Zero-cycles and rational points on some surfaces over a global function field

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Abstract

Let $\mathbf{F}$ be a finite field of characteristic p. We consider smooth surfaces over $\mathbf{F}(t)$ defined by an equation $f + tg = 0$, where f and g are forms of degree d in 4 variables with coefficients in $\mathbf{F}$, with d prime to p. We prove: For such surfaces over $\mathbf{F}(t)$, the Brauer-Manin obstruction to the existence of a zero-cycle of degree one is the only obstruction. For $d = 3$ (cubic surfaces), this leads to the same result for rational points.

Soit $\mathbf{F}$ un corps fini de caractéristique p. Pour une surface lisse sur $\mathbf{F}(t)$ définie par une équation $f + tg = 0$, où f et g sont deux formes de degré d sur $\mathbf{F}$ en 4 variables, avec d premier à p, nous montrons que l'obstruction de Brauer-Manin au principe de Hasse pour les zéro-cycles de degré 1 est la seule obstruction. Pour $d = 3$ (surfaces cubiques), on en déduit le même énoncé pour les points rationnels.

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1 Introduction

Study of the case of curves (Cassels, Tate) and of the case of rational surfaces (Colliot-Thélène et Sansuc [CT/S81]), where a more precise conjecture is made for rational surfaces) has led to the following conjecture for *zero-cycles* on arbitrary varieties over global fields (Kato and Saito [K/S86], Saito [S89], Colliot-Thélène [CT93], [CT99]).

Conjecture 1.1. *Let X be a smooth, projective, geometrically integral variety over a global field k. If there exists a family $\{z_v\}_{v\in\Omega}$ of local zero-cycles of degree 1 (here v runs through the set Ω of places of k) such that for all $A \in \mathrm{Br}(X)$,*

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(z_v)) = 0 \in \mathbf{Q}/\mathbf{Z}$$

holds, then there exists a zero-cycle of degree 1 on X. In other words, the Brauer–Manin obstruction to the existence of a zero-cycle of degree 1 on X is the only obstruction.

Over number fields, this conjecture has been established in special cases in work of (alphabetical order, and various combinations) Colliot-Thélène, Frossard, Salberger, Sansuc, Skorobogatov, Swinnerton-Dyer, Wittenberg (see the introduction of [W10]). None of these results applies to smooth surfaces of degree d at least 3 in 3-dimensional projective space – for $d \geq 5$ these surfaces are of general type. In section 2, we establish the conjecture in the special case of a global field $k = \mathbf{F}(t)$ purely transcendental over a finite field $\mathbf{F}$ and of smooth surfaces $X \subset \mathbf{P}^3_k$ defined by an equation $f + tg = 0$, where f and g are two forms of arbitrary degree d over the field $\mathbf{F}$.

According to a conjecture of Colliot-Thélène and Sansuc ([CT/S80]), the Brauer–Manin obstruction to the existence of a *rational point* on a smooth, geometrically rational surface defined over a global field should be the only obstruction. Such should in particular be the case for smooth cubic surfaces in 3-dimensional projective space $\mathbf{P}^3_k$. In section 3, we establish the conjecture in the special case of a global field $k = \mathbf{F}(t)$ purely transcendental over a finite field $\mathbf{F}$ and of smooth cubic surfaces $X \subset \mathbf{P}^3_k$ defined by an equation $f + tg = 0$, where f and g are two cubic forms over the field $\mathbf{F}$. Simple though they be, such surfaces may fail to obey the Hasse principle.

2 Zero-cycles of degree 1 on surfaces of arbitrary degre

The following theorem is due to S. Saito [S89]. It says that if a strong integral form of the Tate conjecture on 1-dimensional cycles is true, then the above conjecture holds, at least if we stay away from the characteristic of the field. For an alternate proof of Theorem 2.1, see [CT99, Prop. 3.2].

