Multizetas, perinomal numbers, arithmetical dimorphy, and ARI/GARI.

Jean Ecalle (Orsay, CNRS)

Abstract : In a sprawling field like *multizeta arithmetic*, connected with intricate new structures and teeming with 'special objects' (functions, moulds etc), there is room for expositions of all formats : short, medium-sized, huge. Here is a survey on the tiniest scale possible, based on a talk given at the 2002 Luminy conference on Resurgent Analysis.

Résumé : Le texte qui suit, aussi ramassé que possible, reprend un exposé fait à Luminy en novembre 2002. Il présente un panorama des récents progrès en *arithmétique 'dimorphique' des multizêtas* et esquisse les théories (ARI/GARI, objets périnomaux, moules spéciaux) qui ont permis ces progrès.

Contents :

0. Overview. Some notations.

We begin (§1, §2) with a few reminders about *arithmetical dimorphy* and then focus on the prototypical instance of dimorphy : the $\mathbb{Q}$-ring of *multizetas.* The next two sections (§3, §4) outline, for future use, two special theories – one called forth by the study of dimorphy, the other predating it. They are the theory of *perinomal objects* ; and the *flexion structure* – mainly the Lie algebra ARI and its group GARI. Next, we try (§5) to order the field of multizeta arithmetic as a hierarchy of increasingly arduous tasks, with a red thread running through everything : the search for *canonical irreducibles.* We then (§7 through §11) develop the tools (special moulds etc) which make it possible not only to explicitly decompose all multizetas into irreducibles (§12), but also to express these irreducibles *directly* and in a way that truly reflects their *neutral* position, half-way between the two natural bases of multizetas (§13). But before getting started, we must get a few definitions (about moulds, mould operations, and mould symmetries) out of the way.

Moulds $A^\bullet = \{A^{\boldsymbol{\omega}}\} = \{A^{\omega_1,\dots,\omega_r}\}$ are simply functions of a *variable number of variables.* These variables are noted as upper indices, with bold face reserved for sequences, which often get subsumed as a simple dot $\bullet$. Mould addition is trivially defined, but mould multiplication is non-commutative and involves the breaking-up of sequences :

$$\{C^\bullet = A^\bullet \times B^\bullet\} \Longleftrightarrow \{C^{\boldsymbol{\omega}} = \sum_{\boldsymbol{\omega}=\boldsymbol{\omega}^1\boldsymbol{\omega}^2} A^{\boldsymbol{\omega}^1} B^{\boldsymbol{\omega}^2}\} \tag{1}$$

Depending on the context, many other secondary operations may come into play. Moreover, useful moulds tend to fall into one or the other of a few *symmetry types*, which are either preserved by the basic operations, or transformed in transparent manner. Only six symmetry types will be relevant here, to wit : *symmetral/alternal, symmetrel/alternel, symmetril/alternil.*

A mould $A^\bullet$ is said to be symmetral (resp alternal) or again symmetrel (resp. alternel) if the following identities hold for all $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2$:

$$\sum_{\boldsymbol{\omega}\in \mathrm{sha}(\boldsymbol{\omega}^1,\boldsymbol{\omega}^2)} A^{\boldsymbol{\omega}} \;=\; A^{\boldsymbol{\omega}^1} A^{\boldsymbol{\omega}^2}\,(\mathit{resp}\ = 0) \qquad \mathit{(symmetral/alternal)} \tag{2}$$

$$\sum_{\boldsymbol{\omega}\in \mathrm{she}(\boldsymbol{\omega}^1,\boldsymbol{\omega}^2)} A^{\boldsymbol{\omega}} \;=\; A^{\boldsymbol{\omega}^1} A^{\boldsymbol{\omega}^2}\,(\mathit{resp}\ = 0) \qquad \mathit{(symmetrel/alternel)} \tag{3}$$

with $sha(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2)$ (resp. $she(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2)$) denoting the set of all ordinary (resp. contracting) shufflings [1] of $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2$. The last pair *symmetril/alternil* applies

[1] under *ordinary/contracting* shufflings, adjacent indices ω_i, ω_j stemming from different

only to moulds with a double-storeyed indexation, i.e. with indices of the form $w_i = \binom{u_i}{v_i}$. It resembles *symmetrel/alternel*, except that the straightforward addition $(\omega_i, \omega_j) \mapsto \omega_i + \omega_j$ makes way for the subtler contractions :

$$\left(A^{\binom{\ldots,u_i,\ldots}{\ldots,v_i,\ldots}}\,,\,A^{\binom{\ldots,u_j,\ldots}{\ldots,v_j,\ldots}}\right) \;\mapsto\; \frac{1}{v_i - v_j}\left(A^{\binom{\ldots,u_i+u_j,\ldots}{\ldots,v_i,\ldots}} - A^{\binom{\ldots,u_i+u_j,\ldots}{\ldots,v_j,\ldots}}\right) \tag{4}$$

Throughout, we shall use the following abbreviations for sums/differences :

$$u_{i,j} := u_i + u_j \quad , \quad u_{i,j,k} := u_i + u_j + u_k \quad etc \quad ; \quad v_{i:j} := v_i - v_j \tag{5}$$

1. Multizeta numbers and numerical dimorphy.

Some extremely important $\mathbb{Q}$-rings of transcendental numbers are *dimorphic*, i.e. possess two *natural* $\mathbb{Q}$-bases [2] $\{\alpha_m\}$, $\{\beta_n\}$ with a simple *conversion rule* and two independent *multiplication tables*, all of which involve only rational coefficients and finite sums :

$$\begin{array}{llll} \alpha_m = \sum^* H^n_m \;\beta_n & , \quad \beta_n = \sum^* K^m_n \;\alpha_m & & (H^n_m\,,\,K^m_n \in \mathbb{Q}) \\ \alpha_{n_1}\,\alpha_{n_2} = \sum^* A^{n_3}_{n_1,n_2}\,\alpha_{n_3} & , \quad \beta_{n_1}\,\beta_{n_2} = \sum^* B^{n_3}_{n_1,n_2}\,\beta_{n_3} & & (A^{n_3}_{n_1,n_2}\,,\,B^{n_3}_{n_1,n_2} \in \mathbb{Q}) \end{array}$$

The simplest, most basic of all such rings is $\mathbb{Z}$eta , which is not only *multiplicatively* generated but also *linearly* spanned by the so-called *multizetas*.[3]

In the *first basis*, the multizetas are given by polylogarithmic integrals :

$$\mathrm{Wa}_*^{\alpha_1,\ldots,\alpha_l} := (-1)^{l_0} \int_0^1 \frac{dt_l}{(\alpha_l - t_l)} \cdots \int_0^{t_3} \frac{dt_2}{(\alpha_2 - t_2)} \int_0^{t_2} \frac{dt_1}{(\alpha_1 - t_1)} \tag{6}$$

with indices α_j that are either 0 or unit roots[4].

In the *second basis*, multizetas are expressed as familiar-looking sums :

$$\mathrm{Ze}_*^{\binom{\epsilon_1\;,\ldots,\;\epsilon_r}{s_1\;,\ldots,\;s_r}} := \sum_{n_1 > \cdots > n_r > 0} n_1^{-s_1} \ldots n_r^{-s_r}\; e_1^{-n_1} \ldots e_r^{-n_r} \tag{7}$$

---

sequences are *forbidden/allowed* to merge into $\omega_{i,j} := \omega_i + \omega_j$. Thus, for a pair $\boldsymbol{\omega^1} = (\omega_1)$ and $\boldsymbol{\omega^2} = (\omega_2, \omega_3)$, we have $sha(\boldsymbol{\omega^1}, \boldsymbol{\omega^2}) = \{(\omega_1,\omega_2,\omega_3), (\omega_2,\omega_1,\omega_3), (\omega_2,\omega_3,\omega_1)\}$ but $she(\boldsymbol{\omega^1}, \boldsymbol{\omega^2}) = \{(\omega_1,\omega_2,\omega_3), (\omega_2,\omega_1,\omega_3), (\omega_2,\omega_3,\omega_1), (\omega_1+\omega_2,\omega_3), (\omega_1,\omega_2+\omega_3)\}$.

[2] with some natural countable indexation $\{m\}, \{n\}$, not necessarily on $\mathbb{N}$ or $\mathbb{Z}$.

[3] or MZV, short for *multiple zeta values.*

[4] l_0 is the number of zeros in the sequence $\{\alpha_1, \ldots, \alpha_l\}$.

with $s_j \in \mathbb{N}^*$ and unit roots $e_j := \exp(2\pi i \epsilon_j)$ with 'logarithms' $\epsilon_j \in \mathbb{Q}/\mathbb{Z}$.

The stars $*$ means that the integrals or sums are provisionally assumed to be convergent or semi-convergent: for $Wa_*^{\boldsymbol{\alpha}}$ this means that $\alpha_1 \neq 0$ and $\alpha_l \neq 1$, and for $Ze_*^{\binom{\boldsymbol{\epsilon}}{\mathbf{s}}}$ this means that $\binom{\epsilon_1}{s_1} \neq \binom{0}{1}$ i.e. $\binom{e_1}{s_1} \neq \binom{1}{1}$.

The corresponding moulds $Wa_*^\bullet$ and $Ze_*^\bullet$ turn out to be respectively *symmetral* and *symmetrel*:

$$\mathrm{Wa}_*^{\boldsymbol{\alpha^1}}\ \mathrm{Wa}_*^{\boldsymbol{\alpha^2}} = \sum_{\boldsymbol{\alpha} \in \mathrm{sha}(\boldsymbol{\alpha^1},\boldsymbol{\alpha^2})} \mathrm{Wa}_*^{\boldsymbol{\alpha}} \qquad \forall \boldsymbol{\alpha^1}, \forall \boldsymbol{\alpha^2} \tag{8}$$

$$\mathrm{Ze}_*^{\binom{\boldsymbol{\epsilon^1}}{\mathbf{s^1}}}\ \mathrm{Ze}_*^{\binom{\boldsymbol{\epsilon^2}}{\mathbf{s^2}}} = \sum_{\binom{\mathbf{s}}{\boldsymbol{\epsilon}} \in \mathrm{she}(\binom{\boldsymbol{\epsilon^1}}{\mathbf{s^1}},\binom{\boldsymbol{\epsilon^2}}{\mathbf{s^2}})} \mathrm{Ze}_*^{\binom{\boldsymbol{\epsilon}}{\mathbf{s}}} \qquad \forall \binom{\boldsymbol{\epsilon^1}}{\mathbf{s^1}}, \forall \binom{\boldsymbol{\epsilon^2}}{\mathbf{s^2}} \tag{9}$$

These are the so-called *"quadratic relations"*, which express dimorphy. As for the conversion rule, it reads:[5]

$$\mathrm{Wa}_*{}^{e_1,0^{[s_1-1]},\ldots,e_r,0^{[s_r-1]}} := \mathrm{Ze}_*{}^{\left(\begin{smallmatrix} \epsilon_r\ , & \epsilon_{r-1:r}\ , & \ldots, & \epsilon_{1:2} \\ s_r\ , & s_{r-1}\ , & \ldots, & s_1 \end{smallmatrix}\right)} \tag{10}$$

$$\mathrm{Ze}_*{}^{\left(\begin{smallmatrix} \epsilon_1\ , & \epsilon_2\ , & \ldots, & \epsilon_r \\ s_1\ , & s_2\ , & \ldots, & s_r \end{smallmatrix}\right)} =: \mathrm{Wa}_*{}^{e_1\ldots e_r,0^{[s_r-1]},\ldots,e_1e_2,0^{[s_2-1]},e_1,0^{[s_1-1]}} \tag{11}$$

There happen to be unique extensions $Wa_*^\bullet \to Wa^\bullet$ and $Ze_*^\bullet \to Ze^\bullet$ to the divergent case that keep our moulds symmetral/symmetrel while conforming to the 'initial conditions' $Wa^0 = Wa^1 = 0$ and $Ze^{\binom{0}{1}} = 0$. The only price to pay is a slight modification of the conversion rule: see §2 *infra*.

