A remark on Delort's theorem.
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1. Microlocal Defect Measures.

Let (ω_n) be a weakly convergent bounded sequence of the Sobolev space $H^s_{loc}(\Omega)$, where ω is an open subset of $\mathbf{R}^d$. A microlocal defect measure (MDM) for this sequence (or H–measure, according to L.Tartar) is a positive Radon measure μ on $\Omega \times S^{d-1}$ such that, for every pseudodifferential operator on Ω, with kernel compactly supported in $\Omega \times \Omega$ (pdo), A, of order -2s,

$$(1) \qquad (A\omega_n, \omega_n) \to (A\omega, \omega) + \int_{\Omega \times S^{d-1}} \sigma(A)\, d\mu,$$

where ω is the weak limit of (ω_n), and $\sigma(A)$ denotes the principal symbol of A, which is a $(-2s)$–homogeneous C^∞ function on $\Omega \times \mathbf{R}^d \setminus \{0\}$.

Let us make some comments on the above definition. Observe that A maps $H^s_{loc}(\Omega)$ into $H^{-s}_{comp}(\Omega)$, so scalar products in formula (1) make sense.

For example, if $A = \phi(x)a(D)\psi(x)$, where $\phi, \psi \in C_0^\infty(\Omega)$, and $a \in C^\infty(\mathbf{R}^d$ is $(-2s)$–homogeneous for $|\xi| \geq 1$, we have

$$\sigma(x, \xi) = \phi(x)\psi(x)a(\xi),$$

and formula (1) is just Tartar's definition adapted to H^s sequences.

The advantage of dealing with more general pseudodifferential operators is that we dispose of a general "symbolic calculus", namely

i) If A is a pdo of order p and B is a pdo of order q, then AB is a pdo of order $p + q$, and

$$\sigma(AB) = \sigma(A)\sigma(B).$$

ii) If A is a pdo of order p, then A^* is a pdo of order p and

$$\sigma(A^*) = \bar{\sigma}(A).$$

Finally, let us emphasize that, up to the extraction of a subsequence, a given weakly convergent sequence (ω_n) always admits a MDM μ, and that $\mu = 0$ if and only if (ω_n) is relatively compact in $H^s_{loc}(\Omega)$.

2. An example arising in fluid mechanics.

Let Ω be an open subset in $\mathbf{R}^2$; let (u_n) be a weakly convergent sequence of $L^2_{loc}(\Omega, \mathbf{R}^2)$, satisfying $\mathrm{div} u_n = 0$. Set $\omega_n = \mathrm{rot} u_n$. For every $\phi \in C_0^\infty(\Omega)$, we have, by computing $\mathrm{rot}(\phi u_n)$ and $\mathrm{div}(\phi u_n)$ and using Fourier analysis,

$$\phi u_n = B\omega_n + r_n,$$

when r_n is weakly convergent in H^1 and B is a (vector–valued) pdo of order -1 , with

$$\sigma(B)(x,\xi) = \phi(x)|\xi|^{-2}(-i\xi_2,\xi_1).$$

By symbolic calculus and standard compactness arguments, we obtain, for every quadratic form q on $\mathbf{R}^2$,

$$\int \phi(x)q(u_n(x))\,dx - \int \phi(x)q(u(x))\,dx = (A\omega_n,\omega_n) - (A\omega,\omega) + o(1),$$

where u and ω denote weak limits, and A is a pdo of order -2, with

$$\sigma(A)(x,\xi) = -\phi(x)|\xi|^{-4}q(-\xi_2,\xi_1).$$

Hence we are in the situation of section 1 above, with s=-1; if μ is a MDM for (ω_n), we get

$$\int \phi(x)q(u_n(x))\,dx \to \int \phi(x)q(u(x))\,dx - \int_{\Omega\times S^1} \phi(x)q(-\xi_2,\xi_1)\,d\mu(x,\xi). \tag{2}$$

Delort's theorem precisely says that, if $\omega_n \geq 0$, if q satisfies $\int_{S^1} q\,ds = 0$, then we can pass to the limit in $q(u_n)$, i.e. the last term in formula (2) cancels.

3. A version of Delort's theorem.

In terms of MDM, Delort's theorem can be rephrased as follows,

Theorem 1. *Let Ω be an open subset of $\mathbf{R}^d$ and let (ω_n) be a weakly convergent sequence in $H^{-d/2}_{loc}(\Omega)$. Assume that there exists a sequence (m_n) of positive Radon measures on Ω and a locally uniformly integrable sequence (f_n) in $L^1_{loc}(\Omega)$, such that $\omega_n = m_n + f_n$. Then the MDM μ of (ω_n) satisfies, for some measure ν on Ω,*

$$d\mu(x,\xi) = d\nu(x)\,d\sigma(\xi),$$

where σ denotes the Lebesgue measure on S^{d-1}.

Proof. It does not differ substantially from Delort's one.

