Singular and singularly perturbed systems and multiple resurgence.

Jean Ecalle, Orsay Univ. & CNRS.

November 2018 Moscow Conference on PDE's and Applications in Memory of Professor Boris Yu. STERNIN

Contents.

1. Types of resurgence.

Among the many types of *resurgence*, two stand out:

(i) *Equational resurgence*, so-called because it arises in 'singular' equations (differential; partial diff.; functional etc). It is by now well understood. Prof. B. Yu. Sternin, for one, devoted a number of papers to the subject, esp. about PDE' s.

(ii) *Co-equational resurgence*. It is quite prevalent in theoretical physics and occurs typically in (power series) expansions in a 'singular' small parameter. Many interesting parameters or constants in physics fall into this category[1]. It is far more complex than equational resurgence, yet loosely dual to it.

(iii) To highlight these differences-cum-similarities, we shall focus on a model problem (*non-linear singular and singularly perturbed differential systems*) where both types of resurgence coexist side-by-side.

But before getting into the thick of things, a few ultra-quick reminders are in order.

[1] Cf the Michael Berry 'principle': *the divergence of expansions in a small parameter such as $\hbar$ reflects the non-trivial nature of the transition from a classical theory to its non-classical extension.*

2. Resurgent functions.

Resurgent functions live in three models:
(i) In the *formal model*, as formal power series $\widetilde{\varphi}(z)$ of z^{-1}.
(ii) In the *convolution model* or *Borel plane*, as analytic germs $\widehat{\varphi}(\zeta)$ at 0, endlessly continuable (laterally along any finitely broken line).
(iii) In the *geometric models*, as sectorial germs $\varphi_\theta(z)$ at ∞ in the z-variable.

(i) $\quad \widetilde{\varphi}(z) = \sum a_n z^{-n}$ $\qquad$ *multiplication*

$\downarrow$ *Borel*

(ii) $\quad \widehat{\varphi}(\zeta) = \sum a_n \frac{\zeta^{n-1}}{(n-1)!}$ $\qquad$ *convolution* $\begin{cases} (\widehat{\varphi}_1 * \widehat{\varphi}_2)(\zeta) := \\ \int_0^\zeta \widehat{\varphi}_1(\zeta_1)\, \widehat{\varphi}_2(\zeta - \zeta_1) d\zeta_1 \end{cases}$

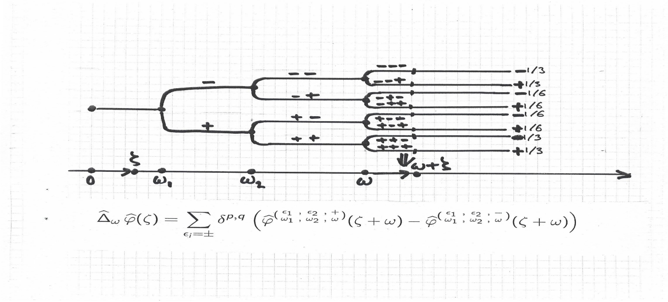
$\downarrow$ *Laplace*

(iii) $\quad \varphi_\theta(z) = \int_{\arg \zeta = \theta} e^{-z\zeta} \widehat{\varphi}(\zeta) d\zeta$ $\quad$ *multiplication*

The singularities of $\widehat{\varphi}(\zeta)$ carry the Stokes constants and are responsible for the divergence of $\widetilde{\varphi}(z)$. So they deserve close attention.
The tools for measuring them are the so-called *alien derivations* Δ_ω.

3. Alien derivations.

The one oustanding fact about resurgent functions is the existence on them of a huge array of exotic derivations – the so-called *alien derivations* Δ_ω $(\omega \in \mathbb{C}_\bullet = \widetilde{\mathbb{C} - \{0\}})$. They are bound by no *a priori* constraints.



$$\widehat{\Delta}_\omega \widehat{\varphi}(\zeta) = \frac{1}{2\pi i} \sum_{\epsilon_i \in \{+,-\}} \frac{p!\, q!}{(p+q+1)!} \begin{cases} +\widehat{\varphi}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_{r-1} & , & + \\ \omega_1 & ,\dots, & \omega_{r-1} & , & \omega \end{smallmatrix}\right)}(\zeta+\omega) \\ -\widehat{\varphi}^{\left(\begin{smallmatrix} \epsilon_1 & ,\dots, & \epsilon_{r-1} & , & - \\ \omega_1 & ,\dots, & \omega_{r-1} & , & \omega \end{smallmatrix}\right)}(\zeta+\omega) \end{cases}$$

3*. Using alien derivations.

$$\widehat{\Delta}_\omega\big(\widehat{\varphi}_1 * \widehat{\varphi}_2\big) \equiv \big(\widehat{\Delta}_\omega\widehat{\varphi}_1\big) * \widehat{\varphi}_2 + \widehat{\varphi}_1 * \big(\widehat{\Delta}_\omega\widehat{\varphi}_2\big)$$

$$\Delta_\omega\big(\widetilde{\varphi}_1.\widetilde{\varphi}_2\big) \equiv \big(\Delta_\omega\widetilde{\varphi}_1\big).\widetilde{\varphi}_2 + \widetilde{\varphi}_1.\big(\Delta_\omega\widetilde{\varphi}_2\big)$$

$$\mathbb{\Delta}_\omega := e^{-\omega z}\Delta_\omega \implies [\mathbb{\Delta}_\omega, \partial_z] \equiv 0$$

The system $\{\Delta_{\omega_r} \dots \Delta_{\omega_1}\widetilde{\varphi}\}$ encodes all the information about the Borel transform $\widehat{\varphi}(\zeta)$, its behaviour on all its Riemann sheets, and its Stokes constants.

$E(\widetilde{\varphi}) = 0$ *gen. diff. or functional equ. or system*

$\downarrow$ *formally*

$E(\widetilde{\varphi}, \mathbb{\Delta}_\omega\widetilde{\varphi}) = 0$ *linear homogeneous in* $\mathbb{\Delta}_\omega\widehat{\varphi}$

$\downarrow$ *formally*

$$\mathbb{\Delta}_\omega\widetilde{\varphi} = A_\omega\widetilde{\varphi}_\omega \qquad \begin{cases} A_\omega = \textit{Stokes constant} \\ \widehat{\varphi}_\omega(\zeta) = \textit{singular germ at, or "over"}, \omega \end{cases}$$

4. The Bridge equation.

(1) $E(Y) = 0$ *gen. diff. or functional equ. or system*

$\downarrow$ *formally*

(2) $Y(z;\boldsymbol{\tau})$ *complete (parameter-saturated) formal solution*

$\downarrow$ *formally* $Y(z;\boldsymbol{\tau})$ *typ.*^{lly} *in* $\mathbb{C}[[z^{-1}; \bigcup_i \tau_i z^{\alpha_i} e^{\omega_i z}]]$

(3) $\boldsymbol{\Omega}$ $\boldsymbol{\Omega} = \{\omega \in \mathbb{C}_\bullet\,;\ \boldsymbol{\Delta}_\omega Y \neq 0\}$

$\downarrow$ *formally*

(4) $\boldsymbol{\Delta}_\omega Y(z;\boldsymbol{\tau}) = \mathbb{A}_\omega Y(z;\boldsymbol{\tau})$ *Bridge Equation*

$\downarrow$ *formally* $\begin{cases} \mathbb{A}_\omega \text{ ord. diff. operator. (exp.-compatible)} \\ \mathbb{A}_\omega \overset{typ^{lly}}{=} e^{-\omega z}\,\boldsymbol{\tau}^{\boldsymbol{n}}\,(A^0_\omega \partial_z + \sum A^i_\omega \tau_i \partial_{t_i}) \end{cases}$

(5) $\boldsymbol{\Delta}_{\omega_r} \ldots \boldsymbol{\Delta}_{\omega_1} Y(z;\boldsymbol{\tau}) = \mathbb{A}_{\omega_1} \ldots \mathbb{A}_{\omega_r} Y(z;\boldsymbol{\tau})$

- Covers a huge range of applications: *singular ODEs; difference equations; resonant vector fields; resonant or identitity tangent diffeos of* $\mathbb{C}^\nu$.
- Often deals with situations that are beyond the reach of geometric methods.
- Keeps the Analysis part down to a minimum.

5. Singular & and singularly perturbed system.

Consider this model instance of a *doubly singular* differential system:

$$0 = \epsilon\, t^2\, \partial_t\, y^i + \lambda_i\, y^i + b^i(t, \epsilon, y^1, \ldots, y^\nu) \quad (1 \le i \le \nu) \begin{cases} t \sim 0 \; (\mathit{variable}) \\ \epsilon \sim 0 \; (\mathit{parameter}) \end{cases}$$

It is advisable, both technically and theoretically, to change to the problem's *'critical variables'* z and *'critical parameter'* x, i.e. to set

$$z := 1/t \sim \infty \qquad , \qquad x := 1/\epsilon \sim \infty$$

so as to prepare for working in the conjugate Borel planes ζ and ξ. This leads to the system:

$$\partial_z Y^i \;=\; Y^i \left(\lambda_i\, x + \sum_{n_j \ge 0 \text{ if } j \ne i}^{1+n_i \ge 0} B^i_{\boldsymbol{n}}(z)\, Y^{\boldsymbol{n}} \right) \qquad (1 \le i \le \nu)$$

with coefficients $B^i_{\boldsymbol{n}}(z) \in \mathbb{C}\{z^{-1}\}$ analytic at infinity and x-free.

5*. Loose duality equational/coequational.

We assume that the multipliers λ_i are neither resonant and nor quasi-resonant (meaning that the combinations $-\lambda_i + \sum_{n_j \geq 0} n_j \lambda_j$) are all $\neq 0$ and do not approximate 0 abnormally fast). The general solution, with its full set $\{\tau_1, \ldots, \tau_\nu\}$ of integration parameters, may be formally expanded in powers of either z^{-1} or x^{-1}:

$$\widetilde{Y} = \widetilde{Y}(z, x, \boldsymbol{\tau}) \in \mathbb{C}[[z^{-1} \; or \; x^{-1}]] \otimes \mathbb{C}\{z^{\rho_1} \tau_1 e^{\lambda_1 zx}, \ldots, z^{\rho_\nu} \tau_\nu e^{\lambda_\nu zx}\}$$

The "residues" $\rho_i \in \mathbb{C}$ are the coefficient of z^{-1} in $B^i_{\mathbf{0}}(z) = B^i_{0,\ldots,0}(z)$. To get rid of the ramifications z^{ρ_i} (which complicate the formal expansions without adding anything of substance to the Analysis) we shall set not only $\rho_i \equiv 0$ but also $B^i_{\mathbf{0}}(z) \equiv 0$.