Basic gradations/filtrations: Four parameters dominate the discussion:
– the *weight* $s := \sum s_i$ (in the $Ze^\bullet$-encoding) or $:= l$ (in the $Wa^\bullet$-encoding)
– the *length* $r :=$ number of ϵ_i's or s_i's or non-zero α_i's.
– the *degree* $d := s - r =$ number of zeros in the $Wa^\bullet$-encoding.[6]
– the *root order* $p :=$ smallest p such that all ϵ_i are in $\frac{1}{p}\mathbb{Z}/\mathbb{Z}$.
Only s defines an (additive and multiplicative) gradation; the other parameters merely induce filtrations.

2. Generating series/functions.

The natural encodings $Wa^\bullet$ and $Ze^\bullet$ being unwieldy and too heterogeneous in their indexations, we must replace them by suitable *generating series*, so

[5] with the usual shorthand for differences: $\epsilon_{i:j} := \epsilon_i - \epsilon_j$.

[6] d is called *degree*, because under the correspondence *scalars* $\to$ *generating series*, the multizetas become coefficients of monomials of total degree d. See (12),(13).

chosen as to preserve the simplicity of the two quadratic relations and that of the conversion rule. This essentially *imposes* the following definitions:

$$\mathrm{Zag}^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} := \sum_{1\le s_j} \mathrm{Wa}^{e_1,0^{[s_1-1]},\dots,e_r,0^{[s_r-1]}}\ u_1^{s_1-1} u_{12}^{s_2-1}\dots u_{12\dots r}^{s_r-1} \tag{12}$$

$$\mathrm{Zig}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{v_1\ ,\dots,\ v_r}} := \sum_{1\le s_j} \mathrm{Ze}^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{s_1\ ,\dots,\ s_r}}\ v_1^{s_1-1}\dots v_r^{s_r-1} \tag{13}$$

These power series are actually convergent: they define *generating functions*[7] that are meromorphic, with multiple poles at simple locations. These functions, in turn, verify simple difference equations, and admit an elementary mould factorisation:

$$\mathrm{Zag}^\bullet := \lim_{k\to\infty} (\mathrm{doZag}_k^\bullet \times \mathrm{coZag}_k^\bullet) \tag{14}$$

$$\mathrm{Zig}^\bullet := \lim_{k\to\infty} (\mathrm{coZig}_k^\bullet \times \mathrm{doZig}_k^\bullet) \tag{15}$$

with dominant parts *doZag*$^\bullet$/*doZig*$^\bullet$ that carry the $\mathbf{u}/\mathbf{v}$-dependence[8]:

$$\mathrm{doZag}_k^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} := \sum_{1\le m_i\le k} e_1^{-m_1}\dots e_r^{-m_r}\ P(m_1-u_1)P(m_{1,2}-u_{1,2})\dots P(m_{1..r}-u_{1..r}) \tag{16}$$

$$\mathrm{doZig}_k^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{v_1\ ,\dots,\ v_r}} := \sum_{k\ge n_1>n_2>\dots n_r\ge 1} e_1^{-n_1}\dots e_r^{-n_r}\ P(n_1-v_1)P(n_2-v_2)\dots P(n_r-v_r) \tag{17}$$

and corrective parts *coZag*$^\bullet$/*coZig*$^\bullet$ that reduce to constants:

$$\mathrm{coZag}_k^{\binom{u_1\ ,\dots,\ u_r}{0\ ,\dots,\ 0}} := (-1)^r \sum_{1\le m_i\le k} P(m_1)P(m_{1,2})\dots P(m_{1\dots r}) \tag{18}$$

$$\mathrm{coZig}_k^{\binom{0\ ,\dots,\ 0}{v_1\ ,\dots,\ v_r}} := (-1)^r \sum_{k\ge n_1\ge n_2\ge\dots n_r\ge 1} \mu^{n_1,\dots,n_r}\ P(n_1)P(n_2)\dots P(n_r) \tag{19}$$

$$\mathrm{coZag}_k^{\binom{u_1\ ,\dots,\ u_r}{\epsilon_1\ ,\dots,\ \epsilon_r}} := 0 \quad \textit{if} \quad (\epsilon_1,\dots,\epsilon_r)\neq(0,\dots,0) \tag{20}$$

$$\mathrm{coZig}_k^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{v_1\ ,\dots,\ v_r}} := 0 \quad \textit{if} \quad (\epsilon_1,\dots,\epsilon_r)\neq(0,\dots,0) \tag{21}$$

with $P(t) := 1/t$ (here and throughout) and with $\mu^{n_1,n_2,..,n_r} := \frac{1}{r_1!\,r_2!..r_l!}$ if the non-increasing sequence $(n_1,\dots,n_r)$ attains r_1 times its highest value, r_2 times its second highest value, etc.

[7] still denoted by the same symbols

[8] with the usual abbreviations $m_{i,j} := m_i + m_j, m_{i,j,k} := m_i + m_j + m_k$ etc

Setting $Mini_k^\bullet := Zig_k^\bullet\|_{\mathbf{v}=0}$ we find :[9]

$$\mathrm{Mini}_k^{\binom{0\ ,\dots,\ 0}{v_1\ ,\dots,\ v_r}} := \sum_{\left[\substack{2\leq r_1\leq r_2\dots\leq r_l \\ r_1+r_2+\dots r_l=r}\right]}^{\left[\substack{1\leq l\leq r/2 \\ 1\leq n_i\leq k}\right]} (-1)^{(r-l)}\mu^{r_1,\dots,r_l}\,\frac{(P(n_1))^{r_1}}{r_1}\dots\frac{(P(n_l))^{r_l}}{r_l} \tag{22}$$

$$\mathrm{Mini}_k^{\binom{\epsilon_1\ ,\dots,\ \epsilon_r}{v_1\ ,\dots,\ v_r}} := 0 \quad \mathit{if} \quad (\epsilon_1,\dots,\epsilon_r)\neq(0,\dots,0) \tag{23}$$

We have an exact equivalence between old and new symmetries :

$$\{\mathrm{Wa}^\bullet\ \ \mathit{symmetral}\} \iff \{\mathrm{Zag}^\bullet\ \ \mathit{symmetral}\} \tag{24}$$

$$\{\mathrm{Ze}^\bullet\ \ \mathit{symmetrel}\} \iff \{\mathrm{Zig}^\bullet\ \ \mathit{symmetril}\} \tag{25}$$

and the old conversion rule for scalar multizetas [10] becomes :

$$\mathrm{Zig}^\bullet = \mathrm{Mini}^\bullet \times \mathrm{swap}(\mathrm{Zag})^\bullet \tag{26}$$

$$\left(\iff \mathrm{swap}(\mathrm{Zig}^\bullet) = \mathrm{Zag}^\bullet \times \mathrm{Mana}^\bullet\right) \tag{27}$$

with the involution *swap* defined as in (43) *infra* and with elementary moulds $Mana^\bullet/Mini^\bullet := \lim_{k\to\infty} Mana_k^\bullet/Mini_k^\bullet$ whose only non-zero components :

$$\mathrm{Mana}^{\binom{u_1\ ,\dots,\ u_r}{0\ ,\dots,\ 0}} \equiv \mathrm{Mini}^{\binom{0\ ,\dots,\ 0}{v_1\ ,\dots,\ v_r}} \equiv \mathrm{Mono}_r \tag{28}$$

due to (22) are expressible in terms of monozetas :

$$1+\sum_{r\geq 2}\ \mathrm{Mono}_r\ t^r := \exp\Big(\sum_{s\geq 2}(-1)^{s-1}\zeta(s)\,\frac{t^s}{s}\Big) \tag{29}$$

To these relations one must add the so-called *self-consistency* relations :

$$\mathrm{Zag}^{\binom{u_1\ ,\dots,\ u_r}{q\,\epsilon_1\ ,\dots,\ q\,\epsilon_r}} \equiv \sum_{q\epsilon_i^*=q\epsilon_i} \mathrm{Zag}^{\binom{q\,u_1\ ,\dots,\ q\,u_r}{\epsilon_1^*\ ,\dots,\ \epsilon_r^*}} \quad \forall q|p\,,\forall u_i\in\mathbb{C}\,,\forall\epsilon_i,\epsilon_i^*\in\frac{1}{p}\mathbb{Z}/\mathbb{Z} \tag{30}$$

which merely reflect trivial identities between unit roots of order p.

3. Perinomal objects : equations, functions, numbers.

---

[9]if we had no factor $\mu^{n_1,\dots,n_r}$ in (19), we would have $Zig_k^\bullet\|_{\mathbf{v}=0} = 0$ and therefore no $Mini_k^\bullet$ terms. But the mould $Zig_k^\bullet$ would fail to be *symmetril*, as required. Here lies the origin of the corrective terms in the conversion rule.

[10]some modified form of (10),(11).

Let $Sl_r(\mathbb{Z})$ denote the 'special group' (integer entries, unit determinant) with its natural action $M : f \mapsto \underline{M}\, f$ on functions of r variables:

$$\underline{M}\, f(\ldots, x_i, \ldots) := f(\ldots, \sum M_{i,j}\, x_j, \ldots) \qquad \forall M \in Sl_r(\mathbb{Z}) \tag{31}$$

A *perinomal system* is a system of equations of the form:

$$\{\underline{\overset{1}{M}}\ f = \overset{1}{\varphi}(x,f), \ldots, \underline{\overset{s}{M}}\ f = \overset{s}{\varphi}(x,f)\} \qquad (f\ \ \textit{unknown}\ ,\ \overset{i}{M} \in Sl_r(\mathbb{Z})) \tag{32}$$

with data $\overset{i}{\varphi}(x,f)$ usually linear or affine in f and 'elementary' in x.

A *perinomal function* is a solution of such a system.

A function is said to have finite *perinomal degrees* $d_{i,j}$ if $f(x_1, .., x_i{+}k\, x_j, .., x_r)$ is *polynomial* in k of degree $d_{i,j}$.

A *perinomal number* is a number attached to a perinomal function – by integration, summation, or taking its Taylor coefficients at the origin, etc.

Perinomal systems are a cross between *difference* and *q-difference* systems, but they also commend themselves to our attention for a number of more specific reasons:

1. *Finiteness properties:* Important spaces of perinomal functions admit a natural gradation by a global degree d, with a finite basis for any given d. The subject is of course closely tied up with the theory of finite linear representations of $Sl_r(\mathbb{Z})$.

