From Posets to Spheres

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- Thin: All interval of length 2 have cardinality 4.

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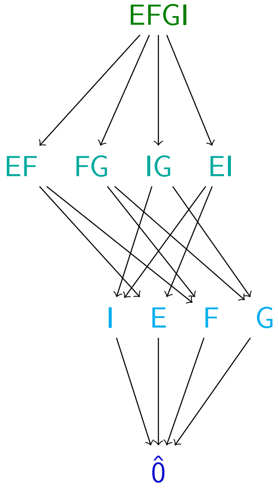
A graded poset



Regular cell complex

Definition

A *regular cell complex* Δ is a finite collection of balls σ in a Hausdorff space $\|\Delta\| = \bigcup_{\sigma \in \Delta} \sigma$ such that

(i) the interiors $\mathring{\sigma}$ partition $\|\Delta\|$ (i.e. every $x \in \|\Delta\|$ lies in exactly one $\mathring{\sigma}$), and

(ii) the boundary $\delta\sigma$ is a union of some members of Δ, for all σ in Δ.

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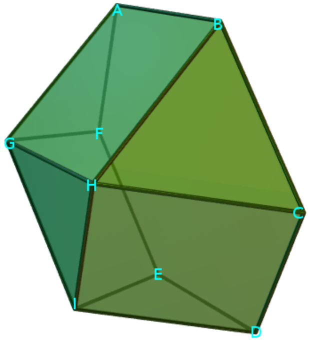


Some regular cell complex terminology

Let Δ be a regular cell complex.

- The balls σ in Δ are called the *closed cells* of Δ, their interiors $\mathring{\sigma}$ are the *open cells*.
- The space $\|\Delta\|$ is called the *underlying space* of Δ.
- If $T \cong \|\Delta\|$, then Δ is said to provide (via the homeomorphism) a *regular cell decomposition* of the space T.
- The *face poset* $\mathcal{F}(\Delta) = (\Delta, \leq)$ is the set of closed cells ordered by containment. The *augmented face poset* $\widehat{\mathcal{F}}(\Delta) = \mathcal{F}(\Delta) \cup \{\hat{0}, \hat{1}\}$ is the face poset enlarged by new elements such that $\hat{0} < \sigma < \hat{1}$ for all σ in Δ.
- The 0-cells and 1-cells are called *vertices* and *edges*, respectively.
- If $\sigma, \tau \in \Delta$ and $\sigma \subseteq \tau$ then σ is said to be a *face* of τ.

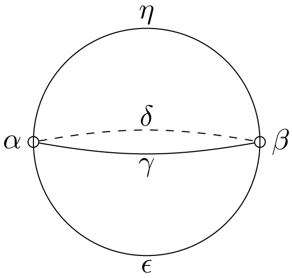
Some regular cell complex terminology

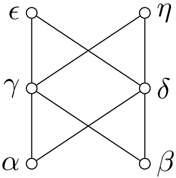
...

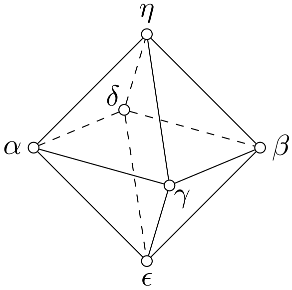
- $\Gamma \subseteq \Delta$ is a *subcomplex* of Δ if $\tau \in \Gamma$ implies that every face of τ also belongs to Γ.
- $\dim \Delta = \max_{\sigma \in \Delta} \dim \sigma$.
- Δ is *pure* if all maximal cells have the same dimension (i.e., every cell is contained in a $(\dim \Delta)$-dimensional cells).

Order complex associated with a poset

With any poset P we assosiate its *order compelex*, $\Delta_{ord}(P)$ as a simplicial complex whose vertices are the elements of P and whose simplices are the chains $x_0 < x_1 < \cdots < x_k$ in P.







Shellability

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Definition

Let Δ be a pure d-dimensional regular cell complex. A linear oredering $\sigma_1, \sigma_2 \dots \sigma_t$ of its maximal cells is called a *shelling* if either $d = 0$, or if $d \geq 1$ and the following conditions are satisfied:

1. $\delta\sigma_j \cap (\bigcup_{i=1}^{j-1} \delta\sigma_i)$ is pure and $(d-1)$-dimensional, for $2 \leq j \leq t$. (in other words, the intersection of the boundary of the j-th closed cell with the union of the boundary of the first $j-1$ cells,
2. $\delta\sigma_j$ has a shelling in which the $(d-1)$-cells of $\delta\sigma_j \cap (\cup_{i=1}^{j-1}\delta\sigma_i)$ come first, for $2 \leq j \leq t$, and
3. $\delta\sigma_1$ has a shelling.

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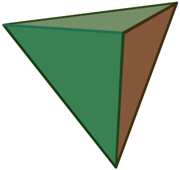

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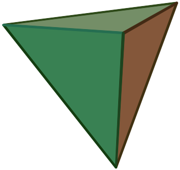

- 0-simplex: Point
- 1-simplex: Line segment
- 2-simplex: Triangle (with the interior)
- 3-simplex: Tertrahedron (with the interior)

Simplicial complex

Intutively, it is a topological space constructed by "gluing together" points, lines, triangles, and their n-dimensional couterparts.

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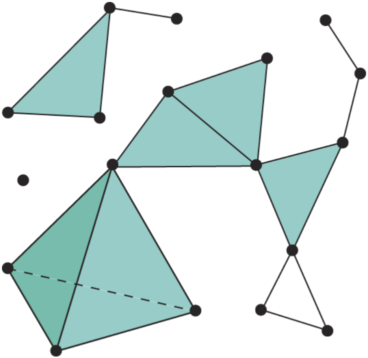


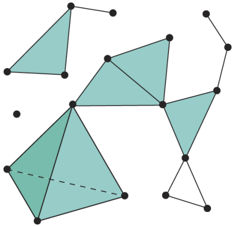
Simplicial complex

Definition

A (geometric) *simplicial complex* Δ is a set of simplices that satifies the following conditions:

(i) Any face of a simplex from Δ is also in Δ.

(ii) The intersection of any two simplices $\sigma_1, \sigma_2 \in \Delta$ is a face of both σ_1 and σ_2.



PL spheres

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Definition

A (geometric) simplicial complex Δ is a *PL d-ball* if it is "PL homeomorhpic" to the d-simplex. It is a *PL d-sphere* if it is PL homeomorphic to the boundary of the $(d + 1)$-simplex.

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Some PL topology results

Theorem

(i) *The union of two PL d-balls, whose intersection is a PL $(d-1)$-ball lying in the boundary of each, is a PL d-ball.*

(ii) *The union of two PL d-balls, which intersect along their boundaries, is a PL d-sphere.*

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