There is bound to be a certain kinship between the z- and x-resurgence, since in the special case when $B^i_{\boldsymbol{n}}(z) = \beta^i_{\boldsymbol{n}}/z$ with $\beta^i_{\boldsymbol{n}}$ scalar, the variable z and the perturbation parameter x coalesce:

$$\widetilde{Y}^i(z, x, \boldsymbol{\tau}) \quad = \quad \widetilde{Y}^i(z\,x) + \sum_{n_j \geq 0}^{j \neq i} \sum_{n_i \geq -1} \widetilde{Y}^i_{\boldsymbol{n}}(z\;x)\; \tau_i \boldsymbol{\tau}^{\boldsymbol{n}}\; e^{(\lambda_i + <\boldsymbol{n}, \boldsymbol{\lambda}>)\, z\, x} \qquad (1)$$

with $\widetilde{Y}^i(zx)$ and $\widetilde{Y}^i_{\boldsymbol{n}}(zx) \in \mathbb{C}[[(zx)^{-1}]]$. A loose kinship, or lax 'duality', survives even in the general case, and justifies the label *equational* for the z-resurgence (z being the variable with respect to which we differentiate in our model system) and *co-equational* for the x-resurgence.

6. Symmetral/alternal moulds.

$$\{S^\bullet \ \ symmetral\} \iff \{ \sum_{\boldsymbol{\omega} \in shuffle(\boldsymbol{\omega}', \boldsymbol{\omega}'')} S^{\boldsymbol{\omega}} \equiv S^{\boldsymbol{\omega}'} S^{\boldsymbol{\omega}''} \quad \forall \boldsymbol{\omega}', \boldsymbol{\omega}''\}$$

$$\{A^\bullet \ \ alternal\} \iff \{ \sum_{\boldsymbol{\omega} \in shuffle(\boldsymbol{\omega}', \boldsymbol{\omega}'')} A^{\boldsymbol{\omega}} \equiv 0 \qquad\qquad \forall \boldsymbol{\omega}', \boldsymbol{\omega}''\}$$

Let the D_{ω_i}'s be (ordinary) formal derivations. Then:

$$\{S^\bullet \ \ symmetral\} \iff \begin{cases} 1 + \sum_{1 \le r} \sum_{\omega_1, \ldots, \omega_r} S^{\omega_1, \ldots, \omega_r} D_{\omega_r} \ldots D_{\omega_1} \\ \text{is a formal automorphism} \end{cases}$$

$$\{A^\bullet \ \ alternal\} \iff \begin{cases} \sum_{1 \le r} \sum_{\omega_1, \ldots, \omega_r} A^{\omega_1, \ldots, \omega_r} D_{\omega_r} \ldots D_{\omega_1} \\ \text{is a formal derivation} \end{cases}$$

7. Normalisers and resurgence monomials.

Replace the general solution $\widetilde{Y}$ by the information-equivalent but more flexible *normalising* operators $\Theta^{\pm 1}$:

$$\Theta = 1 + \sum_{i_k, \boldsymbol{n}_k}^{1 \le r} e^{|\boldsymbol{u}|xz}\, \widetilde{\mathcal{W}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ B^{i_1}_{\boldsymbol{n}_1} & ,\dots, & B^{i_r}_{\boldsymbol{n}_r} \end{smallmatrix}\right)}(z,x)\ \mathbb{D}^{i_r}_{\boldsymbol{n}_r} \dots \mathbb{D}^{i_1}_{\boldsymbol{n}_1}$$

$$\Theta^{-1} = 1 + \sum_{i_k, \boldsymbol{n}_k}^{1 \le r} (-1)^r e^{|\boldsymbol{u}|xz}\, \widetilde{\mathcal{W}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ B^{i_1}_{\boldsymbol{n}_1} & ,\dots, & B^{i_r}_{\boldsymbol{n}_r} \end{smallmatrix}\right)}(z,x)\ \mathbb{D}^{i_1}_{\boldsymbol{n}_1} \dots \mathbb{D}^{i_r}_{\boldsymbol{n}_r}$$

$$\text{with} \quad \begin{cases} u_k := < \boldsymbol{n}_k, \boldsymbol{\lambda} > \quad , & \mathbb{D}^{i_k}_{\boldsymbol{n}_k} := \boldsymbol{\tau}^{\boldsymbol{n}_k}\, \tau^{i_k} \partial_{\tau_{i_k}} \\ 1 \le i_k \le \nu \quad , & \boldsymbol{\tau}^{\boldsymbol{n}}_k\, \tau_{i_k} \in \boldsymbol{\tau}^{\mathbb{N}} \end{cases}$$

and with ‘monomials‘ $\widehat{\mathcal{W}}^{\bullet}$ inductively defined by

$$(\partial_z + |\boldsymbol{u}|x)\widetilde{\mathcal{W}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ B^{i_1}_{\boldsymbol{n}_1} & ,\dots, & B^{i_r}_{\boldsymbol{n}_r} \end{smallmatrix}\right)}(z,x) = -\widetilde{\mathcal{W}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_{r-1} \\ B^{i_1}_{\boldsymbol{n}_1} & ,\dots, & B^{i_{r-1}}_{\boldsymbol{n}_{r-1}} \end{smallmatrix}\right)}(z,x)\, B^{i_r}_{\boldsymbol{n}_r}(z)$$

Or to lighten notations:

$$(\partial_z + |\boldsymbol{u}|x)\widetilde{\mathcal{W}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ b_1 & ,\dots, & b_r \end{smallmatrix}\right)}(z,x) = -\widetilde{\mathcal{W}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_{r-1} \\ b_1 & ,\dots, & b_{r-1} \end{smallmatrix}\right)}(z,x)\, b_r(z)$$

7*. Normalisers and resurgence monomials.

Since $\widetilde{\mathcal{W}}^{\bullet}$ is *symmetral*, the operators Θ and Θ^{-1} are (mutually inverse) formal automorphisms of $\mathbb{C}[[\boldsymbol{\tau}]] := \mathbb{C}[[\tau_1, \ldots, \tau_\nu]]$:

$$\Theta^{\pm 1}\Big(\widetilde{\varphi}_1(\boldsymbol{\tau}).\widetilde{\varphi}_2(\boldsymbol{\tau})\Big) \equiv \Big(\Theta^{\pm 1}\widetilde{\varphi}_1(\boldsymbol{\tau})\Big)\Big(\Theta^{\pm 1}\widetilde{\varphi}_2(\boldsymbol{\tau})\Big) \qquad (\widetilde{\varphi}_i \in \mathbb{C}[[\boldsymbol{\tau}]])$$

Moreover, they exchange the general solution Y of our model system and the elementary general solution Y_{nor} of the corresponding (linear) normal system:

$$\partial_z Y^i = Y^i \left(\lambda_i\, x + \sum B^i_{\boldsymbol{n}}(z)\, Y^{\boldsymbol{n}}\right) \;;\; Y^i(z, x, \boldsymbol{\tau}) \in \mathbb{C}[[z^{-1}]] \otimes \mathbb{C}\{\cup_i \tau_i\, e^{\lambda_i\, x\, z}\}$$

$$\partial_z Y^i_{\mathrm{nor}} = \lambda_i\, x\, Y^i_{nor} \;;\; Y^i_{\mathrm{nor}}(z, x, \boldsymbol{\tau}) = \tau_i\, e^{\lambda_i\, x\, z}$$

$$\begin{cases} \Theta \;\; Y^i(z, x, \boldsymbol{\tau}) \;\; \equiv Y^i_{\mathrm{nor}}(z, x, \boldsymbol{\tau}) \\ \Theta^{-1}\, Y^i_{\mathrm{nor}}(z, x, \boldsymbol{\tau}) \equiv Y^i(z, x, \boldsymbol{\tau}) \end{cases}$$

8. Equational resurgence.

$$(\partial_z + |\boldsymbol{u}|\, x)\ \mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,x) \quad = \quad -\mathcal{W}^{\binom{u_1\ ,\dots,\ u_{r-1}}{b_1\ ,\dots,\ b_{r-1}}}(z,x)\, b_r(z) \qquad (2)$$

Under the z-Borel tranform $\quad \mathcal{B}_z\ :\ z^{-n} \mapsto \frac{\zeta^{n-1}}{(n-1)!} \quad , \quad b(z) \mapsto \widehat{b}(\zeta) \quad , \quad \mathcal{W}^\bullet(z,x) \mapsto \widehat{\mathcal{W}}^\bullet(\zeta,x)$

the induction rule (2) becomes

$$\widehat{\mathcal{W}}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(\zeta,x) = \frac{1}{\zeta - |\boldsymbol{u}|\, x} \int_0^\zeta \widehat{\mathcal{W}}^{\binom{u_1\ ,\dots,\ u_{r-1}}{b_1\ ,\dots,\ \widehat{b}_{r-1}}}(\zeta_1,x)\, \widehat{b}_r(\zeta-\zeta_1)\, dz_1 \qquad (3)$$

and readily yields all the information we need: location of singularities, Stokes constants, pattern of z-resurgence:

$$\Delta_{u\,x}\mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,x) = \sum_{u_1+\ldots+u_i=u} W^{\binom{u_1\ ,\dots,\ u_i}{b_1\ ,\dots,\ b_i}}(x)\, \mathcal{W}^{\binom{u_{i+1}\ ,\dots,\ u_r}{b_{i+1}\ ,\dots,\ b_r}}(z,x) \quad (4)$$

$$\textit{with} \quad \begin{cases} \textit{monomials} \ \ \mathcal{W}^\bullet(z,x) & \textit{symmetral} \ \ \& \ \ \textit{resurgent in } z \\ \quad \textit{monics} \quad\ \ W^\bullet(x) & \textit{alternal} \ \ \& \ \ \textit{entire function of } x \end{cases}$$

9. Co-equational resurgence: four requirements.

Our approach is unabashedly *analytical*, in that it strives to identify and resolve the difficulties **first** at the **most basic level**, i.e. at the level of the monomials $\mathcal{W}^{\binom{u_1\ ,\ldots,\ u_r}{b_1\ ,\ldots,\ b_r}}(z,x)$. But **even at that level**, co-equational resurgence is a hard nut to crack. To completely master it, we shall require four things:

(i) a symmetral *weighted convolution* product *weco*$^\bullet$.
(ii) an alternal *weighted convolution* product *welo*$^\bullet$.
(iii) the (closed) rules for *alien-differentiating* *weco*$^\bullet$ and *welo*$^\bullet$.
(iv) the discrete-valued *tessellation coefficients*, which in this new context shall take the place of the continuous-valued Stokes constants.

10. Co-equational resurgence at the monomial level.

Under the *x-Borel tranform* $\mathcal{B}_x$: $\begin{cases} x^{-n} \mapsto \frac{\xi^{n-1}}{(n-1)!} \\ \mathcal{W}^\bullet(z,x) \mapsto \mathcal{B}_x\mathcal{W}^\bullet(z,\xi) \end{cases}$

things are incomparably more complex than under $\mathcal{B}_z$. The induction rule now assumes the form of a partial differential equation in z and ξ:

$$(\partial_z + |\boldsymbol{u}|\,\partial_\xi)\ \mathcal{B}_x\mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,\xi) \ = \ -\mathcal{B}_x\mathcal{W}^{\binom{u_1\ ,\dots,\ u_{r-1}}{b_1\ ,\dots,\ b_{r-1}}}(z,\xi)\, b_r(z) \quad (5)$$

with for $r \geq 2$ the limit condition : $\mathcal{B}_x\mathcal{W}^{\binom{u_1\ ,\dots,\ u_r}{b_1\ ,\dots,\ b_r}}(z,0) = 0$

For $r = 1$, solving (5) in decreasing powers of x and then applying the Borel transform $x \to \xi$, we find:

$$\mathcal{B}_x\mathcal{W}^{\binom{u_1}{b_1}}(z,\xi) \ = \ -\sum_{n\geq 0} \frac{1}{u_1}\,\frac{(-\xi/u_1)^n}{n!}\,\partial_z^n\, b_1(z) \ = \ -\frac{1}{u_1}\, b_1(z - \frac{\xi}{u_1})$$

But for $r \geq 2$ we shall need a suitably defined *weighted convolution*.

