{t_1,\dots,t_r}_{\omega_*,t_*} \;&:=\; \epsilon_1\dots\epsilon_r\, \mathbf{d}^{\binom{\epsilon_1,\dots,\epsilon_r}{\omega_*,\dots,\omega_*}} && (\text{ “neutral mould” }) \quad (169)\\
\textit{with}\quad \epsilon_i \;&:=\; \textit{sign}(t_i-t_{i-1}) \quad (\forall i<r) \quad \textit{and} \quad \epsilon_r := \textit{sign}(t_r-t_*)
\end{aligned}$$

Due to the self-consistency relations, both the right- and left-lateral moulds encapsulate all the information about the entire weight system $\{\mathbf{d}^{\binom{\epsilon_1,\dots,\epsilon_r}{\omega_1,\dots,\omega_r}}\}$, and each one can be deduced from the other in a simple manner. As for the ‘neutral’ mould, due to the second self-consistency relation (165), it is actually independent of the second lower index t_* which consequently we shall drop.

[61]or more precisely, of the equation or system of which φ happens to be the solution.

The initial consonants **r/l/n** in the names of our moulds stand of course for *right/left/neutral* ; the inner vowels **e/a** stand respectively for *alternel/al* (for that's what our moulds will have to be) ; and the final **d** stands for *derivation.*

The four following conditions are equivalent:
D1 : the alien operators Δ_ω in (166) are (first-order) *alien derivations*
$\mathbf{D1_1}$: the right-lateral mould $\mathbf{red}^\bullet$ is *alternel*
$\mathbf{D1_2}$: the left-lateral mould $\mathbf{led}^\bullet$ is *alternel*
$\mathbf{D1_3}$: the neutral mould $\mathbf{nad}^\bullet_{\omega_*}$ is *alternal* for all ω_*

The two following conditions are equivalent:
D2 : the alien derivations Δ_ω *respect realness*
$\mathbf{D2_1}$: the moulds $\mathbf{red}^\bullet$ and $\mathbf{led}^\bullet$ are complex conjugate

The seven following conditions are equivalent:
D3 : the system Δ_ω in (166) of (first-order) alien derivations is *well-behaved*
$\mathbf{D3_1}$: we have universal bounds $|\,\mathbf{red}^{\boldsymbol{\omega}^{\prec\!\!\prec}}\,| \leq C_1\,(D_1)^r$
$\mathbf{D3_2}$: we have universal bounds $|\,\mathbf{red}^{\boldsymbol{\omega}^{\succ\!\!\succ}}\,| \leq C_2\,(D_2)^r$
$\mathbf{D3_3}$: we have universal bounds $|\,\mathbf{led}^{\boldsymbol{\omega}^{\prec\!\!\prec}}\,| \leq C_3\,(D_3)^r$
$\mathbf{D3_4}$: we have universal bounds $|\,\mathbf{led}^{\boldsymbol{\omega}^{\succ\!\!\succ}}\,| \leq C_4\,(D_4)^r$
$\mathbf{D3_5}$: we have universal bounds $|\,\mathbf{nad}^{\mathbf{t}^{\prec}}_{\omega_*}| \leq C_5\,(D_5)^{\omega_*\,r}$
$\mathbf{D3_6}$: we have universal bounds $|\,\mathbf{nad}^{\mathbf{t}^{\succ}}_{\omega_*}| \leq C_6\,(D_6)^{\omega_*\,r}$
with the ordinary/contracting arborification rules of §2.1 and positive constants C_i, D_i.

The four following conditions are equivalent:
D4 : the system $\{\Delta_\omega\}$ of alien derivations *is scale-invariant.*
$\mathbf{D4_1}$: the mould $\mathbf{red}^\bullet$ is a homogeneous function of $\boldsymbol{\omega}$.
$\mathbf{D4_2}$: the mould $\mathbf{led}^\bullet$ is a homogeneous function of $\boldsymbol{\omega}$.
$\mathbf{D4_3}$: the mould $\mathbf{nad}^\bullet_{\omega_*,t_*}$ is independent of its first lower index ω_*

Tempting answer: the standard alien derivations.

In view of the Leibniz relations (56) verified by the right- and left-lateral differential operators $\mathbb{\Delta}^{\mp}_\omega$, their sums $\mathbf{rul}, \mathbf{lur}$ are mutually inverse alien automorphisms :

$$\mathbf{rul}\,.\,\mathbf{lur} = \mathbf{1} \quad , \quad \mathbf{rul} := 1 + \sum_{\omega>0} \mathbb{\Delta}^-_\omega \quad , \quad \mathbf{lur} := 1 + \sum_{\omega>0} \mathbb{\Delta}^+_\omega \tag{170}$$

$$\mathbf{rul}(\hat\varphi_1 \star \hat\varphi_2) \;\equiv\; \mathbf{rul}(\hat\varphi_1) * \mathbf{rul}(\hat\varphi_2) \tag{171}$$

$$\mathbf{lur}(\hat\varphi_1 \star \hat\varphi_2) \;\equiv\; \mathbf{lur}(\hat\varphi_1) * \mathbf{lur}(\hat\varphi_2) \tag{172}$$

whose common logarithm resolves into a sum of true (ie first-order) alien derivations $\mathbb{\Delta}_\omega = \mathbf{dun}_\omega$:

$$+log(\mathbf{lur}) \;\equiv\; -log(\mathbf{rul}) \;=:\; 2\pi i\,\mathbf{dun} \;\equiv\; 2\pi i \sum_{\omega>0} \mathbf{dun}_\omega \tag{173}$$

They are the so-called *standard* or *uniform* alien derivations. A simple calculation shows that their weights are indeed the ones given in (57). As already pointed out, they are independent of the gaps ω_i and assign symmetric roles to the $\pm$ signs[62]. From the way they are constructed, the standard alien derivations clearly verify **D1** , **D2** , **D4**, but not **D3**.

Let us explain why. Return to Example 1 of §3.1. Write the resurgence equation (72) successively for

$$\mathbb{\Delta}_n := \mathbf{mun}_n \quad , \quad \mathbb{\Delta}_n^+ := \mathbf{lur}_n \quad , \quad \mathbb{\Delta}_n^- := \mathbf{rul}_n$$

and denote by A_n , A_n^+ , A_n^- the corresponding coefficients that appear on the right-hand side. Whereas, as we saw in §4, the 'lateral' coefficients $\mathbb{A}_n^\pm$ give rise to *convergent* Fourier mappings :

$$z \;\mapsto\; z + \sum_{0<n} A_n^\pm\, e^{-n\,z} \quad (\Re z \geqq 1) \quad ; \quad z \;\mapsto\; z + \sum_{n<0} A_n^\pm\, e^{-n\,z} \quad (\Re z \leqq -1)$$

the 'median' coefficients $\mathbb{A}_n^\pm$ give rise to formal series :

$$2\pi i \sum_{0<n} A_n\, e^{-n\,z} \quad (\Re z \geqq 1) \quad ; \qquad 2\pi i \sum_{n<0} A_n\, e^{-n\,z} \quad (\Re z \leqq -1)$$

which are the *infinitesimal generators* of the former, and therefore generically *divergent*, with Gevrey-1 bounds $|A_n| \le C_0\, e^{C_1\,n}\, n^n$ instead of the exponential bounds $|A_n| \le C_0\, e^{C_1\,n}$ which *well-behaved* alien derivations ought to ensure.

6.4 Challenge 3 : Searching for well-behaved resurgence monomials.

In §2.4 we already pointed to the existence of two basic sorts of resurgence monomials, the ∂- and Δ-friendly sorts, and we produced some elementary examples. Here, we are specifically interested in the the notion of Δ-friendly *and* 'well-behaved' resurgence monomials. So let us recall, and sharpen, the main demands we are making on these systems $\{\mathcal{U}^{\boldsymbol{\omega}}(z) = \mathcal{U}^{\omega_1,\dots,\omega_r}(z)\}$ of 'atom-like' resurgent functions. We want our monomials :

(1) to be as elementary as possible, and free of unnecessary parameters

[62]These two properties even characterise the standard derivations

(2) to behave simply under multiplication or convolution[63]
(3′) to behave simply under alien differentiation, and not too badly under natural differentiation
(4′) to be "complete" in the sense of enabling us to expand (or approximate) any given resurgent function φ :

$$\varphi(z) \quad \text{“} = \text{”} \quad \sum_{\boldsymbol{\omega}} c_{\boldsymbol{\omega}}(z)\ \mathcal{U}^{\boldsymbol{\omega}}(z) \quad := \quad \sum_{r\geq 0}\sum_{\omega_i} c_{\omega_1,\dots,\omega_r}(z)\ \mathcal{U}^{\omega_1,\dots,\omega_r}(z) \tag{174}$$

with coefficients $c_{\boldsymbol{\omega}}(z)$ that are either ordinary constants or "resurgence constants", that is to say functions with only vanishing alien derivatives: $\Delta_{\omega_0}\, c_{\boldsymbol{\omega}}(z) \equiv 0$, $\forall \omega_0$.

Condition (2) means that:

$$\mathcal{U}^{\boldsymbol{\omega}'}\,\mathcal{U}^{\boldsymbol{\omega}''} \equiv \sum_{\boldsymbol{\omega}\,\in\,\mathbf{sha}(\boldsymbol{\omega}',\,\boldsymbol{\omega}'')} \mathcal{U}^{\boldsymbol{\omega}} \tag{175}$$

with a sum extending to all sequences $\boldsymbol{\omega}$ obtained by *shuffling* the two factor sequences $\boldsymbol{\omega}'$, $\boldsymbol{\omega}''$. In other words, the mould $\mathcal{U}^\bullet$ should be *symmetral.*[64]

Condition (3') is relative to a given basis $\{\Delta_\omega\}$ of the algebra ALIEN of alien derivations. In concrete terms this condition stipulates that:

$$\begin{aligned} \Delta_{\omega_0}\mathcal{U}^{\omega_1,\dots,\omega_r} &\equiv \mathcal{U}^{\omega_2,\dots,\omega_r} && \textit{if}\ \ \omega_0 = \omega_1 \\ &\equiv 0 && \textit{if}\ \ \omega_0 \neq \omega_1 \end{aligned} \tag{176}$$

Clearly, there exist infinitely many multiplicative systems of resurgence monomials. Indeed, if $\mathcal{U}^\bullet(z)$ is one such system, so will be the system $\mathcal{U}^\bullet_{\mathcal{C}}(z) := \mathcal{U}^\bullet(z)\times\mathcal{C}^\bullet(z)$ derived therefrom by postmultiplication[65] by any symmetral, resurgence-constant mould $\mathcal{C}^\bullet(z)$. Multiplicative systems of resurgence monomials are extremely useful for solving resurgence equations, or systems of such equations, and to express their solutions in the form of expansions of type (174), often with constant coefficients $c_{\boldsymbol{\omega}}$. Thus, if we revert to Example 1 and try to solve the system of resurgence equations (113) that characterise the normalising transformation $f^\star$, we find:

$$f^\star(z) \quad := \quad z - \sum_{r}\sum_{n_i} A_{n_1}\dots A_{n_1}\ \ \Gamma_{n_1,\dots,n_r}\ \ \mathcal{U}^{n_1,\dots,n_r}(z) \tag{177}$$

$$\textit{with} \qquad \Gamma_{n_1,\dots,n_r} \quad := \quad (n_1)\,(n_1+n_2)\dots(n_1+n_2+\dots+n_{r-1}) \tag{178}$$

[63]depending on the model.
[64]see §6.1
[65]in the sense of moulds.

But the real issue of course is convergence. We might try to solve it on an *ad hoc* basis, ie by choosing our resurgence monomials differently for each problem. But we are more ambitious: we ask for resurgence monomials that work in *all* cases. [66] That may seem a tall order, but it is feasible! The answer lies in the notion of *well-behaved* systems of resurgence monomials. And not only do such systems exist, but there is a canonical choice!

To any given system of resurgence monomials we may associate a *right-lateral* mould ${}^{re}\mathcal{U}^{\bullet}(z)$ and a *left-lateral* mould ${}^{le}\mathcal{U}^{\bullet}(z)$ characterised by the orthogonality conditions:

$$\{{}^{re}\mathcal{U}^{\omega_1,\dots,\omega_r}\} \quad \textit{orthogonal to} \quad \{\Delta^+_{\omega_1,\dots,\omega_r} := \Delta^+_{\omega_r}\dots\Delta^+_{\omega_1}\} \tag{179}$$

$$\{{}^{le}\mathcal{U}^{\omega_1,\dots,\omega_r}\} \quad \textit{orthogonal to} \quad \{\Delta^-_{\omega_1,\dots,\omega_r} := \Delta^-_{\omega_r}\dots\Delta^-_{\omega_1}\} \tag{180}$$

The three following conditions are equivalent:
M1 : the system of resurgence monomials is *multiplicative*
$\mathbf{M1_1}$: the right-lateral mould ${}^{re}\mathcal{U}^{\bullet}$ is *symmetrel*[67]
$\mathbf{M1_2}$: the left-lateral mould ${}^{le}\mathcal{U}^{\bullet}$ is *symmetrel*

Observe that the criterion here is *symmetrel* and not *symmetral* as in §3.6.1 This is because the lateral alien operators $\Delta^{\pm}_{\omega}$ are not first-order alien derivations (see §3.5.1) and verify instead the Leibniz rules (56).

The two following conditions are equivalent:
M2 : the system of multiplicative resurgence monomials is *real*
$\mathbf{M2_1}$: the moulds ${}^{re}\mathcal{U}^{\boldsymbol{\omega}}$ and ${}^{le}\mathcal{U}^{\boldsymbol{\omega}}$ are complex conjugate

The five following conditions are equivalent:
M3 : the system of multiplicative resurgence monomials is *well-behaved*
$\mathbf{M3_1}$: we have universal bounds $\|\,{}^{re}\mathcal{U}^{\boldsymbol{\omega}^{\nless}}\,\| \le C_1\,(D_1)^{r(\boldsymbol{\omega}^{\nless})}$
$\mathbf{M3_2}$: we have universal bounds $\|\,{}^{re}\mathcal{U}^{\boldsymbol{\omega}^{\ngtr}}\,\| \le C_2\,(D_2)^{r(\boldsymbol{\omega}^{\ngtr})}$
$\mathbf{M3_3}$: we have universal bounds $\|\,{}^{le}\mathcal{U}^{\boldsymbol{\omega}^{\nless}}\,\| \le C_3\,(D_3)^{r(\boldsymbol{\omega}^{\nless})}$
$\mathbf{M3_4}$: we have universal bounds $\|\,{}^{le}\mathcal{U}^{\boldsymbol{\omega}^{\ngtr}}\,\| \le C_4\,(D_4)^{r(\boldsymbol{\omega}^{\ngtr})}$
for some suitable norm $\|.\|$ and with contracting arborification.

The three following conditions are equivalent:
M4: the system of multiplicative resurgence monomials is *scale invariant*
$\mathbf{M4_1}$: $\mathcal{U}^{\omega_1,\dots,\omega_r}(z) \equiv \mathcal{U}^{l\,\omega_1,\dots,l\,\omega_r}(l^{-1}z)$ $\quad$ ($\forall l$; in the multiplicative models)
$\mathbf{M4_2}$: $\hat{\mathcal{U}}^{\omega_1,\dots,\omega_r}(z) \equiv l\,.\hat{\mathcal{U}}^{l\,\omega_1,\dots,l\,\omega_r}(l\,\zeta)$ $\quad$ ($\forall l$; in the convolutive model)

[66] ie for all systems of resurgence equations that do admit solutions.
[67] see §6.1.

Tempting answer: the standard resurgence monomials.

The *standard* or *hyperlogarithmic* monomials introduced in §2.4 (see also §4.1) cannot be beaten for simplicity. They also verify $\mathbf{M1}, \mathbf{M2}, \mathbf{M4}$. But, as we saw in §4 when attempting to harness them for canonical synthesis, they don't verify **M3**. So they are not the real answer.

6.5 Challenge 4 : Searching for the proper notion of ramified exponential growth.

Conditions **R2** and **R4** (respecting realness and scale invariance) are automatically fulfilled for all sensible growth conditions.

Conditions **R1** means stability under the convolution product: if $\overset{\diamond}{\varphi}_1$ and $\overset{\diamond}{\varphi}_2$ possess minors that verify our growth condition, so too should $\overset{\diamond}{\varphi}_1 \star \overset{\diamond}{\varphi}_2$. This condition has two sides to it. In the absence of singularities, or in the presence of a finite number of them, or again of soft (eg integrable) singularities, the stability of the growth regimen at infinity is wholly unproblematic : it is automatically verified under very mild 'convexity assumptions' on the growth modulus. Difficulties arise only in the presence of fiercely explosive singularities (of a type which fortunately doesn't occur in most applications, such as *analysability theory*, [E6],[E7]) and then only on paths that keep dangerously close to an infinite number of such singularities. This – largely academic – problem is examined in §9.3. It is essentially a question of controlling singularity explosiveness, or of keeping clear of singularities. But the "growth at infinity" aspect is wholly unproblematic : with the above reservations, naive exponential growth is stable under convolution.

So the crux is **R3**. It should be rephrased here as "stability under natural operations" such as inversion (ie convolutive inversion in the Borel plane) or the solving of differential equations (ie their translations in the convolutive model) etc. But actually, for most growth conditions, including the one we shall settle for in §9, the seemingly weaker requirement of "stability under (convolutive) inversion" suffices to ensure the other ones. The main difficulty here has nothing to do with the explosiveness or density of the singularities in the Borel plane. It originates, rather, in the phenomenon of "faster-than-lateral" growth which was already pointed out in §6.1 and which comes into play practically each time we have a Borel axis $\arg \zeta = \theta$ that carries an infinity of singular points.

Tempting answer: straightforward exponential bounds.

The naive answer would be to define *ramified-exponential growth* as ordinary exponential growth along any broken line Γ starting at the origin, stearing clear of singularities ω in a uniform way (ie coming no closer to them than, say, ρ or $\rho\,|\omega|$), and with segments remaining within given angular bounds $\theta_1 \leq \theta \leq \theta_2$, with uniform bounds in θ_1, θ_2, ρ[68]:

$$|\hat{\varphi}(\zeta)| \leq C_{\theta_1,\theta_2,\rho}\ \exp\left(D_{\theta_1,\theta_2,\rho}\,|\zeta|\right) \tag{181}$$

or in the case of *ramified-accelerable growth*, to impose some analogous, but weaker bounds, like for instance:

$$|\hat{\varphi}(\zeta)| \leq C_{\theta_1,\theta_2,\rho}\ \exp\left(D_{\theta_1,\theta_2,\rho}\,|\zeta|^{\frac{1}{1-\alpha}}\right) \qquad (0<\alpha<1) \tag{182}$$

for elementary accelerations $z \mapsto z^{\frac{1}{\alpha}}$

But all these definitions are unsuitable because, while meeting conditions $\mathbf{R1}, \mathbf{R2}, \mathbf{R4}$, they completely fail with respect to $\mathbf{R3}$: see the introductory para §6.1.

6.6 Proofs.

The alien operators $\mathbf{op}$ in this section are all of the "exponential-bearing" sort, like $\mathbb{\Delta}_\omega$, $\mathbb{\Delta}_\omega^\pm$, etc. Their interpretation in the *multiplicative models* offers no difficulty, except that they act internally, not on the algebra $RESUR$ of resurgent functions, but on $RESUR$ tensored by exponential symbols:

$$RESUR \otimes EXP \ \ := \ \ \oplus_\omega\ RESUR_\omega \ \ := \ \ \oplus_\omega\ e^{-\omega\, z}\ RESUR \tag{183}$$

In the *convolutive model* these operators are called "stationary" because, unlike with the ordinary alien operators, their action there involves no translation: $\mathbf{op}\ \hat{\varphi}(\zeta)$ is always a finite sum of type $\sum \gamma_i\, \hat{\varphi}(\zeta_i)$ with points ζ_i that are all located *over* ζ. The exact interpretation in this model demands some care and a number of auxiliary constructions (see [E8]) which we won't recall here, because we require only a few elementary facts about them.

The first fact is the existence, for each $\mathbf{op}$, of a unique decomposition into a sum[69] of 'homogeneous' components $\mathbf{op}_\omega$ characterised by:

$$\mathbf{op} = \sum_{0\leq\omega} \mathbf{op}_\omega \qquad \textit{with} \quad \mathbf{op}_0 \in \mathbb{C}\,.\,\mathrm{id} \qquad \textit{and} \tag{184}$$

$$\{\omega_0 > 0\ ,\ \hat{\varphi}\ \textit{regular over}\ \omega_0\} \ \ \Rightarrow \ \ \{\mathbf{op}_{\omega_0}\,\varphi \equiv 0\} \tag{185}$$

[68] uniform bounds in θ_1, θ_2 are indispensible even in the case of ordinary, non-ramified exponential growth.

[69] the sum may be infinite, but when applied to any given test function, it produces only finitely many non-zero terms.

The second fact is the possibility of characterising each **op** by a system of *weights* $\mathbf{op}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_r}}$ subject to the self-consistency relations (164),(165)[70].