2. *Closure properties:* Perinomal functions tend to be stable under partial differentiation, multiplication, various types of convolution, etc

3. *Self-duality under Fourier/Borel/Laplace:* This tends to be the case whenever any of these transforms applies. For instance, we have a correspondence between *homogeneous linear perinomal systems* of the form:

$$\{e^{x_i \partial_{x_j}}\ f(x) = L_{i,j}(f),\ \ \forall i,j\} \ \ \overset{\mathcal{F}/\mathcal{B}/\mathcal{L}}{\longleftrightarrow}\ \ \{e^{-\partial_{\xi_j}\,\xi_i}\ \hat{f}(\xi) = L_{i,j}(\hat{f}),\ \ \forall i,j\} \tag{33}$$

which is reminiscent of the self-duality properties for *homogeneous linear differential* equations with polynomial coefficients:

$$\Big(\sum a_{m,n}\, x^m\, \partial_x^n\big).f(x)\Big) \ \ \overset{\mathcal{B}/\mathcal{L}}{\longleftrightarrow}\ \ \Big(\sum(-1)^n\, a_{m,n}\, \partial_\xi^m\, \xi^n\big).\hat{f}(\xi)\Big) \tag{34}$$

4. *Duality between perinomal meromorphic functions and their residues:* This applies in particular to *eupolar* meromorphic functions f with multipoles

of maximal order, located 'at' multi-integers $\mathbf{n}$ and carrying multiresidues ρ. Thus, for a rather trivial type of eupolarity :[11]

$$f(x) \overset{ess}{:=} \sum_{n_i \in \mathbb{Z}} \frac{\rho(n_1, ..., n_r)}{(n_1 - x_1)...(n_r - x_r)} \ : \ \{f \ \textit{perinomal}\} \Leftrightarrow \{\rho \ \textit{perinomal}\} \qquad (35)$$

5. *Link with multizeta arithmetic :* Multizeta arithmetic makes extensive use of *perinomal numbers* $\rho^{\#}$ attached to discrete perinomal functions ρ via the series :

$$\rho^{\#}(s_1, \ldots, s_r) \ \overset{ess}{:=} \ \sum_{n_i \in \mathbb{N}^*} \rho(n_1, \ldots, n_r)\, n_1^{-s_1} \ldots n_r^{-s_r} \qquad (36)$$

or, equivalently, attached to meromorphic perinomal functions f under the taking of Taylor coefficients.

To put some flesh on these definitions, let us give two simple examples :

Example 1 : Fix $(k_1, ..., k_r)$ in $\mathbb{N}^r$. The perinomal function f :

$$f(x) \ \overset{ess}{:=} \ \sum_{n_i \,\in \mathbb{Z}^*} \frac{\rho(n_1, ..., n_r)}{(n_1 - x_1)...(n_r - x_r)} \qquad (37)$$

$$\begin{aligned} \textit{with} \qquad \rho(n_1, ..., n_r) \ &:= \ |n_1|^{k_1} \ldots |n_r|^{k_r} \ \ \textit{if} \ \ (n_1, \ldots, n_r) \ \ \textit{coprime} \\ &:= \ 0 \qquad\qquad\qquad \textit{otherwise} \end{aligned}$$

has Taylor coefficients $\rho^{\#}$ of the form[12] :

$$\begin{aligned} \rho^{\#}(s_1, \ldots, s_r) \ &= \ 2^r \, \frac{\zeta(s_1 - k_1) \ldots \zeta(s_r - k_r)}{\zeta(\sum s_i - \sum k_i)} \quad \textit{if all} \quad s_i \ \ \textit{are even} \\ &= \ 0 \qquad\qquad\qquad\qquad\qquad \textit{otherwise.} \end{aligned}$$

Despite being given as infinite sums, these $\rho^{\#}$ are clearly *rational* whenever all data k_i are even integers. This phenomenon shall be pivotal to the construction of the rational-coefficiented bimould $loma^\bullet/lomi^\bullet$.

Example 2 : Consider now the less simplistic perinomal function :

$$f(x) \ := \ \sum_{n_i \in \mathbb{Z}^\bullet} \frac{\rho(n_1, n_2)}{(n_1 - x_1)(n_2 - x_2)} \qquad (38)$$

$$\{\, \rho(n_1 + n_2, n_2) = \rho(n_1, n_2) + 1 \ \ , \ \ \rho(n_1, n_1 + n_2) = \rho(n_1, n_2) - 1 \,\} \qquad (39)$$

$$\rho(n_1, n_2) \ = \ \mathrm{sign}(n_1)\, \mathrm{sign}(n_2)\, (c_1 - c_2 + c_3 - c_4 + \ldots) \qquad (40)$$

$$\textit{if} \ \left|\frac{n_1}{n_2}\right| \ = \ [c_1, c_2, c_3, c_4, \ldots] = \textit{continued fraction} \qquad (41)$$

---

[11] in (35) and *throughout the sequel*, the warning *"ess(entially)"* shall mean : *up to the addition of simple (usually constant) corrective terms* that ensure absolute convergence, or *after suitable regroupings* that ensure semi-convergence.

[12] for homogeneity reasons, $\rho^{\#}(s_1, ..., s_r)$ always denotes the coefficient of $x_1^{s_1-1}...x_r^{s_r-1}$.

with residues ρ defined by the perinomal system (39). What about the arithmetical nature of the corresponding *perinomal numbers*?

$$\rho^{\#}(s_1,s_2) \overset{ess}{:=} \sum_{n_i \in \mathbb{N}^*} \rho(s_1,s_2)\, n_1^{-s_1}\, n_2^{-s_2} \tag{42}$$

Whether we start from (39) or (40), that nature is far from clear. But the moment we form the functions *fa* and *fi*:

$$\begin{array}{lcl}
 & f(x_1,x_2) & \\
fa(u_1,u_2) := & \swarrow \qquad \searrow & fi(v_1,v_2) := \\
+f(u_1,u_2) - f(u_2,u_1) & \| & +f(v_1,v_2) - f(v_2,v_1) \\
+f(u_1+u_2,u_1) - f(u_1,u_1+u_2) & \| & +f(v_1-v_2,v_2) - f(v_2,v_1-v_2) \\
-f(u_1+u_2,u_2) + f(u_2,u_1+u_2) & \| & -f(v_1,v_2-v_1) + f(v_2-v_1,v_1)
\end{array}$$

we see that the multiresidues simplify dramatically, and that the $\rho^{\#}(s_1,s_2)$ are in fact simple rational combinations of multizetas. Furthermore, *fa* links the $\rho^{\#}(s_1,s_2)$ to the $Wa^\bullet$-basis, while *fi* links them to the $Ze^\bullet$-basis. This example is but the tip of a mighty iceberg – namely the *direct-impartial* expression of the multizeta irreducibles.

4. The adequate structure: ARI/GARI and AXI/GAXI.

The starting point is the algebra BIMU. Its elements are bimoulds, ie moulds $A^\bullet = \{A^{w_1,\dots,w_r}\} = \{A^{\binom{u_1\ ,\dots,\ u_r}{v_1\ ,\dots,\ v_r}}\}$ with double-storeyed indices $w_i = \binom{u_i}{v_i}$. BIMU is endowed with the ordinary mould product $\times$, which is often noted *mu* to avoid confusion with a host of other operations on bimoulds. All these operations involve simultaneous additions of the u_i-variables and subtractions of the v_i-variable, which makes it expedient to systematically use the abbreviations (5) for sums and differences.

There is on BIMU a basic involution, the *swap*, which exchanges both sets of variables:

$$B^\bullet = \mathrm{swap}(A^\bullet) \iff B^{\binom{u_1\ ,\ \dots\ ,\ u_r}{v_1\ ,\ \dots\ ,\ v_r}} = A^{\binom{v_r\ ,\ v_{r-1:r}\ ,\ \dots\ ,\ v_{2:3}\ ,\ v_{1:2}}{u_{1,\dots,r}\ ,\ u_{1,\dots,r-1}\ ,\ \dots\ ,\ u_{1,2}\ ,\ u_1}} \tag{43}$$

and a basic shift operator[13], the *push*, which acts as follows:

$$C^\bullet = \mathrm{push}(A^\bullet) \iff C^{\binom{u_1\ ,\ \dots\ ,\ u_r}{v_1\ ,\ \dots\ ,\ v_r}} = A^{\binom{-u_{1,\dots,r}\ ,\ u_1\ ,\ u_2\ ,\ \dots\ ,\ u_{r-1}}{v_r\ ,\ v_{1:r}\ ,\ v_{2:r}\ ,\ \dots\ ,\ v_{r-1:r}}} \tag{44}$$

All further operations involve *'sequence flexions'* $\mathbf{a}.\mathbf{b} \mapsto \mathbf{a}\rceil\lfloor\mathbf{b}$ or $\mathbf{a}\rfloor\lceil\mathbf{b}$. Thus, relative to the factorisation $\mathbf{w} = ...\mathbf{a}.\mathbf{b}... = ...\binom{u_3\ ,\ u_4\ ,\ u_5}{v_3\ ,\ v_4\ ,\ v_5}\binom{u_6\ ,\ u_7\ ,\ u_8\ ,\ u_9}{v_6\ ,\ v_7\ ,\ v_8\ ,\ v_9}...$ the

[13]of order $r+1$ when restricted to components of length r.

flexion markers $\rceil$, $\lceil$, $\rfloor$, $\lfloor$ should be interpreted as follows:

$$\mathbf{a}\rceil := \left(\begin{smallmatrix} u_3 \,,\, u_4 \,,\, u_{5,6,7,8,9} \\ v_3 \,,\, v_4 \,,\, v_5 \end{smallmatrix}\right) \qquad \lfloor\mathbf{b} := \left(\begin{smallmatrix} u_6 \,,\, u_7 \,,\, u_8 \,,\, u_9 \\ v_{6:5} \,,\, v_{7:5} \,,\, v_{8:5} \,,\, v_{9:5} \end{smallmatrix}\right) \tag{45}$$

$$\mathbf{a}\rfloor := \left(\begin{smallmatrix} u_3 \,,\, u_4 \,,\, u_5 \\ v_{3:6} \,,\, v_{4:6} \,,\, v_{5:6} \end{smallmatrix}\right) \qquad \lceil\mathbf{b} := \left(\begin{smallmatrix} u_{3,4,5,6} \,,\, u_7 \,,\, u_8 \,,\, u_9 \\ v_6 \,,\, v_7 \,,\, v_8 \,,\, v_9 \end{smallmatrix}\right) \tag{46}$$

The binary operation *ari* defined by the flexions [14] :

$$C^\bullet = \mathrm{ari}(A^\bullet, B^\bullet) \iff C^{\mathbf{w}} = \sum_{\mathbf{w}=\mathbf{b}.\mathbf{c}} (A^{\mathbf{b}}B^{\mathbf{c}} - B^{\mathbf{b}}A^{\mathbf{c}})$$

$$+ \sum_{\mathbf{w}=\mathbf{b}.\mathbf{c}.\mathbf{d}} (A^{\lfloor\mathbf{c}}B^{\mathbf{b}\rceil\mathbf{d}} - B^{\lfloor\mathbf{c}}A^{\mathbf{b}\rceil\mathbf{d}}) + \sum_{\mathbf{w}=\mathbf{a}.\mathbf{b}.\mathbf{c}} (A^{\mathbf{a}\lceil\mathbf{c}}B^{\mathbf{b}\rfloor} - B^{\mathbf{a}\lceil\mathbf{c}}A^{\mathbf{b}\rfloor}) \tag{47}$$

turns BIMU_*[15] into a Lie algebra known as ARI.
Likewise, the binary operation *gari* defined by the flexions [16]:

$$C^\bullet = \mathrm{gari}(A^\bullet, B^\bullet) \iff C^{\mathbf{w}} =$$

$$\sum_{\mathbf{w}=\mathbf{a^1}.\mathbf{b^1}.\mathbf{c^1}\ldots\mathbf{a^s}.\mathbf{b^s}.\mathbf{c^s}.\mathbf{a^{s+1}}} A^{\lceil\mathbf{b^1}\rceil\ldots\lceil\mathbf{b^s}\rceil} B^{\mathbf{a^1}\rfloor}\ldots B^{\mathbf{a^s}\rfloor} B^{\mathbf{a^{s+1}}\rfloor} B_\star^{\lfloor\mathbf{c^1}}\ldots B_\star^{\lfloor\mathbf{c^s}} \tag{48}$$

turns BIMU^*[17] into a Lie group GARI, with ARI as its Lie algebra.