11. Symmetral weighted convolution.

For $u_i \in \mathbb{C}$ and $\widehat{c}_i(\xi) \in \mathbb{C}\{x\}$, the following integrals

$$\mathrm{weco}^{\binom{u_1}{\widehat{c}_1}}(\xi) \;=\; \frac{1}{u_1}\,\widehat{c}_1\!\left(\frac{\xi}{u_1}\right)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\mathrm{weco}^{\binom{u_1\ ,\ldots,\ u_r}{\widehat{c}_1\ ,\ldots,\ \widehat{c}_r}}(\xi) \;=\; \begin{cases} \int_0^{\theta_*}\ \widehat{c}_r(\xi_r)\,d\xi_r\ \int_{\xi_r}^{\theta_r}\ \widehat{c}_{r-1}(\xi_{r-1})\,d\xi_{r-1}\cdots \\ \cdots\int_{\xi_4}^{\theta_4}\ \widehat{c}_3(\xi_3)\,d\xi_3\ \int_{\xi_3}^{\theta_3}\ \widehat{c}_2(\xi_2)\,d\xi_2\,\widehat{c}_1(\xi_1)\,\frac{1}{u_1} \end{cases}$$

$$\textit{with}\qquad \begin{cases} u_1\,\xi_1 + \cdots + u_r\,\xi_r = \xi \\ \theta_i := (\xi - (u_i\,\xi_i + \cdots + u_r\,\xi_r))(u_1 + \cdots + u_{i-1})^{-1} \\ \theta_* := \xi\,(u_1 + \cdots + u_r)^{-1} \end{cases}$$

unambiguously define germs $weco^{\binom{u_1\ ,\ldots,\ u_r}{\widehat{c}_1\ ,\ldots,\ \widehat{c}_r}}(\xi) \in \mathbb{C}\{\xi\}$ provided that $u_1 + \ldots + u_i \neq 0$. The mould $weco^\bullet$ is symmetral relative to the (ordinary) convolution product.
A more symmetric definition reads

$$\mathrm{weco}^{\binom{u_1\ ,\ldots,\ u_r}{\widehat{c}_1\ ,\ldots,\ \widehat{c}_r}}(\xi) \quad := \quad \int_{W^{u_1,\ldots,u_r}} \widehat{c}_1(\xi_1)\ldots\widehat{c}_r(\xi_r)\,d\xi_1\ldots d\xi_r$$

with integration on a contorted multi-path:

$$\mathcal{P}^{u_1,\ldots,u_r} = \begin{cases} 0 < \xi_r < \xi_{r-1} < \cdots < \xi_2 < \xi_1 \\ (u_1 + \cdots + u_i)\,\xi_i + (u_{i+1}\xi_{i+1} + \cdots + u_r\xi_r) < \xi \qquad (2 \leq i \leq r) \\ u_1\xi_1 + \ldots u_r\xi_r = \xi \end{cases}$$

11*. Relevance of the weighted convolution product.

The Borel transforms $x \to \xi$ of the biresurgent monomials $\mathcal{W}^\bullet$ can be expressed in terms of *weighted convolution* products

$$\mathcal{B}_x \mathcal{W}^{\binom{u_1 \; ,\dots, \; u_r}{b_1 \; ,\dots, \; b_r}}(z,\xi) = \mathrm{weco}^{\binom{u_1 \; ,\dots, \; u_r}{\hat{c}_1 \; ,\dots, \; \hat{c}_r}}(\xi) \quad \textit{with} \quad \hat{c}_i(\xi) := -b_i(z-\xi)$$

with z chosen close enough to ∞ for $\hat{c}_i(\xi)$ to be regular at $\xi = 0$.

- Since $\hat{c}_i(\xi) := -b_i(z-\xi)$, the singularities of the $b_i(z)$ are going to dominate co-equational resurgence.
- We note here the characteristic interference of the multiplicative z-plane and the convolutive ξ-plane.

12. The weighted multiplication behind weighted convolution.

Just as ordinary convolution is the Borel image of ordinary multiplication, weighted convolution *weco* is the Borel image of a weighted multiplication *wemu*:

$$c_1(x), \ldots, c_r(x) \quad \overset{\text{Borel}}{\longrightarrow} \quad \widehat{c}_1(\xi), \ldots, \widehat{c}_r(\xi)$$

$$\mathrm{wemu}^{\binom{u_1, \ldots, u_r}{c_1, \ldots, c_r}}(x) \quad \overset{\text{Borel}}{\longrightarrow} \quad \mathrm{weco}^{\binom{u_1, \ldots, u_r}{\widehat{c}_1, \ldots \widehat{c}_r}}(\xi)$$

For $u_i > 0$ and $\Re x$ positive and large, weighted multiplication is defined by the integrals:

$$\mathrm{wemu}^{\binom{u_1, \ldots, u_r}{c_1, \ldots, c_r}}(x) := \frac{1}{(2\pi i)^r} \int_{-i\infty}^{+i\infty} \frac{c_1(x_1) \ldots c_r(x_r)\; dx_1 \ldots dx_r}{\prod_{i=1}^{i=r} \left((u_1 + \ldots + u_i)\, x - (x_1 + \ldots + x_i)\right)}$$

Integration is along vertical axes $\Im x_j = \alpha_j < u_j \Re x$ but with α_j large enough for $c_j(x_j)$ to be holomorphic on $\alpha_j \leq \Re x_j$. The definition is then extended for general weights u_i by continuous contour deformation, which is always feasible provided the partial sums $u_1 + \cdots + u_j$ remain $\neq 0$.

13. Alternal weighted convolution.

The 'alternal marking' *altmark* is a mould operation:

$$\mathrm{altmark}(M)^{\boldsymbol{\omega}',\omega_i^\dagger,\boldsymbol{\omega}''} \;:=\; (-1)^{r''} \sum_{\boldsymbol{\omega}'''\in \mathrm{sha}(\boldsymbol{\omega}',\widetilde{\boldsymbol{\omega}}'')} M^{\boldsymbol{\omega}''',\boldsymbol{\omega}_i^\dagger}$$

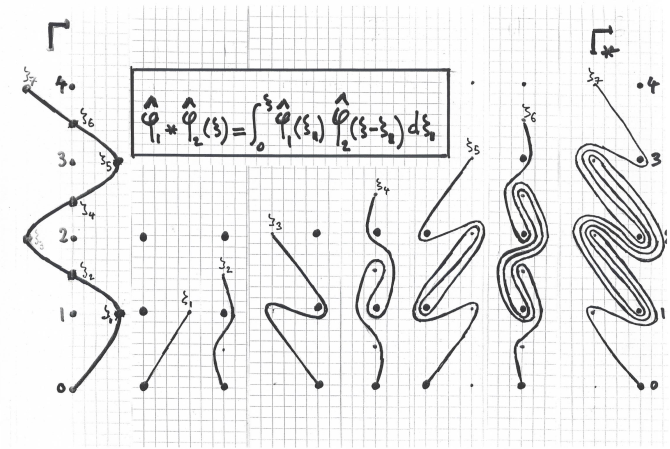
that turns *any* mould $M^\bullet$ into a *marked* mould $\underline{M}^\bullet$ of alternal type.

To get closure under alien differentiation, we must supplement the symmetral convolution $weco^\bullet$ by an alternate convolution $welo^\bullet = altmark(weco^\bullet)$. The corresponding multiplications $wemu^\bullet$ and $welu^\bullet$ have rather similar kernels

$$\begin{cases} \mathrm{wemu}^{\binom{u_1,\,\dots,\,u_i,\,\dots,\,u_r}{c_1,\,\dots,\,c_i,\,\dots,\,c_r}}(x) = \frac{1}{(2\pi i)^r}\int S^{\binom{u_1,\,\dots,\,u_i,\,\dots,\,u_r}{x_1,\,\dots,\,x_i,\,\dots,\,x_r}}(x)\prod c_i(x_i)dx_i \\ \mathrm{welu}^{\binom{u_1,\,\dots,\,u_i^\dagger,\,\dots,\,u_r}{c_1,\,\dots,\,c_i\dagger,\,\dots,\,c_r}}(x) = \frac{1}{(2\pi i)^r}\int \underline{S}^{\binom{u_1,\,\dots,\,u_i\dagger,\,\dots,\,u_r}{x_1,\,\dots,\,x_i\dagger,\,\dots,\,x_r}}(x)\prod c_i(x_i)dx_i \end{cases}$$

$$\begin{cases} S^{\binom{u_1,\,\dots,\,u_i,\,\dots,\,u_r}{x_1,\,\dots,\,x_i,\,\dots,\,x_r}}(x) = \prod_{i=1}^{i=r}\Big((u_1+..+u_i)\,x-(x_1+..+x_i)\Big)^{-1} \\ \underline{S}^{\binom{u_1,\,\dots,\,u_i\dagger,\,\dots,\,u_r}{x_1,\,\dots,\,x_i\dagger,\,\dots,\,x_r}}(x) = \begin{cases} (-1)^{r-j}S^{\binom{u_1,\,\dots,\,u_{i-1}}{x_1,\,\dots,\,x_{i-1}}}(x)\; S^{\binom{u_r,\,\dots,\,u_{i+1}}{x_r,\,\dots,\,x_{i+1}}}(x)\times \\ \Big((u_1+..+u_r)\,x-(x_1+..+x_r)\Big)^{-1} \end{cases} \end{cases}$$

14 Impracticability of the integration multipaths.



Even for ordinary convolution we get impossibly contorted paths. The position is still worse with the *weighted multipaths*. Hence the need for a combinatorial approach.

15. Hyperlogarithmic monomials: stability and density.

We are facing here a highly unusual but inescapable interference of two structures:
(i) the *multiplicative* structure, which leaves the singularities in place,
(ii) the *convolutive* structure, which *adds* singularities, in the sense that:
(singularity over ω_1)$*$(singularity over ω_2)$\Rightarrow$ (singularities over $\omega_1 + \omega_2$).