The third fact is the existence of "complete" operators $\mathbf{op}^*$ whose components $\mathbf{op}^*_\omega$ generate $ALIEN$, so that any other **op** can be expressed uniquely in terms of $\mathbf{op}^*$ with the help of a "transit mould" $\big\langle \mathbf{op}\,,\ \mathbf{op}^*\big\rangle^\bullet$ like this:

$$\mathbf{op} = \sum \mathbf{op}_\omega = \sum \big\langle \mathbf{op}, \mathbf{op}^*\big\rangle^{\omega_1,\ldots,\omega_r}\ \mathbf{op}^*_{\omega_r}\ldots\mathbf{op}^*_{\omega_1} \tag{186}$$

The fourth fact is the existence of uniquely defined operators $\mathbf{op} = \binom{\mathbf{m}_1}{\mathbf{m}_2}$ connecting by post-composition any pair $\mathbf{m}_1, \mathbf{m}_2$ of uniformising averages:

$$\mathbf{m}_2 = \mathbf{m}_1 \binom{\mathbf{m}_1}{\mathbf{m}_2} \tag{187}$$

The fifth fact is the existence of the special, elementary operators:

$$\mathbf{lur} \ := \ \binom{\mathbf{mul}}{\mathbf{mur}} \ = \ 1 + \sum_{0<\omega} \mathbb{\Delta}^+_\omega \tag{188}$$

$$\mathbf{rul} \ := \ \binom{\mathbf{mur}}{\mathbf{mul}} \ = \ 1 + \sum_{0<\omega} \mathbb{\Delta}^-_\omega \tag{189}$$

which are *automorphisms* (see (56)) and *complete.*

The sixth fact is the simple interpretaion of the transit moulds $< .,. >^\bullet$ relative to **lur** or **rul** in terms of operator weights [71]:

$$\big\langle \mathbf{op}\,, \mathbf{rul}\big\rangle^{\omega_1,\ldots,\omega_r} \ = \ (-1)^r\ \mathbf{op}^{\binom{+\ ,\ldots,\ +}{\omega_1\ ,\ldots,\ \omega_r}} \tag{190}$$

$$\big\langle \mathbf{op}\,, \mathbf{lur}\big\rangle^{\omega_1,\ldots,\omega_r} \ = \ (-1)^r\ \mathbf{op}^{\binom{-\ ,\ldots,\ -}{\omega_1\ ,\ldots,\ \omega_r}} \tag{191}$$

If we now recall the constrution of the right/left lateral moulds, we find for an alien derivation **d** these expressions:

$$\mathbf{red}^\bullet \ = \ \big\langle \mathbf{d}\,, \mathbf{rul}\big\rangle^\bullet \tag{192}$$

$$\mathbf{led}^\bullet \ = \ \big\langle \mathbf{d}\,, \mathbf{lur}\big\rangle^\bullet \tag{193}$$

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[70] and *not* (151),(152). On the other hand, due to "stationarity", the action of **op** in the convolutive model is not like (166) but more like (153), ie without ω shift.

[71] Heed the signs! In (188),(189) **lur** , **rul** are associated with the signs $+, -$ respectively, but in (190),(191) it is the reverse.

and for a uniformising average $\mathbf{m}$ we find these :

$$\mathbf{rem}^{\bullet} \;=\; \Big\langle \binom{\mathbf{mul}}{\mathbf{m}}, \mathbf{rul} \Big\rangle^{\bullet} = \Big\langle \binom{\mathbf{mul}}{\mathbf{m}}, \binom{\mathbf{mur}}{\mathbf{mul}} \Big\rangle^{\bullet} \tag{194}$$

$$\mathbf{lem}^{\bullet} \;=\; \Big\langle \binom{\mathbf{mur}}{\mathbf{m}}, \mathbf{lur} \Big\rangle^{\bullet} = \Big\langle \binom{\mathbf{mur}}{\mathbf{m}}, \binom{\mathbf{mul}}{\mathbf{mur}} \Big\rangle^{\bullet} \tag{195}$$

Criteria for C1 :

Using the interpretation of the four basic symmetry types for moulds (see §2.1.3 and below ; of course, we may replace *field* by *derivation* and *diffeo* by *automorphism*) and remembering that the alien operators $\mathbf{lur}, \mathbf{rul}$ are automorphisms, we see that an alien operator $\mathbf{d}$ is a derivation if and only if the transit moulds (192),(193) are alternel; and that a uniformising average $\mathbf{m}$ respects convolution if and only if the transit moulds (194),(195) are symmetrel. This establishes the equivalent criteria given in §6 for **A1** and **D1**. As for **M1** and **E1**, the equivalent criteria are immediate.

Criteria for C2 and C4 :

All proofs are totally elementary here – the criteria are mere rephrasings.

Criteria for C3 :

Respecting "lateral growth" [72] means that $\mathbf{m}\,\hat{\varphi}(\zeta)$ should not grow significantly faster than the right and left determinations $\mathbf{mur}\,\hat{\varphi}(\zeta)$ or $\mathbf{mul}\,\hat{\varphi}(\zeta)$ (ie *exponentially* in a monocritical problem). But in view of what precedes, the three averages are interrelated as follows :

$$\mathbf{m}\,\hat{\varphi}\,(\zeta) \;=\; \sum (\mathbf{rem}^{\omega_1,\ldots,\omega_r})\;\; (\mathbf{mul}\;\; \Delta^{-}_{\omega_r} \ldots \Delta^{-}_{\omega_1}\;\; \hat{\varphi}\,(\zeta)) \tag{196}$$

$$\mathbf{m}\,\hat{\varphi}\,(\zeta) \;=\; \sum (\mathbf{lem}^{\omega_1,\ldots,\omega_r})\;\; (\mathbf{mur}\;\; \Delta^{+}_{\omega_r} \ldots \Delta^{+}_{\omega_1}\;\; \hat{\varphi}\,(\zeta)) \tag{197}$$

or, in compact mould-comould form :

$$\mathbf{m}\,\hat{\varphi}\,(\zeta) \;=\; \sum (\mathbf{rem}^{\bullet})\;\; (\mathbf{mul}\;\; \Delta^{-}_{\bullet}\;\; \hat{\varphi}\,(\zeta)) \tag{198}$$

$$\mathbf{m}\,\hat{\varphi}\,(\zeta) \;=\; \sum (\mathbf{lem}^{\bullet})\;\; (\mathbf{mur}\;\; \Delta^{+}_{\bullet}\;\; \hat{\varphi}\,(\zeta)) \tag{199}$$

The trouble is that in practically all non-linear problems, the comould part $\mathbf{mur} /\!\!/ \mathbf{mul}\;\; \Delta^{\pm}_{\bullet}\;\; \hat{\varphi}\,(\zeta)$ has faster-than-lateral growth and there is no way we can choose $\mathbf{m}$ to make $\mathbf{rem}^{\bullet} /\!\!/ \mathbf{lem}^{\bullet}$ small enough to offset this effect[73]. So

[72] in the Borel plane, along axis $\arg\zeta = \theta_0$. Think of θ_0 as 0.

[73] more precisely *if* – as we must here – we impose **A1** *and* **A2** on $\mathbf{m}$: see §7.6.

instead of asking for the normal convergence of (198),(199) we should ask for that of the (formally equivalent) arborified expansions:

$$\mathbf{m}\,\hat\varphi\,(\zeta) \;=\; \sum (\mathbf{rem}^{\bullet^{\prec\!\!\!<}}) \;\; (\mathbf{mul}\;\; \mathbb{\Delta}^{-}_{\bullet^{\prec\!\!\!<}} \;\; \hat\varphi\,(\zeta)) \tag{200}$$

$$\mathbf{m}\,\hat\varphi\,(\zeta) \;=\; \sum (\mathbf{lem}^{\bullet^{\prec\!\!\!<}}) \;\; (\mathbf{mur}\;\; \mathbb{\Delta}^{+}_{\bullet^{\prec\!\!\!<}} \;\; \hat\varphi\,(\zeta)) \tag{201}$$

Of course it has to be *contracting arborification* since $\mathbf{rem}^\bullet /\!\!/ \mathbf{lem}^\bullet$ shall be chosen *symmetrel* (due to **A1**) and since $\mathbb{\Delta}^-_\bullet /\!\!/ \mathbb{\Delta}^+_\bullet$ is *co-symmetrel* anyway.

The ultimate justification behind this choice is that *it works.* And it works because, in practically all natural situations, the resurgence is governed by some form or other of the Bridge Equation, which says that alien differentiation $\mathbb{\Delta}_\bullet$ or $\mathbb{\Delta}^\pm_\bullet$ amounts to ordinary differentiation $\mathbb{A}_\bullet$ or $\mathbb{A}^\pm_\bullet$ relative to the resurgence variable z and a finite number of parameters u_i [74].

Now, it is a general fact that coarborification succeeds in drastically *decreasing the norm* of ordinary differential operators. And it is another fact that we can almost always manage to construct moulds – no matter what *natural* constraints they are subject to, like here *symmetrelity* plus **A2** etc – in such a way as to *prevent a norm increase* under arborification.

This, roughly, is the motivation behind the decision to interpret the (vague) condition **A3** in the form of the (precise and mutually equivalent) conditions $\mathbf{A3_1}, \ldots, \mathbf{A3_6}$ [75].

The same applies for *alien derivations* and *resurgence monomials.* In the coming sections, we shall show how to construct *uniformising averages*, *alien derivations* and *resurgence monomials* in conformity with these conditions. Then we shall tackle a particular application – Canonical Object Synthesis – and show in detail how arborification-coarborification works.

[74] ie those parameters that enter the complete, parameter-saturated solution of the problem. The operators $\mathbb{A}_\bullet$ themselves depend on the problem, and so does the precise form of the Bridge Equation. In that deeper sense, and despite the *'Bridge phenomenon'*, alien calculus is totally irreducible to ordinary differential calculus, no matter what some adepts of Galois theory may claim.

[75] one point should be emphasised though: the conditions in the pairs $\mathbf{A3_1}$ and $\mathbf{A3_2}$, or $\mathbf{A3_3}$ and $\mathbf{A3_4}$, or $\mathbf{A3_5}$ and $\mathbf{A3_6}$, would not be equivalent if they were to bear on *general armoulds.* But here, majorising the arborification or the antiarborification amounts to the same because the armoulds in question are induced by moulds. These moulds, moreover, are either symmetrel (like the lateral moulds) or symmetral (like the neutral mould), which further simplifies things, by essentially reducing the passage *arborification* $\to$ *coarborification* to *multiplicative inversion.*

7 Well-behaved uniformising averages.

7.1 Reminder about the standard or median average.

The standard or median average $\mathbf{mun} = \mathbf{mu}_{\frac{1}{2},\frac{1}{2}}$ can be embedded into a one-parameter family of so-called 'uniform' averages $\mathbf{mu}_{\alpha,\beta}$ (with $\alpha+\beta=1$) that join the right-lateral average $\mathbf{mur} = \mathbf{mu}_{1,0}$ and the left-lateral average $\mathbf{mul} = \mathbf{mu}_{0,1}$:

$$\begin{array}{llllll}
\mathbf{mu}_{\alpha,\beta} & := & \mathbf{mur}\,.\,(\mathbf{rul})^{\beta} & \equiv & \mathbf{mul}\,.\,(\mathbf{lur})^{\alpha} & (\alpha+\beta=1) \\
\mathbf{mu}_{\alpha,\beta}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} & := & \frac{\Gamma(p+\alpha)\,\Gamma(q+\beta)}{\Gamma(r+1)\ \Gamma(\alpha)\,\Gamma(\beta)} & & & (p+q=r) \\
\mathbf{mun}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} & := & \mathbf{mu}_{\frac{1}{2},\frac{1}{2}}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} & = & \frac{(2p)!\,(2q)!}{4^{p+q}\,p!\,q!\,(p+q)!} & (p+q=r)
\end{array}$$

We call these averages 'uniform' because their weights depend on the number (p,q) of $(+,-)$ signs, but not on the gaps ω_i. Quite clearly, they always verify **A1**, **A4**, but they verify **A2** only if $\alpha=\bar{\beta}$ ($\Rightarrow \Re\,\alpha = \Re\,\beta = 1/2$), and **A3** only if $\alpha,\beta\in\mathbb{Z}$ (see §7.5 , §8.7). These last two conditions being incompatible, there *exists no well-behaved average in the uniform family.*

7.2 Diffusion-induced averages.

Let $\{g_\omega(\bullet)\,,\,\omega>0\}$ be some multiplicative semigroup and consider its Fourier transform, the convolution semigroup $\{f_\omega(\bullet)\,,\,\omega>0\}$:

$$g_\omega(y) \;:=\; e^{-\omega\,\gamma(y)} \qquad (y\in\mathbb{R}) \tag{202}$$

$$f_\omega(x) \;:=\; \frac{1}{2\pi}\int_{\mathbb{R}} g_\omega(y)\,e^{ixy}\,dy \qquad (x\in\mathbb{R}) \tag{203}$$

$$g_{\omega_1}(y)\,g_{\omega_2}(y) \;\equiv\; g_{\omega_1+\omega_2}(y) \tag{204}$$

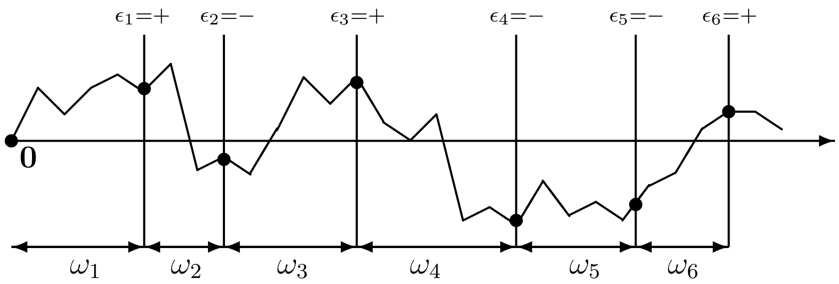
$$(f_{\omega_1} * f_{\omega_2})(x) \;:=\; \int_{-\infty}^{+\infty} f_{\omega_1}(x_1)\,f_{\omega_2}(x-x_1)\,dx_1 \equiv f_{\omega_1+\omega_2}(x) \tag{205}$$

Let $\gamma(y)$ be analytic on $\mathbb{R}$, vanish at $y=0$ and have a fast increasing real part as $y\to\pm\infty$ so that $g_\omega(.), f_\omega(.)\in\mathbb{L}^1(\mathbb{R})$ with $g_\omega(0)=1$ and $\|f_\omega\|_{\mathbb{L}^1}=1$. Let us view each function $f_\omega(.)$ as defining the 'probability distribution'[76] at the time $t=\omega$, on the vertical axis $\omega+i\,\mathbb{R}$, of a particle starting from

[76]strictly speaking, this applies only if $f_\omega(y)\geq 0$ but we can drop this restriction, because the numbers we shall proceed to define with the help of $\{f_\omega(.)\}$ neednot be real – they must simply verify certain algebraic relations (the ones implied by **A1**, **A2**, **A4**) and possess the right sort of bounds. That's why we put 'probability' within inverted commas.

the origin of $\mathbb{C}$ at $t = 0$, moving along $\mathbb{R}^+$ at uniform horizontal speed, and diffusing randomly in the vertical direction.

We may then define $\mathbf{m}^{\binom{\boldsymbol{\epsilon_1},\dots,\boldsymbol{\epsilon_r}}{\boldsymbol{\omega_1},\dots,\boldsymbol{\omega_r}}}$ as the probability of our particle's successively crossing $\omega_1 + i\,\mathbb{R}^{\epsilon_1},\ \omega_1 + \omega_2 + i\,\mathbb{R}^{\epsilon_2}, \dots, \omega_1 + \omega_2 + \dots \omega_r + i\,\mathbb{R}^{\epsilon_r}$.



Since these numbers $\mathbf{m}^{\binom{\epsilon_1,\dots,\epsilon_r}{\omega_1,\dots,\omega_r}}$ verify the self-consistency relations (151) and (152), they may be regarded as *weights* defining a uniformising average $\mathbf{m}$. That average clearly respects *realness* (condition **A2**) *iff* γ is an even function. Less obviously, it also respects *convolution* and *lateral growth* (conditions **A1** and **A3**): see §7.5 infra.

7.3 Scale-invariance.

If we now select some $\tau > 0$ and set $\gamma(y) = {}^\tau\gamma(y) := |y|^\tau$, the variables x, y coalesce with the parameter ω:

$$
\begin{aligned}
{}^\tau g_\omega(y) \;&\equiv\; {}^\tau g_1(\omega^{\frac{1}{\tau}}\, y) \;:=\; \exp(-\omega\,|y|^\tau) && (y \in \mathbb{R})(206)\\
{}^\tau f_\omega(x) \;&\equiv\; \omega^{-\frac{1}{\tau}}\; {}^\tau f_1(\omega^{-\frac{1}{\tau}}\, x) \;:=\; \frac{1}{2\pi}\int_{\mathbb{R}} {}^\tau g_\omega(y)\, e^{ixy}\, dy && (x \in \mathbb{R})(207)
\end{aligned}
$$

This automatically ensures the last missing condition **A4**: invariance under a simultaneous, uniform dilation of all gaps ω_i.

As a consequence, *for any* $\tau \in]0, +\infty[$ *the corresponding uniformising average* ${}^\tau\mathbf{m}$ *is well-behaved.*

Although this one-parameter family $\{{}^\tau\mathbf{m}\}$ does contain all well-behaved, diffusion-induced aveages, there exist many well-behaved, averages which are not of this form, like for instance those of type:

$$\mathbf{m} := {}^{\tau_0}\mathbf{m} \left(\frac{{}^{\tau_1^*}\mathbf{m}}{{}^{\tau_1}\mathbf{m}}\right)^{n_1} \dots \left(\frac{{}^{\tau_k^*}\mathbf{m}}{{}^{\tau_k}\mathbf{m}}\right)^{n_k} \qquad (\tau_i, {\tau_i}^* \in \mathbb{R}^+\ ,\ n_i \in \mathbb{Z}\ ;\ k \in \mathbb{N}) \quad (208)$$

where $\binom{\bullet}{\bullet}$ denotes the unique alien operator that connects any given pair of uniformising averages:

$$ {}^{\tau_i}\mathbf{m} := {}^{\tau_i^*}\mathbf{m} \binom{{}^{\tau_i^*}\mathbf{m}}{{}^{\tau_i}\mathbf{m}} \qquad (\tau_i, \tau_i^* \in \mathbb{R}^+) \tag{209} $$

But the most interesting average corresponds to the limit-case $\lim_{\tau \downarrow 0}\ {}^{\tau}\mathbf{m}$.

7.4 The standard and organic averages as limit-cases.

For $\tau \to +\infty$, ${}^{\tau}\mathbf{m}$ tends to the so-called "standard" average **mun** *which as we saw is not well-behaved.*
For $\tau \to +0$, ${}^{\tau}\mathbf{m}$ tends to the so-called "organic" average **mon** *which as we shall see is well-behaved.*

Though slightly more complex than the standard average, **mon** is by a long shot the simplest of all well-behaved averages. Its weights are given by the elementary recursion:

$$ \mathbf{mo}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)} = \mathbf{mo}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_{r-1}\\ \omega_1 & ,\dots, & \omega_{r-1}\end{smallmatrix}\right)} P_r \tag{210} $$

with

$$ P_r := 1 - \frac{1}{2}\frac{\omega_r}{\omega_1 + \dots \omega_r} \quad if \quad \epsilon_{r-1} = \epsilon_r \tag{211} $$

$$:= \frac{1}{2}\frac{\omega_r}{\omega_1 + \dots \omega_r} \quad if \quad \epsilon_{r-1} \neq \epsilon_r \tag{212} $$

Like **mun**, the organic **mon** can be imbedded into a one-parameter family of similar averages $\mathbf{mo}_{\alpha,\beta}$ $(\alpha + \beta = 1)$ whose weights obey the following recursion:

$$
\begin{aligned}
\mathbf{mo}_{\alpha,\beta}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)} &:= \mathbf{mo}_{\alpha,\beta}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_{r-1}\\ \omega_1 & ,\dots, & \omega_{r-1}\end{smallmatrix}\right)} \tfrac{\omega_1+\dots\omega_{r-1}+\alpha\,\omega_r}{\omega_1+\dots+\omega_r} && if\ \ (\epsilon_{r-1},\epsilon_r)=(+,+)\\
&:= \mathbf{mo}_{\alpha,\beta}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_{r-1}\\ \omega_1 & ,\dots, & \omega_{r-1}\end{smallmatrix}\right)} \tfrac{\beta\,\omega_r}{\omega_1+\dots+\omega_r} && if\ \ (\epsilon_{r-1},\epsilon_r)=(+,-)\\
&:= \mathbf{mo}_{\alpha,\beta}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_{r-1}\\ \omega_1 & ,\dots, & \omega_{r-1}\end{smallmatrix}\right)} \tfrac{\alpha\,\omega_r}{\omega_1+\dots+\omega_r} && if\ \ (\epsilon_{r-1},\epsilon_r)=(-,+)\\
&:= \mathbf{mo}_{\alpha,\beta}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_{r-1}\\ \omega_1 & ,\dots, & \omega_{r-1}\end{smallmatrix}\right)} \tfrac{\omega_1+\dots\omega_{r-1}+\beta\,\omega_r}{\omega_1+\dots+\omega_r} && if\ \ (\epsilon_{r-1},\epsilon_r)=(-,-)\\
\mathbf{mon}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)} &:= \mathbf{mo}_{1/2,1/2}^{\left(\begin{smallmatrix}\epsilon_1 & ,\dots, & \epsilon_r\\ \omega_1 & ,\dots, & \omega_r\end{smallmatrix}\right)} = 2^{-r} \prod_{i=2}^{i=r} && \left(|\epsilon_{i-1}+\epsilon_i| - \tfrac{\epsilon_{i-1}\,\epsilon_i\,\omega_i}{\omega_1+\dots+\omega_i}\right)
\end{aligned}
$$

with $|\epsilon_{i-1}+\epsilon_i|$ standing for 2 (resp. 0) if $\epsilon_{i-1}=\epsilon_i$ (resp. $\epsilon_{i-1}\neq\epsilon_i$).

Clearly $\mathbf{mo}_{1,0}=\mathbf{mu}_{1,0}=\mathbf{mur}$ and $\mathbf{mo}_{0,1}=\mathbf{mu}_{0,1}=\mathbf{mul}$. Moreover:

All averages $\mathbf{mo}_{\alpha,\beta}$ verify $\mathbf{A1},\mathbf{A3},\mathbf{A4}$. In order for them to verify $\mathbf{A2}$ and be well-behaved, we must have $\alpha=\bar{\beta}$ and so $\Re\,\alpha=\Re\,\beta=1/2$.

Let us add that F. Menous has devoted an entire PhD thesis ([Me]) to the subject of well-behaved uniformising averages. It contains in particular a meticulous investigation of the so-called *Catalan average*, which is not discussed here.