Central bimoulds, by definition, *gari*-commute with, and *ari*-annihilate, everyone else. They are of the form:

$$C^{w_1,\ldots,w_r} := c(r) \in \mathbb{C} \quad \textit{if} \quad (v_1,\ldots,v_r) = (0,\ldots,0) \quad (\forall (u_1,\ldots,u_r)) \tag{49}$$

$$:= 0 \quad \textit{if} \quad (v_1,\ldots,v_r) \neq (0,\ldots,0) \quad (\forall (u_1,\ldots,u_r)) \tag{50}$$

The following are important subalgebras/subgroups of GARI/ARI:

$$\begin{array}{lllll}
\mathrm{ARI_{push}} & := \{A^\bullet \;;\; A^\bullet \ \text{push-}\textit{invariant} & & & \} \\
\mathrm{ARI_{al}} & := \{A^\bullet \;;\; A^\bullet \ \textit{alternal} & & & \} \\
\mathrm{ARI}_{\underline{\mathrm{al}}/\underline{\mathrm{al}}} & := \{A^\bullet \;;\; A^\bullet \ \textit{alternal} & , \ \mathrm{swap}(A^\bullet) & \textit{alternal} & , \textit{IP} \} \\
\mathrm{ARI}_{\underline{\mathrm{al}}/\underline{\mathrm{il}}} & := \{A^\bullet \;;\; A^\bullet \ \textit{alternal} & , \ \mathrm{swap}(A^\bullet) & \textit{alternil} & , \textit{IP} \} \\
\\
\mathrm{GARI_{spush}} & := \{A^\bullet \;;\; A^\bullet \ \text{spush-}\textit{invariant} & & & \} \\
\mathrm{GARI_{as}} & := \{A^\bullet \;;\; A^\bullet \ \textit{symmetral} & & & \} \\
\mathrm{GARI}_{\underline{\mathrm{as}}/\underline{\mathrm{as}}} & := \{A^\bullet \;;\; A^\bullet \ \textit{symmetral} & , \ \mathrm{swap}(A^\bullet) & \textit{symmetral} & , \textit{IP} \} \\
\mathrm{GARI}_{\underline{\mathrm{as}}/\underline{\mathrm{is}}} & := \{A^\bullet \;;\; A^\bullet \ \textit{symmetral} & , \ \mathrm{swap}(A^\bullet) & \textit{symmetril} & , \textit{IP} \}
\end{array}$$

---

[14] with $\mathbf{b} \neq \emptyset, \mathbf{c} \neq \emptyset$ in all three sums; but $\mathbf{a}$ and $\mathbf{d}$ may be empty.

[15] i.e. the set of all $A^\bullet$ with vanishing length-0 component ($A^\emptyset = 0$).

[16] with $s \geq 2$ and all factor sequences $\mathbf{b^i} \neq \emptyset$ and $\mathbf{c^i}.\mathbf{a^{i+1}} \neq \emptyset$. The factors $\mathbf{c^i}$ et $\mathbf{a^{i+1}}$ may turn empty but *separately so* and the extreme factors $\mathbf{a^1}$,$\mathbf{c^s}$,$\mathbf{a^{s+1}}$ may also turn empty, *separately or jointly.* As for $B_\star^\bullet$, it denotes the inverse *invmu*$(B^\bullet)$ of $B^\bullet$ with respect to ordinary mould multiplication *mu* (same as $\times$).

[17] i.e. the set of all $A^\bullet$ with unit length-0 component ($A^\emptyset = 1$).

Initial parity clause (IP) : in the above definitions, we demand that bimoulds in $ARI_{\underline{al}/\underline{al}}$ or $ARI_{\underline{al}/\underline{il}}$ (resp. $GARI_{\underline{as}/\underline{as}}$ or $GARI_{\underline{as}/\underline{is}}$) should have as their length-1 component an *even* function of w_1, but we allow for the addition of (resp. multiplication by) a *central bimould* $C^\bullet$ before taking the *swap*.

We have the non-trivial inclusions and isomorphisms :

$$\begin{array}{ccccc}
\mathrm{ARI}_{\mathrm{push}} & \supset & \mathrm{ARI}_{\underline{\mathrm{al}}/\underline{\mathrm{al}}} & \overset{\text{algebra isom.}}{\longleftrightarrow} & \mathrm{ARI}_{\underline{\mathrm{al}}/\underline{\mathrm{il}}} \\
\downarrow_{expari} & & \downarrow_{expari} & & \downarrow_{expari} \\
\mathrm{GARI}_{\mathrm{spush}} & \supset & \mathrm{GARI}_{\underline{\mathrm{al}}/\underline{\mathrm{as}}} & \overset{\text{group isom.}}{\longleftrightarrow} & \mathrm{GARI}_{\underline{\mathrm{as}}/\underline{\mathrm{is}}}
\end{array}$$

and a non-trivial action *arit/garit* of ARI/GARI on the *mu*-algebra BIMU.

Though ARI/GARI traces its origins to *singularity theory*, its double series of variables u_i and v_i as well as its property of *accommodating* and *reproducing* double symmetries, makes it an ideal tool for investigating *arithmetical dimorphy*. ARI/GARI is actually part of a larger umbrella structure, AXI/GAXI, which regroups all *flexion* derivations/automorphisms[18] of BIMU.

5. Multizeta arithmetic : the main steps.

R1 . Formalisation : from numbers to symbols.

Formalising the scalar multizetas means replacing the familiar systems of numbers $Wa^\bullet/Ze^\bullet$ by symbols $wa^\bullet/ze^\bullet$ subject to the same quadratic relations, conversion rule, and self-consistency constraints. In terms of generating series, it means replacing $Zag^\bullet/Zig^\bullet$ by the most general pair $zag^\bullet/zig^\bullet$ of symmetral/symmetril bimoulds connected under the *swap* :[19]

$$\mathrm{swap}(\mathrm{zig})^\bullet = \mathrm{gari}(\mathrm{zag}^\bullet, \mathrm{mana}^\bullet) \qquad (\textit{with}\ \mathrm{mana}^\bullet\ \ \textit{central}) \tag{51}$$

and subject to the old self-consistency constraints (30) but with components that are arbitrary power series instead of well-defined meromorphic functions.

R2 . Free generation.

It says that the $\mathbb{Q}$-rings of (scalar) formal multizetas are *polynomial rings* in

[18]i.e. of all those derivations or automorphisms of the *mu*-algebra BIMU that can be expressed in terms of the *flexions* (45),(46). Elements of AXI/GAXI are determined not by single bimoulds $A^\bullet$, but by *pairs* $(A^\bullet_L, A^\bullet_R)$ consisting of a left and a right bimould.

[19]due to *mana*$^\bullet$ being a *central bimould*, we have in fact :
$\mathrm{gari}(\mathrm{mana}^\bullet, \mathrm{zag}^\bullet) = \mathrm{gari}(\mathrm{zag}^\bullet, \mathrm{mana}^\bullet) = \mathrm{mu}(\mathrm{zag}^\bullet, \mathrm{mana}^\bullet)\Big(\textit{but} \neq \mathrm{mu}(\mathrm{mana}^\bullet, \mathrm{zag}^\bullet)\Big)$

countably many indeterminates – the so-called irreducibles.

R3. From the atomic to the subatomic level.
It means re-interpreting the apparently *unbreakable* irreducibles as elements of a Lie algebra $\mathrm{ARI}^{\underline{\mathrm{reg}},\underline{\mathrm{disc}}}_{\underline{\mathrm{al}},\underline{\mathrm{il}}}$ [20], which opens the way to a finer analysis.

R4. Free subgeneration.
It says that the finer and truly ultimate building blocks[21], or *subgenerators*, are either *free* (for small values of the root order p) or "very nearly free" (for larger p). The various dimensions (additive/multiplicative) that can be attached to subspaces/subrings of multizetas and that depend on the various filtrations/gradations (by s, r, d, p... etc) mostly follow from that.

R5. Decomposition into irreducibles: constructive.
It should be fully constructive, i.e. amenable to effective computation.

R6. Decomposition into irreducibles: explicit.
In a context such as this, if we are to maintain a meaningful distinction between *constructive* and *explicit*, the latter can mean only one thing, namely: given by direct formulas which, though inevitably complex, are nonetheless perspicuous enough [22] and reasonably compact [23]. Above all, *explicit* means that we are not required to solve larger and larger linear systems as the natural filtration/gradation parameters (s, r, d, p etc) increase.

R7. Decomposition into irreducibles: canonical.
Though redolent of subjectivity, the notion of *canonicity* matters immensely. Here, we are fortunate in being able to construct, among all possible, more or less natural systems of irreducibles, one that is indisputably canonical.

R8. Direct and 'impartial' expression of the irreducibles.
It goes way beyond the mere reversing of the canonical-explicit decomposition of multizetas into irreducibles; rather, it asks for a direct and 'impartial' (i.e. 'equidistant' from the two competing bases $wa^\bullet$ and $ze^\bullet$) expression of the irreducibles. This is where perinomal algebra comes in.

R9. Materialisation: from symbols to numbers.
That would mean: showing that the $\mathbb{Q}$-ring of 'actual' or 'genuine' multizetas is actually isomorphic to its formalisation. This is the one great challenge

[20] the upper index *reg* means *regular in* $\mathbf{u}$ *at the origin*, i.e. without Laurent terms etc; the upper index *disc* for *discrete* means *with* $\mathbf{v}$*-variables in* $\mathbb{Q}/\mathbb{Z}$; the underlining of both upper indices means *subject to the self-consistency constraints (30).*

[21] in ARI.

[22] to make the main features, symmetries etc of the objects at hand easily detectable.

[23] both as mathematical text or as computation programmes.

that still lies ahead. It apparently defies extant mathematical tools, but the advent of *direct*[24] *numerical derivations*[25] might change that.

6. The general scheme.

To simplify, we set out the general procedure for *ordinary multizetas*[26].
The $\mathbb{Q}$-ring of multizetas splits into two/three factors rings:

$$\begin{aligned}
\mathbb{Z}\mathrm{eta} &= \mathbb{Z}\mathrm{eta}_{\mathrm{I+II}} \otimes \mathbb{Z}\mathrm{eta}_{\mathrm{III}} && (52)\\
\mathbb{Z}\mathrm{eta} &= \mathbb{Z}\mathrm{eta}_{\mathrm{I}} \otimes \mathbb{Z}\mathrm{eta}_{\mathrm{II+III}} && (53)\\
\mathbb{Z}\mathrm{eta} &= \mathbb{Z}\mathrm{eta}_{\mathrm{I}} \otimes \mathbb{Z}\mathrm{eta}_{\mathrm{II}} \otimes \mathbb{Z}\mathrm{eta}_{\mathrm{III}} \qquad \textit{with} \qquad \mathbb{Z}\mathrm{eta}_{\mathrm{I}} = \mathbb{Q}[\pi^2] && (54)
\end{aligned}$$

The factor-ring $\mathbb{Z}\mathrm{eta}_{\mathrm{I}}$ is generated by π^2.
The factor-ring $\mathbb{Z}\mathrm{eta}_{\mathrm{II}}$ contains all irreducibles of *even* weight *and* 'length'.
The factor-ring $\mathbb{Z}\mathrm{eta}_{\mathrm{III}}$ contains all irreducibles of *odd* weight *and* 'length'.

