Resurgent functions and alien calculus.

Here is a lapidary characterization that hits the nail right on the head: *a resurgence algebra is a function algebra on which there act alien derivations.*

Less cryptically, resurgent 'functions' may be said to simultaneously live in three models:

(i) as formal, usually divergent, series $\widetilde{\varphi}(z) = \sum_{\sigma\uparrow} a_\sigma \, z^{-\sigma}$ or as more general transseries $\widetilde{\varphi}(z) = \sum_{\epsilon\downarrow} a_\epsilon \, \epsilon(z)$ with subexponential transmonomials $\epsilon(z)$;

(ii) as germs $\widehat{\varphi}(\zeta)$ analytic-ramified at $0_\bullet \in \mathbb{C}_\bullet$ and capable of endless analytic continuation (no 'analytic boundaries', only isolated singularities) with at most exponential growth at ∞;

(iii) as analytic germs $\varphi_\theta(z)$ defined in sectorial neighbourhoods of ∞ straddling the axes $arg(z^{-1}) = \theta$.

The passage from (ii) to (iii) is via the Laplace transform $\mathcal{L}$ along $arg(\zeta) = \theta$. The passage from (i) to (ii) is via the term-by-term Borel transform $\mathcal{B}$ (which formally reverses $\mathcal{L}$).

The alien derivations $\widehat{\Delta}_\omega$ are linear operators that measure the singular behaviour of $\widehat{\varphi}(\zeta)$ *at*, or rather *over*, $\omega \in \mathbb{C}_\bullet$. They are indeed derivations relative to the convolution $*$ (convolution being the natural product in the ζ-plane or *Borel plane*) and their pull-backs Δ_ω in the models (i) and (iii) are derivations relative to ordinary multiplication.

Resurgent functions $\widetilde{\varphi}(\zeta)$ of natural origin tend to self-replicate (creatively rather than slavishly) at their singular points. That generic phenomenon, to which they owe their name, is described by *resurgence equations* $\widehat{\mathcal{R}}_\omega(\widehat{\varphi}, \widehat{\Delta}_\omega \widehat{\varphi}) = 0$.

Resurgence is truly ubiquitous: barring ultra-specific factors of divergence (such as Liouvillian small denominators), any divergent power series spontaneously arising from an all-analytic context is likely to prove resurgent.

As for the alien derivations, they generate a calculus of stupendous breadth and richness, with two faces: *differential* and *integral*.