7.5 Proofs and comments.

Let us begin with the diffusion-induced averages. Applying the definition of §7.2 we get for the weights and the lateral moulds two types of integral expressions:

$$\mathbf{m}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_r}} := \int_{\mathbb{R}^r} f_{\omega_1}(x_1)\ldots f_{\omega_r}(x_r)\,\sigma_{\epsilon_1}(\check{x}_1)\ldots\sigma_{\epsilon_r}(\check{x}_r)\,dx_1\ldots dx_r \tag{213}$$

$$:= \frac{\epsilon_1\ldots\epsilon_r}{(2\pi i)^r}\int_{\mathbb{R}^r_{\epsilon_1,\ldots,\epsilon_r}} \frac{g_{\omega_1}(y_1)\ldots g_{\omega_r}(y_r)\;dy_1\ldots dy_r}{(y_1-y_2)\,(y_2-y_3)\ldots(y_{r-1}-y_r)\,y_r} \tag{214}$$

$$\mathbf{rem}^{\omega_1,\ldots,\omega_r} := \int_{\mathbb{R}^r} f_{\omega_1}(x_1)\ldots f_{\omega_r}(x_r)\;\mathrm{sofo}_{+}^{x_1,\ldots,x_r}\;dx_1\ldots dx_r \tag{215}$$

$$:= \frac{(-1)^r}{(2\pi i)^r}\int_{\mathbb{R}^r_{+,\ldots,+}} g_{\omega_1}(y_1)\ldots g_{\omega_r}(y_r)\;\mathrm{tas}_{\infty,0}^{y_1,\ldots,y_r}\;dy_1\ldots dy_r \tag{216}$$

$$\mathbf{lem}^{\omega_1,\ldots,\omega_r} := \int_{\mathbb{R}^r} f_{\omega_1}(x_1)\ldots f_{\omega_r}(x_r)\;\mathrm{sofo}_{-}^{x_1,\ldots,x_r}\;dx_1\ldots dx_r \tag{217}$$

$$:= \frac{1}{(2\pi i)^r}\int_{\mathbb{R}^r_{-,\ldots,-}} g_{\omega_1}(y_1)\ldots g_{\omega_r}(y_r)\;\mathrm{tas}_{\infty,0}^{y_1,\ldots,y_r}\;dy_1\ldots dy_r \tag{218}$$

The sign functions $\sigma_\pm$ are defined as in §2.1. The x-integrals are a direct rendition of the definition in §7.2, and always valid. The y-integrals are derived by Fourier transposition. They are easiest to interpret if g_ω is analytic on $\mathbb{R}$, for then y_i can be allowed to describe, in the course of integration, a minute half-circle around y_{i+1}, in the positive or negative direction depending on whether $\epsilon_i=+$ or $-$. When g_ω is merely $\mathcal{C}^\infty$ on $\mathbb{R}$, this circumvention rule should of course be re-interpreted in terms of distributions.

Now let us turn to the uniform and organic averages. We leave the fact of their being limit-cases of ${}^{\tau}\mathbf{m}$ as a exercise, and choose instead to regard them as defined by their weights (210), which clearly satisfy the self-compatibility conditions. This leads for the lateral moulds to the following expressions:

$$\begin{array}{llllll}
\mathrm{remu}^{\boldsymbol{\omega}}_{\alpha,\beta} & := & (-1)^r \frac{\Gamma(r+\alpha)}{(r+1)!\,\Gamma(\alpha)} & \overset{vol}{\leftrightarrow} \ \mathrm{lemu}^{\boldsymbol{\omega}}_{\alpha,\beta} & := & (-1)^r \frac{\Gamma(r+\beta)}{(r+1)!\,\Gamma(\beta)} \\
\mathrm{remun}^{\boldsymbol{\omega}} & := & (-1)^r\, 4^{-r}\, \frac{(2\,r)!}{r!\,r!} & \overset{vol}{\leftrightarrow} \ \mathrm{lemun}^{\boldsymbol{\omega}} & := & (-1)^r\, 4^{-r}\, \frac{(2\,r)!}{r!\,r!} \\
\mathrm{remo}^{\boldsymbol{\omega}}_{\alpha,\beta} & := & \prod_{i=1}^{i=r}\big(\frac{\beta\,\omega_i}{\omega_1+\cdots+\omega_i}-1\big) & \overset{vol}{\leftrightarrow} \ \mathrm{lemo}^{\boldsymbol{\omega}}_{\alpha,\beta} & := & \prod_{i=1}^{i=r}\big(\frac{\alpha\,\omega_i}{\omega_1+\cdots+\omega_i}-1\big) \\
\mathrm{remon}^{\boldsymbol{\omega}} & := & \prod_{i=1}^{i=r}\big(\frac{\frac{1}{2}\,\omega_i}{\omega_1+\cdots+\omega_i}-1\big) & \overset{vol}{\leftrightarrow} \ \mathrm{lemon}^{\boldsymbol{\omega}} & := & \prod_{i=1}^{i=r}\big(\frac{\frac{1}{2}\,\omega_i}{\omega_1+\cdots+\omega_i}-1\big)
\end{array}$$

We can now proceed to check the good-behaviour criteria.

Condition **A1** is easiest to check in the form $\mathbf{A1_1}$ or $\mathbf{A1_2}$. But the symmetrelity of $\mathbf{rem}^\bullet$ or $\mathbf{lem}^\bullet$ follows from the x-integral expressions (215),(217) and the symmetrelity of the moulds $\mathrm{sofo}^\bullet_\pm$. One may also reason on the y-integrals (216),(218), noting that, due to the circumvention rules (of y_{i+1} by y_i), the symmetrality of the mould $tas^\bullet_{\infty,0}$ translates into the symmetrelity of the integrals. This disposes of the diffusion-induced averages. For the limit cases **mun** and **mon** and their 'tilted' variants $\mathbf{mu}_{\alpha,\beta}$ and $\mathbf{mo}_{\alpha,\beta}$, the symmetrelity of the lateral moulds can be checked directly.

Conditions **A2** and **A4** are immediate to check directly on the weights.

This leaves **A3**. Being a *non-algebraic* growth condition, it is the most difficult one. However, by resorting to the criteria $\mathbf{A3_2}$ or $\mathbf{A3_4}$, it can be settled *algebraically* by observing that the mould $\mathrm{sofo}^\bullet_\pm$ *retain their form under contracting anti-arborification*: they simply become armoulds $\mathrm{sofo}^{\bullet\!\!\twoheadrightarrow}_\pm$ that are still defined by the formula in §2.1.9, except that the sums $\check{x}_i$ must now be extended to all x_j anterior to x_i under the anti-arborescent order[77]. Checking this in a simple exercise: reason inductively on the number of branches and make repeated use of the identities:

$$\sigma_\pm(a)\,\sigma_\pm(a+b)+\sigma_\pm(b)\,\sigma_\pm(a+b)=\sigma_\pm(a)\,\sigma_\pm(b)+\sigma_\pm(a+b) \qquad (219)$$

Remark: We have been using the right- and left-lateral moulds simply for the sake of symmetry, but they are essentially the same. Indeed, as soon as as condition **A1** is fulfilled, they are related under the simple involution:

$$\mathrm{rem}^\bullet \ \leftrightarrow \ \mathrm{lem}^\bullet \ : \quad \mathrm{lem}^\bullet \ \equiv \ (\mathrm{rem}^\bullet \circ \mathbf{J}^\bullet) \times \mathbf{J}^\bullet$$

[77] x_i itself should be included in the sum.

with $\mathbf{J}^{\omega_1,\dots,\omega_r} := (-1)^r$. The elementary mould $\mathbf{J}^\bullet$ is itself symmetrel and involutive in the sense that $\mathbf{J}^\bullet \circ \mathbf{J}^\bullet = \mathbf{1}^\bullet + \mathbf{I}^\bullet$.

7.6 When exactly must we take recourse to well-behaved averages ?

Oft-crossing paths.

Let $\hat{\varphi}$ possess infinitely many singularities over the zero axis $\arg\zeta = 0$ or $\mathbb{R}^+$ for short, for instance at all integral points n. As our example in §6.1 suggests, and as a careful examination of the non-linear examples in §3 would confirm, the growth of $\hat{\varphi}(\zeta)$ is *bad* on oft-crossing paths, and *worst* on the ever-crossing path $\Gamma^{+-+-\cdots}$ that intersects every interval between two consecutive singularites. There $\hat{\varphi}(\zeta)$ grows roughly like $e^{C_1|\zeta|}\,\Gamma(\zeta)$. But it so happens that on these very paths some (not all) well-behaved averages, in particular the organic average **mon**, possess weights that decrease precisely like $e^{C_1|\zeta|}/\Gamma(\zeta)$. Thus, on these *ever-crossing* paths, *functions* and *weights* are perfectly matched, their product having exponential growth at most. The same function-weight matching also holds, trivially, for the *never-crossing* ie lateral paths. This raises the question : don't there exist *'super well-behaved'* or *'magic'* averages, such that not only $|\mathbf{m}\,\hat{\varphi}(\zeta)|$ grows exponentially, but also the sum $\sum_\Gamma |\mathbf{m}_\Gamma\,\hat{\varphi}^\Gamma(\zeta)|$ over all the branches Γ ?

Role of sign compensations.

This, however, is impossible, as was shown in [E8] : *There exist no magic averages* , capable of acting by "brute force" alone. Strangely, the trouble comes from paths that cross $\mathbb{R}^+$ neither too often nor too rarely[78].

Anyway, the upshot is that some quite systematic branch-to-branch *compensation* must come into play. This compensation, however, which explains the success of well-behaved averages **m**, doesn't come from the average's weights (all of which may be real positive, as with **mon**) but from the functions $\hat{\varphi}(\zeta)$ which, constrained as they are by the Bridge Equation, display a precise pattern of sign alternation on their different branches.

Frequency of faster-than-lateral growth but infrequency of non-medianisable growth.

One should carefully distinguish between the *quite frequent phenomenon of faster than lateral growth* on the Borel axis of direction θ, which tends to occur (in non-linear problems) as soon as there is *one*[79] *active alien derivation* $\mathbb{\Delta}_{\omega_1}$ on that axis (ie $\arg\omega_1 = \theta$), and the *far rarer need to take recourse*

[78] see [E8] Prop A.5.5.

[79] or several, but finitely many.

to a well-behaved average in order to Laplace-sum (or accelerate) over that same axis, which need arises only if there are infinitely many active alien derivations $\mathbb{\Delta}_{\omega_i}$ on the axis (ie $\arg \omega_i = \theta$). In the first case we speak of *medianisable*[80] faster-than-lateral growth; and in the second case, of *non-medianisable* faster-than-lateral growth.

Four instances of medianisable growth.

med1 : *Convolution inverses.*
This is the function $\hat{B}$ in §6.1. The axis direction is $\theta = 0$ and the only active derivation is $\mathbb{\Delta}_{\omega_1} = \mathbb{\Delta}_1$

med2 : *Euler-like equations in the "negative direction".*
This is Ex 2 of §3.2 with $\theta = \pi \mod 2\pi$ and $\mathbb{\Delta}_{\omega_1} = \mathbb{\Delta}_{-1}$.

med3 : *Singular Riccati equations in "both directions".*
This is again Example 2 of §3.2, but with only three non-zero coefficients $b_{-1}(z), b_0(z), b_1(z)$. Then we have medianisable faster-than-lateral growth in two directions: for $\theta = \pi \mod 2\pi$ with $\mathbb{\Delta}_{\omega_1} = \mathbb{\Delta}_{-1}$ and for $\theta = 0 \mod 2\pi$ with $\mathbb{\Delta}_{\omega_1} = \mathbb{\Delta}_1$.

med4 : *Resonant systems in the "negative directions".*
This is Ex 3 of §3.3 with $\theta = \arg(-\lambda_j) \mod 2\pi$ and $\mathbb{\Delta}_{\omega_1} = \mathbb{\Delta}_{-\lambda_j}$.

In all four cases, despite the presence of infinitely many singular points (in arithmetical progression) on the Borel axis of direction θ, the alien automorphism $\mathbf{lur}_\theta$ which links the lateral averages acts, due to the Bridge equation, like the convergent differential operator on the far right of (220), and so its square root acts like the equally convergent differential operator on the far right of (221).

$$\begin{pmatrix}\mathbf{mur}_\theta \\ \mathbf{mul}_\theta\end{pmatrix} = \mathbf{lur}_\theta = \exp\big(2\pi i \sum_{\arg \omega = \theta} \mathbb{\Delta}_\omega\big) \sim \exp\big(2\pi i\, \mathbb{A}_{\omega_1}\big) \tag{220}$$

$$\begin{pmatrix}\mathbf{mul}_\theta \\ \mathbf{mur}_\theta\end{pmatrix}^{\frac{1}{2}} = \mathbf{lur}_\theta^{\frac{1}{2}} = \exp\big(\pi i \sum_{\arg \omega = \theta} \mathbb{\Delta}_\omega\big) \sim \exp\big(\pi i\, \mathbb{A}_{\omega_1}\big) \tag{221}$$

We may therefore[81] in all these cases, apply the median or standard average $\mathbf{mun}_\theta$: its weights will automatically combine with the superexponentially large values of $\hat{\varphi}(\zeta)$ encountered on the various branches in such a way as to produce (at most) an exponentially large $\mathbf{mun}_\theta\, \hat{\varphi}(\zeta)$

Four instances of non-medianisable growth.

[80]because in that case the *median average* **mun** does respect lateral growth.
[81]in view of the interpretation (160),(161) of the median average **mun**.

The same would still hold for a finite number of active derivations on our Borel axis, but *not for an infinite number.* The reason is this: take ordinary first-order differential operators $\mathbb{A}_\omega$ and their formal sum $\mathbb{A} = \sum_{\arg\omega=\theta} \mathbb{A}_\omega$.[82] Then, if the sum $\mathbb{A}$ has an infinite number of terms, the convergence of the formal automorphism $exp(2\pi i\,\mathbb{A})$ in no way implies that of its infinitesimal generators $\mathbb{A}$ or even that of its square root $exp(\pi i\,\mathbb{A})$. This explains why, in all such cases, we will generically be facing *non-medianisable* faster-than-exponential growth.

We may of course use the lateral averages $\mathbf{mur}_\theta$ or $\mathbf{mur}_\theta$ if we don't object to imaginary parts, but if the Borel axis is $\mathbb{R}^\pm$ and if the context makes imaginary parts inacceptable (as in physics, or real geometry, or the theory of analysable functions), there there is no alternative to the well-behaved averages. Here are four such examples.

non-med1: *unitary iteration of unitary diffeomorphisms.*
This is Ex 1 of §3.1, but for "unitary" diffeos, whose inverse coincides with their complex conjugate: $f \circ \bar{f} = id$. If we want to define sectorial-regular fractional iterates of f in right or left half-planes, then we must Laplace integrate on $\mathbb{R}^+$ or $\mathbb{R}^-$ with respect to a well-behaved average[83].

non-med2: *Euler-like equations on the "positive directions".*
This is once more Ex 2 of §2 but in the direction $\theta = 0$. If we consider the particular formal solution $\tilde{y}(z, 0)$ there is no probblem, since $\hat{y}(\zeta, 0)$ has no singularities on $\mathbb{R}^+$. But if we ask for a real resummation of the full solution $\tilde{y}(z, u)$ [84] then we have to contend with an infinity of active alien derivations and must use a well-behaved average.[85]

non-med3: *hyperbolic transit maps.*
Let ϖ be a real, local-analytic differential form formally conjugate to $\varpi^{\mathrm{nor}} := \frac{dx_2}{x_2} + (1 + \rho\, x_1)\frac{dx_1}{x_1^2}$. Its integral curves are hyperbolae that draw closer and closer to the coordinate axes (in the first and third quadrants). The local "transit map" $g : x_1 \mapsto x_2$ which link the distance to the axes (on suitable transversals) of the incoming and outcoming hyperbolae branches, corresponds to a formal transseries that commingles powers and exponentials and generically displays *non-medianisable* faster-than-lateral growth. These local transit maps play a part in the resummation-theoretic proof of the Dulac conjecture (about the finiteness of limit cycles for polynomial planar vector fields). For a detailed discussion, see [E8],[EM]. Here, however, it should be

[82] for example, take the $\mathbb{A}_n$ as in (74) or (79)
[83] see [EM], §4
[84] in domains where $|z^{-1}|$ and $|u\, e^z|$ are both small.
[85] a detailed discussion may be found in [E6], §3.10.

noted that the well-behaved average is needed to go from the geometric germ g (which is given by ϖ) to its formalisation $\tilde{g}$. In other words, the usual process (from formal to geometric) is reversed.

non-med4 : *Resonant systems on the "positive directions".*
This is again Ex 3 of §3.3 but this time on a "positive axis" $\theta = \arg \lambda_j$. The picture is much the same as in **non-med2** : we have non-medianisable faster-than-lateral growth for $\hat{y}(\zeta, u)$ though not for $\hat{y}(\zeta, 0)$ or indeed for any 'partial' solution that leaves out the parameter u_j.

8 Well-behaved alien derivations.

8.1 Reminder about the standard alien derivations.

In §2.3 we introduced, and then dismissed, the lateral alien operators $\Delta^{\pm}_{\omega}$ which do indeed measure singularities over ω and also satisfy **D3**, **D4**, but infringe **D1** (they are not first-order derivations) and **D2**.

Then from the $\Delta^{\pm}_{\omega}$ we constructed (§6.3, (172)) the more satisfactory *standard*, or *uniform*, alien derivations Δ_{ω}. These verify **D1**, **D2**, **D4**, but not **D3** : they are not *well-behaved*. But they have very simple weights :

$$\mathbf{d}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} := \frac{\epsilon_r}{2\pi i}\,\frac{p!\,q!}{(p+q+1)!} = \frac{\epsilon_r}{2\pi i}\,\frac{p!\,q!}{r!} \qquad \begin{array}{l} p := \#\{1 \leq i < r\,;\ \epsilon_i = +\} \\ q := \#\{1 \leq i < r\,;\ \epsilon_i = -\} \end{array}$$

and they relate to the standard averages as follows[86] :

$$\partial_{\alpha} \mathbf{mu}_{\alpha,\beta} \ \equiv\ 2\pi i\, \mathbf{mu}_{\alpha,\beta}\, \mathbf{dun} \qquad \textit{with} \quad \alpha+\beta = 1 \ \ , \ \ \mathbf{dun} := \sum_{\omega>0} \mathbb{\Delta}_{\omega} \tag{222}$$

Weight-wise, and after due order reversal, this translates into :

$$\partial_{\alpha} \mathbf{mu}^{\bullet}_{\alpha,\beta} \ \equiv\ 2\pi i\, \mathbf{dun}^{\bullet} \times \mathbf{mu}^{\bullet}_{\alpha,\beta} \qquad \textit{ie} : \tag{223}$$

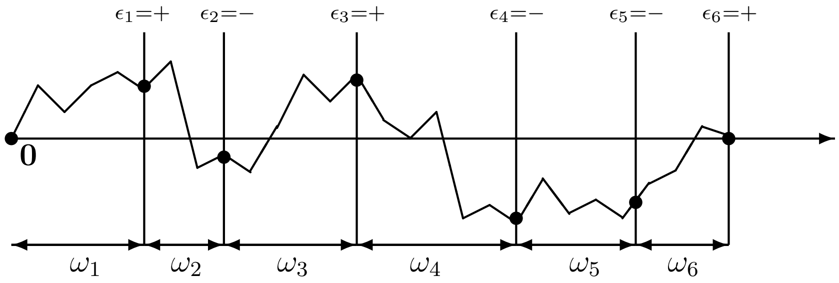
$$\partial_{\alpha} \mathbf{mu}_{\alpha,\beta}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}} \ \equiv\ 2\pi i \sum_{1\leq i\leq r} \mathbf{dun}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_i}{\omega_1\ ,\dots,\ \omega_i}}\, \mathbf{mu}_{\alpha,\beta}^{\binom{\epsilon_{i+1}\ ,\dots,\ \epsilon_r}{\omega_{i+1}\ ,\dots,\ \omega_r}} \tag{224}$$

8.2 Diffusion-induced alien derivations.

We revert to the 'diffusion process' of §7.2, but now we define the number $2\pi i\, \epsilon_r\, \mathbf{d}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{\omega_1\ ,\dots,\ \omega_r}}$ as the conditional probability that a particle starting from 0

[86] here, $\alpha + \beta \equiv 1$ and so the derivation $\partial_{\alpha} = -\partial_{\beta}$ acts on both α and β.

and crossing the real axis at the point $\omega_1 + \ldots \omega_r$ should successively cross the vertical half-axes $\omega_1 + i\,\mathbb{R}^{\epsilon_1}$, $\omega_1 + \omega_2 + i\,\mathbb{R}^{\epsilon_2}$, ..., $\omega_1 + \omega_2 + \ldots \omega_{r-1} + i\,\mathbb{R}^{\epsilon_{r-1}}$.



Since these numbers $\mathbf{d}^{\binom{\epsilon_1\;,\ldots,\;\epsilon_r}{\omega_1\;,\ldots,\;\omega_r}}$ verify the self-consistency relations (164),(165), they may be looked upon as the *weights* of a well-defined alien operators $\mathbf{d}$. That operator clearly respects *realness* (condition **D2**) *iff* γ is an even function. Less obviously, it is also a first-order *alien derivation* (condition **D1**) and it even respects *lateral growth* (condition **D3**): see §7.5 infra.

8.3 Scale-invariance.

If we now switch to $\{{}^{\tau}f_\omega\,,\,{}^{\tau}g_\omega\}$ for $0 < \tau < +\infty$ as we did in §7.3, then the corresponding systems $\{{}^{\tau}\mathbf{d}_\omega\}$, while retaining **D1**,**D2**,**D3**, acquire scale-invariance (condition **D3**) : in other words, *they are well-behaved systems of alien derivations.*

From these, many other well-behaved systems can be constructed : not only can we resort to the same trick (208) as with the uniformising averages[87] but, unlike with the averages, we can also take advantage of the stability of well-behaved alien derivations under addition and the Lie bracket.[88]

8.4 Standard and organic alien derivations as limit-cases.

For $\tau \to +\infty$, the system ${}^{\tau}\mathbf{d} = \{{}^{\tau}\mathbf{d}_\omega\}$ tends to the system $\mathbf{dun} = \{\mathbf{dun}_\omega\}$ of so-called "standard" alien derivations, which as we saw is not well-behaved.

For $\tau \to +0$, the system ${}^{\tau}\mathbf{d} = \{{}^{\tau}\mathbf{d}_\omega\}$ tends to the system $\mathbf{don} = \{\mathbf{don}_\omega\}$ of so-called "organic" alien derivations, which as we shall see is well-behaved.

[87] with $\mathbf{d}, {}^{\tau_0}\mathbf{d}$ in place of $\mathbf{m}, {}^{\tau_0}\mathbf{m}$, but without changing the left-side factors $\left(\frac{{}^{\tau_i^*}\mathbf{m}}{{}^{\tau_i}\mathbf{m}}\right)^{n_i}$.