This splitting of the ring $\mathbb{Z}\mathrm{eta}$ stems from a corresponding factorisation of the generating functions:

$$\begin{aligned}
\mathrm{zag}^\bullet &= \mathrm{gari}(\mathrm{zag}^\bullet_{\mathrm{I+II}}, \mathrm{zag}^\bullet_{\mathrm{III}}) && (\mathrm{zag}^\bullet_{\mathrm{III}} \in \mathrm{GARI}^{\mathrm{o.l.}}_{\underline{\mathrm{as}}/\underline{\mathrm{is}}}) && (55)\\
\mathrm{zag}^\bullet &= \mathrm{gari}(\mathrm{zag}^\bullet_{\mathrm{I}}, \mathrm{zag}^\bullet_{\mathrm{II+III}}) && (\mathrm{zag}^\bullet_{\mathrm{II+III}} \in \mathrm{GARI}_{\underline{\mathrm{as}}/\underline{\mathrm{is}}}) && (56)\\
\mathrm{zag}^\bullet &= \mathrm{gari}(\mathrm{zag}^\bullet_{\mathrm{I}}, \mathrm{zag}^\bullet_{\mathrm{II}}, \mathrm{zag}^\bullet_{\mathrm{III}}) && (\mathrm{zag}^\bullet_{\mathrm{II}} \in \mathrm{GARI}^{\mathrm{e.l.}}_{\underline{\mathrm{as}}/\underline{\mathrm{is}}}) && (57)
\end{aligned}$$

The factors $zag^\bullet_{\mathrm{I}}$ and $zag^\bullet_{\mathrm{II}}$ carry only terms of *even* weight. As a consequence, their components of *even/odd* length are *even/odd* functions of $\mathbf{u}$.

The factor $zag^\bullet_{\mathrm{III}}$ carries only terms of *odd* weight. As a consequence, its components of *even/odd* length are *odd/even* functions of $\mathbf{u}$.

The factor $zag^\bullet_{\mathrm{I}}$ is symmetral/il[27] but doesn't verify the initial parity condition *IP* (see §4). Therefore, its *gari*-logarithm is *not* alternal/il[28]. The factors $zag^\bullet_{\mathrm{II}}$ and $zag^\bullet_{\mathrm{III}}$, on the other hand, do verify that condition and so belong to the proper symmetral/il group $\mathrm{GARI}_{\underline{\mathrm{as}}/\underline{\mathrm{is}}}$. Consequently, their *gari*-logarithms $lozag^\bullet_{\mathrm{II}}$ and $lozag^\bullet_{\mathrm{III}}$ belong to the proper alternal/il algebra $\mathrm{ARI}_{\underline{\mathrm{al}}/\underline{\mathrm{il}}}$ and can be *further analysed* therein. Actually, it turns out that $lozag^\bullet_{\mathrm{II}}$ and $lozag^\bullet_{\mathrm{III}}$ can be uniquely generated by the $\mathbf{u}$-homogeneous parts of a crucial

[24] all the emphasis here is on *direct*; other derivations are useless chaff.
[25] that annihilate $\mathbb{Q}$ but act non-trivially on $\mathbb{Q}$-rings of transcendental numbers
[26] i.e. multizetas without unit roots.
[27] i.e. symmetral and with a symmetral *swappee*.
[28] i.e. more exactly, it is alternal all right, but its *swappee* is not.

bimould, $loma^\bullet \in \mathrm{ARI}_{\underline{al}/\underline{il}}$, with well-defined coefficients in front of the multi-brackets. These coefficients are none other than the *formal irreducibles.*

Isolating the factor $zag^\bullet_{\mathrm{III}}$ from the other two is quite easy. Indeed:[29]

$$\mathrm{gari}(\mathrm{zag}^\bullet_{\mathrm{III}}, \mathrm{zag}^\bullet_{\mathrm{III}}) = \mathrm{gari}(\mathrm{nepar}(\mathrm{invgari}(\mathrm{zag}^\bullet)), \mathrm{zag}^\bullet) \tag{58}$$

But separating $zag^\bullet_{\mathrm{I}}$ from $zag^\bullet_{\mathrm{II}}$ is a far more arduous undertaking, especially if we insist, as we do, on getting a 'canonical' separation. This requires an elaborate construction, with the introduction of three special bimoulds, leading to a subfactorisation of $zag^\bullet_{\mathrm{I}}$:

$$\begin{aligned} \mathrm{zag}^\bullet_{\mathrm{I}} &= \mathrm{gari}^\bullet(\mathrm{tal}^\bullet, \mathrm{invgari}(\mathrm{pal}^\bullet), \mathrm{expari}(\mathrm{roma}^\bullet)) &(59)\\ &= \mathrm{gari}^\bullet(\mathrm{tal}^\bullet, \mathrm{expari}(\mathrm{viroma}^\bullet), \mathrm{invgari}(\mathrm{pal}^\bullet)) &(60)\end{aligned}$$

– with a 'eupolar' factor $pal^\bullet \in \mathrm{GARI}_{\mathrm{as/as}}$ but $\notin \mathrm{GARI}_{\underline{as}/\underline{as}}$ (see §7);
– with a 'eutrigonometric' factor $tal^\bullet \in \mathrm{GARI}_{\mathrm{as/as}}$ but $\notin \mathrm{GARI}_{\underline{as}/\underline{as}}$ (see §7);
– with a 'corrective factor' $roma^\bullet \in \mathrm{ARI}_{\underline{al}/\underline{il}}$ or its variant $viroma^\bullet \in \mathrm{ARI}_{\underline{al}/\underline{al}}$.

In sum, everything begins with the construction of two special bimoulds[30] *:*
– $pal^\bullet/pil^\bullet$ and $tal^\bullet/til^\bullet$
– both *symmetral/symmetral* (as bimoulds)
– both relatively elementary (as functions of **u**)

but the really sensitive part consists in constructing and understanding two further, even more crucial, and highly non-elementary, bimoulds:
– $loma^\bullet/lomi^\bullet$ and $roma^\bullet/romi^\bullet$
– both *alternal/alternil* (as bimoulds)
– both with rational coefficients[31] (as formal series in **u**)
– both strongly transcendental (as meromorphic functions of **u**) and actually of *perinomal* and *eupolar* type.

7. The bisymmetral bimoulds $pal^\bullet/pil^\bullet$ and $tal^\bullet/til^\bullet$.

The two semi-elementary factors in the decomposition of $Zag^\bullet_I$ are built from

[29] *invgari* denotes the *gari*-inversion; and *nepar* multiplies each length-r component by $(-1)^r$ while changing the signs of all u_i's and v_i's.

[30] as usual, the *swappee* of a bimould bears the same name, with i instead of a.

[31] in the case of $roma^\bullet/romi^\bullet$, the coefficients become rational after an elementary rescaling $\pi^2 \mapsto 1$.

the following simple ingredients:

$$
\begin{aligned}
P(t) &:= 1/t && (61)\\
Q(t) &:= \pi/\tan(\pi t) && (62)\\
Qa^{w_0} = Qa^{\binom{u_0}{v_0}} &:= \frac{1}{p}\sum_{1\le k\le p} e^{2\pi i k v_0}\ Q(u_0+\frac{k}{p}) \quad \mathit{if} \quad v_0\in\frac{1}{p}\mathbb{Z}/\mathbb{Z} && (63)\\
Ea^{w_0} = Ea^{\binom{u_0}{v_0}} &:= \pi \ \mathit{if}\ v_0 = 0_{\mathbb{Q}/\mathbb{Z}} \quad (\mathit{resp.}\ 0\ \mathit{if}\ v_0\neq 0_{\mathbb{Q}/\mathbb{Z}}) && (64)
\end{aligned}
$$

The simpler bimould, $pal^\bullet$, depends only on the $\mathbf{u}$-variables. It is called *eupolar* because of the very specific form of its poles. Its length-r component is a homogeneous polynomial of total degree r in simple P-expressions :

$$\mathrm{pal}^{w_1,\dots,w_r}\in\mathbb{Q}\,[P(u_1),P(u_2),\dots,P(u_r)\,,\,P(u_{1,2}),P(u_{1,2,3}),\dots,P(u_{1,..,r})]$$

The second bimould, $tal^\bullet$, is called *eutrigonometric.* For vanishing $\mathbf{v}$-variables, it closely resembles the eupolar bimould since its length-r component is again a homogeneous polynomial[32] of degree r in Q-expressions and π :

$$\mathrm{tal}^{w_1,\dots,w_r}\ \ \in\ \ \mathbb{Q}\,[\pi\,,\,Q(u_1),Q(u_2),\dots,Q(u_r)\,,\,Q(u_{1,2}),Q(u_{1,2,3}),\dots,Q(u_{1,..,r})]$$

For general $\mathbf{v}$-variables, $tal^{w_1,\dots,w_r}$ is still a homogeneous polynomial of total degree r, but in the variables $Ea^{w_i^*}$ and $Qa^{w_i^*}$, with double indices $w_i^*=\binom{u_i^*}{v_i^*}$ subject to $\sum u_i^*v_i^* = \sum u_iv_i$ and of the form $\binom{u_j}{v_j}$ or $\binom{u_j}{v_{j:k}}$ or $\binom{u_{1,\dots,j}}{v_k}$.

Both bimoulds verify the *self-consistency constraints* and both are *bissymmetral*[33] with, in the case of $tal^\bullet$, an elementary connection factor $mana_I^\bullet$ much like the 'global' $mana^\bullet$ in (26) but carrying only even powers of π :

$$
\begin{aligned}
\mathrm{swap}(\mathrm{pil}^\bullet) &= \mathrm{pal}^\bullet = \mathit{symmetral} && (65)\\
\mathrm{swap}(\mathrm{til}^\bullet) &= \mathrm{gari}(\mathrm{tal}^\bullet,\mathrm{mana}_{\mathrm{I}}^\bullet) = \mathrm{gari}(\mathrm{mana}_{\mathrm{I}}^\bullet,\mathrm{tal}^\bullet) = \mathit{symmetral} && (66)
\end{aligned}
$$

These properties[34] completely determine $pal^\bullet/pil^\bullet$ and $tal^\bullet/til^\bullet$. These bimoulds admit fully explicit expressions and enjoy an incredible number of properties. As far as multizeta algebra is concerned, they intervene at three critical junctures :

– $tal^\bullet$ describes *'almost all'* poles of $Zag_I^\bullet$. It is therefore the mainstay of the *canonical-rational Drinfel'd associator.*

– $pal^\bullet$ provides us with a canonical-explicit isomorphism between the two

[32] but with only *even* powers of π in it.

[33] however, $pal^\bullet, tal^\bullet \notin \mathrm{GARI}_{\underline{\mathrm{as}},\underline{\mathrm{as}}}$ since pal^{w_1} and tal^{w_1} are *odd* functions of w_1.

[34] together with the initial conditions $pal^{w_1} = -\frac{1}{2\,u_1}$ and $tal^{w_1} = -\frac{1}{2}Qa^{w_1}$.

'doubly symmetric' or 'dimorphic' algebras $\mathrm{ARI}_{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$ and $\mathrm{ARI}_{\underline{\mathrm{al}}/\underline{\mathrm{il}}}$.
– $pal^\bullet$ enables us to construct the so-called "singulators" and these in turn make it possible to remove all the unwanted singularities at $\mathbf{u} = \mathbf{0}$ which appear during the inductive construction of the moulds $loma^\bullet$ and $roma^\bullet$.

