ON THE CONSERVATION LAWS OF THE DEFOCUSING CUBIC NLS EQUATION

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The defocusing cubic NLS equation

$$i\partial_t u + \partial_x^2 u = 2|u|^2 u \tag{1}$$

is well known to enjoy a Lax pair structure, discovered by Zakharov and Shabat in [4].

Given $u \in C^\infty(\mathbb{T}, \mathbb{C})$, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, we consider the following differential operators acting on $L^2(\mathbb{T}, \mathbb{C}^2)$,

$$L_u = \begin{pmatrix} -D & u \\ \overline{u} & D \end{pmatrix} \ , \ D := -i\partial_x \ ,$$

$$B_u = \begin{pmatrix} 2i\partial_x^2 - i|u|^2 & u' + 2u\partial_x \\ \overline{u}' + 2\overline{u}\partial_x & -2i\partial_x^2 + i|u|^2 \end{pmatrix} \ .$$

Notice that L_u is a selfadjoint operator with domain $H^1(\mathbb{T}, \mathbb{C}^2)$, therefore, for every $h > 0$, the operator $(I + h^2 L_u^2)^{-1}$ is well defined on $L^2(\mathbb{T}, \mathbb{C}^2)$ and valued into $H^2(\mathbb{T}, \mathbb{C}^2)$, hence is trace class as an operator on $L^2(\mathbb{T}, \mathbb{C}^2)$.

If $u = u(t, x)$ is a solution of (1), it also satisfies the Lax pair identity

$$\frac{d}{dt} L_{u(t)} = [B_{u(t)}, L_{u(t)}] \ .$$

From the above Lax pair identity, the quantity $\mathrm{Tr}\left[(I + h^2 L_{u(t)}^2)^{-1}\right]$ is independent on t. We are going to expand this quantity in powers of h as h goes to 0. The coefficients of this expansion are therefore conservation laws of (1). In particular, we are going to prove the following result.

Theorem 1. *For every $p \in \mathbb{N}$, there exists a polynomial function*

$$F_p = F_p(u, \dots, u^{(p-1)}) : (\mathbb{C} \simeq \mathbb{R}^2)^p \to \mathbb{R},$$

at most quadratic in the variable $u^{(p-1)}$, such that

$$E_p(u) := \int_{\mathbb{T}} \left[|u^{(p)}(x)|^2 + F_p(u(x), \dots, u^{(p-1)}(x)) \right] dx$$

is a conservation law of (1).

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Remarks

- *Using the Sobolev inequalities, Theorem 1 implies that, for every $p \geq 1$, for every H^p solution u of (1), the family $(u(t))_{t\in\mathbb{R}}$ is bounded in $H^p(\mathbb{T})$. In fact, using the Birkhoff coordinates introduced in Grébert–Kappeler [2], we know that u is an almost periodic function on $\mathbb{R}$ valued into $H^p(\mathbb{T})$. As a consequence, the family $(u(t))_{t\in\mathbb{R}}$ is relatively compact in $H^p(\mathbb{T})$.*
- *The boundedness of trajectories in $H^s(\mathbb{T})$ if $s \geq 1$ is not an integer, has been recently tackled in reference [3].*
- *The quantities $E_p(u)$ correspond to the conservation laws $I_{2p+1}(u)$ described in the book by Faddeev–Takhtajan [1].*

Proof. The operator

$$I + h^2 L_u^2 = (1 + h^2 D^2) I + h^2 M(x) \ , \ M(x) := \begin{pmatrix} |u|^2 & iu' \\ -i\overline{u}' & |u|^2 \end{pmatrix}$$

is a semiclassical differential operator $P(x, hD, h)$ with

$$P(x, \xi, h) = (1 + \xi^2) I + h^2 M(x) \ ,$$

therefore we can expand its inverse in powers of h as

$$(I + h^2 L_u^2)^{-1} \sim \sum_{j\geq 0} h^j A_j(x, hD) \ ,$$

where matrix valued symbols A_j are given by semiclassical pseudo-differential calculus. They are characterized by the following infinite system,

$$\begin{aligned} (1+\xi^2)A_0(x,\xi) &= I \ , \\ (1+\xi^2)A_1(x,\xi) - 2i\xi\partial_x A_0(x,\xi) &= 0 \ , \\ (1+\xi^2)A_j(x,\xi) - 2i\xi\partial_x A_{j-1}(x,\xi) + (M(x) - \partial_x^2)A_{j-2}(x,\xi) &= 0 \ , \ j \geq 2 \ . \end{aligned}$$