Theorem 2.1. *(Saito) Let* $\mathbf{F}$ *be a finite field and* $C/\mathbf{F}$ *a smooth, projective, geometrically integral curve over* $\mathbf{F}$*. Let* $k = \mathbf{F}(C)$ *be its function field. Let* $\mathcal{X}$ *be a smooth, projective, geometrically integral* $\mathbf{F}$*-variety of dimension* n *and* $f : \mathcal{X} \to C$ *a faithfully flat map whose generic fibre* X/k *is smooth and geometrically integral.*

Assume:

(1) For each prime $l \neq \mathrm{char}(\mathbf{F})$*, the cycle map*

$$T_X : \mathrm{CH}^{n-1}(\mathcal{X}) \otimes \mathbf{Z}_l \to H^{2n-2}_{\acute{e}t}(\mathcal{X}, \mathbf{Z}_l(n-1))$$

from the Chow group of dimension 1 cycles on $\mathcal{X}$ *to étale cohomology is onto.*

(2) There exists a family $\{z_v\}_{v\in\Omega}$ *of local zero-cycles of degree 1 (here* v *runs through the set* Ω *of places of* k*) such that for all* $A \in \mathrm{Br}(X)$*,*

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(z_v)) = 0 \in \mathbf{Q}/\mathbf{Z}.$$

Then there exists a zero-cycle on X *of degree a power of* $\mathrm{char}(\mathbf{F})$*.*

In this statement, $A(z_v)$ is the element of the Brauer group of the local field k_v obtained by evaluation of A on the zero-cycle z_v. The map $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbf{Q}/\mathbf{Z}$ is the local invariant of class field theory.

Here is one case where assumption (1) in the previous theorem is fulfilled.

Theorem 2.2. *Let* $\mathbf{F}$ *be a finite field and* l *a prime,* $l \neq \mathrm{char}(\mathbf{F})$*. For a smooth, projective, geometrically integral threefold* $\mathcal{X}$ *over* $\mathbf{F}$ *which is birational to* $\mathbf{P}^3_F$*, the cycle map* $T_{\mathcal{X}} : \mathrm{CH}^2(\mathcal{X}) \otimes \mathbf{Z}_l \to H^4_{\acute{e}t}(\mathcal{X}, \mathbf{Z}_l(2))$ *is onto.*

Proof. If $\mathcal{X} = \mathbf{P}^3_{\mathbf{F}}$, then $\mathrm{CH}^2(\mathcal{X}) = \mathbf{Z}$ and one easily checks that the cycle map

$$T_{\mathcal{X}} : \mathrm{CH}^2(\mathcal{X}) \otimes \mathbf{Z}_l \to H^4_{\acute{e}t}(\mathcal{X}, \mathbf{Z}_l(2))$$

is simply the identity map $\mathbf{Z}_l = \mathbf{Z}_l$. Using the standard formulas for the computation of Chow groups and of cohomology for a blow-up along a smooth projective subvariety, as well as the vanishing of Brauer groups of smooth projective curves over a finite field, one shows: For $\mathcal{X}$ a smooth projective threefold, the cokernel of the above cycle map $T_{\mathcal{X}}$ is invariant under blow-up of smooth projective subvarieties on $\mathcal{X}$.

By a result of Abhyankar ([Abh66, Thm. 9.1.6]), there exists a smooth projective variety $\mathcal{X}'$ which is obtained from $\mathbf{P}^3_{\mathbf{F}}$ by a sequence of blow-ups along smooth projective $\mathbf{F}$-subvarieties, and which is equipped with a birational $\mathbf{F}$-morphism $p : \mathcal{X}' \to \mathcal{X}$.

There are push-forward maps π_* and pull-back maps π^* both for Chow groups and for étale cohomology, and for the birational map π we have $\pi_* \circ \pi^* = \mathrm{id}$. Moreover these maps are compatible with the cycle class map. Thus the cokernel of $T_{\mathcal{X}}$ is a subgroup of the cokernel of $T_{\mathcal{X}'}$, hence is zero. □

Combining Theorems 2.1 and 2.2, we get:

Theorem 2.3. *Let $\mathbf{F}$ be a finite field and $C/\mathbf{F}$ a smooth, projective, geometrically integral curve over $\mathbf{F}$. Let $k = \mathbf{F}(C)$ be its function field. Let $\mathcal{X}$ be a smooth, projective, geometrically integral $\mathbf{F}$-variety of dimension n and $f : \mathcal{X} \to C$ a faithfully flat map whose generic fibre X/k is smooth and geometrically integral.*

Assume:

(1) $\dim \mathcal{X} = 3$ *and $\mathcal{X}$ is $\mathbf{F}$-rational;*

(2) there exists a family $\{z_v\}_{v\in\Omega}$ of local zero-cycles of degree 1 (here v runs through the set Ω of places of k) such that for all $A \in \mathrm{Br}(X)$,

$$\sum_{v\in\Omega} \mathrm{inv}_v(A(z_v)) = 0 \in \mathbf{Q}/\mathbf{Z}.$$

Then there exists a zero-cycle on X of degree a power of $\mathrm{char}(\mathbf{F})$.