Then, messing up things still further, we must contend with the *weighted* convolution *weco*, which also *adds* singularities, but via weighted rather than straightforward sums. This forces us to juggle two systems of notation:

- *incremental*, with sequences $(\omega_1, \ldots, \omega_r)$ $\quad (\omega_i = \alpha_i - \alpha_{i-1})$
- *positional*, with sequences $[\alpha_1, \ldots, \alpha_r]$ $\quad (\alpha_i = \omega_1 + \ldots + \omega_i)$

The ideal tool for understanding this hybrid structure is the *hyperlogarithms*, with their *two encodings* (*positional* and *incremental*) their stability under *two products* (ordinary *pointwise multiplication* and *convolution*) and *two sets of exotic derivations* and, not least, their *density* property: any given resurgent function in the Borel plane is the limit, uniformly on any compact set of its Riemann surface, of a suitable series of hyperlogarithms. Here are the main definitions and properties:

15*. Hyperlogarithmic monomials: dimorphy.

$$\widehat{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(\tau) \quad := \quad \int_0^{\tau} \frac{d\tau_r}{\tau_r - \alpha_r} \cdots \int_0^{\tau_3} \frac{d\tau_2}{\tau_2 - \alpha_2} \int_0^{\tau_2} \frac{d\tau_1}{\tau_1 - \alpha_1}$$

$$\widehat{\mathcal{V}}^{\omega_1,\ldots,\omega_r}(\tau) \quad \equiv \quad \widehat{\mathcal{V}}^{[\alpha_1,\ldots,\alpha_r]}(\tau) \quad \textit{with} \quad \alpha_i \equiv \omega_1 + \ldots + \omega_i \quad (\forall i)$$

To express the multiplication-convolution dimorphy we require the *upper convolution* $\widehat{*}$, which has the same unit 1 as pointwise multiplication. Its definition is: $(\widehat{\varphi}_1 \,\widehat{*}\, \widehat{\varphi}_2)(\tau) := \int_0^\tau \widehat{\varphi}_1(\tau_1)\, \widehat{\varphi}_1(\tau - \tau_1) d\tau_1$

$$\times\text{-}\textbf{symmetrality}: \quad (\widehat{\mathcal{V}}^{[\boldsymbol{\alpha}']} \,.\, \widehat{\mathcal{V}}^{[\boldsymbol{\alpha}'']})(\tau) \quad \equiv \quad \sum_{\boldsymbol{\alpha} \in \mathrm{sha}(\boldsymbol{\alpha}',\boldsymbol{\alpha}'')} \widehat{\mathcal{V}}^{[\boldsymbol{\alpha}]}(\tau) \qquad (6)$$

$$\widehat{*}\text{-}\textbf{symmetrality}: \quad (\widehat{\mathcal{V}}^{\boldsymbol{\omega}'} \,\widehat{*}\, \widehat{\mathcal{V}}^{\boldsymbol{\omega}''})(\tau) \quad \equiv \quad \sum_{\boldsymbol{\omega} \in \mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} \widehat{\mathcal{V}}^{\boldsymbol{\omega}}(\tau) \qquad (7)$$

(6) says that $\widehat{\mathcal{V}}^{[\bullet]}$ is symmetral relative to pointwise multiplication.

(7) says that $\widehat{\mathcal{V}}^{\bullet}$ is symmetral relative to the convolution $\widehat{*}$.

15**. Hyperlogarithmic monomials and monics.

The **hyperlogarithmic monomials** $\widehat{\mathcal{V}}^{\bullet}$ most useful in the present context are defined by:

$$\widehat{\mathcal{V}}^{\,\omega_1,\dots,\omega_r}(\zeta) \;:=\; \frac{1}{\zeta-(\omega_1+\dots+\omega_r)}\int_0^{\zeta}\frac{d\zeta_{r-1}}{\zeta_{r-1}-(\omega_1+\dots+\omega_{r-1})}\cdots\int_0^{\zeta_2}\frac{d\zeta_1}{\zeta_1-\omega_1}$$

and verify
$$\begin{cases}\widetilde{\mathcal{V}}^{\,\boldsymbol{\omega}'}(z)\,.\,\widetilde{\mathcal{V}}^{\,\boldsymbol{\omega}''}(z) & \equiv \sum_{\boldsymbol{\omega}\in\mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')}\widetilde{\mathcal{V}}^{\,\boldsymbol{\omega}}(z)\\ (\widehat{\mathcal{V}}^{\,\boldsymbol{\omega}'}\ast\widehat{\mathcal{V}}^{\,\boldsymbol{\omega}''})(\tau) & \equiv \sum_{\boldsymbol{\omega}\in\mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')}\widehat{\mathcal{V}}^{\,\boldsymbol{\omega}}(\zeta)\end{cases}$$

The corresponding **hyperlogarithmic monics** $V^{\bullet}$ are defined inductively by:

$$\Delta_{\omega_1+\dots+\omega_r}\mathcal{V}^{\omega_1,\dots,\omega_r}(z) = V^{\omega_1,\dots,\omega_r} + \sum_{\omega_{i+1}+\ldots+\omega_r=0} V^{\omega_1,\dots,\omega_i}\,\mathcal{V}^{\omega_{i+i},\dots,\omega_r}(z) \qquad (8)$$

and verify the alternility relation $\sum_{\boldsymbol{\omega}\in\mathrm{sha}(\boldsymbol{\omega}',\boldsymbol{\omega}'')} V^{\boldsymbol{\omega}} \equiv 0$. They are univalued, piecewise analytic functions of their indices ω_i.

15***. Hyperlogarithmic monomials and monics.

Index differentiation for the hyperlogarithmic monomials:

$$
\begin{aligned}
\omega_1(\partial_{\omega_1}+z)\,\mathcal{V}^{\omega_1,\dots,\omega_r}(z) &= -\mathcal{V}^{\omega_1+\omega_2,\dots,\omega_r}(z)\\
\omega_j(\partial_{\omega_j}+z)\,\mathcal{V}^{\omega_1,\dots,\omega_r}(z) &= +\mathcal{V}^{\omega_1,\dots,\omega_{j-1}+\omega_j,\dots,\omega_r}(z)-\mathcal{V}^{\omega_1,\dots,\omega_j+\omega_{j+1},\dots,\omega_r}(z)\\
\omega_r(\partial_{\omega_r}+z)\,\mathcal{V}^{\omega_1,\dots,\omega_r}(z) &= +\mathcal{V}^{\omega_1,\dots,\omega_{r-1}+\omega_r}(z)-\mathcal{V}^{\omega_1,\dots,\omega_{r-1}}(z)\\
z(\partial_z+|\boldsymbol{\omega}|)\,\mathcal{V}^{\omega_1,\dots,\omega_r}(z) &= -\mathcal{V}^{\omega_1,\dots,\omega_{r-1}}(z)
\end{aligned}
$$

Index differentiation for the hyperlogarithmic monics:

$$
\begin{aligned}
\omega_1\partial_{\omega_1}V^{\omega_1,\dots,\omega_r} &= -V^{\omega_1+\omega_2,\dots,\omega_r}\\
\omega_j\partial_{\omega_j}V^{\omega_1,\dots,\omega_r} &= +V^{\omega_1,\dots,\omega_{j-1}+\omega_j,\dots,\omega_r}-V^{\omega_1,\dots,\omega_j+\omega_{j+1},\dots,\omega_r}\\
\omega_r\partial_{\omega_r}V^{\omega_1,\dots,\omega_r} &= +V^{\omega_1,\dots,\omega_{r-1}+\omega_r}
\end{aligned}
$$

Jump rules for the hyperlogarithmic monics:
The monics $V^{\bullet}$ are univalued, piecewise analytic functions with cuts along the hypersurfaces $\frac{\omega_1+\cdots+\omega_i}{\omega_{i+1}+\cdots+\omega_r}V^{\omega_1,\dots,\omega_r}\in\mathbb{R}^+$ and determination discontinuities given by the *jump formula*:

$$
\begin{cases}
D_{\frac{\omega_1+\cdots+\omega_i}{\omega_{i+1}+\cdots+\omega_r}}V^{\omega_1,\dots,\omega_r}\equiv 2\pi i\;V^{\omega_1,\dots,\omega_i}\;V^{\omega_{i+1},\dots,\omega_r}\\
D_x\,F(x):=\lim_{\epsilon\to 0}(F(x+i\,\epsilon)-F(x-i\,\epsilon)) \qquad (t,\epsilon\in\mathbb{R}^+)
\end{cases}
$$

16 Weighted convolution with polar inputs.

Setting $\widehat{\mathcal{S}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)}(\xi) := weco^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ \widehat{c}_1 & ,\dots, & \widehat{c}_r \end{smallmatrix}\right)}(\xi)$ with $\widehat{c}_i(\xi) := \frac{1}{\xi - v_i}$, we get

$$\mathcal{S}^{\left(\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right)}(x) := \mathcal{V}^{u_1\, v_1}(x)$$

$$\mathcal{S}^{\left(\begin{smallmatrix} u_1 & , & u_2 \\ v_1 & , & v_2 \end{smallmatrix}\right)}(x) := \begin{cases} +\mathcal{V}^{u_1\, v_1 \;,\; u_2\, v_2}(x) \\ -\mathcal{V}^{(u_1+u_2)\, v_1 \;,\; u_2\, (v_2-v_1)}(x) \\ +\mathcal{V}^{(u_1+u_2)\, v_2 \;,\; u_1\, (v_1-v_2)}(x) \end{cases}$$

$$\mathcal{S}^{\left(\begin{smallmatrix} u_1 & , & u_2 & , & u_3 \\ v_1 & , & v_2 & , & v_3 \end{smallmatrix}\right)}(x) := \begin{cases} +\mathcal{V}^{u_1\, v_1 \,,\, u_2\, v_2 \,,\, u_3\, v_3}(x) \\ +\mathcal{V}^{u_1\, v_1 \,,\, (u_2+u_3)\, v_3 \,,\, u_2\, (v_2-v_3)}(x) \\ -\mathcal{V}^{u_1\, v_1 \,,\, (u_2+u_3)\, v_2 \,,\, u_3\, (v_3-v_2)}(x) \\ +\mathcal{V}^{(u_1+u_2)\, v_2 \,,\, u_1\, (v_1-v_2) \,,\, u_3\, v_3}(x) \\ -\mathcal{V}^{(u_1+u_2)\, v_1 \,,\, u_2\, (v_2-v_1) \,,\, u_3\, v_3}(x) \\ +\mathcal{V}^{(u_1+u_2)\, v_2 \,,\, u_3\, v_3 \,,\, u_1\, (v_1-v_2)}(x) \\ -\mathcal{V}^{(u_1+u_2)\, v_1 \,,\, u_3\, v_3 \,,\, u_2\, (v_2-v_1)}(x) \\ +\mathcal{V}^{(u_1+u_2+u_3)\, v_1 \,,\, (u_2+u_3)\, (v_2-v_1) \,,\, u_3\, (v_3-v_2)}(x) \\ -\mathcal{V}^{(u_1+u_2+u_3)\, v_1 \,,\, (u_2+u_3)\, (v_3-v_1) \,,\, u_2\, (v_2-v_3)}(x) \\ +\mathcal{V}^{(u_1+u_2+u_3)\, v_1 \,,\, u_3\, (v_3-v_1) \,,\, u_2\, (v_2-v_1)}(x) \\ -\mathcal{V}^{(u_1+u_2+u_3)\, v_2 \,,\, u_1\, (v_1-v_2) \,,\, u_3\, (v_3-v_2)}(x) \\ -\mathcal{V}^{(u_1+u_2+u_3)\, v_2 \,,\, u_3\, (v_3-v_2) \,,\, u_1\, (v_1-v_2)}(x) \\ +\mathcal{V}^{(u_1+u_2+u_3)\, v_3 \,,\, u_1\, (v_1-v_3) \,,\, u_2\, (v_2-v_3)}(x) \\ -\mathcal{V}^{(u_1+u_2+u_3)\, v_3 \,,\, (u_1+u_2)\, (v_1-v_3) \,,\, u_2\, (v_2-v_1)}(x) \\ +\mathcal{V}^{(u_1+u_2+u_3)\, v_3 \,,\, (u_1+u_2)\, (v_2-v_3) \,,\, u_1\, (v_1-v_2)}(x) \end{cases}$$

$\widehat{\mathcal{S}}^{\left(\begin{smallmatrix} u_1 & ,\dots, & u_r \\ v_1 & ,\dots, & v_r \end{smallmatrix}\right)}(\xi)$ has $r!! := 1.3.5\ldots(2r-1)$ hyperlogarithmic summands.