This leads to

$$\begin{aligned} A_0(x,\xi) &= \frac{1}{1+\xi^2} I \ , \ A_1(x,\xi) = 0 \ , \\ A_2(x,\xi) &= -\frac{1}{(1+\xi^2)^2} M(x) \ , \ A_3(x,\xi) = -\frac{2i\xi}{(1+\xi^2)^3}\partial_x M(x) \ , \end{aligned}$$

with the following induction formula,

$$A_j(x,\xi) = \frac{2i\xi}{1+\xi^2}\partial_x A_{j-1}(x,\xi) + \frac{1}{1+\xi^2}(\partial_x^2 - M(x))A_{j-2}(x,\xi) \ , \ j \geq 2 \ .$$

Then we get, as h tends to 0,

$$\mathrm{Tr}\left[(I + h^2 L_u^2)^{-1}\right] \sim \sum_{j\geq 0} h^j \, \mathrm{Tr} A_j(x, hD) \ .$$

Notice that

$$\mathrm{Tr}A(x,hD)=\sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{\mathbb{T}}\mathrm{tr}A(x,hn)\,dx=\frac{1}{2\pi h}\int_{\mathbb{T}}\mathrm{tr}A(x,\xi)\,dx\,d\xi+O(h^k)$$

for every $k\in\mathbb{N}$. Therefore each quantity

$$T_j(u):=\frac{1}{2\pi}\int_{\mathbb{T}\times\mathbb{R}}\mathrm{tr}A_j(x,\xi)\,dx\,d\xi$$

is a conservation law. In view of the above formulae, T_0 does not depend on u, $T_1=0$, while

$$T_2(u)=-\frac{1}{4}\int_{\mathbb{T}}\mathrm{tr}M(x)\,dx=-\frac{1}{2}\int_{\mathbb{T}}|u(x)|^2\,dx\ .$$

Then $T_3(u)=0$, while

$$T_4(u)=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{(1+\xi^2)^3}\,d\xi\int_{\mathbb{T}}\mathrm{tr}(M(x)^2)\,dx\ .$$

Since the eigenvalues of $M(x)$ are $|u(x)|^2\pm|u'(x)|$, we conclude

$$T_4(u)=\frac{3}{8}\int_{\mathbb{T}}(|u'(x)|^2+|u(x)|^4)\,dx\ .$$

This proves the above theorem for $p=0,1$, which corresponds to the mass and energy conservation laws. Let us compute the next conservation law. Again we have $T_5(u)=0$, but

$$\begin{aligned}T_6(u)&=-\frac{1}{2\pi}\int_{\mathbb{T}\times\mathbb{R}}\mathrm{tr}(M(x)A_4(x,\xi))\,dx\,d\xi\\&=\left[\frac{1}{2\pi}\int_{\mathbb{R}}\left(\frac{3}{(1+\xi^2)^4}-\frac{4}{(1+\xi^2)^5}\right)d\xi\right]\int_{\mathbb{T}}\mathrm{tr}(M'(x)^2)\,dx\\&\quad-\left[\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{(1+\xi^2)^4}\,d\xi\right]\int_{\mathbb{T}}\mathrm{tr}(M(x)^3)\,dx\ .\end{aligned}$$

We conclude

$$T_6(u)=-\frac{5}{32}\int_{\mathbb{T}}\left(|u''(x)|^2+(\partial_x(|u(x)|^2))^2+6|u(x)u'(x)|^2+2|u(x)|^6\right)\,dx\ .$$

This proves the case $p=2$. In order the prove the general case, we need the following lemma.

Lemma 1. *For every $j\geq 2$, there exist functions $c_{j,\alpha_1,\dots,\alpha_r}(\xi)$ such that*

$$A_j(x,\xi)=\sum_{2r+\alpha_1+\dots\alpha_r=j}c_{j,\alpha_1,\dots,\alpha_r}(\xi)\,\partial_x^{\alpha_1}M(x)\dots\partial_x^{\alpha_r}M(x)\ .$$

The proof of Lemma 1 is an easy induction on j, in view of the above induction formula for $A_j(x,\xi)$ and of the above formulae for $A_2(x,\xi)$, $A_3(x,\xi)$.