We may now prove the main result of this section.

Theorem 2.4. *Let $\mathbf{F}$ be a finite field, let f, g be two nonproportional homogeneous forms in 4 variables, of degree d prime to the characteristic of $\mathbf{F}$. Let $k = \mathbf{F}(t)$. Suppose the k-surface $X \subset \mathbf{P}^3_k$ defined by $f + tg = 0$ is smooth. If there is no Brauer–Manin obstruction to the Hasse principle for zero-cycles of degree 1 on X, then*

(i) there exists a zero-cycle of degree 1 on the k-surface X;

(ii) there exists a zero-cycle of degree 1 on the $\mathbf{F}$-curve Γ defined by $f = g = 0$ in $\mathbf{P}^3_{\mathbf{F}}$.

Proof. Let $\mathcal{X}_1 \subset \mathbf{P}^3_{\mathbf{F}} \times_F \mathbf{P}^1_{\mathbf{F}}$ be the schematic closure of $X \subset \mathbf{P}^3_{\mathbf{F}(t)}$. The $\mathbf{F}$-variety $\mathcal{X}_1$ has an affine birational model with equation

$$\phi(x, y, z) + t\psi(x, y, z) = 0,$$

hence t is determined by x, y, z, thus $\mathcal{X}$ is $\mathbf{F}$-birational to $\mathbf{P}^3_{\mathbf{F}}$. Since $\mathcal{X}_1$ admits a smooth projective model over $\mathbf{F}$, a result of Cossart ([Co92, Théorème, p. 115]) shows that there exists a smooth projective threefold $\mathcal{X}/\mathbf{F}$ and an $\mathbf{F}$-birational morphism $\mathcal{X} \to \mathcal{X}_1$ which is an isomorphism over the smooth locus of $\mathcal{X}_1$, hence in particular which induces an isomorphism over Spec $\mathbf{F}(t) \subset \mathbf{P}^1_{\mathbf{F}}$. That is, the generic fibre of $\mathcal{X} \to \mathbf{P}^1_{\mathbf{F}}$ is k-isomorphic to X/k.

Statement (i) then follows from Thm. 2.3. Statement (ii) follows from (i) as a special application of a result of Colliot-Thélène and Levine ([CT/L09, Théorème 1, p. 217]). □

Remark 2.5. Theorem 2.4 is of interest only in the case where the $\mathbf{F}$-curve Γ does not contain a geometrically integral component. Otherwise the two statements immediately follow from the Weil estimates for the number of points on geometrically integral curves. These estimates actually provide more: they show that if there exists such a component, then on any field extension $\mathbf{F}'$ of $\mathbf{F}$ of high enough degree, there exists an $\mathbf{F}'$-point on Γ, hence for any such field there exists an $\mathbf{F}'(t)$-point on the $\mathbf{F}(t)$-surface X.

Remark 2.6. One could try to circumvent the cohomological machinery, i.e. Theorems 2.1 and 2.2. For this, in each of the special cases where there are zero-cycles of degree 1 everywhere locally on X but there is no zero-cycle of degree one on the curve Γ, one should:

(i) Check that the Brauer group is not trivial, find generators.

(ii) Check that there is a Brauer–Manin obstruction.

Already when the common degree of f and g is 3, which we shall now more particularly examine, this seems no easy enterprise.

3 Rational points on cubic surfaces

The proof of the following theorem is independent of the previous results.

Theorem 3.1. *Let* $\mathbf{F}$ *be a finite field, let* f, g *be two nonproportional cubic forms over* $\mathbf{F}$ *in 4 variables. Assume the characteristic of* $\mathbf{F}$ *is not 3. Let* $k = \mathbf{F}(t)$. *Suppose the* k*-surface* $X \subset \mathbf{P}^3_k$ *defined by* $f + tg = 0$ *is smooth.*

Let $\Gamma \subset \mathbf{P}^3_{\mathbf{F}}$ be the complete intersection curve defined by $f = g = 0$. The following conditions are equivalent:

(i) There exists a k-rational point on the k-variety X.