[88] thus, setting $\mathbf{d}'_\omega := a\ {}^{\tau_1}\mathbf{d}_\omega + b\ {}^{\tau_2}\mathbf{d}_\omega$ and $\mathbf{d}''_\omega := \sum_{\omega_1+\omega_2=\omega} c(\omega_1/\omega_2)\ [{}^{\tau_1}\mathbf{d}_{\omega_1}\,,\ {}^{\tau_2}\mathbf{d}_{\omega_2}]$, with a bounded c, we get two new well-behaved systems $\{\mathbf{d}'_\omega\}$ and $\{\mathbf{d}''_\omega\}$.

Like the organic average **mon**, the organic system of alien derivations **don** has weights that are simple[89] rational functions of the gaps ω_i and it can be embedded into a one-parameter family $\mathbf{don}_{\alpha,\beta}$ (with $\alpha+\beta=1$) of systems which are charcterised by:

$$\partial_\alpha\, \mathbf{mo}_{\alpha,\beta} = -\partial_\beta\, \mathbf{mo}_{\alpha,\beta} = 2\pi i\; \mathbf{mo}_{\alpha,\beta}\, \mathbf{do}_{\alpha,\beta} \tag{225}$$

and which are well-behaved *iff* $\alpha=\bar{\beta}$. The "tilted" systems $\mathbf{do}_{1,0}$, $\mathbf{do}_{0,1}$ are particularly simple, since they load (each) only r out of 2^r paths. Unfortunately, they are not well-behaved (they offend against **D2**) but their half-sum **dom** is well-behaved, like **don**, only much simpler. In fact, there exists no simpler choice of well-behaved derivations than **dom**. As for the weights of the organic family, simple calculations lead to the following formulas:

$$\begin{array}{lcll}
\mathbf{don}^{\binom{\epsilon_1\ \cdots\ \epsilon_r}{\omega_1\ \cdots\ \omega_r}} & := & \mathbf{do}_{1/2,1/2}^{\binom{\epsilon_1\ \cdots\ \epsilon_r}{\omega_1\ \cdots\ \omega_r}} & \textit{with}\ \ \mathbf{do}^{\bullet}_{\alpha,\beta}\ \textit{characterised by} \\
\partial_\alpha\, \mathbf{mo}^{\bullet}_{\alpha,\beta} & = & 2\pi i\; \mathbf{do}^{\bullet}_{\alpha,\beta} \times \mathbf{mo}^{\bullet}_{\alpha,\beta} & \textit{and}\ \ \alpha+\beta\equiv 1\ \ ,\ \ \partial_\alpha \equiv -\partial_\beta \\
\mathbf{dom}^{\bullet} & := & \frac{1}{2}\,(\mathbf{do}^{\bullet}_{1,0}+\mathbf{do}^{\bullet}_{0,1}) & \\
\mathbf{dom}^{\binom{\epsilon_1\ \cdots\ \epsilon_r}{\omega_1\ \cdots\ \omega_r}} & := & \frac{1}{2}\frac{\epsilon_r}{2\pi i}\,\frac{\omega_{p+1}}{\omega_1+\cdots+\omega_r} & \textit{if}\ \ (\epsilon_1,\ldots,\epsilon_r)=((+)^p,(-)^q,\epsilon_r) \\
 & := & \frac{1}{2}\frac{\epsilon_r}{2\pi i}\,\frac{\omega_{q+1}}{\omega_1+\cdots+\omega_r} & \textit{if}\ \ (\epsilon_1,\ldots,\epsilon_r)=((-)^q,(+)^p,\epsilon_r) \\
 & := & 0 & \qquad \textit{otherwise}
\end{array}$$

8.5 Proofs and comments.

We begin with the diffusion-induced derivations. Applying the definition of §8.2 we get for the weights and the lateral moulds two types of integral expressions:

$$\begin{aligned}
\mathbf{d}^{\binom{\epsilon_1\ ,\ldots,\ \epsilon_r}{\omega_1\ ,\ldots,\ \omega_r}} &:= \frac{\epsilon_r}{2\pi i}\int_{\mathbb{R}^r} f_{\omega_1}(x_1)\ldots f_{\omega_r}(x_r)\,\sigma_{\epsilon_1}(\check{x}_1)\ldots\sigma_{\epsilon_{r-1}}(\check{x}_{r-1})\delta(\check{x}_r)\,dx_1\ldots dx_r \\
&:= \frac{\epsilon_1\ldots\epsilon_r}{(2\pi i)^{r+1}}\int_{\mathbb{R}^r_{\epsilon_1,\ldots,\epsilon_r}} \frac{g_{\omega_1}(y_1)\ldots g_{\omega_r}(y_r)\;dy_1\ldots dy_r}{(y_1-y_2)(y_2-y_3)\ldots(y_{r-1}-y_r)}
\end{aligned} \tag{226}$$

[89]but less simple than with **mon**, for they admit no closed expression as a product of r elementary factors.

$$\begin{aligned}
\mathbf{red}^{\omega_1,\dots,\omega_r} &:= \frac{1}{2\pi i}\int_{\mathbb{R}^r} f_{\omega_1}(x_1)\dots f_{\omega_r}(x_r)\ \mathrm{lefo}_+^{x_1,\dots,x_r}\ dx_1\dots dx_r && (227)\\
&:= \frac{(-1)^r}{(2\pi i)^{r+1}}\int_{\mathbb{R}^r_{+,\dots,+}} g_{\omega_1}(y_1)\dots g_{\omega_r}(y_r)\ \mathrm{tas}_{**}^{y_1,\dots,y_r}\ dy_1\dots dy_r && (228)
\end{aligned}$$

$$\begin{aligned}
\mathbf{led}^{\omega_1,\dots,\omega_r} &:= \frac{-1}{2\pi i}\int_{\mathbb{R}^r} f_{\omega_1}(x_1)\dots f_{\omega_r}(x_r)\ \mathrm{lefo}_-^{x_1,\dots,x_r}\ dx_1\dots dx_r && (229)\\
&:= \frac{-1}{(2\pi i)^{r+1}}\int_{\mathbb{R}^r_{-,\dots,-}} g_{\omega_1}(y_1)\dots g_{\omega_r}(y_r)\ \mathrm{tas}_{**}^{y_1,\dots,y_r}\ dy_1\dots dy_r && (230)
\end{aligned}$$

Now let us turn to the alien derivations of uniform or organic type and form their lateral moulds. We find :

$$\begin{aligned}
2\pi i\ \mathrm{redun}^{\boldsymbol{\omega}} &:= (-1)^r\,\tfrac{1}{r} &\overset{vol}{\leftrightarrow}\quad 2\pi i\ \mathrm{ledun}^{\boldsymbol{\omega}} &:= (-1)^{r-1}\,\tfrac{1}{r}\\
2\pi i\ \mathrm{redon}^{\bullet} &:= \mathrm{dremon}^{\bullet}\times\mathrm{vremon}^{\bullet} &\overset{vol}{\leftrightarrow}\quad 2\pi i\ \mathrm{ledon}^{\bullet} &:= \mathrm{dlemon}^{\bullet}\times\mathrm{vlemon}^{\bullet}\\
2\pi i\ \mathrm{redom}^{\boldsymbol{\omega}} &:= (-1)^r\,\tfrac{1}{2}\,\tfrac{\omega_1+\omega_r}{\omega_1+\dots+\omega_r} &\overset{vol}{\leftrightarrow}\quad 2\pi i\ \mathrm{ledom}^{\boldsymbol{\omega}} &:= (-1)^{r-1}\,\tfrac{1}{2}\,\tfrac{\omega_1+\omega_r}{\omega_1+\dots+\omega_r}
\end{aligned}$$

$$\begin{aligned}
\mathrm{dremo}^{\bullet}_{\alpha,\beta} &:= \partial_\alpha\ \mathrm{remo}^{\bullet}_{\alpha,\beta}\\
\mathrm{dlemo}_{\bullet} &:= \partial_\alpha\ \mathrm{remo}^{\bullet}_{\alpha,\beta} = -\,\partial_\beta\ \mathrm{lemo}^{\bullet}_{\alpha,\beta}\\
\mathrm{vremo}^{\bullet}_{\alpha,\beta} &:= (\mathrm{remo}^{\bullet}_{\alpha,\beta})^{-1} \qquad \text{(mould inversion)}\\
\mathrm{vlemo}^{\bullet}_{\alpha,\beta} &:= (\mathrm{lemo}^{\bullet}_{\alpha,\beta})^{-1} \qquad \text{(mould inversion)}\\
\mathrm{dremo}^{\omega_1,\dots,\omega_r}_{\alpha,\beta} &:= +\,(\textstyle\sum_{i=1}^{i=r}\frac{\omega_i}{\omega_1+\dots+\omega_{i-1}+\alpha\,\omega_i})\ \mathrm{remo}^{\omega_1,\dots,\omega_r}_{\alpha,\beta}\\
\mathrm{dlemo}^{\omega_1,\dots,\omega_r}_{\alpha,\beta} &:= -\,(\textstyle\sum_{i=1}^{i=r}\frac{\omega_i}{\omega_1+\dots+\omega_{i-1}+\beta\,\omega_i})\ \mathrm{lemo}^{\omega_1,\dots,\omega_r}_{\alpha,\beta}\\
\mathrm{vremo}^{\omega_1,\dots,\omega_r}_{\alpha,\beta} &:= +\,\alpha\,\tfrac{\omega_1}{\omega_1+\dots+\omega_r}\textstyle\prod_{i=2}^{i=r}(\alpha\,\frac{\omega_i}{\omega_1+\dots+\omega_i}-1)\\
\mathrm{vlemo}^{\omega_1,\dots,\omega_r}_{\alpha,\beta} &:= +\,\beta\,\tfrac{\omega_1}{\omega_1+\dots+\omega_r}\textstyle\prod_{i=2}^{i=r}(\beta\,\frac{\omega_i}{\omega_1+\dots+\omega_i}-1)
\end{aligned}$$

We can now proceed to check the good-behaviour criteria.

Condition **D1** is easiest to check in the form $\mathbf{D1_1}$ or $\mathbf{D1_2}$. But the alternelity of $\mathbf{red}^{\bullet}$ or $\mathbf{led}^{\bullet}$ follows from the x-integral expressions (227),(229) and the alternelity of the moulds $\mathrm{lefo}^{\bullet}_{\pm}$. One may also reason on the y-integrals (228),(229) by heeding the circumvention rules already mentioned in §7.5 and by observing that they have the effect of turning the alternal mould $tas^{\bullet}_{**}$ into alternel integrals ! In the limit cases **dun** and **don**, as well as with **dom**, it is best to check the alternelity directly on the above formulas.

Conditions **D2** and **D4** are immediate to check directly on the weights.

This leaves the "difficult" growth condition **D3**. Again, though *non-algebraic*, this condition, once rephrased as criterion $\mathbf{D3_2}$ or $\mathbf{D3_4}$, can be settled *algebraically*, by observing that the mould $\mathrm{lefo}^{\bullet}_{\pm}$ *retain their form under contracting anti-arborification*: they simply become armoulds $\mathrm{lefo}^{\bullet^{\succ}}_{\pm}$ that are still defined as in §2.1.9, except that the sums $\check{x}_i$ must now be extended to all x_j anterior to x_i under the anti-arborescent order[90]. In particular, $\mathrm{lefo}^{\boldsymbol{\omega}^{\succ}}_{\pm} = 0$ as soon as the anti-arborescent sequence $\boldsymbol{\omega}^{\succ}$ has more than one anti-root or maximal element: this is the so-called property of *separativity* for alternel armoulds. Checking these identities is a simple, but useful exercise.[91]

Remark: We have been using the right- and left-lateral moulds simply for the sake of symmetry, by they are essentially the same. Indeed, as soon as as condition **D1** is fulfilled, they are related under the simple involution:

$$\mathrm{red}^{\bullet} \;\leftrightarrow\; \mathrm{led}^{\bullet} \quad : \quad \mathrm{led}^{\bullet} \;\equiv\; \mathrm{red}^{\bullet} \circ \mathbf{J}^{\bullet}$$

with $\mathbf{J}^{\omega_1,\dots,\omega_r} := (-1)^r$, $\forall \omega_i$.

8.6 Tables of averages and derivations.

The following tables contrast the behaviour of averages and derivations from the organic (well-behaved) and uniform (non w.-b.) families. The latter satisfy all conditions **Ci** except **C3** ("proper growth"). This is reflected in the fact that for strongly alternating sign sequences $\boldsymbol{\epsilon}$, the weights of the organic (resp uniform) operators tend to be small (resp not so small). The difference would be even more striking if it were possible to print the tables for larger lengths (We had to stop at $r = 6$).

To simplify, we set all gaps ω_i equal to 1. So we mention only the $\pm$ signs ϵ_i. For derivations, we drop the $\frac{1}{2\pi i}$ factor. We also take advantage of the left-right symmetries to retain only the sign sequences ending with $\epsilon_r = +$:

$$\mathbf{m}^{\bar{\epsilon}_1,\dots,\bar{\epsilon}_r} \equiv +\, \mathbf{m}^{\epsilon_1,\dots,\epsilon_r} \qquad ; \qquad \mathbf{d}^{\bar{\epsilon}_1,\dots,\bar{\epsilon}_r} \equiv -\, \mathbf{d}^{\epsilon_1,\dots,\epsilon_r} \tag{231}$$

[90] x_i itself should be included in the sum.

[91] alternatively, one may reason on the y-integrals and observe *first* that the mould $tas^{\bullet}_{**}$ retains its form under *ordinary (ie non-contracting) anti-arborification* and *second* that, under the circumvention rules, this translates into *contracting anti-arborification* for the integrals. But, on balance, reasoning on the x-integrals is simpler.

object	*average*	*average*	*derivation*	*derivation*	*derivation*
species	*uniform*	*organic*	*uniform*	*organic*	*organic*
nature	*"bad"*	*"good"*	*"bad"*	*"good"*	*"good"*
name	**mun**	**mun**	**dun**	**don**	**dom**
$(+)$	1/2	1/2	1	1	1
sums	**1/2**	**1/2**	**1**	**1**	**1**
$(+,+)$	3/8	3/8	1/2	1/2	1/2
$(-,+)$ *	1/8	1/8	1/2	1/2	1/2
sums	**1/2**	**1/2**	**1**	**1**	**1**
$(+,+,+)$	5/16	5/16	1/3	1/3	1/3
$(-,+,+)$	1/16	5/48	1/6	1/6	1/6
$(+,-,+)$ *	1/16	1/48	1/6	1/6	1/6
$(-,-,+)$	1/16	1/16	1/3	1/3	1/3
sums	**1/2**	**1/2**	**1**	**1**	**1**
$(+,+,+,+)$	35/128	35/128	1/4	1/4	1/4
$(-,+,+,+)$	5/128	35/384	1/12	5/48	1/8
$(+,-,+,+)$	5/128	7/384	1/12	1/24	0
$(-,-,+,+)$	3/128	7/128	1/12	5/48	1/8
$(+,+,-,+)$	5/128	1/128	1/12	5/48	1/8
$(-,+,-,+)$ *	3/128	1/384	1/12	1/24	0
$(+,-,-,+)$	3/128	5/384	1/12	5/48	1/8
$(-,-,-,+)$	5/128	5/128	1/4	1/4	1/4
sums	**1/2**	**1/2**	**1**	**1**	**1**
$(+,+,+,+,+)$	63/256	63/256	1/5	1/5	1/5
$(-,+,+,+,+)$	7/256	21/256	1/20	37/480	1/10
$(+,-,+,+,+)$	7/256	21/1280	1/20	11/480	0
$(-,-,+,+,+)$	3/256	63/1280	1/30	1/16	1/10
$(+,+,-,+,+)$	7/256	9/1280	1/20	11/480	0
$(-,+,-,+,+)$	3/256	3/1280	1/30	1/120	0
$(+,-,-,+,+)$	3/256	3/256	1/30	7/240	0
$(-,-,-,+,+)$	3/256	9/256	1/20	37/480	1/10
$(+,+,+,-,+)$	7/256	1/256	1/20	37/480	1/10
$(-,+,+,-,+)$	3/256	1/768	1/30	7/240	0
$(+,-,+,-,+)$ *	3/256	1/3840	1/30	1/120	0
$(-,-,+,-,+)$	3/256	1/1280	1/20	11/480	0
$(+,+,-,-,+)$	3/256	7/1280	1/30	1/16	1/10
$(-,+,-,-,+)$	3/256	7/3840	1/20	11/480	0
$(+,-,-,-,+)$	3/256	7/768	1/20	37/480	1/10
$(-,-,-,-,+)$	7/256	7/256	1/5	1/5	1/5
sums	**1/2**	**1/2**	**1**	**1**	**1**

name	**mun**	**mun**	**dun**	**don**	**dom**
$(+,+,+,+,+,+)$	231/1024	231/1024	1/6	1/6	1/6
$(-,+,+,+,+,+)$	21/1024	77/1024	1/30	71/1152	1/12
$(+,-,+,+,+,+)$	21/1024	77/5120	1/30	23/1440	0
$(-,-,+,+,+,+)$	7/1024	231/5120	1/60	29/640	1/12
$(+,+,-,+,+,+)$	21/1024	33/5120	1/30	11/960	0
$(-,+,-,+,+,+)$	7/1024	11/5120	1/60	23/5760	0
$(+,-,-,+,+,+)$	7/1024	11/1024	1/60	47/2880	0
$(-,-,-,+,+,+)$	5/1024	33/1024	1/60	29/640	1/12
$(+,+,+,-,+,+)$	21/1024	11/3072	1/30	23/1440	0
$(-,+,+,-,+,+)$	7/1024	11/9216	1/60	11/1920	0
$(+,-,+,-,+,+)$	7/1024	11/46080	1/60	1/720	0
$(-,-,+,-,+,+)$	5/1024	11/15360	1/60	23/5760	0
$(+,+,-,-,+,+)$	7/1024	77/15360	1/60	47/2880	0
$(-,+,-,-,+,+)$	5/1024	77/46080	1/60	11/1920	0
$(+,-,-,-,+,+)$	5/1024	77/9216	1/60	13/576	0
$(-,-,-,-,+,+)$	7/1024	77/3072	1/30	71/1152	1/12
$(+,+,+,+,-,+)$	21/1024	7/3072	1/30	71/1152	1/12
$(-,+,+,+,-,+)$	7/1024	7/9216	1/60	13/576	0
$(+,-,+,+,-,+)$	7/1024	7/46080	1/60	11/1920	0
$(-,-,+,+,-,+)$	5/1024	7/15360	1/60	47/2880	0
$(+,+,-,+,-,+)$	7/1024	1/15360	1/60	23/5760	0
$(-,+,-,+,-,+)$ *	5/1024	1/46080	1/60	1/720	0
$(+,-,-,+,-,+)$	5/1024	1/9216	1/60	11/1920	0
$(-,-,-,+,-,+)$	7/1024	1/3072	1/30	23/1440	0
$(+,+,+,-,-,+)$	7/1024	3/1024	1/60	29/640	1/12
$(-,+,+,-,-,+)$	5/1024	1/1024	1/60	47/2880	0
$(+,-,+,-,-,+)$	5/1024	1/5120	1/60	23/5760	0
$(-,-,+,-,-,+)$	7/1024	3/5120	1/30	11/960	0
$(+,+,-,-,-,+)$	5/1024	21/5120	1/60	29/640	1/12
$(-,+,-,-,-,+)$	7/1024	7/5120	1/30	23/1440	0
$(+,-,-,-,-,+)$	7/1024	7/1024	1/30	71/1152	1/12
$(-,-,-,-,-,+)$	21/1024	21/1024	1/6	1/6	1/6
sums	**1/2**	**1/2**	**1**	**1**	**1**

8.7 Pinpointing the difference between "good" and "bad".

We first set some notations:

$$\mathrm{cnb}(r_1, r_2) \ := \sum_{k=0}^{k=r_2} \ (-1)^k \ \frac{r_2!}{k!\,(r_2-k)!} \ (r_2-k)^{r_1} \tag{232}$$

In particular:

$$\mathrm{cnb}(r,1) \ = \ 1 \tag{233}$$

$$\mathrm{cnb}(r,r) \ = \ r! \tag{234}$$

$$\mathrm{cnb}(r,r_\star) \ \geq \ 2 \quad if \quad 1 < r_\star < r \tag{235}$$

$$\mathrm{cnb}(r,r_\star) \ = \ 0 \quad if \quad r < r_\star \tag{236}$$

Then we set:

$$\mathrm{f}(r_0, r_1) = \sum_{r_2=1}^{r_2=r_1} \ \mathrm{f}(r_0+r_2) \ \ \mathrm{cnb}(r_1, r_2) \tag{237}$$

Next, let $\boldsymbol{\omega}^{\prec}_{r_0,r_1}$ (resp. $\boldsymbol{\omega}^{\succ}_{r_0,r_1}$) be the arborescent (resp. antiarborescent) sequence obtained by *suffixing* (resp *prefixing*) the *totally non-ordered* sequence $(\omega'_1,\dots,\omega'_{r_1})$ to the *totally ordered* sequence $(\omega_1,\dots,\omega_{r_0})$.

Assume now that $F^\bullet$ is some *constant-type mould* like $tu^\bullet_a$ (§6.1.4 *supra*), ie a mould whose values depend solely on the sequence length r, so that $F^{\omega_1,\dots,\omega_r} \equiv f(r)$. In view of what precedes, it is clear that after a contracting arborification or antiarborification we get:

$$\mathrm{F}^{\boldsymbol{\omega}^{\prec}_{r_0,r_1}} \equiv \mathrm{F}^{\boldsymbol{\omega}^{\succ}_{r_0,r_1}} \equiv \mathrm{f}(r_0,r_1) \qquad\qquad \text{with} \quad \mathrm{f}(r_0,r_1) \text{ as in } (284) \tag{238}$$

If we take $F^\bullet := tu^\bullet_a$ with $a \in \mathbb{Z}$, then $tu^\bullet_a$ is **well-behaved** and indeed we can see (trivially for $a<0$, less so for $a>0$) that:

$$\limsup_{r_1 \to +\infty} \Big(\frac{log|\mathrm{f}(r_0,r_1)|}{r_0+r_1} \Big) \ < \ +\infty \qquad (\forall\, r_0 \text{ fixed and } \geq 1) \tag{239}$$

But if $a \notin \mathbb{Z}$, then $tu^\bullet_a$ is **not well-behaved** and we can show that:

$$\limsup_{r_1 = +\infty} \Big(\frac{log|\mathrm{f}(r_0,r_1)|}{r_0+r_1} \Big) \ = \ +\infty \qquad (\forall\, r_0 \text{ fixed and } \geq 1) \tag{240}$$

and in fact:

$$\limsup_{r_1 \to +\infty} \Big(\frac{log|\mathrm{f}(r_0,r_1)|}{(r_0+r_1)\,log(r_0+r_1)} \Big) \ > \ 0 \qquad (\forall\, r_0 \text{ fixed and } \geq 1) \tag{241}$$

9 Proper notion of ramified-exponential growth.

Four cases of increasing difficulty have to be considered, corresponding to four types of resurgence algebras :
(1) the trivial, singularity-free case.
(2) the case with bounded singularities.
(3) the case with 'manageable' singularities.
(4) the case with absolutely arbitrary singularities.