17. Weighted convolution with hyperlogarithmic inputs.

The weighted convolution of r hyperlogs of depths $d_1, \ldots, d_r$ is a sum of $\mu(d_1, \ldots, d_r)$ hyperlogs each of depth $\sum d_i$. The number $\mu(\bullet)$ tends to be huge. Thus:

$\mu(\overbrace{1,\ldots,1}^{r\ \text{times}})$	$=$	$1.3.5\ldots(2r-1)$	$=$	$r!!$		*polar inputs*
$\mu(5,5,5)$	$=$	29 135 106	$\sim$	29	10^6	*hyperlog. inputs*
$\mu(4,5,6)$	$=$	22 855 560	$\sim$	23	10^6	
$\mu(6,5,4)$	$=$	23 963 940	$\sim$	24	10^6	
$\mu(4,4,4,4)$	$=$	10 050 665 625	$\sim$	10	10^9	
$\mu(1,3,5,7)$	$=$	349 098 750	$\sim$	0.4	10^9	
$\mu(7,5,3,1)$	$=$	539 188 650	$\sim$	0.5	10^9	
$\mu(3,3,3,3,3)$	$=$	60 575 515 000	$\sim$	60	10^9	
$\mu(1,2,3,4,5)$	$=$	6 067 061 000	$\sim$	6	10^9	
$\mu(5,4,3,2,1)$	$=$	9 641 071 440	$\sim$	10	10^9	

Thus, for a linear system as simple as (*):

$$(*) \quad (\partial_z + \omega_i\, x)\, Y_i(z,x) = Y_{i-1}(z,x)\, b_i(z) \qquad \begin{cases} (1 \le i \le 4,\ Y_0 \equiv 1) \\ b_i \ \textit{hyperlog. of depth}\ 4 \end{cases}$$

we have only 4 singularities in the ζ-plane, but close to 10^{10} in the ξ-plane.

18. Disappearance of the Stokes constants.

Applying the rules $\begin{cases} \Delta_{\omega_0} \mathcal{V}^{\omega_1,\dots,\omega_r}(x) = \sum_{\omega_1+\dots+\omega_i=\omega_0} V^{\omega_1,\dots,\omega_i} \mathcal{V}^{\omega_{i+1},\dots,\omega_r}(x) \\ V^{\omega_1} \equiv 1 \ , \ V^{\omega_1,\omega_2} = \textit{suitable determination of } \log \frac{\omega_2}{\omega_1} \end{cases}$

to the weighted convolution product: $\mathcal{S}^{\binom{u_1\ ,\ u_2}{v_1\ ,\ v_2}}(x) := \begin{cases} +\mathcal{V}^{u_1\, v_1\ ,\ u_2\, v_2}(x) \\ -\mathcal{V}^{(u_1+u_2)\, v_1\ ,\ u_2\, (v_2-v_1)}(x) \\ +\mathcal{V}^{(u_1+u_2)\, v_2\ ,\ u_1\, (v_1-v_2)}(x) \end{cases}$

we find that the continuous-valued Stokes constants disappear. Indeed:

$$
\begin{aligned}
\Delta_{u_1\, v_1} \mathcal{S}^{\binom{u_1\ ,\ u_2}{v_1\ ,\ v_2}}(x) &= \mathcal{V}^{u_2\, v_2}(x) &&= \mathcal{S}^{\binom{u_2}{v_2}}(x) \\
\Delta_{(u_1+u_2)\, v_1} \mathcal{S}^{\binom{u_1\ ,\ u_2}{v_1\ ,\ v_2}}(x) &= -\mathcal{V}^{u_2\, (v_2-v_1)}(x) &&= -\mathcal{S}^{\binom{u_2}{v_2-v_1}}(x) \\
\Delta_{(u_1+u2)\, v_2} \mathcal{S}^{\binom{u_1\ ,\ u_2}{v_1\ ,\ v_2}}(x) &= \mathcal{V}^{u_1\, (v_1-v_2)}(x) &&= \mathcal{S}^{\binom{u_1}{v_1-v_2}}(x) \\
\Delta_{u_1\, v_1+u_2\, v_2} \mathcal{S}^{\binom{u_1\ ,\ u_2}{v_1\ ,\ v_2}}(x) &= \mathrm{tes}\binom{u_1\ ,\ u_2}{v_1\ ,\ v_2} &&= \log \frac{u_2 v_2}{u_1 v_1} - \log \frac{u_2(v_2-v_1)}{(u_1+u2)v_1} + \log \frac{u_1(v_1-v_2)}{(u_1+u2)v_2}
\end{aligned}
$$

with a locally constant *tessellation coefficient* $\mathrm{tes}\binom{u_1\ ,\ u_2}{v_1\ ,\ v_2} \in \{0, \pm 2\pi i\}$.

The phenomenon is general and holds for all values of r.

Caveat: The disappearance of Stokes constants is incomplete in the case of v_i-repetitions.

19. The tessellation coefficients: hyperlog. expansions.

At depths $r \geq 3$, local constancy still holds: differentiate the following $tes^{\bullet}$ in any u_i or any v_i, and you get ... 0.

$$\mathrm{tes}\binom{u_1\,,\,u_2\,,\,u_3}{v_1\,,\,v_2\,,\,v_3} := \begin{cases} +V^{\,u_1\,v_1\,,\,u_2\,v_2\,,\,u_3\,v_3} \\ +V^{\,u_1\,v_1\,,\,(u_2+u_3)\,v_3\,,\,u_2\,(v_2-v_3)} \\ -V^{\,u_1\,v_1\,,\,(u_2+u_3)\,v_2\,,\,u_3\,(v_3-v_2)} \\ +V^{\,(u_1+u_2)\,v_2\,,\,u_1\,(v_1-v_2)\,,\,u_3\,v_3} \\ -V^{\,(u_1+u_2)\,v_1\,,\,u_2\,(v_2-v_1)\,,\,u_3\,v_3} \\ +V^{\,(u_1+u_2)\,v_2\,,\,u_3\,v_3\,,\,u_1\,(v_1-v_2)} \\ -V^{\,(u_1+u_2)\,v_1\,,\,u_3\,v_3\,,\,u_2\,(v_2-v_1)} \\ +V^{\,(u_1+u_2+u_3)\,v_1\,,\,(u_2+u_3)\,(v_2-v_1)\,,\,u_3\,(v_3-v_2)} \\ -V^{\,(u_1+u_2+u_3)\,v_1\,,\,(u_2+u_3)\,(v_3-v_1)\,,\,u_2\,(v_2-v_3)} \\ +V^{\,(u_1+u_2+u_3)\,v_1\,,\,u_3\,(v_3-v_1)\,,\,u_2\,(v_2-v_1)} \\ -V^{\,(u_1+u_2+u_3)\,v_2\,,\,u_1\,(v_1-v_2)\,,\,u_3\,(v_3-v_2)} \\ -V^{\,(u_1+u_2+u_3)\,v_2\,,\,u_3\,(v_3-v_2)\,,\,u_1\,(v_1-v_2)} \\ +V^{\,(u_1+u_2+u_3)\,v_3\,,\,u_1\,(v_1-v_3)\,,\,u_2\,(v_2-v_3)} \\ -V^{\,(u_1+u_2+u_3)\,v_3\,,\,(u_1+u_2)\,(v_1-v_3)\,,\,u_2\,(v_2-v_1)} \\ +V^{\,(u_1+u_2+u_3)\,v_3\,,\,(u_1+u_2)\,(v_2-v_3)\,,\,u_1\,(v_1-v_2)} \end{cases}$$

20. The tessellation coefficients: elementary induction.

Local constancy is an invitation to search for a more elementary expression of $tes^{\bullet}$.

Limiting hypersurfaces $\mathcal{H}^{+}_{i,j} = \{\boldsymbol{w} \in \mathbb{C}^{2r}\ ;\ H_{i,j}(\boldsymbol{w}) \in \mathbb{R}^{+}\}$ (there are $r^2 - 1$ of them):

$$
\begin{aligned}
H_{i,j}(\boldsymbol{w}) &:= Q^{*}_{i,j}(\boldsymbol{w})/Q^{**}_{i,j}(\boldsymbol{w}) \qquad (i-j \neq 0;\, i,j \in \mathbb{Z}_{r+1}) \\
Q^{*}_{i,j}(\boldsymbol{w}) &:= \sum_{\mathrm{circ}(i<q\leq j)} u^{\sharp}_{q}\,(v^{\sharp}_{q} - v^{\sharp}_{j}) \\
Q^{**}_{i,j}(\boldsymbol{w}) &:= \sum_{\mathrm{circ}(j<q\leq j)} u^{\sharp}_{q}\,(v^{\sharp}_{q} - v^{\sharp}_{j}) \;=\; <\boldsymbol{u},\boldsymbol{v}> - Q^{*}_{i,j}(\boldsymbol{w})
\end{aligned}
$$

The jump rule for $tes^{\boldsymbol{w}}$: It is only when $\boldsymbol{w}$ crosses a hypersurface $\mathcal{H}^{+}_{i,j}$ that $\mathrm{tes}^{\boldsymbol{w}}$ can change its value. Let $\boldsymbol{w}$ be any point on $\mathcal{H}^{+}_{i,j}$ and let $\boldsymbol{w}^{+}, \boldsymbol{w}^{-}$ be two points close by, with $\pm\Im H_{i,j}(\boldsymbol{w}^{\pm}) > 0$. Then

$$
\boldsymbol{tes}^{\boldsymbol{w}^{+}} - \boldsymbol{tes}^{\boldsymbol{w}^{-}} = \boldsymbol{tes}^{\boldsymbol{w}^{*}}\ \boldsymbol{tes}^{\boldsymbol{w}^{**}}
$$

$$
\textit{with}\quad
\begin{cases}
\boldsymbol{w}^{*} := \begin{pmatrix} u_{i+1} & ,\dots, & u_{p} & ,\dots, & u_{j} \\ v_{i+1}-v_{i} & ,\dots, & v_{p}-v_{i} & ,\dots, & v_{j}-v_{i} \end{pmatrix} & (\mathrm{circ}(i < p \leq j)\ \in \mathbb{Z}_{r+1}) \\
\boldsymbol{w}^{**} := \begin{pmatrix} u_{j+1} & ,\dots, & u_{q} & ,\dots, & u_{i-1} \\ v_{j+1}-v_{i} & ,\dots, & v_{q}-v_{i} & ,\dots, & v_{i-1}-v_{i} \end{pmatrix} & (\mathrm{circ}(j < q < i)\ \in \mathbb{Z}_{r+1})
\end{cases}
$$

20*. The tessellation coefficients: elementary expression.