(ii) There exists a zero-cycle of degree 1 on the k-variety X.

(iii) There exists a zero-cycle of degree 1 on the $\mathbf{F}$-curve Γ.

(iv) There exists a closed point of degree prime to 3 on the $\mathbf{F}$-curve Γ.

(v) There exists a closed point of degree a power of 2 on the $\mathbf{F}$-curve Γ.

Proof. That (i) implies (ii) is trivial. That (ii) implies (iii) is a special case of [CT/L09]. Statements (iii) and (iv) are equivalent, since Γ is a curve of degree 9. If (v) holds, then Γ has a point in a tower of quadratic extensions of $\mathbf{F}$, hence the cubic surface X has a point in a tower of quadratic extensions of k. An extremely well known argument shows that if a cubic surface over a field has a point in a separable quadratic extension of that field, then it has a rational point: the line joining two conjugate points is defined over the ground field, either it is entirely contained in the cubic surface or it meets it in a third, rational point. Iterating this remark, we see that X has a rational point, i.e. (i) holds.

Let us prove that (iii) implies (v). To prove this, one may replace $\mathbf{F}$ by its maximal pro-2-extension extension F, which we now do. For an odd integer n, we let F_n/F be the unique, cyclic, field extension of F of degree n.

For Z/L a variety over a field L, the index $\mathrm{ind}(Z) = \mathrm{ind}(Z/L)$ is the gcd of the L-degrees of closed points on Z. The index of an L-variety is equal to the index of its reduced L-subvariety. The index of an L-variety which is a finite union of L-varieties is the gcd of the indices of each of them. The assumption made in (iii) is precisely that the index of the curve Γ is 1.

Since F has no quadratic or quartic extension, an effective zero-cycle of degree $1, 2, 4$ contains an F-rational point, and an effective zero-cycle of degree 3, 6 or 9 either contains an F-point or has index a multiple of 3.

If Γ contains a geometrically integral component, then $\Gamma(F) \neq \emptyset$ (Weil estimates, see the remark after Theorem 2.4).

Suppose Γ does not contain a geometrically integral component. One then easily checks that the degree 9 curve $\overline{\Gamma}$ can break up only in one of the following ways:

$9 = 3(1 + 1 + 1)$

$9 = 2(1 + 1 + 1) + (1 + 1 + 1)$

$9 = (2 + 2 + 2) + (1 + 1 + 1)$

$9 = (1 + 1 + 1) + (1 + 1 + 1) + (1 + 1 + 1)$

$9 = (1 + \cdots + 1)$ (9 times)

$9 = (3+3+3)$

Here $m(a+a+a)$ means the sum of three conjugate integral curves of degree a over $\overline{F}$ with multiplicity m.

An integral curve of degree 2 over $\overline{F}$ is a smooth plane conic, contained in a well-defined plane. An integral curve of degree 3 over $\overline{F}$ is either a plane cubic or a smooth twisted cubic.

Let the integral curve $C \subset \mathbf{P}^3_F$ break up as (1+1+1). The singular set consists of at most 3 points. Then either $C(F) \neq \emptyset$ or 3 divides $\mathrm{ind}(C)$.

Let the integral curve $C \subset \mathbf{P}^3_F$ break up as (2+2+2). Each conic is defined over F_3. Two distinct smooth conics on $f = 0$ define two distinct planes, hence they intersect in at most 2 geometric points. Such points must already be in F_3. Thus any closed point in the singular locus of C has degree 1 or 3. One concludes that either $C(F) \neq \emptyset$ or 3 divides $\mathrm{ind}(C)$.

Let the integral curve $\Gamma \subset \mathbf{P}^3_F$ break up as $(1 + \cdots + 1)$ (9 times). The 9 lines are defined over F_9, the degree 9 extension of F. So are their intersection points. This implies that any singular closed point on Γ has degree a power of 3. Thus $\Gamma(F) \neq \emptyset$ or 3 divides $\mathrm{ind}(\Gamma)$.

Let the integral curve $\Gamma \subset \mathbf{P}^3_F$ break up as $(3+3+3)$, and assume that this corresponds to a decomposition as three conjugate plane cubics. Each of these is defined over F_3. The intersection number of two of these cubics is 3. The points of intersection of two such curves are thus defined over F_9. We conclude that the singular locus of Γ splits over F_9. This implies that the degree of any closed point in that locus is a power of 3. Thus either $\Gamma(F) \neq \emptyset$ or 3 divides $\mathrm{ind}(\Gamma)$.