In the trivial, singularity-free case, ie for minors that have no singularity other than $0_\bullet$, the only sensible definition of exponential growth is :

$$|\hat{\varphi}(\zeta)| \;\leq\; C_{\theta\,,\varphi}\,\exp\left(D_{\theta\,,\varphi}\,|\zeta|\right) \qquad (for\;\;\zeta\;\;large) \tag{242}$$

with θ*-continuous* constants $C_{\theta\,,\varphi}$, $D_{\theta\,,\varphi}$. The latter continuity condition is essential to ensure *uniform* exponential growth inside sectors of finite aperture. Clearly, some uniformity condition of this sort will have to be there in the other cases also but, as we saw in §6.5, the simple imposition of bounds, exponential or otherwise, on *single paths*, won't do [92]: some subtle, path-to-path compensation conditions must also come into play. The simplest way to express these constraints is *dually to the well-behaved averages*: see §9.1.

9.1 Bounded singularities.

This is the case when singularities at the origin are of integrable type and when, close to any singular point ω, the minors remain bounded on any sector of finite aperture and with apex ω.

Let us fix some well-behaved average $\mathbf{m}$ – preferably the organic average - – and let us denote $\mathbf{m}_\theta$ its action on the half-axis $\arg\zeta = \theta$ of the Borel plane. *If we then define ramified-exponential growth as the existence of finite*

[92]Indeed, the diemma is this : *either* we impose uniform exponential bounds (242) on all paths, and then such simple functions as $\hat{B}$ in §6.1 won't verify these bounds, *or* we relax the growth condition on oft-crossing paths in such as way as to accommodate functions like $\hat{B}$, and in that case it is easy to see that there are going to be functions $\hat{\varphi}$ that meet those relaxed conditions and yet *cannot be Laplace-summed over* $\mathbb{R}^+$ *whichever realness preserving averages* $\mathbf{m}\,\hat{\varphi}$ *we choose to consider.* So we are in a fix here, unless we reconcile ourselves to the idea of restricting the growth, not of single determinations $\hat{\varphi}(\zeta)$, but of suitable averagings of several determinations $\hat{\varphi}(\zeta_i)$, for points ζ_i with identical projections on $\mathbb{C}_\bullet$.

bounds[93] *(243) (resp (243)+(244)):*

$$|\mathbf{m}_\theta\, \hat{\varphi}(\zeta)| \;\leq\; C_{\theta\,,\varphi}\, \exp\left(D_{\theta\,,\varphi}\,|\zeta|\right) \tag{243}$$
$$|\mathbf{m}_\theta\, \Delta_{\omega_r} \ldots \Delta_{\omega_1}\, \hat{\varphi}(\zeta)| \;\leq\; C_{\theta\,,\boldsymbol{\omega}\,,\varphi}\, \exp\left(D_{\theta\,,\boldsymbol{\omega}\,,\varphi}\,|\zeta|\right) \tag{244}$$

then the growth notion so defined meets all four conditions $\mathbf{R1}, \mathbf{R2}, \mathbf{R3}, \mathbf{R4}$.

9.2 Manageable singularities.

A singularity at a point $\omega \in \mathbb{C}_\bullet$ of argument $\arg\omega = \theta$ is said to be "manageable" if, by choosing a suitably *slow time* z_θ within the critical time class $\{z\}$, the singularity in question can be smoothened (ie rendered bounded, even $\mathcal{C}^\infty$, or even smoother, but always short of *cohesive*[94]) on a whole *closed, notched* disk centered at ω but whose right and left radii $[\omega, \omega + r\, e^{i\theta}]^+$ and $[\omega, \omega + r\, e^{i\theta}]^-$ have to be regarded as *distinct.*

Manageable singularities (which can be extremely violent) cover all singularities liable to occur in practice, and in particular all those one encounters in the construction of *analysable functions.* For resurgent functions with such singularities, ramified-exponential growth is still defined by the existence of bounds (243) or (244), but relative to slow times z_θ that may depend on θ and with the usual continuity conditions on θ.

9.3 Arbitrary singularities. An open question.

That leaves only the question of arbitrary, ie possibly *unmanageable* singularities. Not only do these never occur in natural situations, but it is not at all clear whether they permit a satisfactory, convolution-stable notion of ramified-exponential growth. This, at any rate, would presuppose, among other things, that the implication (248) *infra* holds for all finite sums (247) obtained by convoluting resurgent functions $\overset{\diamond}{A}_i$ whose minors are regular (say, sectorially bounded at ∞) away from $0_\bullet$ but which may possess horrendous singularities at $0_\bullet$, and functions $\overset{\diamond}{B}_i$, whose singular points ω_i are $\neq 0_\bullet$ and possibly quite dense, but with singularities there that are fairly tame (say, integrable, or sectorially bounded, or simple poles), and with uniform

[93]with constants $C_{\theta\,,\varphi}$, $D_{\theta\,,\varphi}$ or $C_{\theta\,,\boldsymbol{\omega}\,,\varphi}$, $D_{\theta\,,\boldsymbol{\omega}\,,\varphi}$ *continuous* in θ and $\boldsymbol{\omega}$.

[94]see [E7] , Lectures 3 and 4.

exponential growth at ∞.

$$\begin{aligned}
\overset{\diamond}{A}_i &= \{\check{A}_i, \hat{A}_i\} \quad with \quad \hat{A}_i \;\; regular\text{-}bounded\, away\, from\; 0_{\bullet} && (245)\\
\overset{\diamond}{B}_i &= \{\check{B}_i, \hat{B}_i\} \quad with \quad \hat{B}_i \;\; of \;\; integrable\text{-}exponential \;\; type. && (246)\\
\overset{\diamond}{C} &:= \overset{\diamond}{A}_1 * \overset{\diamond}{B}_1 + \cdots \overset{\diamond}{A}_n * \overset{\diamond}{B}_n && (247)
\end{aligned}$$

$$\{\overset{\diamond}{C}= 0\} \quad \overset{?}{\Longrightarrow} \quad \{C = 0\} \tag{248}$$

That, however, appears to be an open question, even for such quite simple pairs as:

$$\begin{aligned}
\check{A}_i(\zeta) &:= \textstyle\sum_{0\le n,m<c.n} \; a_{i,n,m} \; \zeta^{-n} \, (\log\zeta)^m\\
\hat{B}_i(\zeta) &:= \textstyle\sum_{0\le n} \; b_{i,\omega_i} \; (1-\frac{\zeta}{\omega_i}) \; \log(1-\frac{\zeta}{\omega_i})
\end{aligned}$$

or even:

$$\begin{aligned}
\check{A}_i(\zeta) &:= \textstyle\sum_{0\le n} \; a_{i,n} \; \zeta^{-n}\\
\hat{B}_i(\zeta) &:= \textstyle\sum_{0\le n} \; b_{i,\omega_i} \; (\zeta-\omega_i)^{-1}
\end{aligned}$$

10 Well-behaved resurgence monomials.

10.1 Reminder about the standard resurgence monomials.

Assume for a start that all ω_i are > 0 and consider once again the Δ-friendly hyperlogarithmic monomials ${}^{\pm}\mathcal{U}e^{\bullet}(z)$, but equipped with their exponential factor $e^{\|\boldsymbol{\omega}\|.z}$ and expressed in the bases orthogonal to the right/left lateral alien operators $\blacktriangle^{\pm}_{\omega}$. In other words:

$$\begin{aligned}
\blacktriangle^{\pm}_{\omega_0} \; {}^{\pm}\mathcal{U}e^{\,\omega_1,\dots,\omega_r}(z) &= {}^{\pm}\mathcal{U}e^{\,\omega_2,\dots,\omega_r}(z) \quad if \quad \omega_0=\omega_1\\
&= \quad 0 \qquad\qquad\qquad if \quad \omega_0\neq\omega_1
\end{aligned}$$

Let us re-write the integral (106) for these monomials, but in a suggestive, easily generalisable form. Setting $go_\omega(y) := e^{-\omega\, y}$ we get:

$$\begin{aligned}
{}^{+}\mathcal{U}e^{\,\omega_1,\dots,\omega_r}(z) &:= \frac{1}{(2\pi i)^r}\int_0^{+\infty} \frac{go_{\omega_1}(y_1)\dots go_{\omega_r}(y_r)}{go_{\omega_1}(z)\dots go_{\omega_r}(z)} \; \frac{dy_1\dots dy_r}{(y_r-y_{r-1})\dots(y_1-z)}\\
{}^{-}\mathcal{U}e^{\,\omega_1,\dots,\omega_r}(z) &:= \frac{1}{(2\pi i)^r}\int_0^{-\infty} \frac{go_{\omega_1}(y_1)\dots go_{\omega_r}(y_r)}{go_{\omega_1}(z)\dots go_{\omega_r}(z)} \; \frac{dy_1\dots dy_r}{(y_r-y_{r-1})\dots(y_1-z)}
\end{aligned}$$

with lateral integration right (resp left) of $\arg\zeta = 0$ in the first (resp second) integral[95].

[95] it is the change from $\blacktriangle$-orthogonality to $\blacktriangle^{\pm}$-orthogonality that is responsible for the appearance of the factor $(2\pi i)^{-r}$ in front of the integrals.

10.2 Well-behaved resurgence monomials induced by prodiffusions.

Instead of pairs $\{f_\omega, g_\omega\}$ connected by the Fourier transform, we are now dealing we pairs $\{fo_\omega, go_\omega\}$ connected by the Borel-Laplace transform:

$$fo_\omega(x) = \int_0^\infty e^{-x\,y}\, go_\omega(y)\, dy \tag{249}$$

$$go_\omega(x) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{+x\,y}\, fo_\omega(x)\, dx \tag{250}$$

We still have to consider multiplicative semigroups $\{go_\omega(\bullet)\,,\, \omega > 0\}$ and convolution semigroups $\{fo_\omega(\bullet)\,,\, \omega > 0\}$ but the latter are now relative to the vertical, open-ended convolution $\divideontimes$:

$$go_\omega(y) \equiv (go_1(y))^\omega \qquad \forall\omega \qquad (go_1(y) > 0) \tag{251}$$

$$fo_{\omega_1} \divideontimes fo_{\omega_2} \equiv fo_{\omega_1+\omega_2} \qquad \forall\omega_i \qquad \textit{with} \tag{252}$$

$$(fo_{\omega_1} \divideontimes fo_{\omega_2})(x) := \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} fo_{\omega_1}(x_1)\, fo_{\omega_2}(x-x_1)\, dx_1 \tag{253}$$

The function $go_1(y)$ should be analytic on $\mathbb{C}_\bullet$ or, better still, uniform-analytic on $\mathbb{C}^*$, and it should fulfill too additional conditions at least:

(i) $\log(go_1(y)) \sim -y$ as $y \to +\infty$
(ii) $go_1(y))$ should be real for $\arg y = 0$
(iii) $go_1(y)$ should decrease fast enough as $y \downarrow +0$ for $\int_{-i\infty}^{+i\infty} |fo_1(x)||dx| < \infty$
(iv) if $go_\omega(y) = go_{\omega,c}(y)$ has to depend on a parameters c, then there should be invariance under some simple change $(y, c, \omega) \mapsto (l\,y, l^{-1}\,\omega, l^k\,c)$ [96].

Condition (i) is there to ensure that the integrals of §10.1 retain the right asymptotic behaviour at ∞ in the geometric model and sum up to proper resurgence monomials orthogonal to the $\Delta^{\pm}_{\omega}$

Condition (ii) is there to ensure the 'realness' property **M2** .

Condition (iii), which of course wasn't verified for the earlier choice of $go_1(y) := e^{-y}$ in §10.1, is there to ensure the crucial property **M3** of 'proper growth' in the $\boldsymbol{\omega}$ variables.

Condition (iv) is there to ensure the homogeneity property **M4** .

[96] $\forall l > 0$.

10.3 Canonical well-behaved resurgence monomials.

The four conditions (i)-(iv) leave a huge freedom of choice, but let us go straight for the simplest and most economical choice, which corresponds to a kernel function:

$$g_{\omega_0}(y) := \exp(-\omega y - c^2 \bar{\omega} y^{-1}) \qquad (\omega \in \mathbb{C}^\star \,; c \geq 0) \tag{254}$$

that involves only one free parameter $c > 0$ and meets all four conditions above, including (iv) with $k = 1$.

This leads to the formulas:

$$\mathcal{U}a_c^{\omega_1,\dots,\omega_r}(z) \;:=\; SPA \int_0^\infty \frac{e^{-\sum_1^r (\omega_i\, y_i + c^2 \bar{\omega}_i\, y_i^{-1})}\; dy_1 \dots dy_r}{(y_r - y_{r-1}) \dots (y_2 - y_1)(y_1 - z)} \tag{255}$$

$$\mathcal{U}e_c^{\omega_1,\dots,\omega_r}(z) \;:=\; \mathcal{U}a_c^{\omega_1,\dots,\omega_r}(z)\; e^{(\sum_1^r \omega_i)\, z + (\sum_1^r \bar{\omega}_i)\, c^2\, z^{-1}} \tag{256}$$

$$\mathcal{U}_c^{\omega_1,\dots,\omega_r}(z) \;:=\; \mathcal{U}a_c^{\omega_1,\dots,\omega_r}(z)\; e^{(\sum_1^r \bar{\omega}_i)\, c^2\, z^{-1}} \tag{257}$$

with integration along the rays $\arg(\omega_i\, y_i) = \arg(\bar{\omega}_i / y_i) = 0$.

The $\mathcal{U}a_c^{\boldsymbol{\omega}}$ are auxiliary expressions. The resurgence monomials proper are the $\mathcal{U}_c^{\boldsymbol{\omega}}$ (orthogonal to the ordinary alien derivations Δ_ω) and the exponential-carrying $\mathcal{U}e_c^{\boldsymbol{\omega}}$ (orthogonal to the exponential-carrying alien derivations $\mathbb{\Delta}_\omega$).

Interpretation of "*SPA*"**.**
SPA in front of the integral means *suitable path average.* If we integrate first in y_1, then y_2, etc, the question arises[97] as to how (ie on which side) y_i should bypass the next (yet unused) variable y_{i+1}. If to the right, we set $\epsilon_i := +$. If to the left, we set $\epsilon_i := -$. To each choice $\{\epsilon_1, \epsilon_2, \dots\}$ there corresponds a different integration path, and *SPA* means that one should take a precise average of all such paths, depending on which system Δ_ω of alien derivations one wishes the $\mathcal{U}^{\boldsymbol{\omega}}$ to be orthogonal to. But for the right- or left-lateral moulds (characterised by orthogonality to $\Delta_\omega^\pm$) the *SPA* average reduces to one single path, with all ϵ_i identical (either $+$ or $-$). In that case, however, one should add the factor $(2\pi i)^{-r}$ in front of the integrals, as in §10.1.

Interpretation of $1/(y_1 - z)$
The integral (134) defines $\mathcal{U}_c^{\boldsymbol{\omega}}$ in all three models (formal, geometric, convolutive) at one stroke, depending on how we construe $1/(y_1 - z)$:
– as a power series in z^{-1},

[97] at least when two consecutive integration axes coincide, ie when $\arg \omega_i = \arg \omega_{i+1}$.

– or as a function germ at ∞,
– or again as its own Borel tranform.

$$\begin{array}{llll} Formal\ model: & \frac{1}{y_1-z} \rightarrow & -\sum_0^\infty z^{-n-1}\, y_1^n & \\ \Longrightarrow & \tilde{\mathcal{U}}_c^{\boldsymbol{\omega}}(z) & as\ a\ formal\ power\ series & \\ Geometric\ model: & \frac{1}{y_1-z} \rightarrow & z\text{-}germ\ at\ \infty & \\ \Longrightarrow & \mathcal{U}_c^{\boldsymbol{\omega}}(z) & as\ a\ sectorial\ z\text{-}germ\ at\ \infty & \\ Convolutive\ model: & \frac{1}{y_1-z} \rightarrow & -\exp(y_1\,\zeta) & \\ \Longrightarrow & \hat{\mathcal{U}}_c^{\boldsymbol{\omega}}(\zeta) & as\ a\ full\ \zeta\text{-}germ\ at\ 0 & \end{array}$$

Main result:
For positive values $c > 0$ of the twist, $\mathcal{U}_c^\bullet$ constitutes a well-behaved, multiplicative system of resurgence monomials.

The limit-case $c = 0$.
In the limit-case $c = 0$ the integrals (134) remain well-defined and we still have a multiplicative system $\mathcal{U}_0^\bullet$ of resurgence monomials, but it is no longer *well-behaved.* In fact, it coincides with the much more ancient system $\mathcal{U}^\bullet$ of hyperlogarithmic monomials, so-called because their dependence on the ω_i's is indeed of hyperlogarithmic type. In contradistinction, the $\mathcal{U}_c^\bullet$ and the host of special functions attached to them (see §6.7) are called *paralogarithmic.*

Why "twisted", why "spherical", why "canonical" ?
The presence of a free parameter c slightly detracts from the "canonicity" of our system, but this cannot be helped: no system of well-behaved resurgence monomials can suffice for all problems unless there is at least one free parameter that can be adjusted from case to case. The miracle is rather that *one* parameter should be enough! So much for the twist. As for "spherical", it refers to the striking symmetry of behaviour which our monomials $\mathcal{U}_c^\bullet$ exhibit at the antipodes 0 and ∞ of the Riemann sphere when $c > 0$, and which, remarkably enough, disappears when we "untwist" them, ie when $c = 0$.

10.4 Proofs and comments.

Let ${}^{\pm}\mathcal{U}a_c^{\omega_1,\dots,\omega_r}(z,\eta)$ denote the $\mathbf{\Delta}^{\pm}$ orthogonal variant of our canonical monomials, in the geometric-sectorial model that corresponds to Borel-Laplace integration right (resp left) of the axis $\arg\zeta = 0$ if $\eta = +$ (resp -). The orthogonality relations to *ALIEN* can easily be checked on the y-integral representations

$$ {}^{\epsilon}\mathcal{U}e_c^{\omega_1,\dots,\omega_r}(z,\eta) \;:=\; \frac{(\epsilon 1)^r}{(2\pi i)^r}\int_0^{\epsilon,\eta\,\infty} \frac{go_{\omega_1}(y_1)\dots go_{\omega_r}(y_r)}{go_{\omega_1}(z)\dots go_{\omega_r}(z)}\,\frac{dy_1\dots dy_r}{(y_r-y_{r-1})\dots(y_1-z)} $$

with 'lateral circumvention' or, if you prefer, with integration on close but distinct axes $\theta_i \sim 0$:

$$ \epsilon\,(\theta-\theta_1)>0\;;\;\eta\,(\theta_i-\theta_{i+1})>0 \qquad (\arg z=\theta\;;\;\arg y_i=\theta_i\;;\;\epsilon,\eta\in\{+,-\}) $$

The finite' conditions **M1**, **M2**, **M4** offer no difficulties. **M1** is easily checked on the y-integrals. **M2** is immediate. So is **M4**: its expression in the multiplicative models (geometric or formal) is simply the invariance of the monomials under dilatations $(y,c,\omega_i)\mapsto(l\,y,l\,c,l^{-1}\,\omega_i)\;,\forall l>0$. In a sense, this is less than the "exact" homogeneousness (namely: invariance under the changes $(y,\omega_i)\mapsto(l\,y,l^{-1}\,\omega_i)$ as with the *standard* or *hyperlogarithmic* monomials of §2.4) that we might wish for, but this cannot be helped as soon as a parameter c comes into play. Moreover, this slight 'defect' is more than offset by the emergence of a new symmetry – under the *antipodal involution*, see §11.4, §12.4 – which has no 'standard' equivalent.

But the difficult bit is of course the growth condition **M3**. Let us first use the elementary moulds of §2.1 to rewrite the y-integral and its companion x-integral in a form reminiscent[98] of the formulas for the lateral moulds of the uniformising averages:

$$ \begin{aligned} {}^{\pm}\mathcal{U}a_c^{\omega_1,\dots,\omega_r}(z,\eta) &= \frac{(\pm 1)^r}{(2\pi i)^r}\int_0^{\infty} go_{\omega_1}(y_1)\dots go_{\omega_1}(y_1)\;\mathrm{tas}_{z,\infty}^{y_1,\dots,y_r}\;dy_1\dots dy_r \\ &= \mp\,\eta\,\frac{(-1)^r}{(2\pi i)^r}\int_{-i\infty}^{+\infty}\sigma_\eta(\hat{x}_1)\,\sigma_\mp(\hat{x}_2)\dots\sigma_\mp(\hat{x}_2) \\ &\qquad\qquad \times\; fo_{\omega_1}(x_1)\dots fo_{\omega_r}(x_r)\;e^{\|\mathbf{x}\|\,z}dx_1\dots dx_r \end{aligned} $$

Here, it is more convenient to use the criteria **M3.1** or **M3.3**, which involve *contracting arborification*. As with the uniformising averages, we have the choice between two strategies.

Either we choose to work with the x-integrals. This means subjecting the moulds *antisofo*$^\bullet$ or *antisefo*$^\bullet$ (depending on η) to contracting arborification

[98] except that the y-sequence must now be inversed, and the $\check{x}_i$-sums (22) replaced by $\hat{x}_i$-sums (23).

and observing that this operation leaves their outward expression unchanged: they are still given by the formulas of §2.1.9, but with sums $\hat{x}_i := \sum_{i\leq j} x_j$ now relative to the arborescent order.