8. From the atomic to subatomic level. Free generation and subgeneration.

$\mathrm{ARI}_{\underline{\mathrm{al}}/\underline{\mathrm{al}}}$ is easily shown to be closed under the *ari* bracket. Under the adjoint action of $pal^\bullet$ in ARI, this closure property carries over to $\mathrm{ARI}_{\underline{\mathrm{al}}/\underline{\mathrm{il}}}$:

$$\mathrm{adari(pal^\bullet)} \ : \quad \mathrm{ARI}_{\underline{\mathrm{al}}/\underline{\mathrm{al}}} \overset{\textit{algebra isom.}}{\longrightarrow} \mathrm{ARI}_{\underline{\mathrm{al}}/\underline{\mathrm{il}}} \tag{67}$$

Further, using the factorisation (or stability) property:

$$\mathrm{GARI}_{\mathrm{as/is}} \ = \ \mathrm{GARI}_{\mathrm{as/is}} \,.\, \mathrm{GARI}_{\underline{\mathrm{as}}/\underline{\mathrm{is}}} \tag{68}$$

$$\mathrm{zag}^\bullet \ = \ \mathrm{gari}(\mathrm{zag}_0^\bullet \,,\, \mathrm{expari}(\mathrm{A}^\bullet)) \quad \textit{with} \quad \mathrm{A}^\bullet \in \mathrm{ARI}^{\mathrm{ent/disc}}_{\underline{\mathrm{al}}/\underline{\mathrm{il}}} \tag{69}$$

we get the general $zag^\bullet$ verifying all the constraints of *R1* in §5 by postcomposing any particular solution[35] $zag_0^\bullet$ by the *ari*-exponential of the generic element $A^\bullet$ of $\mathrm{ARI}^{\mathrm{ent/disc}}_{\underline{\mathrm{al}}/\underline{\mathrm{il}}}$. Expanding $A^\bullet := \sum c_J\, A_J^\bullet$ along any rational linear basis $\{A_J\}$ of $\mathrm{ARI}^{\mathrm{ent/disc}}_{\underline{\mathrm{al}}/\underline{\mathrm{il}}}$, we get one degree of freedom per basis element. The corresponding scalar coefficients c_J are none other than the sought-after *irreducibles.* Thus, the $\mathbb{Q}$-ring $\mathbb{Z}\mathrm{eta}^{form}$ is seen to be isomorphic to the polynomial ring $\mathbb{Q}[\pi^2] \otimes \mathbb{Q}[c_{J_1}, c_{J_2}, \dots]$: this is the *free generation theorem.*

Now, as *scalars*, the irreducibles a_J are *'atoms'*, i.e. incapable of further analysis. But they correspond one-to-one to dual objects $A_J^\bullet$ which, being elements of the Lie algebra $\mathrm{ARI}^{\mathrm{ent/disc}}_{\underline{\mathrm{al}}/\underline{\mathrm{il}}}$, may be broken down as Lie brackets of still 'simpler' and much less numerous *'subatoms'* $B_J^\bullet$. Moreover, for small values of the root-order p, in particular for $p = 1, 2, 3$, the subatoms in question *freely generate* $\mathrm{ARI}^{\mathrm{ent/disc}}_{\underline{\mathrm{al}}/\underline{\mathrm{il}}}$ as an algebra. This is the so-called *free subgeneration theorem.* And even for general values of p, the relations between the subatoms $B_J^\bullet$ are *few* in number and *easy to describe*, so that we may speak of a *nearly free subgeneration.* In all instances, knowledge of the subatoms implies knowledge of all the relevant *dimensions*: additive, multiplicative, etc (see §5).

[35] For example the ready-made $Zag^\bullet$ constructed from the genuine multizetas, or the factor $zag_I^\bullet$ in (56).

We already saw how the change *scalars* $\rightarrow$ *generating series/functions* entails a far-going *restoration of symmetry* between the two fundamental encodings, and a welcome 'compactification' of the *conversion rule* and *quadratic constraints*. We now register another, more decisive gain : the possibility of moving from the *atomic* to the *subatomic level*, leading to a complete understanding of the irreducibles. As for the further shift, from *generating series* to *generating functions*, the specific reward it brings is the notion of *perinomal function*, with the attendant direct-impartial description of the irreducibles.

9. Constructing $\boldsymbol{loma}^\bullet/\boldsymbol{lomi}^\bullet$ and $\boldsymbol{roma}^\bullet/\boldsymbol{romi}^\bullet$: the easy steps.

This leaves us with two main tasks :

– constructing in $\mathrm{ARI}^{\mathrm{ent}/\underline{\mathrm{disc}}}_{\underline{\mathrm{al}}/\underline{\mathrm{il}}}$ a mould $loma^\bullet := \sum_s loma^\bullet_s$, regular in $\mathbf{u}$ at the origin, and with weight-homogeneous summands [36] $loma^\bullet_s$ that shall yield the afore-mentioned sub-atoms B_J and support the description of all irreducibles but π^2.

– constructing in $\mathrm{ARI}^{\mathrm{sing}/\underline{\mathrm{disc}}}_{\underline{\mathrm{al}}/\underline{\mathrm{il}}}$ a mould $roma^\bullet$, singular in $\mathbf{u}$ at the origin, but with singularities that exactly compensate those of the factors $pal^\bullet$ and $tal^\bullet$ in (59),(60) so as to produce a *regular* factor $zag^\bullet_I$, leading to an automatic *separation* of the 'rogue' irreducible π^2 from all others.

Both constructions rely on an induction on the component length r [37] but there is a sharp dichotomy between *easy steps* :

– for $loma^\bullet$: going from length r odd to length $r+1$ even

– for $roma^\bullet$: going from length r even to length $r+1$ odd

which are automatic under the ARI/GARI machinery, and *tricky steps* (the alternate ones !) which involve a complex compensation mechanism.

[36] the summand $loma^\bullet_s$ of weight s has s non-vanishing components, and its component of length r is a polynomial $loma_s^{w_1,\dots,w_r} \in \mathbb{Q}[u_1,\dots,u_r]$ of total degree $d := s-r$.

[37] in the sequel, $M^\bullet\|_r$ denotes $M^\bullet$ defined *modulo* all components of length $> r$.

Here is a table setting forth the *easy* induction steps for $loma^\bullet$:

$$\begin{array}{lll}
\mathrm{loma}^\bullet\|_r & \in \mathrm{ARI}_{\underline{al}/\underline{il}} & \textit{and regular at } 0. \\
\quad\downarrow \mathrm{adari}(\mathrm{pal}^\bullet)^{-1} & & \\
\mathrm{viloma}^\bullet\|_r & \in \mathrm{ARI}_{\underline{al}/\underline{al}} & \textit{and singular at } 0. \\
\quad\downarrow \textit{trivial extension} & & \\
\mathrm{viloma}^\bullet\|_{r+1} & \in \mathrm{ARI}_{\underline{al}/\underline{al}} & \textit{and singular at } 0. \\
\quad\downarrow \mathrm{adari}(\mathrm{pal}^\bullet) & & \\
\mathrm{loma}^\bullet\|_{r+1} & \in \mathrm{ARI}_{\underline{al}/\underline{il}} & \textit{and regular at } 0.
\end{array}$$

The trivial extension $viloma^\bullet\|_r \mapsto viloma^\bullet\|_{r+1}$ consists of course in setting the $r+1^{th}$ component of $viloma^\bullet\|_{r+1}$ equal to 0.

The same basic procedure holds for $roma^\bullet$, but with *even/odd* exchanged and with the *regular/singular* dichotomy applying not to $roma^\bullet$ itself, but to the product $zag_I^\bullet$.

10. Singulators and the removal of singularities : the tricky steps.

If we apply the same procedure for the *'tricky steps'*, the machinery will work all right and still produce an extension $loma\|^\bullet_{r+1}$ with the proper symmetries, but with unwanted singularities at the origin. To removed these, some finely honed terms have to be added. The key notion here is that of *singulator*. These are operators $slank_{r_0}$ and $slang_{r_0}$ that turn elementary bimoulds (regular in $\mathbf{u}$ at 0, and with a single non-zero component, for $r = 1$) into bimoulds with the proper symmetries (either $\underline{al}/\underline{al}$ or $\underline{al}/\underline{il}$) and the proper singularities ('eupolar') at the component of length r_0.

Here is how they are defined :

$$\mathrm{slank}^\bullet_{\mathrm{r_0}}\,\mathrm{H}^\bullet \;:=\; \mathrm{neginvar}\,.\,\mathrm{leng}_{\mathrm{r_0}}\,.\mathrm{adari}(\mathrm{pal}^\bullet)^{-1}\,.\,\mathrm{mu}(\mathrm{mupaj}^\bullet,\,\mathrm{leng}_1\,.\mathrm{H}^\bullet,\mathrm{paj}^\bullet) \tag{70}$$

$$\equiv\; \mathrm{pushinvar}_{\mathrm{r}}\,.\,\mathrm{leng}_{\mathrm{r}}\,.\mathrm{mu}(\mathrm{anti}\,.\,\mathrm{pal}^\bullet,\mathrm{garit}(\mathrm{pal}^\bullet).\,\mathrm{leng}_1\,.\mathrm{H}^\bullet,\mathrm{pari}.\,\mathrm{pal}^\bullet) \tag{71}$$

$$\mathrm{slang}^\bullet_{\mathrm{r_0}}\,\mathrm{H}^\bullet \;:=\; \mathrm{adari}(\mathrm{pal}^\bullet)\,.\,\mathrm{slank}^\bullet_{\mathrm{r_0}}\,\mathrm{H}^\bullet \tag{72}$$

$$\mathrm{slank}^\bullet_{\mathrm{r_0}}\,\mathrm{H}^\bullet \;\in\; \mathrm{ARI}_{\underline{al}/\underline{al}} \qquad ; \qquad \mathrm{slang}^\bullet_{\mathrm{r_0}}\,\mathrm{H}^\bullet \;\in\; \mathrm{ARI}_{\underline{al}/\underline{il}} \qquad\qquad (r_0\in\mathbb{N}^*) \tag{73}$$

and here is how they affect their argument $H^\bullet$:

$$\begin{array}{rcll} H^\bullet & \in & \mathrm{ARI}^{\mathrm{reg/disc}}_{\underline{al}/\underline{al}} & \| \ \textit{sole non-zero component for } r=1, \textit{ regular in } \mathbf{u} \textit{ at } \mathbf{0}. \\ \downarrow & & & \\ \mathrm{slank}_{r_0} H^\bullet & \in & \mathrm{ARI}^{\mathrm{sing/disc}}_{\underline{al}/\underline{al}} & \| \ \textit{sole non-zero component for } r=r_0, \textit{ polar part in } \mathbf{u} \textit{ of order } r_0-1 \textit{ at } \mathbf{0}. \\ \downarrow & & & \\ \mathrm{slang}_{r_0} H^\bullet & \in & \mathrm{ARI}^{\mathrm{sing/disc}}_{\underline{al}/\underline{il}} & \| \ \textit{sole non-zero components for } r\geq r_0, \textit{ polar part in } \mathbf{u} \textit{ of order } r-1 \textit{ at } \mathbf{0}. \end{array}$$

In the above relations $leng_r$, *neginvar*, *pushinvar* denote projectors on $BIMU$:
– $leng_r$ retains the component of length r and annihilates all others.
– *neginvar* turns any bimould into one that is an *even* function of $\mathbf{w}$.
– *pushinvar* turns any bimould into one that is *push*-invariant (see (44)).
The elementary symmetral bimoulds $paj^\bullet$, $mupaj^\bullet$ are mutually inverse under the ordinary mould product *mu* and depend only on the $\mathbf{u}$-variables :

$$\mathrm{paj}^{w_1,\ldots,w_r} = P(u_1)P(u_{1,2})\ldots P(u_{1,\ldots,r}) \tag{74}$$
$$\mathrm{mupaj}^{w_1,\ldots,w_r} = (-1)^r P(u_{1,\ldots,r})P(u_{2,\ldots,r})\ldots P(u_r) \tag{75}$$

The operator *pari* multiplies components of length r by $(-1)^r$ and *anti* reverses the order in index sequences. Lastly, *garit* denotes the natural action of GARI on BIMU.[38]

11. Explicit formulae for $loma^\bullet/lomi^\bullet$.

Starting from the two elementary bimoulds $H^\bullet_s$ and $K^\bullet_n$ with length-1 components of the form $H^{w_1}_s := u_1^{s-1}$ and $K^{w_1}_n := P(n-u_1)$ and applying $slang_r$, we get two series of bimoulds :

$$H^\bullet_{[{s \atop r}]} := \mathrm{slang_r}\, H^\bullet_s \in \mathrm{ARI}^{\mathrm{sing/disc}}_{\underline{al}/\underline{il}} \quad , \quad K^\bullet_{[{n \atop r}]} := \mathrm{slang_r}\, K^\bullet_n \in \mathrm{ARI}^{\mathrm{sing/disc}}_{\underline{al}/\underline{il}} \tag{76}$$

that make it possible to compensate the unwanted singularities produced at each *tricky step* of the induction. This leads to two parallel expansions.