We fix some $c \in \mathbb{C}^*$ and set $\Re_c : z \in \mathbb{C} \mapsto \Re(c\,z) \in \mathbb{R}$. Then we define:

$$f_{\boldsymbol{w}}^{\boldsymbol{w}'} := <\boldsymbol{u}', \boldsymbol{v}'><\boldsymbol{u}, \boldsymbol{v}>^{-1} \quad , \quad g_{\boldsymbol{w}}^{\boldsymbol{w}'} := <\boldsymbol{u}', \Re_\theta \boldsymbol{v}'><\boldsymbol{u}, \Re_\theta \boldsymbol{v}>^{-1} \tag{9}$$

From these scalars we construct the crucial sign factor *sig* which takes its values in $\{-1, 0, 1\}$. Here, the abbreviation $si(.)$ stands for $sign(\Im(.))$.

$$\mathrm{sig}^{\boldsymbol{w}', \boldsymbol{w}''} = \mathrm{sig}_c^{\boldsymbol{w}', \boldsymbol{w}''} := \frac{1}{8} \begin{cases} \left(\mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}'} - f_{\boldsymbol{w}}^{\boldsymbol{w}''}) - \mathrm{si}(g_{\boldsymbol{w}}^{\boldsymbol{w}'} - g_{\boldsymbol{w}}^{\boldsymbol{w}''})\right) \times \\ \left(1 + \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}'} / g_{\boldsymbol{w}}^{\boldsymbol{w}'}) \;\; \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}'} - g_{\boldsymbol{w}}^{\boldsymbol{w}'})\right) \times \\ \left(1 + \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}''} / g_{\boldsymbol{w}}^{\boldsymbol{w}''}) \; \mathrm{si}(f_{\boldsymbol{w}}^{\boldsymbol{w}''} - g_{\boldsymbol{w}}^{\boldsymbol{w}''})\right) \end{cases} \tag{10}$$

Next, from the pair $(\boldsymbol{w}', \boldsymbol{w}'')$ we derive a pair $(\boldsymbol{w}^*, \boldsymbol{w}^{**})$ by setting:

$$\boldsymbol{u}^* := \boldsymbol{u}' \quad , \quad \boldsymbol{v}^* := \boldsymbol{v}' \;\; <\boldsymbol{u}, \boldsymbol{v}>^{-1} \Im g_{\boldsymbol{w}}^{\boldsymbol{w}'} \; - \; \Re_c \boldsymbol{v}' \;\; <\boldsymbol{u}, \Re_c \boldsymbol{v}>^{-1} \Im f_{\boldsymbol{w}}^{\boldsymbol{w}'} \tag{11}$$

$$\boldsymbol{u}^{**} := \boldsymbol{u}'' \quad , \quad \boldsymbol{v}^{**} := \boldsymbol{v}'' \; <\boldsymbol{u}, \boldsymbol{v}>^{-1} \Im g_{\boldsymbol{w}}^{\boldsymbol{w}''} \; - \; \Re_c \boldsymbol{v}'' \;\; <\boldsymbol{u}, \Re_c \boldsymbol{v}>^{-1} \; \Im f_{\boldsymbol{w}}^{\boldsymbol{w}''} \tag{12}$$

or more symmetrically:

$$\boldsymbol{v}^* := \det \begin{pmatrix} \frac{\boldsymbol{v}'}{<\boldsymbol{u},\boldsymbol{v}>} & \frac{\Re_c \boldsymbol{v}'}{<\boldsymbol{u},\Re_c \boldsymbol{v}>} \\ \Im \frac{<\boldsymbol{u}',\boldsymbol{v}'>}{<\boldsymbol{u},\boldsymbol{v}>} & \Im \frac{<\boldsymbol{u}',\Re_c \boldsymbol{v}'>}{<\boldsymbol{u},\Re_c \boldsymbol{v}>} \end{pmatrix} \quad , \quad \boldsymbol{v}^{**} := \det \begin{pmatrix} \frac{\boldsymbol{v}''}{<\boldsymbol{u},\boldsymbol{v}>} & \frac{\Re_c \boldsymbol{v}''}{<\boldsymbol{u},\Re_c \boldsymbol{v}>} \\ \Im \frac{<\boldsymbol{u}'',\boldsymbol{v}''>}{<\boldsymbol{u},\boldsymbol{v}>} & \Im \frac{<\boldsymbol{u}'',\Re_c \boldsymbol{v}''>}{<\boldsymbol{u},\Re_c \boldsymbol{v}>} \end{pmatrix}$$

Lastly, from all these ingredients, we construct an auxilliary bimould $\mathrm{urtes}_{\mathrm{nor}}^{\bullet}$ by setting:

$$\mathrm{urtes}_{\mathrm{nor}}^{\boldsymbol{w}} = \sum_{\boldsymbol{w}'\boldsymbol{w}''=\boldsymbol{w}} \mathrm{sig}^{\boldsymbol{w}'\boldsymbol{w}''} \; \mathrm{tes}_{\mathrm{nor}}^{\boldsymbol{w}^*} \; \mathrm{tes}_{\mathrm{nor}}^{\boldsymbol{w}^{**}} \qquad \left((\boldsymbol{w}', \boldsymbol{w}'') \neq (\boldsymbol{w}^*, \boldsymbol{w}^{**})\right) \tag{13}$$

Then the tessellation bimould can be inductively calculated from:

$$\mathrm{tes}_{\mathrm{nor}}^{\bullet} = \sum_{0 \le n \le r(\bullet)} \mathrm{push}^n \, \mathrm{urtes}_{\mathrm{nor}}^{\bullet} \qquad (\forall c \in \mathbb{C}^*) \tag{14}$$

21. The tessellation coefficients. Main properties.

$\boldsymbol{P_1}$: $tes^\bullet$ is invariant under the involution *swap* and the iden-potent *push*:

$$\mathrm{swap}.A^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} = A^{\binom{v_r\ ,\ \dots,\ v_3-v_4\ ,\ v_2-v_3\ ,\ v_1-v_2}{u_1+\dots+u_r\ ,\ \dots,\ u_1+u_2+u_3\ ,\ u_1+u_2\ ,\ u_1}} \qquad (\mathrm{swap}^2 = \mathrm{iden})$$

$$\mathrm{push}.A^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}} = A^{\binom{-u_1\dots-u_r\ ,\ u_1\ ,\ u_2\ ,\ \dots,\ u_{r-1}}{-v_r\ ,\ v_1-v_r\ ,\ v_2-v_r\ ,\ \dots,\ v_{r-1}-v_r}} \qquad (\mathrm{push}^{r+1} = \mathrm{iden})$$

$\boldsymbol{P_2}$: the bimould $tes^\bullet$ is *bialternal*, i.e. alternal and of alternal *swappee*.

$\boldsymbol{P_3}$: $tes^\bullet_{nor}$ assumes all its sole values in $\mathbb{Z}$ and $|tes^{w_1,\dots,w_r}| < (r-1)!(r+1)!$ (far from sharp)

$\boldsymbol{P_4}$: As r increases, the set where $\mathrm{tes}^\bullet \neq 0$ has surprisingly small Lebesgue measure.

$$\begin{array}{lll}
\mathrm{tes}^{w_1} \equiv 1 & & \\
\mathrm{tes}^{w_1,w_2} \in \{0,\pm1\} & \mathcal{P}(\mathrm{tes}^{w_1,w_2} = \pm1) \sim 0.21 & \\
\mathrm{tes}^{w_1,w_2,w_3} \in \{0,\pm1\} & \mathcal{P}(\mathrm{tes}^{w_1,w_2,w_3} = \pm1) \sim 0.026 & \\
\mathrm{tes}^{w_1,\dots,w_4} \in \{0,\pm1,\pm2\} & \mathcal{P}(\mathrm{tes}^{w_1,\dots,w_4} = \pm1) \sim 0.0037 & \mathcal{P}(\mathrm{tes}^{w_1,\dots,w_4} = \pm2) \sim 0.0000037
\end{array}$$

$\boldsymbol{P_5}$: in presence of vanishing u_i-sums, we no longer have local constancy in the v_j's.

$\boldsymbol{P_6}$: conversely, in presence of v_i-repetitions, we no longer have local constancy in the u_j's.

$\boldsymbol{P_7}$: in the *semi-real* case, i.e. when *either* all u_i's *or* all v_i's are aligned with the origin, the tessellation coefficients altogether exit the picture, since in that case $tes^{w_1,\dots,w_r} \equiv 0$ as soon as $2 \leq r$.

22. Weighted convolution under alien derivations.

The only alien derivatives Δ_{ω_0} acting effectively on $\mathrm{wemu}^{\left(\begin{smallmatrix} u_1 & \dots, & u_r \\ c_1 & ,\dots, & c_r \end{smallmatrix}\right)}(x)$ correspond either to simple ($s = 1$) or composite ($s > 1$) indices ω_0 of the form

$$\omega_0 = |\boldsymbol{u^1}|\, v^1_{i_1} + \dots + |\boldsymbol{u^s}|\, v^s_{i_s} \quad \textit{with} \begin{cases} \boldsymbol{u^1}\,\boldsymbol{u^2}\dots\boldsymbol{u^{s-1}}\,\boldsymbol{u^s}\,\boldsymbol{u^*} = \boldsymbol{u} \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \textit{ and } \left(\begin{smallmatrix} u^k_{i_k} \\ c^k_{i_k} \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \boldsymbol{u^k} \\ \boldsymbol{c^k} \end{smallmatrix}\right) \end{cases}$$

with each factor sequence $\left(\begin{smallmatrix} \boldsymbol{u^k} \\ \boldsymbol{c^k} \end{smallmatrix}\right)$ re-indexed for convenience as $\left(\begin{smallmatrix} u^k_1 & ,\dots, & u^k_{r_k} \\ c^k_1 & ,\dots, & c^k_{r_k} \end{smallmatrix}\right)$. The corresponding alien derivative is given by:

$$\Delta_{\omega_0}\mathrm{wemu}^{\left(\begin{smallmatrix} u_1 & \dots, & u_r \\ c_1 & ,\dots, & c_r \end{smallmatrix}\right)}(x) = \begin{cases} \sum_{\check{v}^k_j \text{ over } v^k_{i_k}} \mathrm{Tes}^{\left(\begin{smallmatrix} |\boldsymbol{u^1}| & ;\dots; & |\boldsymbol{u^s}| \\ \check{v}^1_1,\dots,\check{v}^1_{r_1} & ;\dots; & \check{v}^s_1,\dots,\check{v}^s_{r_s} \end{smallmatrix}\right)} \times \\ \prod_{1\le k\le s} \mathrm{welu}^{\left(\begin{smallmatrix} u^k_1 & ,\dots, & \left(\begin{smallmatrix} u^k_{i_k} \\ \Delta_{\check{v}^k_{i_k}} c^k_{i_k} \end{smallmatrix}\right)^{\dagger} & ,\dots, & u^k_{r_k} \\ \check{v}^k_1\, c^k_1 & ,\dots, & & ,\dots, & \check{v}^k_{r_k}\, c^k_{r_k} \end{smallmatrix}\right)}(x) \times \\ \mathrm{wemu}^{\left(\begin{smallmatrix} u^*_1 & \dots, & u^*_{r_*} \\ c^*_1 & ,\dots, & c^*_{r_*} \end{smallmatrix}\right)}(x) \end{cases}$$

22*. Weighted convolution under alien derivations.