Or we can work instead with the y-integrals. This means subjecting the mould $tas^\bullet_{z,\infty}$:

$$\mathrm{tas}^{y_1,\ldots,y_r}_{z,\infty} := \prod_{1\leq i\leq r} \frac{1}{y_{i\_} - y_i} \qquad \text{with} \tag{258}$$

$$y_{i\_} := y_{i-1} \ \text{ if } \ i > 1 \quad , \quad y_{i\_} := z \ \text{ if } \ i = 1 \tag{259}$$

to *ordinary* arborification (which, due to the circumvention rules, translates into *contracting* arborification for the integrals) and observe that this operation doesn't alter the factorisation of §2.1.8, but merely changes the interpretation of $y_{i\_}$: if y_i is not a root (minimal element), then $y_{i\_}$ must be interpreted as the (unique) antecedent of y_i within the arborescent sequence $\mathbf{y}^{\prec}$, and if y_i is a root , $y_{i\_}$ should be set equal to z.

Both methods not only yield the required bounds $\mathbf{M3_1}$,$\mathbf{M3_3}$, but there even appears an important factor $e^{-C_* c}$, $C_* > 0$ in front of the constants. Checking the fact of "form-preserving arborification" lemma (see §2.1.11) is easier with the polar mould $tas^\bullet_{z,\infty}$ than with the flat moulds *antisofo*$^\bullet$ or *antisefo*$^\bullet$, but with the first method (x-integration) the bounds $\mathbf{M3_1}$,$\mathbf{M3_3}$ are immediate to establish, since the 'flat' part of the integrand is either 0 or ± 1 and each $fo_{\omega_i}(x_i)$ is bounded on $i\,\mathbb{R}$, whereas with the second method (y-integration) one should carefully select (near 0) the multipath of integration in order to control the smallness of the denominators $y_{i\_} - y_i$.

10.5 Extension to the multicritical case.

General monocritical paralogaritms.

In the 6 examples discussed in §3, we have on purpose chosen the simplest normal forms $\mathbf{Ob}^{\mathrm{nor}}$, thereby ensuring that all objects $\mathbf{Ob} \sim \mathbf{Ob}^{\mathrm{nor}}$ could be represented by entire power series (and exponentials). But this doesn't hold for all conjugacy classes. Take for instance Example 2 in §3.2 and replace the normal form (75) by $d_x\, y^{nor} = y^{nor} + \sigma\, z^{-1}\, y^{nor}$ with $\sigma \notin \mathbb{Z}$. The formal integral (77) is no longer in $\mathbb{C}[[z^{-1}\,,\, u\, e^z]]$ but in $\mathbb{C}[[z^{-1}\,,\, u\, e^z\, z^\sigma]]$, which means that we are stuck with non-entire powers of z. This has three main consequences:

(i) *as far as alien calculus is concerned*, the indices ω of our alien derivations must now be regarded as elements of $\mathbb{C}_\bullet$ rather than $\mathbb{C}^*$.

(ii) *as far as object analysis is concerned*, the total "quantity" of (independent) analytic invariants doesn't increase, because for each pair $\omega, \omega' \in \mathbb{C}_\bullet$ with the same projections $\dot{\omega}, \dot{\omega}' \in \mathbb{C}^*$, the corresponding invariants $\mathbb{A}_\omega$ and $\mathbb{A}_{\omega'}$ are simply related (one is deducible from the other).

(iii) *as far as object synthesis is concerned*, we must now work with the ramified variants of our Δ-friendly twisted monomials, obtained by systematically replacing in all three integrals (255),(256),(257) the elements $\varpi(y) := \omega\, y + c^2\, \bar{\omega}\, y^{-1}$ by their ramified counterparts $\varpi(y) := \omega\, y + c^2\, \bar{\omega}\, y^{-1} + \sigma\, \log y$. That apart, nothing much changes, the new monomials being every bit as *well-behaved* as the unramified prototypes.

General polycritical paralogaritms.

In polycritical situations (exemplified by Ex 4 and Ex 6 in §3) object synthesis calls for polycritical twisted Δ-friendly monomials. We construct them by replacing the binomial elements $\varpi(y) := \omega\, y + c^2\, \bar{\omega}\, y^{-1}$ by the bipolynomial elements:

$$\varpi(y) := \sum_p (\omega_p\, y^p + c_p^2\, \bar{\omega}_p\, y^{-p})$$

or, in ramified situations, by:

$$\varpi(y) := \sum_p (\omega_p\, y^p + c_p^2\, \bar{\omega}_p\, y^{-p}) + \sigma\, \log y$$

Of course, for each *higher* critical time class $z_p := \{z^p\}$, the monomials' expression in the corresponding geometric model (resp Borel plane) involves a polarisation θ_q for each *lower* (resp *strictly lower*) critical time classe $z_q := \{z^q\}$ with $q \leq p$ (resp $q < p$).

As for the several twist parameters c_p attached to the various classes z_p, we may of course collapse them into one by setting $c_p := c$ or $c_p := p\, c$ or $c_p := p^{-1}\, c$ etc, but there seems to be no compelling reason for preferring one choice to the other.

∂-friendly paralogarithms.

The thorough symmetry which obtains, in the standard, twistless context, between the Δ-friendly and ∂-friendly monomials $\mathcal{U}^\bullet$ and $\mathcal{V}^\bullet$ and their monics $U^\bullet$ and $V^\bullet$, and which comes through graphically in the diagram of §4.1 and the formulas of §12.2.2 – that symmetry does survive for the twisted monomials, but in less perfect a manner. Some of the relevant formulas are

mentioned in §12.3.6. It should be noted, however, that the ∂-friendly monomials serve no purpose in *object analysis* (the standard, twistless monomials fully suffice there). Their *raison d'être* is rather :
(i) to lead to a better understanding of the twisted Δ-friendly monomials
(ii) to lead to simple formulas for the corresponding monics $U^\bullet$ and $V^\bullet$.

The corresponding developments would fill too much space here, but one curious feature perhaps deserves mention : whereas the classical gamma function $\Gamma(\sigma) := \int_0^\infty e^{-t}\, t^{\sigma-1}\, dt$ totally dominates the subject of *ramified hyperlogarithms*, for the *ramified paralogarithms* it should be replaced by a *twisted* look-alike $\Gamma_c(\sigma) := \int_0^\infty e^{-t-c^2\, t^{-1}}\, t^{\sigma-1}\, dt$ whose properties, however (entireness, asymptotics etc) couldn't be more different as soon as $c > 0$.

11 Applications to canonical Object Synthesis.

11.1 Outline of the construction.

Basically, with the twisted monomials at our disposal, Object Synthesis becomes a purely mechanical affair. This is precisely what we had set out to achieve : to reduce the whole process to a succession of formal manipulations. We saw in §4 why the early attempts, based on the standard monomials $\mathcal{U}^\bullet(z)$, couldn't fully succeed. But with their twisted counterparts $\mathcal{U}_c^\bullet(z)$, everything works fine.

Let us outline the six main steps :

Step 1: select a formal class of local analytic objects, characterised by a formal normal form $\mathbf{Ob}^{\mathrm{nor}}$, and start from any given *admissible* system of analytic invariants $\mathbb{A} = \{\mathbb{A}_\omega\, ;\ \omega \in \mathbf{\Omega}\}$

Step 2: choose a *well-behaved system of alien derivations*, preferably the *organic* system $\Delta^{\mathrm{org}} = \{\Delta^{\mathrm{org}}_\omega\, ;\ \omega \in \mathbb{C}^\star\}$, and express the analytic invariants in the corresponding basis $\mathbb{A}^{\mathrm{org}} = \{\mathbb{A}^{\mathrm{org}}_\omega\, ;\ \omega \in \mathbf{\Omega}\}$

Step 3: solve *"mechanically"*[99] the system of resurgence equations that

[99]ie without worrying about convergence. Mark the choice of words: *mechanically*, ie by means of expansions into series (127) of abstract resurgence monomials, rather than *formally*, which would suggest solving the problem in the ring of formal power series. To highlight the difference, we might also, as in §4, speak of *semi-formal expansions.*

characterise the direct or inverse normaliser $\Theta^{\pm 1}$. For instance, in the case of simply resonant fields we find these expansions[100] :

$$\Theta \overset{always}{:=} 1 + \sum_{1 \le r} \sum_{\omega_i \in \Omega} (-1)^r \, \mathcal{U}\!e_{\mathrm{org}}^{\omega_1,\ldots,\omega_r}(z) \; \mathbb{A}_{\omega_r}^{\mathrm{org}} \ldots \mathbb{A}_{\omega_1}^{\mathrm{org}} \qquad (260)$$

$$\Theta^{-1} \overset{conditionally}{:=} 1 + \sum_{1 \le r} \sum_{\omega_i \in \Omega} \mathcal{U}\!e_{\mathrm{org}}^{\omega_r,\ldots,\omega_1}(z) \; \mathbb{A}_{\omega_r}^{\mathrm{org}} \ldots \mathbb{A}_{\omega_1}^{\mathrm{org}} \qquad (261)$$

Step 4: replace in that "mechanical" solution the abstract monomials $\mathcal{U}_{\mathrm{org}}^{\boldsymbol{\omega}}(z)$ by the *twisted* or *spherical* monomials $\mathcal{U}_{c,\mathrm{org}}^{\boldsymbol{\omega}}(z)$ for a large enough twist c.

Step 5: *re-order* the above expansions for $\Theta^{\pm 1}$ so as to make them absolutely *convergent* in the space of resurgent functions, under the standard arborification-coarborification scheme: ie subject simultaneously the mould $\mathcal{U}\!e_{\mathrm{org}}^{\bullet}$ to the arborification rules in §2.1.4, and the co-mould $\mathbb{A}_{\bullet}^{\mathrm{org}}$ to the dual rule for "homogeneous" co-arborification spelt out in §11.2 below.

Step 6: Construct the sought-after analytic object **Ob** from its normaliser by using $\mathbf{Ob} = \Theta\, \mathbf{Ob}^{\mathrm{nor}}\, \Theta^{-1}$

The reader may easily work this out in the case of our four monocritical examples of §3 and §4 (Ex 1 through 4). The polycritical examples (Ex 5 and 6) also respond to the same treatment, except that *acceleration theory*[101] is needed. Here, to avoid drowning in secondary details, we shall focus only on the central point, namely the *convergence*, after arborification, of the *mechanical expansions* into series of twisted monomials.

11.2 Proof of the convergence for a positive or large enough twist c.

Generic divergence prior to arborification.

Let us work with the direct normalisers. Prior to arborification, their expansions, whether we write them in a well-behaved basis, say the *organic*

[100] The second expansion, for the reverse normaliser, is valid only if all the invariants $\mathbb{A}_\omega$ have no ∂_z-component and so commute with the resurgence monomials $\mathcal{U}\!e^{\boldsymbol{\omega}}(z)$. When this is not the case, the expansion (261) should be slightly modified, but one can also be content to work with (260), which is always valid, and then derive Θ^{-1} by straightforward inversion of Θ.

[101] see for instance [E4][E6][E7].

basis[102]:

$$\Theta = \sum_{\boldsymbol{\omega}} \mathcal{U}e_c^{\boldsymbol{\omega}}(z)\ \mathbb{A}_{\boldsymbol{\omega}} = \sum_r \sum_{\omega_i} \mathcal{U}e_c^{\omega_1,\ldots,\omega_r}(z)\ \mathbb{A}_{\omega_r}\ldots\mathbb{A}_{\omega_1} \tag{262}$$

or in one of the (right/left) lateral bases:

$$\Theta = \sum_{\boldsymbol{\omega}} {}^{\pm}\mathcal{U}e_c^{\boldsymbol{\omega}}(z)\ \mathbb{A}^{\pm}_{\boldsymbol{\omega}} = \sum_r \sum_{\omega_i} {}^{\pm}\mathcal{U}e_c^{\omega_1,\ldots,\omega_r}(z)\ \mathbb{A}^{\pm}_{\omega_r}\ldots\mathbb{A}^{\pm}_{\omega_1} \tag{263}$$

are hopelessly divergent[103] since we cannot expect better bounds than:

$$\|\mathcal{U}e_c^{\boldsymbol{\omega}}\| \leq c_1^r\, e^{-C_0\,c\,r} \quad ; \quad \|\mathbb{A}_{\boldsymbol{\omega}}\| \leq c_2^r\, r! \quad (0 < c_i < \infty) \tag{264}$$
$$\|{}^{\pm}\mathcal{U}e_c^{\boldsymbol{\omega}}\| \leq c_3^r\, e^{-C_0\,c\,r} \quad ; \quad \|\mathbb{A}^{\pm}_{\boldsymbol{\omega}}\| \leq c_4^r\, r! \quad (0 < c_i < \infty) \tag{265}$$

r being the length of the totally ordered sequence $\boldsymbol{\omega}$. So we have to arborify these expansions:

$$\sum_{\boldsymbol{\omega}} \mathcal{U}e_c^{\boldsymbol{\omega}}(z)\ \mathbb{A}_{\boldsymbol{\omega}} \quad \to \quad \sum_{\boldsymbol{\omega}^{\prec}} \mathcal{U}e_c^{\boldsymbol{\omega}^{\prec}}(z)\ \mathbb{A}_{\boldsymbol{\omega}^{\prec}} \qquad \text{(ordinary arborification)}$$
$$\sum_{\boldsymbol{\omega}} {}^{\pm}\mathcal{U}e_c^{\boldsymbol{\omega}}(z)\ \mathbb{A}^{\pm}_{\boldsymbol{\omega}} \quad \to \quad \sum_{\boldsymbol{\omega}^{\prec\!\!\!\prec}} {}^{\pm}\mathcal{U}e_c^{\boldsymbol{\omega}^{\prec\!\!\!\prec}}(z)\ \mathbb{A}^{\pm}_{\boldsymbol{\omega}^{\prec\!\!\!\prec}} \qquad \text{(contracting arborification)}$$

Under the dual definitions for arborification and coarborification (see §2.1.4), the operation leaves our expansions for Θ formally unchanged: it merely rearranges their terms. But instead of the unsatisfactory bounds (264), (265) we shall now get new ones:

$$\|\mathcal{U}e_c^{\boldsymbol{\omega}^{\prec}}\| \leq C_1^r\, e^{-C_0\,c\,r} \quad ; \quad \|\mathbb{A}_{\boldsymbol{\omega}^{\prec}}\| \leq C_2^r \quad (0 < C_i < \infty) \tag{266}$$
$$\|{}^{\pm}\mathcal{U}e_c^{\boldsymbol{\omega}^{\prec\!\!\!\prec}}\| \leq C_3^r\, e^{-C_0\,c\,r} \quad ; \quad \|\mathbb{A}^{\pm}_{\boldsymbol{\omega}^{\prec\!\!\!\prec}}\| \leq C_4^r \quad (0 < C_i < \infty) \tag{267}$$

that shall ensure absolute convergence.

The *removability* of the factor $r!$ in the comould part doesn't come as a great surprise, since the coarborification constraints[104] *split* each $\mathbb{A}_{\boldsymbol{\omega}}$ into a sum of roughly $r!$ terms. But we might fear, dually, the *appearance* of a factor $r!$ in the mould part, since the arborification rules[105] *regroup* roughly $r!$ terms on the right-hand side. This is indeed what arborification does to a 'random' mould: it creates a factor of magnitude $r!$. However, most applications

[102]for brevity we drop the indes "org" in the sequel.
[103]generically – except in linear problems like Example 3.
[104]see §2.1.4.
[105]see §2.1.4.

involve moulds which are far from 'random': not only do they fall into one of the four basic symmetry types, but they usually verify additional identities, which ensure that they retain their 'size' (and sometimes even their 'form') under arborification.

Arborification here doesn't increase mould norms.

This applies to most of the moulds catalogued in §2.1, with only a handful of exceptions that are mentioned there[106]. This applies in particular, as we showed in §10.4, to the twisted resurgence monomials, as soon as their twist c is > 0.

Homogeneous coarborification.

Let $\{\mathbb{B}_\omega, \omega \in \Omega\}$ be any system of ordinary differential operators in the variables $x_1, \dots, x_\nu$ and define the comould $\mathbb{B}_\bullet$ as usual by setting:

$$\mathbb{B}_{\omega_1,\dots,\omega_r} := \mathbb{B}_{\omega_r} \dots \mathbb{B}_{\omega_1} \tag{268}$$

Then *there exists a privileged arborescent comould* $\mathbb{B}_{\bullet^\prec}$ *, the so-called homogeneous co-arborification of* $\mathbb{B}_\bullet$ *which is entirely characterised by the following three properties:*
P1 $\mathbb{B}_{\bullet^\prec}$ is co-separative[107], ie:

$$\mathbb{B}_{\boldsymbol{\omega}^\prec}(\varphi_1\,\varphi_2) \equiv \sum_{\boldsymbol{\omega^1}^\prec \oplus \boldsymbol{\omega^2}^\prec = \boldsymbol{\omega}^\prec} \mathbb{B}_{\boldsymbol{\omega^1}^\prec}(\varphi_1)\,\mathbb{B}_{\boldsymbol{\omega^2}^\prec}(\varphi_2) \tag{269}$$

P2 If $deg(\boldsymbol{\omega}^\prec) = d$ ie if the tree $\boldsymbol{\omega}^\prec$ has exactly d roots, then the operator is homogeneous in the $\partial_i := \partial_{x_i}$ with total degree d
P3 If $\boldsymbol{\omega} = \omega_1\,\boldsymbol{\omega}^*$ (in other words, if $\boldsymbol{\omega}$ is of degree one, with a root element ω_1 followed by some arborescent sequence $\boldsymbol{\omega}^{*\prec}$) the corresponding operator factors as:

$$\mathbb{B}_{\boldsymbol{\omega}^\prec}\ x_j \ \equiv\ \mathbb{B}_{\boldsymbol{\omega}^{*\prec}}\ \mathbb{B}_{\omega_1}\ \log x_j \qquad (j = 1, 2, \dots, \nu) \tag{270}$$

Moreover, if $\mathbb{B}_\bullet$ *is co-symmetral*[108] *(resp co-symmetrel*[109]*), then* $\mathbb{B}_\bullet$ *and* $\mathbb{B}_\bullet^\prec$ *are indeed correlated according to* $B_{\boldsymbol{\omega}} := \sum_{\boldsymbol{\omega}^\prec < \boldsymbol{\omega}} \ B_{\boldsymbol{\omega}^\prec}$ *(resp* $B_{\boldsymbol{\omega}} :=$

[106]see also §8.7.
[107] $\boldsymbol{\omega^1}^\prec \oplus \boldsymbol{\omega^2}^\prec$ denotes the tree obtained by juxtaposition of $\boldsymbol{\omega^1}^\prec$ and $\boldsymbol{\omega^2}^\prec$, with no other order relations than those inherited from the sub-trees $\boldsymbol{\omega^i}$. The sum (279) extends also to the trivial juxtapositions, with one summand $\boldsymbol{\omega^i}^\prec$ equal to $\boldsymbol{\omega}^\prec$ and the other one empty.
[108]see (17).
[109]see(17).

$\sum_{\boldsymbol{\omega}^{\not\prec} \ll \boldsymbol{\omega}} B_{\boldsymbol{\omega}^{\not\prec}}$). *In other wordss, whereas symmetral and symmetrel moulds obey different arborification rules (simple versus contracting), the homogeneous co-arborification rules are exactly the same for a co-symmetral comould (like* $\mathbb{A}_{\bullet}$ *above) and a co-symmetrel one (like* $\mathbb{A}^{\pm}_{\bullet}$ *above).*

Let us check, by induction on the length r of $\boldsymbol{\omega}^{\prec}$, the fact that **P1, P2, P3** together do determine $\mathbb{B}_{\boldsymbol{\omega}^{\prec}}$.

Either $d(\boldsymbol{\omega}^{\prec}) = 1$, which means that $\boldsymbol{\omega}^{\prec}$ is of the form (271), in which case $\mathbb{B}_{\boldsymbol{\omega}^{\prec}}$ is as below:

$$\begin{aligned} \boldsymbol{\omega}^{\prec} &= (\omega_1, \boldsymbol{\omega}^{*\prec}) \\ \mathbb{B}_{\boldsymbol{\omega}^{\prec}} &= \sum_{1 \le i \le \nu} (\mathbb{B}_{\omega^{*\prec}} . \mathbb{B}_{\omega_1{}^{\prec}} . \log x_j)(x_j\, \partial_j) \end{aligned} \tag{271}$$

Or $deg(\boldsymbol{\omega}^{\prec}) = d \geq 2$, which means that $\boldsymbol{\omega}^{\prec}$ is of the form (272), with s clusters of $d_1, \ldots, d_s$ identical, irreducible summands $\boldsymbol{\omega}^{\boldsymbol{i_1}\prec}, \ldots, \boldsymbol{\omega}^{\boldsymbol{i_s}\prec}$, in which case $\mathbb{B}_{\boldsymbol{\omega}^{\prec}}$ is as below:

$$\begin{aligned} \boldsymbol{\omega}^{\prec} &= \boldsymbol{\omega}^{\mathbf{1}\prec} \oplus \cdots \oplus \boldsymbol{\omega}^{\boldsymbol{d}\prec} && (\boldsymbol{\omega}^{\boldsymbol{i}\prec} \neq \emptyset \ ,\ deg(\boldsymbol{\omega}^{\boldsymbol{i}\prec}) = 1) \\ &= (\boldsymbol{\omega}^{\boldsymbol{i_1}\prec})^{\oplus d_1} \oplus \cdots \oplus (\boldsymbol{\omega}^{\boldsymbol{i_s}\prec})^{\oplus d_s} && (d_1 + \cdots + d_s = d) \\ \mathbb{B}_{\boldsymbol{\omega}^{\prec}} &= \frac{1}{d_1! \ldots d_s!} \sum_{\substack{1 \le s \le d \\ 1 \le j_s \le \nu}} (\mathbb{B}_{\boldsymbol{\omega}^{1\prec}} . \log x_{j_1}) \ldots (\mathbb{B}_{\boldsymbol{\omega}^{d\prec}} . \log x_{j_d})\ (x_{j_1}\, \partial_{j_1}) \ldots (x_{j_d}\, \partial_{j_d}) \end{aligned} \tag{272}$$

Coarborification diminishes comould norms.