First expansion of $\mathrm{loma}^\bullet$ *and each* $\mathrm{loma}^\bullet_s$ *: as power series in* $\mathbf{u}$.

$$\begin{array}{rcl} act(\mathrm{loma}^\bullet_s) & := & +act(H^\bullet_{[{s \atop 1}]}) \\ (r=3) & \| & + \sum\limits_{[{s_1+s_2=s \atop 1+2=3}]} \beta^{[{s_1,\,s_2 \atop 1,\,2}]}\ act(H^\bullet_{[{s_1 \atop 1}]})\ act(H^\bullet_{[{s_2 \atop 2}]}) \\ & \| & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (r\ odd\ \geq 5) & \| & + \sum\limits_{[{s_1+\ldots s_q=s \atop r_1+\ldots r_q=r}]} \beta^{[{s_1\ ,\ldots,\ s_q \atop r_1\ ,\ldots,\ r_q}]}\ act(H^\bullet_{[{s_1 \atop r_1}]})\ \ldots\ act(H^\bullet_{[{s_q \atop r_q}]}) \\ & \| & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

[38]which of course is distinct from the adjoint action of GARI in ARI.

Second expansion of loma$^\bullet$ *: as meromorphic-perinomal functions in* $\mathbf{u}$.

$$\begin{array}{rcll} act(\mathrm{loma}^\bullet) & \overset{ess}{:=} & +act(K^\bullet_{[^1_1]}) & \\ (r=3) & \| & + \displaystyle\sum_{[^{(n_1,n_2)\ coprime}_{1+2=3}]} & \rho^{[^{n_1,s_2}_{1\ ,\ 2}]}\ act(K^\bullet_{[^{n_1}_1]})\ act(K^\bullet_{[^{n_2}_2]}) \\ & \| & \cdots\cdots\cdots & \\ (r\ odd\ \geq 5) & \| & + \displaystyle\sum_{[^{(n_1,..,n_q)\ coprime}_{r_1+\ldots r_q=r}]} & \rho^{[^{n_1\ ,\ldots,\ n_q}_{r_1\ ,\ldots,\ r_q}]}\ act(K^\bullet_{[^{n_1}_{n_1}]})\ \ldots\ act(K^\bullet_{[^{n_q}_{r_q}]}) \\ & \| & \cdots\cdots\cdots & \end{array}$$

The second expansion is unique. It involved well-defined, rational multiresidues $\rho^\bullet$ that are discrete perinomal functions of the integers n_i.

The first expansion is not unique, but becomes so if we want it to coincide with the second one. It then involves well-defined coefficients $\beta^\bullet$ which are, unexpectedly but crucially, rational numbers. Each one of them is expressible as a ratio of finitely many *hyper-bernoullian numbers.* In fact, for $r=3$, they are quotients $\beta'\beta''/\beta'''$ of just three *ordinary Bernouilly numbers.*[39]

As moulds, both $\rho^\bullet$ and $\beta^\bullet$ are alternal.

We have similar expansions for the bimould *roma*$^\bullet$.

12. Explicit-canonical decomposition of multizetas into irreducibles.

Let τ^s be the projector on $\mathrm{ARI}^{\mathrm{reg/disc}}_{\underline{al}/\underline{il}}$ which, when applied to a bimould $M^\bullet$, retains only the part of weight s. For a component of length r, this means retaining only the part of degree $d = s-r$ in the $\mathbf{u}$-variables.

$$\tau^s\, M^{(^{u_1\ ,\ldots,\ u_r}_{v_1\ ,\ldots,\ v_r})} \quad := \quad M^{(^{u_1\ ,\ldots,\ u_r}_{v_1\ ,\ldots,\ v_r})}\|_{\boldsymbol{u}\text{-part of degree } s-r} \tag{77}$$

For the generating series of the multizetas, whether 'genuine'[40] or formal[41], this leads to unique decompositions, with a *loma*$^\bullet$-part that carries only ratio-

[39] the step $3 \mapsto 4$ requiring no corrections, these harmless quotients $\beta'\beta''/\beta'''$ already yield the explicit-canonical decomposition of all multizetas of length $r \leq 4$ and of any weight s, up to infinity!

[40] with upper-case initials.

[41] with lower-case initials.

nal coefficients, and coefficients *Irr*$^\bullet$/*irr*$^\bullet$ that absorb all the transcendence:

$$act(\mathrm{Zag}^\bullet_{_{\rm II}}) \;=\; 1+\sum \;\; \mathrm{Irr}_{_{\rm II}}{}^{s_1,\dots,s_r}\; act(\tau^{s_1}\mathrm{loma}^\bullet)\dots act(\tau^{s_r}\mathrm{loma}^\bullet) \quad (78)$$

$$act(\mathrm{Zag}^\bullet_{_{\rm III}}) \;=\; 1+\sum \;\; \mathrm{Irr}_{_{\rm III}}{}^{s_1,\dots,s_r}\; act(\tau^{s_1}\mathrm{loma}^\bullet)\dots act(\tau^{s_r}\mathrm{loma}^\bullet) \quad (79)$$

$$act(\mathrm{zag}^\bullet_{_{\rm II}}) \;=\; 1+\sum \;\; \mathrm{irr}_{_{\rm II}}{}^{s_1,\dots,s_r}\; act(\tau^{s_1}\mathrm{loma}^\bullet)\dots act(\tau^{s_r}\mathrm{loma}^\bullet) \quad (80)$$

$$act(\mathrm{zag}^\bullet_{_{\rm III}}) \;=\; 1+\sum \;\; \mathrm{irr}_{_{\rm III}}{}^{s_1,\dots,s_r}\; act(\tau^{s_1}\mathrm{loma}^\bullet)\dots act(\tau^{s_r}\mathrm{loma}^\bullet) \quad (81)$$

In all four sums, the indices s_i are odd integers ≥ 3 and *"act"* denotes *any* transitive action of ARI/GARI in BIMU – it doesn't matter which. The *gari*-factorisations between symmetral bimoulds:

$$\mathrm{Zag}^\bullet_{_{\rm II+III}} \;:=\; \mathrm{gari}^\bullet\,(\mathrm{Zag}^\bullet_{_{\rm II}}\,,\,\mathrm{Zag}^\bullet_{_{\rm III}}) \quad (82)$$

$$\mathrm{zag}^\bullet_{_{\rm II+III}} \;:=\; \mathrm{gari}^\bullet\,(\mathrm{zag}^\bullet_{_{\rm II}}\,,\,\mathrm{zag}^\bullet_{_{\rm III}}) \quad (83)$$

induce corresponding *mu*-factorisations for the symmetral, scalar moulds:

$$\mathrm{Irr}^\bullet_{_{\rm II+III}} \;:=\; \mathrm{Irr}^\bullet_{_{\rm II}} \times \mathrm{Irr}^\bullet_{_{\rm III}} \quad (84)$$

$$\mathrm{irr}^\bullet_{_{\rm II+III}} \;:=\; \mathrm{irr}^\bullet_{_{\rm II}} \times \mathrm{irr}^\bullet_{_{\rm III}} \quad (85)$$

Moreover, due to the *parity* which governs everything here, the only non-zero components in the mould logarithms $logmu(Irr^\bullet_{_{\rm II}})$ and $logmu(irr^\bullet_{_{\rm II}})$ (resp. $logmu(Irr^\bullet_{_{\rm III}})$ and $logmu(irr^\bullet_{_{\rm III}})$) are those of *even* (resp. *odd*) length r. As an easy consequence, the mould $irr^\bullet_{_{\rm II+III}}$, or $irr^\bullet$ for short, actually *determines* its two factors $irr^\bullet_{_{\rm II}}$ and $irr^\bullet_{_{\rm III}}$. Summing up, we may say:

Together with the symbol $\mathrm{irr}^2_{_I} \sim$ "π^2", *the symmetral mould*[42] *:*

$$\mathrm{irr}^\bullet_{_{\rm II+III}} = \mathrm{irr}^\bullet = \{\mathrm{irr}^{s_1,s_2,\dots,s_r}\in\mathbb{C}\}\;,\; r=1,2,3\dots\;,\; s_i\in\{3,5,7,9\dots\} \quad (86)$$

constitutes a system, both complete and free[43]*, of canonical irreducibles for the (ordinary or 'rootless') formal multizetas. More precisely, any such multizeta may be uniquely linearised as a sum:*

$$\sum_{r\geq 0}\;\sum_{s_1,\dots,s_r\; odd\geq 3}^{s_0\; even\geq 0} \gamma^{s_0\,;\,s_1,\dots,s_r}\;\pi^{s_0}\;\; irr^{s_1,\dots,s_r} \qquad (\gamma^\bullet\in\mathbb{Q}) \qquad (87)$$

[42]i.e. subject to no other constraints than symmetrality.

[43]that is, free *up to the symmetrality constraints.* These constraints could easily be removed by working with the alternal mould $logmu(irr^\bullet)$ and picking some Lyndon basis in the corresponding Lie algebra, but that would entail a slight loss of canonicity. The truth of the matter is that the irreducibles *spontaneously* present themselves in the shape of a symmetral mould – and there is no going against that. The whole point of the *reduction*, of course, lies in the change from a mould $ze^\bullet$ with a double symmetry and indices s_i running through $\{1,2,3,4...\}$, to a mould $irr^\bullet$ with a single symmetry and indices s_i running through $\{3,5,7,9...\}$.

. Analogous results hold for the *rooted* multizetas.

13. 'Impartial' expression of the irreducibles as perinomal numbers.

Plugging (59),(80),(81) into the factorisation (57) and then picking the Taylor coefficients of either $zag^\bullet$ or $zig^\bullet$, we get the formal multizetas, in both encodings $wa^\bullet$ and $ze^\bullet$, automatically[44], uniquely, and explicitly expanded as finite sums of irreducibles $irr^\bullet$. The process of course may be reversed, yielding expressions of $irr^\bullet$ in terms of either $wa^\bullet$ and $ze^\bullet$, but these reverse formulae are trebly defective: they are not particularly explicit; they are decidedly non-unique; and they are 'partial', in the sense of *leaning* towards one or the other of the two natural encodings. To remove these blemishes, we require expansions which, like (78)-(79), express $Zag^\bullet_{II}$, $Zag^\bullet_{III}$ in terms of $loma^\bullet$, but treating these bimoulds as perinomal-meromorphic *functions* and no longer as *power series.* So, instead of breaking up $loma^\bullet$ under the projectors τ^s, we now apply the following dilation automorphisms [45] δ^n:

$$\delta^n \; M^{\binom{u_1\;,\dots,\;u_r}{v_1\;,\dots,\;v_r}} \;\; := \;\; n^{-r} \;\; M^{\binom{u_1/n\;,\dots,\;u_r/n}{v_1\;\;,\dots,\;\;v_r}} \tag{88}$$

The new expansions read (with all n_i running through $\mathbb{N}^*$):

$$act(\mathrm{Zag}^\bullet_{\mathrm{II}}) \;\stackrel{ess}{=}\; 1 + \sum \;\; \mathrm{Urr}_{\mathrm{II}}{}^{n_1,\dots,n_r}\; act(\delta^{n_1}\mathrm{loma}^\bullet)\dots act(\delta^{n_r}\mathrm{loma}^\bullet) \tag{89}$$

$$act(\mathrm{Zag}^\bullet_{\mathrm{III}}) \;\stackrel{ess}{=}\; 1 + \sum \;\; \mathrm{Urr}_{\mathrm{III}}{}^{n_1,\dots,n_r}\; act(\delta^{n_1}\mathrm{loma}^\bullet)\dots act(\delta^{n_r}\mathrm{loma}^\bullet) \tag{90}$$

with symmetral moulds $Urr^\bullet_{\mathrm{II}}, Urr^\bullet_{\mathrm{III}}$ that are rational-valued perinomal functions of the integers n_i.