Let us complete the proof of Theorem 1. Let $p \geq 1$. From the induction formula, we infer

$$T_{2p+2}(u) = -\frac{1}{2\pi}\int\limits_{\mathbb{T}\times\mathbb{R}} \frac{\mathrm{tr}(M(x)A_{2p}(x,\xi))}{1+\xi^2}\,dx\,d\xi\ .$$

Next we apply Lemma 1 with $j = 2p$. The terms $\partial_x^{\alpha_1}M(x)\dots\partial_x^{\alpha_r}M(x)$ in the decomposition of $A_{2p}(x,\xi)$ are of three types :

(1) $r \geq 3$ and $\alpha_1+\dots+\alpha_r \leq 2p-6$. In this case, using integrations by parts, we obtain that the quantity

$$\int\limits_{\mathbb{T}} \mathrm{tr}(M(x)\partial_x^{\alpha_1}M(x)\dots\partial_x^{\alpha_r}M(x))\,dx$$

is a sum of terms of the form

$$\int\limits_{\mathbb{T}} \mathrm{tr}(\partial_x^{\beta_0}M(x)\partial_x^{\beta_1}M(x)\dots\partial_x^{\beta_r}M(x))\,dx$$

where $\beta_k \leq p-3$ for every k. In view of the expression of $M(x)$, these quantities can be written as

$$\int\limits_{\mathbb{T}} G(u(x),\dots,u^{(p-2)}(x))\,dx$$

where G is a polynomial function.

(2) $r = 2$ and $\alpha_1+\alpha_2 = 2p-4$. Then the quantity

$$\int\limits_{\mathbb{T}} \mathrm{tr}(M(x)\partial_x^{\alpha_1}M(x)\partial_x^{\alpha_2}M(x))\,dx$$

is a sum of terms either of the form $\int_{\mathbb{T}} \mathrm{tr}(\partial_x^{\beta_0}M(x)\partial_x^{\beta_1}M(x)\partial_x^{\beta_2}M(x))\,dx$, with $\beta_k \leq p-3$, or of the form $\int_{\mathbb{T}} \mathrm{tr}(M(x)(\partial_x^{p-2}M(x))^2)\,dx$, which can be written as

$$\int\limits_{\mathbb{T}} H(u(x),\dots,u^{(p-1)}(x))\,dx$$

when is a polynomial function, at most quadratic in $u^{(p-1)}$.

(3) $r = 1$ and $\alpha_1 = 2p-2$. Then, after integrating by parts, the contribution to $T_{2p+2}(u)$ is

$$(-1)^p\left[\frac{1}{2\pi}\int\limits_{\mathbb{R}} \frac{c_{2p,2p-2}(\xi)}{1+\xi^2}\,d\xi\right]\int\limits_{\mathbb{T}} \mathrm{tr}((\partial_x^{p-1}M(x))^2)\,dx\ .$$

Notice that

$$\begin{aligned}\mathrm{tr}((\partial_x^{p-1}M(x))^2) &= (\partial_x^{p-1}(|u(x)|^2) - |\partial_x^p u(x)|)^2 + (\partial_x^{p-1}(|u(x)|^2) + |\partial_x^p u(x)|)^2 \\ &= 2[(\partial_x^{p-1}(|u(x)|^2))^2 + |u^{(p)}(x)|^2] \ .\end{aligned}$$

Therefore, in order to complete the proof, we just have to check that the coefficient

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{c_{2p,2p-2}(\xi)}{1+\xi^2}\, d\xi$$

is not 0. Set

$$c_j(\xi) := c_{j,j-2}(\xi) \ , \ j \geq 2 \ .$$

The induction formula for $A_j(x,\xi)$ implies

$$c_j(\xi) = \frac{2i\xi}{1+\xi^2}c_{j-1}(\xi) + \frac{1}{1+\xi^2}c_{j-2}(\xi) \ , \ j \geq 4 \ .$$

Solving this linear induction problem in view of

$$c_2(\xi) = -\frac{1}{(1+\xi^2)^2} \ , \ c_3(\xi) = \frac{-2i\xi}{(1+\xi^2)^3} \ ,$$

we obtain

$$c_j(\xi) = -\frac{1}{2}\left(\frac{1}{(1+i\xi)(1-i\xi)^j} + \frac{(-1)^j}{(1-i\xi)(1+i\xi)^j}\right) \ .$$

Consequently,

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{c_{2p}(\xi)}{1+\xi^2}\, d\xi = -\frac{2p+1}{2^{2p+2}} \neq 0 \ .$$

This completes the proof. □

References

[1] Faddeev, L.D., Takhtajan, L.A., *Hamiltonian Methods in the Theory of Solitons*, Springer series in Soviet Mathematics, Springer, Berlin, 1987.

[2] Grébert, B., Kappeler, T., *The defocusing NLS equation and Its Normal Form*, EMS series of Lectures in Mathematics, European Mathematical Society, 2014.

[3] Kappeler, T., Schaad, B., Topalov, P., *Scattering like phenomena of the periodic defocusing NLS equation*, Preprint, May 2015, `arXiv:1505.07394v1` [math.AP].

[4] Zakharov, V. E., Shabat, A. B., *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media.* Soviet Physics JETP 34 (1972), no. 1, 62–69.

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