The phenomenon takes place for any reasonable norm on local differential operators, for instance:

$$\|\mathbb{B}\| = \|\mathbb{B}\|_{\mathcal{D}_1, \mathcal{D}_2} := \sup_{\varphi \neq 0} \frac{\|\mathbb{B}\, \varphi\|_{\mathcal{D}_1}}{\|\varphi\|_{\mathcal{D}_2}} \qquad \textit{with} \quad 0 \in \mathcal{D}_1 \, ,\, \bar{\mathcal{D}}_1 \subset \mathcal{D}_2 \subset \mathbb{C}^{\nu} \tag{273}$$

with $\mathcal{D}_1, \mathcal{D}_2$ two small open neighbourhoods of 0 and $\|\varphi\|_{\mathcal{D}_i}$ the uniform norm on $\mathcal{D}_i$. To illustrate norm reduction, ie the improvement from (274) to (275):

$$\|\mathbb{B}_{\boldsymbol{\omega}}\| \leq r(\boldsymbol{\omega}^{\prec})!\ C^{N(\boldsymbol{\omega}^{\prec})}\ \|\mathbb{B}_{\omega_1}\| \ldots \|\mathbb{B}_{\omega_r}\| \tag{274}$$

$$\|\mathbb{B}_{\boldsymbol{\omega}^{\prec}}\| \leq C^{N(\boldsymbol{\omega}^{\prec})}\ \|\mathbb{B}_{\omega_1}\| \ldots \|\mathbb{B}_{\omega_r}\| \tag{275}$$

let us fix a non-resonant spectrum $\lambda \in \mathbb{C}^\nu$ and consider first-order differential operators of the form:

$$\mathbb{B}_{\omega_i} := x^{n_i}\,\mathbf{B}_{\omega_i} \qquad \textit{with} \qquad \mathbf{B}_{\omega_i} := \sum_{1\leq j\leq \nu} B^j_{\omega_i}\, x_j\, \partial_{x_j} \ ,\ \omega_i := < n_i, \lambda > \tag{276}$$

Next, let us carry out homogeneous coarborification for three extreme types of arborescent sequences:

$$\begin{array}{lcllll}
\boldsymbol{\omega} & := & (\omega_1,\dots,\omega_r) & ; & \text{total order} & ; \ \text{all } \omega_i \text{ distinct} \\
\boldsymbol{\omega}'^{\prec} & := & (\omega_1,\dots,\omega_r) & ; & \text{total order} & ; \ \text{all } \omega_i \text{ distinct} \\
\boldsymbol{\omega}''^{\prec} & := & (\omega_r \oplus \dots \oplus \omega_1) & ; & \text{no order} & ; \ \text{all } \omega_i \text{ distinct} \\
\boldsymbol{\omega}'''^{\prec} & := & (\omega_r \oplus \dots \oplus \omega_1) & ; & \text{no order} & ; \ \text{all } \omega_i \text{ identical}
\end{array}$$

We find:

$$\mathbb{B}_{\boldsymbol{\omega}} := \mathbb{B}_{\omega_r} \dots \mathbb{B}_{\omega_1} \tag{277}$$

$$\mathbb{B}_{\boldsymbol{\omega}'^{\prec}} := x^{n_r}(\mathbf{B}_{\omega_r}\, x^{n_{r-1}})\,(\mathbf{B}_{\omega_{r-1}}\, x^{n_{r-2}}) \dots (\mathbf{B}_{\omega_3}\, x^{n_2})\,(\mathbf{B}_{\omega_2}\, x^{n_1})\mathbf{B}_{\omega_1} \tag{278}$$

$$\mathbb{B}_{\boldsymbol{\omega}''^{\prec}} := x^{n_1+\dots+n_r}\,\mathbf{B}_{\omega_1} \dots \mathbf{B}_{\omega_r} \tag{279}$$

$$\mathbb{B}_{\boldsymbol{\omega}'''^{\prec}} := \frac{1}{r!}\, x^{n_1+\dots+n_r}\,\mathbf{B}_{\omega_1} \dots \mathbf{B}_{\omega_r} \tag{280}$$

and in all three cases we observe the disappearance of the factor $r!$, though for rather distinct reasons:

– in (278) we have a first-order differential operator $\mathbf{B}_{\omega_1}$ preceded by innocuous scalar factors $\mathbf{B}_{\omega_i}\, x^{n_{i-1}}$

– in (279) we a differential operator $\mathbf{B}_{\omega_1} \dots \mathbf{B}_{\omega_r}$ (all terms commute) of order r and of factorially large norm, but with a more than factorially small front factor $x^{\|n\|}$ since $\|n\| \geq \mathit{const}\,.\, r^{1+\frac{1}{\nu}}$

– in (280) we have again a differential operator $\mathbf{B}_{\omega_1} \dots \mathbf{B}_{\omega_r}$ (all terms are equal) of order r and of factorially large norm, but with a multiplicity factor $\frac{1}{r!}$ in front.

In fact, norm reduction holds for *all* arborescent sequences: see for instance [E5],§4.

Convergence after twisting and arborification.

At this stage, to establish the normal convergence of the arborified expansions for the direct normalisers Θ for a large enough value of the twist c, it is enough:

– to pair the mould-comould estimates in (266) or (267)
– to establish the existence of bounds $q(N) \leq Q_0\, Q_1^N$ where $q(N)$ denotes the total number of arborescent sequences $\boldsymbol{\omega}^{\prec} = (\omega_i)^{\prec}$ corresponding to multi-integer sequences $\mathbf{n}^{\prec} = (n_i)^{\prec}$ with $n_i \in \mathbb{N}^\nu$ of sum $\|\mathbf{n}^{\prec}\| = \sum |n_i| = N$ [110].

The one-sided/two-sided dichotomy.

Actually, the choice of a large enough twist c is necessary only in what we might call problems of *two-sided resurgence* (which, in the context of our 6 examples, would cover the *binary*, *sesquilateral*, and *bilateral* cases) ie in all problems where the index set Ω is such that in the expansions (262) infinitely many $\boldsymbol{\omega} = (\omega_i)$ may contribute to the same total frequency $\omega_0 = \|\boldsymbol{\omega}\| = \sum \omega_i$. But in problems of *one-sided resurgence* (in the context of our 6 examples, this would cover the *unary* and *unilateral* cases) we have convergence for any $c > 0$.[111]

The linear/non-linear dichotomy.

On the other hand, the recourse to arborification is indispensible only for *non-linear* problems. For *linear* or *affine problems* (such as Example 3 and 5), the unarborified expansions are already (normally) convergent. Lastly, in the very exceptional – and elementary – instance of problems that are both *linear/affine and one-sided* (this would correspond to the *unary* subcase in Ex 1 and Ex 2) there is need for neither twist nor arborification.

The total picture can be neatly summed up in the following table:

	linear problems	non-linear problem
one-sided resurgence	$c \geq 0$; *without arbor.*	$c > 0$; *with arbor.*
two-sided resurgence	$c \gg 1$; *without arbor.*	$c \gg 1$; *with arbor.*

11.3 What is so special about the twistless case ?

Basically, we already know the answer: only for $c > 0$ are the "prodistribution" functions $fo_\omega(x)$ absolutely integrable at $x = \pm i\infty$, and this fact in turn is responsible for property **M3**. But we must now look concretely at the difference which a positive c makes to object synthesis. Let us reason on Example 2 (§3.2) and focus on the *unilateral case* (see §4.5.3,§4.6) because, unlike the *bilateral case*, it allows us to take c as small as we wish.

[110] one can take $Q_1 := 4.2^\nu$, see [E5],pp94-95.
[111] but usually not for $c = 0$. See §4.6.

Starting from some admissible unilateral system $\{\mathbb{A}_n := u^{n+1}\partial_u \,,\, n \in \mathbb{N}^+\}$ of analytic invariants, using the mould $tas^{\bullet}_{a,b}$ of §2.1.8, and taking recourse to the usual *SPA* trick (*special path averaging*), let us form the general substitution operators :

$$\begin{aligned} H_{b,a} \quad := \quad (2\pi i)^{-r}\ SPA \int_0^{\infty} \mathrm{tas}_{a,b}^{y_1,\dots,y_r}\ e^{-\sum_{1\le i\le r}(n_i y_i + c^2 n_i\, y_i^{-1})} \times \\ dy_1\dots dy_r\ \ v^{n_1+\dots+n_r}\ \mathbb{A}_{n_r}\dots\mathbb{A}_{n_r} \qquad (281) \end{aligned}$$

which verify[112] :

$$H_{b,a}\ \varphi = \varphi \circ h_{a,b} \qquad with \qquad h_{a,b} := H_{b,a}.u \qquad (282)$$

$$H_{a_3,a_2}\ .\ H_{a_2,a_1} = H_{a_3,a_1} \quad but \quad h_{a_1,a_2} \circ h_{a_2,a_3} = h_{a_1,a_3} \qquad (283)$$

and which, for suitable specialisations of the parameters a, b, v, yield the direct and inverse normalisers :

$$\begin{array}{llllllll} \Theta & \equiv & H_{\infty,z} & , & \theta & \equiv & h_{z,\infty} & for \quad v := e^{\,z+c^2\,z^{-1}} \\ \Theta^{-1} & \equiv & H_{z,\infty} & , & \theta^{-1} & \equiv & h_{\infty,z} & for \quad v := e^{\,z+c^2\,z^{-1}} \end{array}$$

For instance, if we set all invariants equal to 0 and retain only the first one, ie $\mathbb{A}_1 = A_1\, u^2\, \partial_u \neq 0$, we get:

$$\begin{array}{lllll} h_{a,b} & = & \frac{u}{1+A_1\,(\mathcal{U}a^1_c(a)-\mathcal{U}a^1_c(b))\,v\,u} & & (\,for\ \ \mathbb{A}_1 = A_1\, u^{n+1}\,\partial_u\ \ ) \\ \theta(u) & = & \frac{u}{1+A_1\ \mathcal{U}a^1_c(z)\, e^{z+c^2\,z^{-1}}\, u} & = & \frac{u}{1+A_1\ \mathcal{U}e^1_c(z)\, u} \\ \theta^{-1}(u) & = & \frac{u}{1-A_1\ \mathcal{U}a^1_c(z)\, e^{z+c^2\,z^{-1}}\, u} & = & \frac{u}{1-A_1\ \mathcal{U}e^1_c(z)\, u} \end{array}$$

Let us now add infinitesimal, admissible perturbations $\delta\mathbb{A}_n$ $(n \geq 2)$, thus changing our invariant system $\{\mathbb{A}_1, 0, 0, \dots\}$ to $\{\mathbb{A}_1, \delta\mathbb{A}_2, \delta\mathbb{A}_3, \dots\}$. Since :

$$\mathrm{tas}_{a,b}^{y_1,\dots,y_r} \equiv \mathrm{tas}_{a,y_i}^{y_1,\dots,y_{i-1}}\ \ \frac{a-b}{(a-y_i)\,(y_i-b)}\ \ \mathrm{tas}_{y,b}^{y_{i+1},\dots,y_r} \qquad (\forall i, 1 \leq i \leq r)$$

the operator $H_{b,a}$ will undergo the perturbation :

$$\delta H_{b,a} \quad := \quad \frac{1}{2\pi i} \sum_{2\le n} \int_0^{\infty} \mathrm{H}_{b,y}\ \ \frac{(a-b)\ e^{-n\,(y+c^2 y^{-1})}\ v^n}{(a-y)\,(y-b)}\ \ \delta\mathbb{A}_n\ \mathrm{H}_{y,a}\ dy$$

$$\delta h_{a,b} \quad := \quad \frac{1}{2\pi i} \sum_{2\le n} \int_0^{\infty} \frac{(a-b)\ e^{-n\,(y+c^2 y^{-1})}\ v^n}{(a-y)\,(y-b)}\ \ \frac{(h_{y,b}(u))^{n+1}}{\partial_u h_{y,b}(u)}\ \partial_u h_{a,b}(u)\ dy$$

[112] note the index exchange $H_{b,a} \to h_{a,b}$ which reflects the composition rules

Finally, setting $a = z, b = +\infty, v = e^{z+c^2 z^{-1}}$ we get :

$$\begin{aligned}\delta h_{a,b} \;&:=\; \frac{1}{2\pi i} \sum_{2\leq n} \int_0^\infty \frac{e^{-n\,(y+c^2\,y^{-1})}\; e^{+n\,(z+c^2\,z^{-1})} u^n}{(z-y)(1+A_1\,\mathcal{U}a_c^1(y)\,e^{\,z+c^2\,z^{-1}}\,u)^{\,n+1}} \\ &\qquad\qquad \times \frac{dy}{(1+A_1\,\mathcal{U}a_c^1(z)\,e^{\,z+c^2\,z^{-1}}\,u)}\end{aligned} \tag{284}$$

Reverting to §3.2 and (77), we see that the crucial variable are z ($\sim \infty$) and $w := u\,e^z$ (~ 0). More precisely, for canonical synthesis to survive the infinitesimal pertubation we have just performed, the above integral (284) must converge uniformly for z large enough (more precisely : for z in some U-shaped neighbourhood of $+\infty$) and for w small enough (more precisely : in a *full* neighbourhood of 0).

For a vanishing twist ($c = 0$), $\mathcal{U}a_c^1(y)$ reduces to a hyperlogarithmic resurgence monomial and it goes to ∞ as *const.* $\log(1/y)$ when $y \downarrow +0$. So, for z fixed, the denominator $1 + A_1\,\mathcal{U}a_c^1(y)\,e^{\,z+c^2\,z^{-1}}\,u|_{c=0} \equiv 1 + A_1\,\mathcal{U}a_0^1(y)\,w$ vanishes, when the integration variable y decreases to $+0$, for at least one value $w := w(z, y)$ which also goes to 0. So the integral (284) cannot be w-analytic at $w = 0$.

For $c > 0$, on the other hand, the paralogarithmic resurgence monomial $\mathcal{U}a_c^1(y)$ remains bounded on the whole integration axis $\mathbb{R}^+$ and each n-summand in (284) is indeed w-anlytic at $w = 0$. [113]

11.4 The antipodal pairing.

For $c > 0$ the twisted monomials are essentially invariant under the involution:

$$\text{pod} \quad : \qquad z \mapsto z^{-1} \;\; \textit{and} \;\; (\omega_i, c^2\bar{\omega}_i) \mapsto (\bar{\omega}_i, c^2\omega_i) \tag{285}$$

More precisely, setting :

$$\text{pari}(M^{\boldsymbol{\omega}}) \;:=\; (-1)^{r(\boldsymbol{\omega})}\;\; M^{\boldsymbol{\omega}} \tag{286}$$

and with $\mathit{SSU}_c^\bullet$ defined as in §12.3.3, we have :

$${}^{\text{pod}}\mathcal{U}_c^\bullet(z) \times \mathcal{U}_c^\bullet(c^2\,z^{-1}) = \mathit{SSU}_c^\bullet \tag{287}$$

and therefore :

$$\mathcal{U}_c^\bullet(c^2\,z^{-1}) = \text{pari}\,({}^{\text{pod}}\mathcal{U}_c^\bullet(z) \times \mathit{SSU}_c^\bullet) = \text{pari}\,({}^{\text{pod}}\mathcal{U}_c^\bullet(z)) \times \text{pari}\,(\mathit{SSU}_c^\bullet) \tag{288}$$

[113] As for the n-convergence of the series in (284), it offers nondifficulty as long as we take a admissible pertubation $\delta\mathbb{A}_n$. But n-convergence is not the real issue here, and indeed we may get rid of it by assuming that only a finite number of $\delta\mathbb{A}_n$ are $\neq 0$.

$$\Theta_{c,c^2 z^{-1}} = \mathrm{Iv}_u \,.\, K_c \,.\, \Theta_{c,z} \,.\, \mathrm{Iv}_u \tag{289}$$

$$K_c \;:=\; \sum \mathrm{pari}(\mathcal{SSU}_c^\bullet)\;\; \mathbb{A}_\bullet \;=\; \sum (-1)^{r(\bullet)}\; \mathcal{SSU}_c^\bullet\;\; \mathbb{A}_\bullet \tag{290}$$

$$\mathrm{Iv}_u \,.\varphi(u) := -\,\varphi(-u) \qquad ; \qquad \mathrm{Iv}_u \,.u^{n+1}\partial_u \,.\mathrm{Iv}_u = (-1)^n\, u^{n+1}\,\partial_u \tag{291}$$

So, once again we encounter this strange pairing of the "antipodes". In fact, any time we perform canonical synthesis, we get two objects "for the price of one": we simultaneously synthesize one (main) object at $z = \infty$, and another at $z = 0$. These two may, or may not, link up under analytic continuation.

11.5 Iso-invariant deformations.

There exists a closed system of formulae (see §6.7 and [E15]) to describe the exact dependence (partial derivatives, asymptotics, etc) of our canonical resurgence monomials as functions of their variable z, twist c and indices $\omega_i, \bar{\omega}_i$. As a result, one may write down the – often unexpectedly simple – partial differential equations which govern the sundry deformations (iso-invariant, iso-monodromic, iso-resurgent, iso-Galoisian, etc) of our synthesised objects.

Thus, from the c-differentiation rule for the monomials:

$$\frac{1}{2} c\, \partial_c \mathcal{U}e^\bullet(z) = z^{-1}\, \mathcal{U}e^\bullet(z) \times (e^{\,z\,\Box + c^2\, z^{-1}\,\Box}\;\; \mathcal{UR}^\bullet) \tag{292}$$

we get (with the notations of §12.1.3) the c-differentiation rule for the normaliser Θ_c:

$$\begin{aligned} & [\,\frac{1}{2} c\,\partial_c\,,\,\Theta_c\,] && = \quad UK\,.\,\Theta_c && (293)\\ \textit{with} \qquad & UK && := \quad \sum (-1)^{r(\bullet)}\,(e^{\,z\,\Box + c^2\, z^{-1}\,\Box}\;\; \mathcal{UR}^\bullet)\,\mathbb{A}_\bullet && \end{aligned}$$

11.6 Remarks and complements.

Remark 1: Antipodal involution.
As already pointed out, our twisted monomials have much the same behaviour at both poles of the Riemann sphere. The exact correspondence has

just been described in §6.7 using the so-called *antipodal involution* (285) :

In terms of the objects being produced, this means that canonical object synthesis automatically generates two objects : the 'true' object, local at ∞ and with exactly the prescribed invariants, and a 'mirror reflection', local at 0 and with closely related invariants. Depending on the nature of the problem (linear/non-linear, etc) and of the invariants (verification or non-verification of an "overlapping condition") , these two objects may or may not link up under analytic continuation on the Riemann sphere.

Remark 2: Analogy with q-equations.
Authors like Sauloy recently ([S]) observed that q-difference equations are in some sense easier to tackle than difference or differential equations, due to dilations $z \mapsto q\,z$ having two fixed points 0 and ∞, whereas shifts $z \mapsto a + z$ have only one, namely ∞. It is certainly no coincidence that the simplest resurgence monomials that permit object synthesis are precisely the twisted ones ($c \neq 0$), for whom the antipodal symmetry is restored, whereas the twistless monomials ($c = 0$), though apparently more simple, turn out to be inadequate for this particular purpose.

Remark 3: Necessity of a one-parameter freedom.
The necessity of having at least one degree of freedom in object synthesis has been known since the 1980s at least. This holds even for such elementary objects as linear systems (Example 3). Indeed, in most cases, the twist c must exceed a certain lower bound c_{min} that depends on the invariants $\{\mathbb{A}_\omega\}$. However, as already pointed out, there exists an important exception : the so-called *unilateral classes*, when for instance all non-vanishing $\mathbb{A}_\omega$ have their indices on the same half-line. There *any choice* $c > 0$ *will do !* This applies in particular to Example 2 when $\mathbb{A}_{-1} = 0$

Remark 4: WB derivations and WB monomials : unequal status.
Working with well-behaved alien derivations is merely convenient, whereas the recourse to well-behaved resurgence monomials is truly indispensible. There is a subtle difference here, which should be well understood. Indeed, the choice of this or that system of WB alien derivations *does not affect the result*: it simply gives us a comfortable basis of ALIEN to work with. Besides, there is always the lazy option of working with the lateral alien operators $\mathbf{\Delta}^{\pm}_{\omega}$, the only drawback being that the corresponding invariants $\mathbb{A}^{\pm}_{\omega}$ cease to be first-order differential operators. In complete contrast, the synthesised object very much depends on the choice of the system of WB monomials. And in the absence of well-behaved monomials, canonical synthesis would founder altogether.

Let us now collect, in a final section, a number of useful formulas about the twisted monomials and the associated monics.

12 Hyper-, peri-, para-logarithmic monomials and monics: total closure.

12.1 Main objects and notations.

12.1.1 Hyperlogarithms, perilogarithms, paralogarithms.

Perilogarithms have indices $\varpi_i = (\omega_i, \omega_i^\star) \in (\mathbb{C}^\star, \mathbb{C}^\star)$.
Hyperlogarithms have indices $\omega_i \in \mathbb{C}^\star$.
Perilogarithms have indices $\varpi_i = (\omega_i, \omega_i^\star) \in (\mathbb{C}^\star, \mathbb{C}^\star)$ with $\omega_i \omega_i^\star \in \mathbb{R}^+$.
Perilogarithms have indices $\varpi_i = (\omega_i, c^2 \bar{\omega}_i) \in (\mathbb{C}^\star, \mathbb{C}^\star)$.
Usually c is fixed, so that only ω_i is mentioned.

12.1.2 ∂- or Δ-friendly monomials and monics.

Monics depend only on the indices ω_i or ϖ_i.
Monomials depend on a variable z as well.
∂-friendly monomials behave simply under ordinary z-differentiation, but less so under alien z-differentiation: their alien derivatives necessarily involve a number of so-called ∂-friendly monics.
Δ-friendly monomials carry behave simply under alien z-differentiation, but less so under ordinary z-differentiation: their ordinary derivatives necessarily involve a number of so-called Δ-friendly monics.
∂-friendly (resp Δ-friendly) monomials always carry a calligraphic $\mathcal{V}^\bullet$ (resp $\mathcal{U}^\bullet$) as part of their names while the corresponding monics carry an uppercase $V^\bullet$ (resp. $U^\bullet$), sometimes supplemented by a suitable string of pre- or suffixes.

12.1.3 Total closure.

The monomials, as functions of z, are acted upon by *one* ordinary derivation $\partial := \partial_z$ but by *infinitely many* independent alien derivations Δ_{ω_i} or their variants $\mathbb{\Delta}_\omega := e^{-\omega z} \Delta_\omega$. The latter have the advantage of commuting with the ordinary derivation ∂, but at the cost of introducing an exponential factor and thus ceasing to act internally on the ring of formal power series of z^{-1}.