A first fallout (from inverting (90)) is yet another expansion for $loma^\bullet$:

$$act(\mathrm{loma}^\bullet) \;\stackrel{ess}{=}\; \sum \mathrm{Orr}_{\mathrm{III}}{}^{n_1,\dots,n_r} act(\delta^{n_1}\mathrm{Zag}^\bullet_{\mathrm{III}} - 1^\bullet)\dots act(\delta^{n_r}\mathrm{Zag}^\bullet_{\mathrm{III}} - 1^\bullet) \tag{91}$$

which is often referred to as *"wasteful-useful"* [46]. But the main consequence is a *direct-impartial expression for the irreducibles.* Indeed, if for any index

[44] via the ARI/GARI machinery

[45] they are automorphisms of $ARI_{\underline{al}/\underline{al}}$ and, thanks to the factor n^{-r}, of $ARI_{\underline{al}/\underline{il}}$ as well.

[46] *"wasteful"*, because it derives an object with sparse poles and rational Taylor coefficients from one with "dense" poles and transcendental Taylor coefficients; *"useful"*, because it automatically transports important properties (like invariance under the *digonal involution*: see [E2], Appendix) from upper-case $Zag^\bullet$ and $Zag^\bullet_{III}$ over to $loma^\bullet$ and then, via (59),(80),(81), back to lower-case $zag^\bullet$, thus proving that these properties (digonal invariance etc) are algebraically implied by the quadratic relations.

X of the form *II*, *III* or *II+III* we set:

$$\mathrm{Yrr}_{\mathrm{X}}^{s_1,\dots,s_r} \overset{ess}{:=} \sum_{n_i\geq 1} \mathrm{Urr}_{\mathrm{X}}^{n_1,\dots,n_r}\ n_1^{-s_1}\dots n_r^{-s_r} \tag{92}$$

we get three parallel identities between *symmetral* moulds:

$$\mathrm{Zag}^{\bullet}_{\mathrm{II+III}} \quad = \quad \mathrm{gari}^{\bullet}\,(\mathrm{Zag}^{\bullet}_{\mathrm{II}}\,,\,\mathrm{Zag}^{\bullet}_{\mathrm{III}}) \tag{93}$$

$$\mathrm{Urr}^{\bullet}_{\mathrm{II+III}} \quad := \quad \mathrm{Urr}^{\bullet}_{\mathrm{II}} \times \mathrm{Urr}^{\bullet}_{\mathrm{III}} \tag{94}$$

$$\mathrm{Yrr}^{\bullet}_{\mathrm{II+III}} \quad := \quad \mathrm{Yrr}^{\bullet}_{\mathrm{II}} \times \mathrm{Yrr}^{\bullet}_{\mathrm{III}} \tag{95}$$

and we find that the moulds $Irr^{\bullet}_{X}$ and $Yrr^{\bullet}_{X}$ actually *coincide*, though they vastly differ as to the way they are defined.

Remark: there is also a notion of *formal perinomal numbers*, parallel to that of *formal multizetas* (but relative to constraints altogether different from the *quadratic relations*) and the above relations translate into a direct-impartial expression of the irreducibles $irr^{\bullet}_{X}$ attached to the *formal multizetas.*

The whole theory also carries over to the case of *rooted multizetas*, but with the new phenomenon of *retroaction.*[47] Results are particularly simple for the root orders $p = 2$ and $p = 3$. If anything, the case of *Eulerian multizetas* ($p = 2$) is even simpler than that of *ordinary multizetas* ($p = 1$). Perinomal functions still rule the roost. They fall into six main classes:[48]

root order	*Bimoulds in* $\mathrm{ARI}^{\mathrm{reg/disc}}_{\underline{\mathrm{al/il}}}$	*Eupolar functions (meromorphic-perinomal)*	*Residues* ρ *(discrete-perinomal)*	*Taylor coefficients* $\rho^{\#}$ *(perinomal numbers)*
$p = 1$	$\mathrm{loma}^{\bullet}$ $\mathrm{roma}^{\bullet}$	*"sparse" multipoles*	*well-defined perinomal degrees* $\mathrm{d_{i,j}}$	*rational and "Bernoullian"*
$p = 2$	$\mathrm{loma}^{\bullet}$ $\mathrm{roma}^{\bullet}$	*"dense" multipoles*	*well-defined perinomal degrees* $\mathrm{d_{i,j}}$	*rational and "Bernoullian"*
$p \geq 1$	$\mathrm{logari}(\mathrm{Zag}^{\bullet}_{\mathrm{II}})$ $\mathrm{logari}(\mathrm{Zag}^{\bullet}_{\mathrm{III}})$	*"dense" multipoles*	*well-defined perinomal degrees* $\mathrm{d_{i,j}}$	*multizetaic and "impartial"*

15. Conclusion. Looking back/ahead/sideways.

Looking back: Despite interesting work by M. Petitot, J. van der Hoeven, Minh, etc, and some vigorous numerical exploration by Broadhurst,

[47] which means that, for a fixed weight s, some of the constraints binding the multizetas of a given length $r_0 < s$ *do not result* from the double symmetries written down for $r = 1,..,r_0$: the *full* sequence $r = 1,...,s$ must be taken into account.

[48] we say that a meromorphic-eupolar functions on $\mathbb{C}^r$ with multipoles "at" the multi-integers $\in \mathbb{Z}^r$ has "dense" (resp. "sparse") multipoles if the latters' number inside a ball of radius l grows like $\mathcal{O}(l^r)$ (resp. $o(l^r)$).

Borwein, Kreimer etc, plus some inspired conjectures based on these numerical data, the main problems in multizeta arithmetic were still open by the end of the 90s. The intervention in the last months of 2000 of the (pre-existing) ARI/GARI apparatus unfroze the situation. It yielded at once the *free generation theorem* and the basic dimorphic *special bimoulds*, leading to the *canonical-explicit decomposition* into irreducibles. Later still, in 2002, the recourse to *perinomal objects* opened the way for the *direct-impartial expansion* of irreducibles.

Looking ahead : Here are four possible avenues for further research :
– going through the list of known multizetaic constraints and showing that each of them is algebraically derivable from the *quadratic relations.* This has already been done in special instances. The trouble with that process is of course its open-endedness. . .
– attempting to establish once and for all the rigorous arithmetical isomorphism of *formal* and *genuine* multizetas. The most promising approach here is that of *direct numerical derivations* patterned on the *alien derivations* of analysis, which have largely systematised the proving of transcendence results for resurgent functions.
– pursuing the investigation of dimorphy beyond multizetas, in the next main dimorphic $\mathbb{Q}$-rings $\subset \mathbb{C}$ that have already been identified, beginning of course with suitable *formalisations* of these rings.
– exploring the trigebra $\boldsymbol{\mathcal{N}}a$ of *natural analysable germs*[49] and its numerical accompaniment, the ring $\mathbb{N}$a of *natural numbers*, as the (probably) broadest setting for the understanding of numerical dimorphy.

Looking sideways : strangely, multizeta theory is rife with mistakes and misconceptions, some of which persist long after exposure. Here is a sample :
– ***the conversion rule (26)-(28)*** : though long known and immediate to derive (see §2), it sometimes receives needlessly convoluted proofs.
– ***the meromorphic continuation of $Ze^\bullet$ in the* s*-variables*** : this almost self-evident fact (two lines of proof !) has been questioned, even denied, and then given clumsy, roundabout proofs[50]. Nor is there any awareness of the existence of another $\mathbf{s}$-continuable bimould $\mathcal{Z}a^\bullet$, analogous to $Ze^\bullet$ but linked to the $Wa^\bullet$ encoding.[51] The only difficult, and still open, question in this context pertains to the irreducibles $Irr^\bullet$: these are easily definable as holomorphic functions for large positive $\mathbf{s}$-variables, but do they admit a meromorphic continuation on the negative side ?
– ***status of the self-consistency constraints (30)*** : it has been vari-

[49]with its two sets of exotic derivations : *alien* and *foreign.*
[50]which fall well short of a full description of the multipoles and their residues.
[51]it is closesy connected with the Taylor coefficients of the *gari*-inverse of $Zag^\bullet$.

ously stated or suggested that they don't exhaust the *additional constraints* for rooted multizetas[52]. In fact the *exact opposite is the case*: they are always *sufficient* and for small values of the root order p they are even *redundant.*
– ***status of ARI/GARI*** : it has been likened to various constructs, in particular the Ihara algebra, which is downright absurd, if only because:
(i) the Ihara algebra lacks the dual set of variables $\mathbf{u}/\mathbf{v}$ which is indispensable for a 'symmetrical' treatment of dimorphy[53]
(ii) it cannot accommodate the *singular functions*[54] which fit effortlessly into the ARI/GARI framework and on which everything revolves
(iii) it has no place for *any* of the sixty-odd *special moulds*[55] which are essential to the construction and description of irreducibles.
– ***status of numerical dimorphy*** : when not ignored purely and simply, this central fact about transcendental numbers is often discussed within the quite uncongenial framework of "period theory" which, due to its partiality for *integrals* over *series*, distorts at the outset all the symmetries that underpin dimorphy.

16. Some references.

[**B**] D.J. Broadhurst, *Conjectured Enumeration of irreducible Multiple Zeta Values, from Knots and Feynman Diagrams* preprint, Physics Dept., Open University Milton Keynes, MK76AA, UK, Nov. 1996.
[**E1**] J.Ecalle, *A Tale of Three Structures: the Arithmetics of Multizetas, the Analysis of Singularities, the Lie algebra ARI.* in Differential Equations and the Stokes Phenomenon, B.L.J. Braaksma, G.K. Immink, M van der Put, J. Top Eds., World Scient. Publ. 2002, vol. 17, 89-146.
[**E2**] J.Ecalle, *ARI/GARI, la dimorphie et l'arithmétique des multizêtas : un premier bilan.* Jour. de Théorie des Nombres de Bordeaux **15**(2003),411-478.
[**E3**] J.Ecalle, *ARI/GARI and the flexion structure.*
[**E4**] J.Ecalle, *ARI/GARI and its special bimoulds.*
[**E5**] J.Ecalle, *Explicit-canonical decomposition of multizetas into irreducibles.*

[52] i.e. the constraints that must be added to the *quadratic relations* and the *conversion rule* to get an (empirically) complete system of constraints.

[53] as a consequence, even the *bracket* of ARI and that of the Ihara algebra are irredeemably non-isomorphic.

[54] i.e. those bimoulds of ARI/GARI, which, as functions of $\mathbf{u}$ or $\mathbf{v}$, have poles at the origin, or away from it.

[55] the infatuation of algebraists with the Ihara apparatus must be the reason why, after more than a decade of fudging with multizetas, such conspicuous objects as $pal^\bullet/pil^\bullet$ and $tal^\bullet/til^\bullet$ had still escaped discovery, although these moulds were standing *right there*, tall and erect, and almost impossible to miss, at the very entrance of multizeta territory, as its guardians!

[E6] J.Ecalle, *Perinomal numbers and multizeta irreducibles.*
[M.P.H] H.N.Minh, M.Petitot, J.v.d. Hoeven, *Shuffle algebra and polylogaritms* , Disc. Math. **225** (2000), 217-230.

[E3],[E4] shall be available as Orsay preprints in February 2005, and [E5],[E6] shall appear later on in the course of that same year. All four texts [E3,4,5,6] are extremely lengthy, but abridged versions shall be submitted to ordinary mathematical journals. Electronic files of the unabridged versions shall be put on the WEB, along with an assortment of Tables and Maple programmes.