The only alien derivatives Δ_{ω_0} acting effectively on $\mathrm{welu}^{\left(\begin{smallmatrix} u_1 \\ c_1 \end{smallmatrix}, \dots, \left(\begin{smallmatrix} u_j \\ c_j \end{smallmatrix}\right)^\dagger, \dots, \begin{smallmatrix} u_r \\ c_r \end{smallmatrix}\right)}(x)$ correspond either to simple $(s = 1)$ or composite $(s > 1)$ indices ω_0 of three possible types – initial, final, global. Respectively:

$$\omega_0^{ini} = |\boldsymbol{u^1}|\, v^1_{i_1} + \cdots + |\boldsymbol{u^s}|\, v^s_{i_s} \;\; with \;\; \begin{cases} \boldsymbol{u^1} \;\dots\; \boldsymbol{u^s}\, \boldsymbol{u^*} = \boldsymbol{u} \;\; ; \;\; \left(\begin{smallmatrix} u_j \\ c_j \end{smallmatrix}\right)^\dagger \in \left(\begin{smallmatrix} \boldsymbol{u^*} \\ \boldsymbol{c^*} \end{smallmatrix}\right) \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \; and \; \left(\begin{smallmatrix} u^k_{i_k} \\ c^k_{i_k} \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \boldsymbol{u^k} \\ \boldsymbol{c^k} \end{smallmatrix}\right) \end{cases} \tag{15}$$

$$\omega_0^{fin} = |\boldsymbol{u^1}|\, v^1_{i_1} + \cdots + |\boldsymbol{u^s}|\, v^s_{i_s} \;\; with \;\; \begin{cases} \boldsymbol{{}^*u}\, \boldsymbol{u^1} \;\dots\; \boldsymbol{u^s} = \boldsymbol{u} \;\; ; \;\; \left(\begin{smallmatrix} u_j \\ c_j \end{smallmatrix}\right)^\dagger \in \left(\begin{smallmatrix} \boldsymbol{{}^*u} \\ \boldsymbol{{}^*c} \end{smallmatrix}\right) \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \; and \; \left(\begin{smallmatrix} u^k_{i_k} \\ c^k_{i_k} \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \boldsymbol{u^k} \\ \boldsymbol{c^k} \end{smallmatrix}\right) \end{cases} \tag{16}$$

$$\omega_0^{glo} = |\boldsymbol{u^1}|\, v^1_{i_1} + \cdots + |\boldsymbol{u^s}|\, v^s_{i_s} \;\; with \;\; \begin{cases} \boldsymbol{u^1} \;\dots\; \boldsymbol{u^s} = \boldsymbol{u} \\ \Delta_{v^k_{i_k}}\, c^k_{i_k} \neq 0 \; and \; \left(\begin{smallmatrix} u^k_{i_k} \\ c^k_{i_k} \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \boldsymbol{u^k} \\ \boldsymbol{c^k} \end{smallmatrix}\right) \end{cases} \tag{17}$$

with each factor sequence $\left(\begin{smallmatrix} \boldsymbol{u^k} \\ \boldsymbol{c^k} \end{smallmatrix}\right)$ re-indexed for convenience as $\left(\begin{smallmatrix} u^k_1 \\ c^k_1 \end{smallmatrix}, \dots, \begin{smallmatrix} u^k_{r_k} \\ c^k_{r_k} \end{smallmatrix}\right)$. The corresponding alien derivatives are given by:

22**. Weighted convolution under alien derivations.

$$\Delta_{\omega_0^{ini}} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & \dots, \\ c_1 & ,\dots, \end{smallmatrix} \left(\begin{smallmatrix} u_j \\ c_j \end{smallmatrix}\right)^{\dagger} \begin{smallmatrix} \dots, & u_r \\ ,\dots, & c_r \end{smallmatrix}\right)}(x) = \begin{cases} + \sum_{\check{v}_j^k \text{ over } v_{i_k}^k} \mathrm{Tes}^{\left(\begin{smallmatrix} |\boldsymbol{u}^1| & ;\dots; & |\boldsymbol{u}^s| \\ \check{v}_1^1,\dots,\check{v}_{r_1}^1 & ;\dots; & \check{v}_1^s,\dots,\check{v}_{r_s}^s \end{smallmatrix}\right)} \times \\ \prod_{1\le k\le s} \mathrm{welu}^{\left(\begin{smallmatrix} u_1^k & ,\dots, \\ \check{v}_1^k\; c_1^k & ,\dots, \end{smallmatrix} \left(\begin{smallmatrix} u_{i_k}^k \\ \Delta_{\check{v}_{i_k}^k}\; c_{i_k}^k \end{smallmatrix}\right)^{\dagger} \begin{smallmatrix} ,\dots, & u_{r_k}^k \\ ,\dots, & \check{v}_{r_k}^k\; c_{r_k}^k \end{smallmatrix}\right)}(x) \times \\ \mathrm{welu}^{\left(\begin{smallmatrix} u_1^* & \dots, \\ c_1^* & ,\dots, \end{smallmatrix} \left(\begin{smallmatrix} u_j \\ c_j \end{smallmatrix}\right)^{\dagger} \begin{smallmatrix} \dots, & u_{r_*}^* \\ ,\dots, & c_{r_*}^* \end{smallmatrix}\right)}(x) \end{cases}$$

$$\Delta_{\omega_0^{fin}} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & \dots, \\ c_1 & ,\dots, \end{smallmatrix} \left(\begin{smallmatrix} u_j \\ c_j \end{smallmatrix}\right)^{\dagger} \begin{smallmatrix} \dots, & u_r \\ ,\dots, & c_r \end{smallmatrix}\right)}(x) = \begin{cases} - \sum_{\check{v}_j^k \text{ over } v_{i_k}^k} \mathrm{Tes}^{\left(\begin{smallmatrix} |\boldsymbol{u}^1| & ;\dots; & |\boldsymbol{u}^s| \\ \check{v}_1^1,\dots,\check{v}_{r_1}^1 & ;\dots; & \check{v}_1^s,\dots,\check{v}_{r_s}^s \end{smallmatrix}\right)} \times \\ \prod_{1\le k\le s} \mathrm{welu}^{\left(\begin{smallmatrix} u_1^k & ,\dots, \\ \check{v}_1^k\; c_1^k & ,\dots, \end{smallmatrix} \left(\begin{smallmatrix} u_{i_k}^k \\ \Delta_{\check{v}_{i_k}^k}\; c_{i_k}^k \end{smallmatrix}\right)^{\dagger} \begin{smallmatrix} ,\dots, & u_{r_k}^k \\ ,\dots, & \check{v}_{r_k}^k\; c_{r_k}^k \end{smallmatrix}\right)}(x) \times \\ \mathrm{welu}^{\left(\begin{smallmatrix} {}^*u_1 & \dots, \\ {}^*c_1 & ,\dots, \end{smallmatrix} \left(\begin{smallmatrix} u_j \\ c_j \end{smallmatrix}\right)^{\dagger} \begin{smallmatrix} \dots, & {}^*u_{r_*} \\ ,\dots, & {}^*c_{r_*} \end{smallmatrix}\right)}(x) \end{cases}$$

$$\Delta_{\omega_0^{glo}} \mathrm{welu}^{\left(\begin{smallmatrix} u_1 & \dots, \\ c_1 & ,\dots, \end{smallmatrix} \left(\begin{smallmatrix} u_j \\ c_j \end{smallmatrix}\right)^{\dagger} \begin{smallmatrix} \dots, & u_r \\ ,\dots, & c_r \end{smallmatrix}\right)}(x) = \begin{cases} + \sum_{\check{v}_j^k \text{ over } v_{i_k}^k} \mathrm{Tes}^{\left(\begin{smallmatrix} |\boldsymbol{u}^1| & ;\dots; & |\boldsymbol{u}^s| \\ \check{v}_1^1,\dots,\check{v}_{r_1}^1 & ;\dots; & \check{v}_1^s,\dots,\check{v}_{r_s}^s \end{smallmatrix}\right)} \times \\ \prod_{1\le k\le s} \mathrm{welu}^{\left(\begin{smallmatrix} u_1^k & ,\dots, \\ \check{v}_1^k\; c_1^k & ,\dots, \end{smallmatrix} \left(\begin{smallmatrix} u_{i_k}^k \\ \Delta_{\check{v}_{i_k}^k}\; c_{i_k}^k \end{smallmatrix}\right)^{\dagger} \begin{smallmatrix} ,\dots, & u_{r_k}^k \\ ,\dots, & \check{v}_{r_k}^k\; c_{r_k}^k \end{smallmatrix}\right)}(x) \end{cases}$$

23. First, Second, Third Bridge equations.

First Bridge equation: $[\mathbb{\Delta}_\omega, \Theta^{-1}] = \mathbb{A}_\omega \; \Theta^{-1}$

with $\mathbb{\Delta}_\omega := e^{-\omega z}\Delta_\omega$ (z-resurgence) and $\mathbb{A}_\omega = \sum_{(u_1+\cdots+u_r)x=\omega} W^{\binom{u_1\ ,\ldots,\ u_r}{b_1\ ,\ldots,\ b_r}}(x)\mathbb{D}_{\|u_1}\ldots\mathbb{D}_{\|u_r}$

Second Bridge equation: $[\mathbb{\Delta}_\omega, \Theta^{-1}] = \mathbb{P}_\omega \; \Theta^{-1}$

with $\mathbb{\Delta}_\omega := e^{-\omega x}\Delta_\omega$ (x-resurgence) and:

$$\begin{cases} \mathbb{P}_\omega := \sum_{\sum u_i(z-\alpha_i)=\omega} \mathrm{tes}^{\binom{u_1\ ,\ldots,\ u_r}{z-\alpha_1\ ,\ldots,\ z-\alpha_r}} \mathbb{Q}_{[\,{u_1 \atop \alpha_1}\,]}\ldots\mathbb{Q}_{[\,{u_r \atop \alpha_r}\,]} \\ \mathbb{Q}_{[\,{u_0 \atop \alpha_0}\,]} := e^{u_0\alpha_0}\sum_{\sum u_i=u_0} \mathrm{welu}^{\binom{u_1\ ,\ldots,\ (\ u_i\ )^{\dagger}\ ,\ldots,\ u_r}{\bar\alpha_0 \cdot c_1\ ,\ldots,\ (\Delta_{\alpha_0} c_i)\ ,\ldots,\ \bar\alpha_0\cdot c_r}} \mathbb{D}_{\|u_1}\ldots\mathbb{D}_{\|u_r} \end{cases} \tag{18}$$