To highlight the $\partial \leftrightarrow \Delta$ duality, it is sometimes convenient to (formally) regroup all alien derivation into one single symbol:

$$\Delta := \sum \Delta_\omega \quad ; \quad \mathbb{\Delta} := \sum \mathbb{\Delta}_\omega$$

$$\begin{aligned}
\Box A^{\varpi} &:= \|\omega\|\, A^{\varpi} := (\sum \omega_i)\, A^{\varpi} \\
\Box^\star A^{\varpi} &:= \|\omega^\star\|\, A^{\varpi} := (\sum \omega_i^\star)\, A^{\varpi} \\
\Box_{\omega_i} &:= [\partial_{\omega_i}, \Box] \\
\Box^\star_{\omega_i} &:= [\partial_{\omega_i^\star}, \Box^\star]
\end{aligned}$$

12.2 Hyperlogarithmic monomials and monics.

12.2.1 Basic hyperlogarithms.

Δ-*friendly monomials*	$\mathcal{U}^\bullet(z)\,,\mathcal{U}e^\bullet(z)$	*symmetral*
Δ-*friendly monics*	$U^\bullet$	*alternal*
Δ-*friendly monics*	$US^\bullet\,,\,SU^\bullet$	*symmetral*
∂-*friendly monomials*	$\mathcal{V}^\bullet(z)\,,\,\mathcal{V}e^\bullet(z)$	*symmetral*
∂-*friendly monics*	$V^\bullet$	*alternal*
∂-*friendly monics*	$VS^\bullet\,,\,SV^\bullet$	*symmetral*

12.2.2 Basic relations.

$$\begin{aligned}
\mathcal{U}e^\bullet(z) &= \exp(z\,\Box)\,.\,\mathcal{U}^\bullet(z) \\
\mathcal{V}e^\bullet(z) &= \exp(z\,\Box)\,.\,\mathcal{V}^\bullet(z) \\
\mathcal{U}^\bullet(z) &= \mathcal{V}^\bullet(z)\circ U^\bullet \\
\mathcal{V}^\bullet(z) &= \mathcal{U}^\bullet(z)\circ V^\bullet \\
\mathbf{I}^\bullet &= U^\bullet \circ V^\bullet && = V^\bullet \circ U^\bullet \\
\mathbf{1}^\bullet &= US^\bullet \times SU^\bullet && = VS^\bullet \times SV^\bullet \\
U^\bullet &= US^\bullet \times I^\bullet \times SU^\bullet && \textit{if all } \omega_i \in \mathbb{R}^+ \\
V^\bullet &= VS^\bullet \times I^\bullet \times SV^\bullet && \textit{if all } \omega_i \in \mathbb{R}^+
\end{aligned}$$

12.2.3 More relations.

$$\partial_{\omega_i}\mathcal{U}^\bullet(z) = -\mathcal{U}^\bullet(z)\times(\frac{\Box_i}{\Box}\,U^\bullet) - z\,\Box_i\,\mathcal{U}^\bullet(z) \tag{294}$$

$$z\,\partial_z\,\mathcal{U}^\bullet(z) = -z\,\Box\,\mathcal{U}^\bullet(z) - \mathcal{U}^\bullet(z)\times U^\bullet \tag{295}$$

$$\begin{aligned}
\partial_{\omega_i} \mathcal{U}e^{\bullet}(z) &= -\mathcal{U}e^{\bullet}(z) \times (\exp(z\,\square).\frac{\square_i}{\square}.\,U^{\bullet}) && (296)\\
z\,\partial_z \mathcal{U}e^{\bullet}(z) &= -\,\mathcal{U}e^{\bullet}(z) \times (\exp(z\,\square).U^{\bullet}) && (297)
\end{aligned}$$

$$\begin{aligned}
\partial_{\omega_i} U^{\bullet} &= +\,(\frac{\square_i}{\square}\,U^{\bullet}) \times U^{\bullet} - U^{\bullet} \times (\frac{\square_i}{\square}\,U^{\bullet}) && (298)\\
\partial_{\omega_i} \mathcal{US}^{\bullet} &= +\,(\frac{\square_i}{\square}\,U^{\bullet}) \times \mathcal{US}^{\bullet} && (299)\\
\partial_{\omega_i} \mathcal{SU}^{\bullet} &= -\,U^{\bullet} \times (\frac{\square_i}{\square}\,\mathcal{SU}^{\bullet}) && (300)
\end{aligned}$$

$\partial_{\boldsymbol{\omega}} := \sum \omega_i \partial_{\omega_i}$

$$\begin{aligned}
\partial_{\boldsymbol{\omega}}\, U^{\bullet} &= 0 && (301)\\
\partial_{\boldsymbol{\omega}}\, \mathcal{US}^{\bullet} &= +\,U^{\bullet} \times \mathcal{US}^{\bullet} && (302)\\
\partial_{\boldsymbol{\omega}}\, \mathcal{SU}^{\bullet} &= -\,\mathcal{SU}^{\bullet} \times U^{\bullet} && (303)
\end{aligned}$$

12.2.4 Ordinary and alien differentiation. The $\partial \leftrightarrow \Delta$ duality .

$$\begin{aligned}
(z\,\partial_z + z\,\square)\,\mathcal{U}^{\bullet}(z) &= -\mathcal{U}^{\bullet}(z) \times U^{\bullet} && (304)\\
(z\,\partial_z + z\,\square)\,\mathcal{V}^{\bullet}(z) &= -\mathcal{V}^{\bullet}(z) \times I^{\bullet} && (305)
\end{aligned}$$

$$\begin{aligned}
\Delta\,\mathcal{U}^{\bullet}(z) &= I^{\bullet} \times \mathcal{U}^{\bullet}(z) && (306)\\
\Delta\,\mathcal{V}^{\bullet}(z) &= V^{\bullet} \times \mathcal{V}^{\bullet}(z) && (307)
\end{aligned}$$

with $\square$ standing as usual for multiplication by $\| \bullet \| = \sum \omega_i$ and

$$I^{\omega_1} := 1 \qquad ; \qquad I^{\omega_1,\dots,\omega_r} := 0 \quad if \quad r \neq 1 \tag{308}$$

12.3 Perilogarithmic monomials and monics.

12.3.1 Basic perilogarithms.

Perilogarithms have indices $\boldsymbol{\varpi} := (\varpi_1, \dots, \varpi_r)$ with $\varpi_i = (\omega_i, \omega_i^{\star}) \in (\mathbb{C}^{\star}, \mathbb{C}^{\star})$ and a real-positive product: $\omega_i\,\omega_i^{\star} \in \mathbb{R}^{+}$.
Antipodal involution

$$\# : \qquad {}^{\#}M^{\varpi_1,\dots,\varpi_r} := M^{\varpi_r^{\star},\dots,\varpi_1^{\star}} \quad \text{with} \quad \varpi^{\star} = (\omega_i^{\star}, \omega_i) \quad \text{if} \quad \varpi = (\omega_i, \omega_i^{\star})$$

Δ-frienly perilogarithms:

	primary	*secondary*	*type*
monomials	$\mathcal{U}a^\bullet$	$\mathcal{U}^\bullet, \mathcal{U}e^\bullet$	*symmetral*
monics	$U^\bullet$	$UR^\bullet, UL^\bullet$	*alternal*
monics	$USS^\bullet, SSU^\bullet$	$US^\bullet, SU^\bullet$	*symmetral*

D-frienly perilogarithms:

	primary	*secondary*	*type*
monomials	$\mathcal{V}a^\bullet$	$\mathcal{V}^\bullet, \mathcal{V}e^\bullet$	*symmetral*
monics	$V^\bullet$	$VR^\bullet, VL^\bullet$	*alternal*
monics	$VSS^\bullet, SSV^\bullet$	$VS^\bullet, SV^\bullet$	*symmetral*

12.3.2 Basic relations.

$$\begin{aligned}
\mathcal{U}e^\bullet(z) &= \exp(\, z\,\square + \, z^{-1}\,\square^\star).\, \mathcal{U}a^\bullet(z) && (309)\\
\mathcal{U}^\bullet(z) &= \exp(\, z^{-1}\,\square^\star).\, \mathcal{U}a^\bullet(z) && (310)\\
\mathbf{1}^\bullet &= USS^\bullet \times SSU^\bullet = US^\bullet \times SU^\bullet && (311)\\
UR^\bullet &= USS^\bullet \times (\square^\star\, SSU^\bullet) && (312)\\
UL^\bullet &= USS^\bullet \times (\square\, SSU^\bullet) && (313)\\
USS^\bullet &= US^\bullet \times {}^{\#}US^\bullet && (314)\\
SSU^\bullet &= {}^{\#}SU^\bullet \times SU^\bullet && (315)\\
SU^\bullet &= \mathcal{U}a^\bullet(1) && (316)
\end{aligned}$$

12.3.3 Integral formulae for the Δ-friendly monomials and monics.

Their main ingredients are the CCI ("Common Core Integrand") :

$$CCI := \frac{\exp(-\sum \omega_i t_i - \sum \omega_i^\star\, t_i^{-1})}{(t_r - t_{r-1})\ldots(t_3-t_2)(t_2-t_1)} \qquad (317)$$

and the *SPA* rule ("Standard Path Averaging" [114]) for multiple integration.

[114] see §7

Monomials

$$
\begin{array}{llllll}
\mathcal{U}a^{\varpi}(z) & = & SPA\int_0^\infty CCI & (t_1-z)^{-1} & dt_1\dots dt_r \\
\mathcal{U}e^{\varpi}(z) & = & e^{\|\boldsymbol{\omega}\|\, z+\|\boldsymbol{\omega}^\star\|\, z^{-1}} & \mathcal{U}a^{\varpi}(z) & \\
\mathcal{U}^{\varpi}(z) & = & e^{\|\boldsymbol{\omega}^\star\|\, z^{-1}} & \mathcal{U}a^{\varpi}(z) &
\end{array}
$$

Monics

$$
\begin{array}{llllll}
U^{\varpi} & = & SPA\int_0^\infty CCI & (\sum\omega_i) & dt_1\dots dt_r \\
 & = & SPA\int_0^\infty CCI & (\sum\omega_i^\star/t_i^2) & dt_1\dots dt_r \\
UL^{\varpi} & = & SPA\int_0^\infty CCI & (\sum\omega_i/t_i) & dt_1\dots dt_r \\
UR^{\varpi} & = & SPA\int_0^\infty CCI & (\sum\omega_i^\star/t_i) & dt_1\dots dt_r \\
SU^{\varpi} & = & SPA\int_0^\infty CCI & (t_1-1)^{-1} & dt_1\dots dt_r \\
US^{\varpi} & = & SPA\int_0^\infty CCI & (1-t_r)^{-1} & dt_1\dots dt_r \\
SSU^{\varpi} & = & SPA\int_0^\infty CCI & (1/t_1) & dt_1\dots dy_r \\
USS^{\varpi} & = & SPA\int_0^\infty CCI & (-1/t_r) & dt_1\dots dt_r
\end{array}
$$

12.3.4 More relations for the Δ-friendly perilogarithms.

Monomials:

$$\partial_{\omega_i}\mathcal{U}a^\bullet(z) = -\mathcal{U}a^\bullet(z)\times\frac{\square_i}{\square}U^\bullet - z\,\square_i\mathcal{U}a^\bullet(z) \tag{318}$$

$$\partial_{\omega_i^\star}\mathcal{U}a^\bullet(z) = +z^{-1}\mathcal{U}a^\bullet(z)\times USS^\bullet\times\square_i^\star SSU^\bullet - z^{-1}\,\square_i^\star\,\mathcal{U}a^\bullet(z) \tag{319}$$

$$z\,\partial_z\mathcal{U}a^\bullet(z) = (-z\,\square+z^{-1}\,\square^\star)\mathcal{U}a^\bullet(z)-\mathcal{U}a^\bullet(z)\times(U^\bullet+z^{-1}\,UR^\bullet) \tag{320}$$

Monics:

$$\partial_{\omega_i}U^\bullet = +(\frac{\square_i}{\square}\,U^\bullet)\times U^\bullet - U^\bullet\times(\frac{\square_i}{\square}\,U^\bullet)-\square_i\,UR^\bullet \tag{321}$$

$$\partial_{\omega_i^\star}U^\bullet = +\square\,((\square_i^\star\,USS^\bullet)\times SSU^\bullet) \tag{322}$$

$$\partial_{\omega_i}USS^\bullet = +(\frac{\square_i}{\square}\,U^\bullet)\times USS^\bullet \tag{323}$$

$$\partial_{\omega_i^\star}USS^\bullet = +USS^\bullet\times(\frac{\square_i^\star}{\square^\star}\,{}^{\#}U^\bullet) \tag{324}$$

$$\partial_{\omega_i}\,SSU^\bullet = -SSU^\bullet\times(\frac{\square_i}{\square}\,U^\bullet) \tag{325}$$

$$\partial_{\omega_i^\star}\,SSU^\bullet = -(\frac{\square_i^\star}{\square^\star}\,{}^{\#}U^\bullet)\times SSU^\bullet \tag{326}$$

$$
\begin{aligned}
\partial_{\omega_i} US^\bullet &= +(\frac{\Box_i}{\Box} U^\bullet) \times US^\bullet - \Box_i\, US^\bullet && (327)\\
\partial_{\omega_i^\star} US^\bullet &= +US^\bullet \times (\Box_i^\star\, {}^\# US^\bullet) \times SU^\bullet && (328)\\
\partial_{\omega_i} {}^\# US^\bullet &= +SU^\bullet \times (\Box_i\, US^\bullet) \times {}^\# US^\bullet && (329)\\
\partial_{\omega_i^\star} {}^\# US^\bullet &= + {}^\# US^\bullet \times (\frac{\Box_i^\star}{\Box^\star}\, {}^\# U^\bullet) - \Box_i^\star\, {}^\# US^\bullet && (330)
\end{aligned}
$$

$$
\begin{aligned}
\partial_{\omega_i} SU^\bullet &= -SU^\bullet \times (\frac{\Box_i}{\Box} U^\bullet) - \Box_i\, SU^\bullet && (331)\\
\partial_{\omega_i^\star} SU^\bullet &= + {}^\# US^\bullet \times (\Box_i^\star\, {}^\# SU^\bullet) \times SU^\bullet && (332)\\
\partial_{\omega_i} {}^\# SU^\bullet &= + {}^\# SU^\bullet \times (\Box_i\, SU^\bullet) \times US^\bullet && (333)\\
\partial_{\omega_i^\star} {}^\# SU^\bullet &= -(\frac{\Box_i^\star}{\Box^\star}\, {}^\# U^\bullet) \times {}^\# SU^\bullet - \Box_i^\star\, {}^\# SU^\bullet && (334)
\end{aligned}
$$

12.3.5 Yet more relations for the Δ-friendly perilogarithms.

The partial differentiation rules relative to

$$\partial_{\boldsymbol{\omega}} := \sum \omega_i \partial_{\omega_i} \quad \textit{and} \quad \partial_{\boldsymbol{\omega}^\star} := \sum \omega_i^\star \partial_{\omega_i^\star}$$

though deducible from the above, are also worth mentioning. They become particularly useful in the paralogarithmic case, since

$$\partial_{\boldsymbol{\omega}} \equiv \partial_{\boldsymbol{\omega}^\star} \equiv 1/2\; c\, \partial_c \equiv c^2\, \partial_{c^2}$$

$$
\begin{aligned}
\partial_{\boldsymbol{\omega}} \mathcal{U}a^\bullet(z) &= -\mathcal{U}a^\bullet(z) \times U^\bullet - z\,\Box\,\mathcal{U}a^\bullet(z) && (335)\\
\partial_{\boldsymbol{\omega}^\star} \mathcal{U}a^\bullet(z) &= +z^{-1}\, \mathcal{U}a^\bullet(z) \times UR^\bullet - z^{-1}\,\Box\,\mathcal{U}a^\bullet(z) && (336)
\end{aligned}
$$

$$
\begin{array}{ccccccc}
\partial_{\boldsymbol{\omega}} U^\bullet & = & \partial_{\boldsymbol{\omega}^\star} U^\bullet & = & \Box\,((\Box^\star\, USS^\bullet) \times SSU^\bullet & = & -\Box UR^\bullet\\
\partial_{\boldsymbol{\omega}} USS^\bullet & = & \partial_{\boldsymbol{\omega}^\star} USS^\bullet & = & +U^\bullet \times USS^\bullet & = & USS^\bullet \times^\# U^\bullet\\
\partial_{\boldsymbol{\omega}} SSU^\bullet & = & \partial_{\boldsymbol{\omega}^\star} SSU^\bullet & = & -SSU^\bullet \times U^\bullet & = & -{}^\# U^\bullet SSU^\bullet
\end{array}
$$

$$
\begin{aligned}
\partial_{\boldsymbol{\omega}} US^\bullet &= +U^\bullet \times US^\bullet - \Box\, US^\bullet && (337)\\
\partial_{\boldsymbol{\omega}^\star} US^\bullet &= +US^\bullet \times (\Box^\star\, {}^\# US^\bullet) \times^\# SU^\bullet && (338)
\end{aligned}
$$

$$
\begin{aligned}
\partial_{\boldsymbol{\omega}} SU^\bullet &= -SU^\bullet \times U^\bullet - \Box\, SU^\bullet && (339)\\
\partial_{\boldsymbol{\omega}^\star} SU^\bullet &= +{}^\# US^\bullet \times (\Box^\star\, {}^\# SU^\bullet) \times SU^\bullet && (340)
\end{aligned}
$$

12.3.6 A glimpse of the ∂-friendly perilogarithms.

$$(z\,\partial_z + z\,\Box - z^{-1}\,\Box^{\star})\;\mathcal{V}a^{\bullet}(z) = -\mathcal{V}a^{\bullet}(z) \times Ja^{\bullet}(z) \tag{341}$$

with an elementary, one-component mould $Ja^{\bullet}$:

$$\begin{array}{rcl} Ja^{\underline{\varpi}_1}(z) = Ja^{\binom{\varpi_1}{\tau_1}}(z) & := & 1 \quad if \quad \tau_1 = 0 \\ & := & cz^{-1} \quad if \quad \tau_1 = 1 \\ Ja^{\underline{\varpi}_1,\dots,\underline{\varpi}_r}(z) = Ja^{\binom{\varpi_1\ ,\dots,\ \varpi_r}{\tau_1\ ,\dots,\ \tau_r}}(z) & := & 0 \quad if \quad r \neq 1 \end{array}$$

and with discrete indices $\tau_i \in \{0,1\}$.

12.3.7 From ∂- to Δ-friendly .

$$\mathcal{U}a^{\bullet}(z) = \mathcal{V}a^{\bullet} \circ U^{\bullet} \tag{342}$$

which is short-hand for

$$\mathcal{U}a^{\boldsymbol{\varpi}}(z) = \sum_{\left\{ {\boldsymbol{\varpi}^1 \dots \boldsymbol{\varpi}^s = \boldsymbol{\varpi}} \atop {s \geq 1\,,\,\tau_i \in \{0,1\}} \right\}} \mathcal{V}a^{\binom{\|\boldsymbol{\varpi}^1\|\ ,\dots,\ \|\boldsymbol{\varpi}^s\|}{\tau_1\ \ ,\dots,\ \ \tau_s}}(z)\;\; U^{\binom{\boldsymbol{\varpi}^1}{\tau_1}} \dots U^{\binom{\boldsymbol{\varpi}^s}{\tau_s}} \tag{343}$$

with

$$\begin{array}{rcl} U^{\underline{\boldsymbol{\varpi}}} = U^{\binom{\boldsymbol{\varpi}}{\tau_0}} & := & U^{\boldsymbol{\varpi}} \quad if \quad \tau_0 = 0 \\ & := & UR^{\boldsymbol{\varpi}} \quad if \quad \tau_0 = 1 \end{array}$$

12.3.8 Resurgence equations.

$$\Delta\,\mathcal{U}^{\bullet} \;=\; I^{\bullet} \times \mathcal{U}^{\bullet} \qquad (\ with\ indices\ \ \varpi_i) \tag{344}$$

$$\Delta\,\mathcal{V}^{\bullet} \;=\; V^{\bullet} \times \mathcal{V}^{\bullet} \qquad (\ with\ indices\ \ \underline{\varpi}_i := \binom{\varpi_i}{\tau_i}) \tag{345}$$

The endearingly simple relation $I^{\bullet} = V^{\bullet} \circ U^{\bullet}$ connecting the Δ- and ∂-friendly hyperlogarithmic monics carries over to the perilogarithmic monics, but with doubled storeyed indices $\underline{\varpi}_i = \binom{\varpi_i}{\tau_i}$ in the moulds and a double-storeyed mould composition $\circ$ interpreted as above.

12.4 Paralogarithmic monomials and monics.

We now replace the antipodal involution:

$$({}^{\#}M)^{\varpi_1,\dots,\varpi_r} := M^{c^2\bar{\varpi}_r,\dots,c^2\bar{\varpi}_1} \tag{346}$$

by the more convenient variant :

$$({}^{\sharp}M)^{\varpi_1,\dots,\varpi_r} := M^{\bar{\varpi}_r,\dots,\bar{\varpi}_1} \tag{347}$$

and we get these relations for *monics*:

$$\begin{aligned}
{}^{\sharp}SSU_c^{\bullet} &= SSU_c^{\bullet} && (348)\\
{}^{\sharp}USS_c^{\bullet} &= USS_c^{\bullet} && (349)\\
{}^{\sharp}U_c^{\bullet} &= SSU_c^{\bullet} \times U_c^{\bullet} \times USS_c^{\bullet} && (350)\\
{}^{\sharp}UR_c^{\bullet} &= SSU_c^{\bullet} \times UL_c^{\bullet} \times USS_c^{\bullet} && (351)\\
{}^{\sharp}UL_c^{\bullet} &= SSU_c^{\bullet} \times UR_c^{\bullet} \times USS_c^{\bullet} && (352)
\end{aligned}$$

and this key antipodality relation for *monomials*:

$$({}^{\sharp}\mathcal{U}a)_c^{\bullet}(z) \times (\mathcal{U}a)_c^{\bullet}(c^2/z) \equiv SSU_c^{\bullet} \tag{353}$$

The integral formulae of §12.3.3 remain unchanged, except that the extreme factors $(y_1 - 1)^{-1}$ and $(1 - y_r)^{-1}$ become $(y_1 - c)^{-1}$ and $(c - y_r)^{-1}$. One should always integrate along the axes $arg(\omega_i\, y_i) = arg(\bar{\omega}_i/y_i) = 0$ and heed the "SPA" rules of mutual bypassing whenever several consecutive $Arg(\omega_i)$ coincide. The partial differentiation rules for the perilogarithms particularise to the paralogarithms.

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