Let the curve $\Gamma \subset \mathbf{P}^3_F$ break up as $(3+3+3)$, and assume that Γ breaks up as the sum of three conjugate twisted cubics. The curve Γ lies on the smooth cubic surface X over $F(t)$ defined by $f + tg = 0$. Each twisted curve is defined over F_3. Let σ be a generator of $\mathrm{Gal}(F_3(t)/F(t))$. Write $\Gamma = C + \sigma(C) + \sigma^2 C$ on $X_{F_3(t)}$. Using intersection theory on the smooth surface $X_{F_3(t)}$, which is invariant under the action of $\mathrm{Gal}(F_3(t)/F(t))$, and letting H be the class of a plane section, we find $27 = (3H.3H) = (\Gamma.\Gamma) = 3(C.C) + 6(C.\sigma(C))$. The curve C is a twisted cubic, hence a smooth curve of genus zero on the smooth cubic surface X, whose canonical bundle K is given by $-H$. The formula for the arithmetic genus of a curve on a surface, namely $2(p_a(C)-1) = (C.C)+(C.K)$ gives $(C.C) = 1$. This implies $(C.\sigma(C)) = 4$, hence $(\sigma(C).\sigma^2(C)) = 4$ and $(\sigma^2(C).C) = 4$. Since each of these twisted cubics is defined over F_3 and since F_3 has no field extension of degree 2 or 4, this implies that the points of intersection of any two of these twisted cubics are defined over F_9. We conclude that the singular locus of Γ splits over F_9. This implies that the degree of any closed point in that locus is a power of 3. Thus either $\Gamma(F) \neq \emptyset$ or 3 divides $\mathrm{ind}(\Gamma)$.

In all cases we have proved: Either $\Gamma(F) \neq \emptyset$ or 3 divides $\mathrm{ind}(\Gamma)$. The assumption $\mathrm{ind}(\Gamma) = 1$ now implies $\Gamma(F) \neq \emptyset$. □

Remark 3.2. If the order q of the finite field $\mathbf{F}$ is large enough and $f+tg=0$ is soluble in $\mathbf{F}(t)$, a variant of the proof for the equivalence of (iv) and (v) shows that $f+tg=0$ has a solution in polynomials of degree at most 5. This raises the interesting general question whether there are integers $N(d)$ with the following property: Suppose that $G(X_0, \dots, X_4, t)$ is a polynomial defined over $\mathbf{F}$, homogeneous of degree 3 in the X_i and of degree d in t; if $G = 0$ is soluble in $\mathbf{F}(t)$, then it has a solution in polynomials of degree at most $N(d)$.

We may now prove:

Theorem 3.3. *Let $\mathbf{F}$ be a finite field, let f, g be two nonproportional cubic forms in 4 variables. Assume the characteristic of $\mathbf{F}$ is not 3. Let $k = \mathbf{F}(t)$. Suppose the cubic surface $X \subset \mathbf{P}^3_k$ over k defined by $f + tg = 0$ is smooth. If there is no Brauer–Manin obstruction to the Hasse principle for rational points on X, then there exists a k-rational point on X.*

Proof. Combine Theorem 2.4 and Theorem 3.1. □

Remark 3.4. Again, it would be nice to avoid the cohomological machinery, i.e. Theorems 2.1 and 2.2. When X has no rational points over $\mathbf{F}(t)$ but points in all the completions of $\mathbf{F}(t)$ one should exhibit an explicit Brauer–Manin obstruction for X. For this purpose, it would probably be helpful to use [SD93]. Down to earth computations, which we shall not insert here, have led to the following result. If a smooth cubic surface X given by $f+tg = 0$ is a counterexample to the Hasse principle over $\mathbf{F}(t)$, then, after replacing $\mathbf{F}$ by its maximal pro-2-extension F, the following holds: When going over to the algebraic closure of F, the curve Γ in the proof of Theorem 3.1 breaks up as a sum of 9 conjugate lines, or a sum of three twisted cubics, or a sum of three conjugate conics plus a sum of three coplanar conjugate lines; when using the word "conjugate" we mean that the Galois action is transitive. Only in these three cases may we expect a Brauer–Manin obstruction.

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