Third Bridge equation: $\mathbb{\Delta}_\omega \mathbb{Q}_{[\,{u_0 \atop \alpha_0}\,]} = \begin{cases} +\sum_{u_1+u_2=u_0} \mathbb{P}_{\omega,[\,{u_1 \atop \alpha_0}\,]}\mathbb{Q}_{[\,{u_2 \atop \alpha_0}\,]} \\ -\sum_{u_1+u_2=u_0} \mathbb{Q}_{[\,{u_1 \atop \alpha_0}\,]}\mathbb{P}_{\omega,[\,{u_2 \atop \alpha_0}\,]} \end{cases}$

$$\text{with} \quad \mathbb{P}_{\omega,[\,{u_0 \atop \alpha_0}\,]} := \sum_{\sum u_i(\alpha_0-\alpha_i)=\omega}^{\sum u_i=u_0} \mathrm{tes}^{\binom{u_1\ ,\ldots,\ u_r}{\alpha_0-\alpha_1\ ,\ldots,\ \alpha_0-\alpha_r}} \mathbb{Q}_{[\,{u_1 \atop \alpha_1}\,]}\ldots\mathbb{Q}_{[\,{u_r \atop \alpha_r}\,]} \tag{19}$$

24. BE2 and BE3 in the semi-real and other cases.

The semi-real case: In the important instances when the tessellation coefficients $tes^{w_1,\dots,w_r}$ turn trivial (i.e. $\equiv 1$ for $r = 1$ and $\equiv 0$ for $r \neq 1$), the Third Bridge equation simplifies:

$$\textbf{(BE3)} \qquad \mathbb{\Delta}_\omega \mathbb{Q}_{\left[\begin{smallmatrix} u_0 \\ \alpha_0 \end{smallmatrix}\right]} = \sum_{u_1+u_2=u_0}^{u_1(\alpha_0-\alpha_1)=\omega} \left[\mathbb{Q}_{\left[\begin{smallmatrix} u_1 \\ \alpha_1 \end{smallmatrix}\right]}, \mathbb{Q}_{\left[\begin{smallmatrix} u_2 \\ \alpha_0 \end{smallmatrix}\right]}\right] \qquad (20)$$

and one can checks the equality of the exponential factors on both sides:

(i) $\mathbb{\Delta}_\omega$ carries a factor $e^{-\omega\, x} = e^{-u_1(\alpha_0-\alpha_1)\, x}$

(ii) $\mathbb{Q}_{\left[\begin{smallmatrix} u_0 \\ \alpha_0 \end{smallmatrix}\right]}$ carries a factor $e^{u_0\alpha_0\, x} = e^{(u_1+u_2)\alpha_0\, x}$

(iii) $\mathbb{Q}_{\left[\begin{smallmatrix} u_1 \\ \alpha_1 \end{smallmatrix}\right]}$ carries a factor $e^{u_1\alpha_1\, x}$

(iv) $\mathbb{Q}_{\left[\begin{smallmatrix} u_2 \\ \alpha_0 \end{smallmatrix}\right]}$ carries a factor $e^{u_2\alpha_0\, x}$

The most general case: In the opposite direction, the results extend to the case of hyperlogarithmic (instead of meromorpic) or even **absolutely general inputs** $\mathrm{b}_i(z)$ (and thus $\widehat{c}_i(\xi)$), except that we must switch to a multiple indexation $\alpha_i \to \check{\alpha}_i$ and that the **BE3** inherits a third term, corresponding to the case $\Delta_\omega^{glo} welu^\bullet$. We get:

$$\textbf{(BE3)} \qquad \mathbb{\Delta}_\omega \mathbb{Q}_{\left[\begin{smallmatrix} u_0 \\ \check{\alpha}_0 \end{smallmatrix}\right]} = \begin{cases} + \sum_{u_1+u_2=u_0} \mathbb{P}_{\omega,\left[\begin{smallmatrix} u_1 \\ \check{\alpha}_0 \end{smallmatrix}\right]} \mathbb{Q}_{\left[\begin{smallmatrix} u_2 \\ \check{v}_0 \end{smallmatrix}\right]} \\ - \sum_{u_1+u_2=u_0} \mathbb{Q}_{\left[\begin{smallmatrix} u_1 \\ \check{\alpha}_0 \end{smallmatrix}\right]} \mathbb{P}_{\omega,\left[\begin{smallmatrix} u_2 \\ \check{\alpha}_0 \end{smallmatrix}\right]} \\ + \quad \mathbb{P}_{\omega,\left[\begin{smallmatrix} u_0 \\ \check{\alpha}_0 \end{smallmatrix}\right]} \end{cases} \qquad (21)$$

25. Equational vs co-equational resurgence.

- To produce *equational resurgence*, the coefficients $b_i(z)$ need only be analytic germs at ∞ (and verify a uniformity condition).
- To produce *co-equational resurgence*, the $b_i(z)$ must be endlessly continuable over the Riemann sphere (with a uniformity condition).
- **BE1** : The index reservoir Ω_1 is rigidly determined by the *multipliers* λ_i. The Stokes constants are entire functions of x.
- **BE2** : The index reservoir Ω_2 depends linearly on z and the singular points of the coefficients $b_i(z)$. The Stokes constants disappear (*qualification here*) and get replaced by discrete-valued tessellation coefficients. BE2 involves *wemu*$^\bullet$ and *welu*$^\bullet$.
- **BE3** : The index reservoir Ω_3 and the tessellation coefficients cease to depend on z. BE3 involves only *welu*$^\bullet$

25*. Complexity of co-equational resurgence.

At the end of this tour of coequational resurgence, we find a clear four level stratification:

- *The atomic level*, populated by objects such as simple poles or hyperlogarithms.
- *The molecular level*, consisting of huge clusters of atoms, with unsuspected emergent properties.
- *The microscopic level*, consisting of derivation operators $\mathbb{Q}_\omega$, usually infinite chains of molecules contracted by elementary derivation operators.
- *The macroscopic level*, consisting of new derivation operators $\mathbb{P}_\omega$ assembled from the earlier $\mathbb{Q}_\omega$.
- The passage from the atomic to the molecular level is mediated on the Analysis side by *weighted convolution* and on the combinatorial side by the *scrambling transform*.
- The passage from the molecular to the microsopic level is rather mechanical – mere growth by accumulation.
- The passage from the microscopic to the macroscopic level, arguably the most interesting of the three, is mediated by the *tessellation coefficients*. While much is known about them, it would seem that just as much remains to be discovered.

When we have both z- and x-resurgence, there can be no hesitation. But in theoretical physics, the x-resurgence is often all we have.

Cf the July 2015 CERN conference on resurgence (Geneva) or the June 2019 IHES conference (Paris).

26. Emergent properties: the flection structure.

When looking at the *weighted convolution* products of poles or hyperlogarithms, we just caught a glimpse of the strange ways in which the u_i- and v_i-indices interact, as well as of the numerous symmetries and invariance properties of the related *tessellation coefficients*.

By following this lead, I stumbled on a whole new algebraic structure – the so-called *flection structure* – with the Lie algebra ARI and the group GARI as its center piece. This *flection structure* in turn proved quite helpful in the investigation of *arithmetical dimorphy*, i.e. in the study of those $\mathbb{Q}$-rings of transcendental numbers (such as the *multizetas*) that possess *two* natural (and independent) *'multiplication tables'*.

So we have here this minor miracle of a beautiful algebraic structure spontaneously emerging from an *a priori* 'amorphous' Analysis problem.

27. Example: the time-independent Schrödinger equation.

$$\partial_q^2 \psi = \frac{x^2}{4} W(q) \quad \textit{with} \quad \begin{cases} W(q) = q^\nu + \alpha_1 \, q^{\nu-1} + \cdots + \alpha_\nu \\ \alpha_\nu \; = -E \; (\textit{energy}) \quad , \quad x = 2/\hbar \end{cases}$$

$$z = z(q) = \int_0^q \sqrt{W(q')} dq' \sim \frac{2}{2+\nu} q^{\frac{2+\nu}{2}}$$

$$\psi(z,x) = \begin{cases} +C_+(x) \, e^{xz/2} \; (q'(z))^{\frac{1}{2}} \;\; \varphi_+(z,x) \\ +C_-(x) \, e^{-xz/2} \; (q'(z))^{\frac{1}{2}} \;\; \varphi_-(z,x) \end{cases}$$

$$\partial_z^2 \; \varphi_\pm \; \pm x\partial_z \; \varphi_\pm = (H^2(z) - H'(z)) \; \varphi_\pm$$

$$\textit{with} \quad H(z) := \frac{1}{2} \frac{q''(z)}{q'(z)} = -\frac{1}{2} \frac{\nu}{2+\nu} z^{-1} + \ldots$$

27*. Example: the time-independent Schrödinger equation.

BE₁ $$\Delta_{x_k}\ \varphi_{\pm} = S_k(x)\ \varphi_{\mp} \qquad \begin{cases} (\Delta_{x_k} = \Delta^{(z)}_{x_k}) \\ x_k := x\exp(\frac{4\pi i\,k}{2+\nu}) \\ S_k(x) \in \mathbb{C}\{x^{\frac{2}{2+\nu}}\} \end{cases}$$

BE₂ $$\Delta_{\pm z\pm\omega_j}\ \varphi_{\pm} = P_{j\pm}(x)\ \varphi_{\mp} \qquad (\Delta_{\pm z\pm\omega_j} = \Delta^{(x)}_{\pm z\pm\omega_j})$$

The $P_{j\pm}(x)$ are rational functions (whose form depend on the z-area) of $V_1(x), \ldots, V_\nu(x)$ with $V_1(x)V_2(x)\ldots V_\nu(x) \equiv 1$.

BE₃ $$\begin{cases} \Delta_{n\omega_{i,j}} V_k = 0 \qquad (k \neq i,j) \qquad (\Delta_{n\omega_{ij}} = \Delta^{(x)}_{n\omega_{ij}}) \\ \Delta_{n\omega_{i,j}} V_i = +\frac{1}{n}\left(-\frac{V_{i+1}V_{i+2}\cdots V_{j-1}}{V_{j+1}V_{j+2}\cdots V_{i-1}}\right)^n \quad (i \neq j) \\ \Delta_{n\omega_{i,j}} V_j = -\frac{1}{n}\left(-\frac{V_{i+1}V_{i+2}\cdots V_{j-1}}{V_{j+1}V_{j+2}\cdots V_{i-1}}\right)^n \quad (i \neq j) \end{cases}$$

$$\omega_{ij} = \omega_j - \omega_i \quad \textit{with} \quad \omega_i = \int_{\gamma_i} \sqrt{W(q')}\, dq'$$

The main results on the time-independent Schrödinder are due to Y. Sibuya (Stokes constants), A. Voros (resurgence in the ξ-plane), and J.E. (convergence in the ξ-plane).

Thank you!

Спасибо за внимание!

И еще огромное спасибо организаторам...

.

Some references:

- J. E. *Weighted products and parametric resurgence*, in Méthodes résurgentes, Travaux en Cours, **47**, Ed. L. Boutet de Monvel, 1994, pp 7-49.
- J. E. *The scrambling operators applied to multizeta algebra and singular perturbation analysis*, (to appear), Oct. 2018, 156 pp. Preview available on my homepage: <https://www.math.u-psud.fr/~ecalle/>