Step 1. Let ω be a complex Radon measure on Ω. We assume that $\omega \in H^{-d/2}_{loc}(\Omega)$. Then, for every $x_0 \in \Omega$, we have $\omega(\{x_0\}) = 0$.
Indeed, if $\phi \in C_0^\infty(\Omega), \phi(0) = 1$, we have, by the dominated convergence theorem,

$$\int \phi(\frac{x-x_0}{r})\,d\omega(x) \to \omega(\{x_0\}) = 0 \tag{3}$$

as r goes to 0. On the other hand, it is easy to check that $\phi((x-x_0)/r)$ is bounded in $H^{d/2}_{comp}$ as r goes to zero and converges weakly to 0 in the sense of distributions. Hence the right hand side of formula (3) is 0. In particular, if $\omega = m + f$, where m is a positive Radon measure and f is locally integrable, then for every $x_0 \in \Omega$, we have $m(\{x_0\}) = 0$.

Step 2. Let A be a pdo of order $-d$ on Ω. Then the kernel of A is $K(x,y) = k(x, x-y)$, where $k(x,z)$ is a compactly supported distribution and is smooth for $z \neq 0$. Explicitly, if $A = a(x, D)$, we have

$$k(x,z) = \int_{\mathbf{R}}^{d} e^{iz\xi} a(x,\xi)\, d\xi/(2\pi)^d$$

in the sense of Fourier transform of temperate distributions, where a is a smooth function such that, for $|\xi|$ large enough,

$$|a(x,\xi) - \sigma(A)(x,\xi)| \leq C|\xi|^{-d-1}.$$

An elementary calculation using polar coordinates gives

$$|k(x,z) - (2\pi)^{-d} \log(1/|z|) \int_{S^{d-1}} \sigma(A)\, d\sigma| \leq C'.$$

Assume that $\int_{S^{d-1}} \sigma(A)\, d\sigma = 0$. Then the kernel of A is bounded.

Step 3. Let ω, A be a in steps 1 and 2. By step 1 and Fubini's theorem, $|\omega| \otimes |\omega|(\{x = y\}) = 0$. Therefore, since by step 2 the kernel K of A is bounded, compactly supported and continuous outside $\{x = y\}$, we can define the quantity

$$(4). \qquad Q(\omega) = \int\int K(x,y)\, d\omega(x)\, d\bar{\omega}(y)$$

If ω is smooth, we have clearly $(A\omega, \omega) = Q(\omega)$. We claim that this identity still holds if ω is a complex measure belonging to $H_{loc}^{-d/2}(\Omega)$. Indeed, this is a consequence of the following easy lemma,

Lemmma 1. *Let X be a locally compact, σ-compact, metrizable space. Let (μ_n) be a sequence of complex Radon measures on X, such that $\mu_n \to \mu$ for the vague topology. Assume that $|\mu_n| \leq \nu_n$, where (ν_n) is a sequence of positive Radon measures on X, weakly convergent to ν. Then, for any bounded Borel function h, compactly supported in X, continuous outside of some closed set F such that $\nu(F) = 0$, we have*

$$\int_X h\, d\mu_n \to \int_X h\, d\mu.$$

We leave the proof of this lemma to the reader. To prove the claim, just apply the lemma to $X = \Omega \times \Omega, h = K, F = \{x = y\}, \mu_n = \rho_n * \omega \otimes \rho_n * \omega, \nu_n = \rho_n * |\omega| \otimes \rho_n * |\omega|$, where ρ_n is a smooth nonnegative approximation of δ.

Step 4. We are now ready to prove the theorem. Denote by ω the weak limit of (ω_n). By Dunford–Pettis criterion, we may assume that f_n tends to f and $|f_n|$ tends to g, weakly in L^1_{loc}. Then, for the vague topology of positive measures, we have $m_n \to m = \omega - f$. To prove the theorem, it is enough to check that, for every pdo A of order $-d$, such that, for every $x \in \Omega$, the integral of $\int_{S^{d-1}} \sigma(x,\xi)\, d\sigma(\xi) = 0$, we have

$$\int_{\Omega \times S^{d-1}} \sigma(x,\xi)\, d\mu(x,\xi) = 0.$$

By step 3 and the definition of an MDM, this is equivalent to

$$\int\int K(x,y)\, d\omega_n(x)\, d\bar{\omega}_n(y) \to \int\int K(x,y)\, d\omega(x)\, d\bar{\omega}(y).$$

But the latter formula is a consequence of lemma 1 above, with $X = \Omega \times \Omega, h = K, F = \{x = y\}, \mu_n = \omega_n \otimes \omega_n, \nu_n = (m_n + |f_n|) \otimes (m_n + |f_n|), \nu = (m+g) \otimes (m+g)$, since by step 1 and Fubini's theorem, $\nu(\{x = y\}) = 0$. This completes the proof.

4. An elementary example.

Set $u_\epsilon(x) = \epsilon^{-d}\psi(x/\epsilon)$, where $\psi \in L^2(\mathbf{R}^d)$ is compactly supported. Then an elementary computation shows that

$$||u_\epsilon||_{H^{-d/2}} = (2\pi)^{-d}|\hat{\psi}(0)|^2 \log(1/\epsilon) + O(1).$$

Hence, if $\int \psi\, dx = 0$, (u_ϵ) is a bounded sequence of $H^{-d/2}$ and L^1, and another calculation shows that its MDM is given by

$$\mu = \delta(x) \otimes (2\pi)^{-d} f(\xi)\, d\xi, \quad f(\xi) = \int_0^\infty |\hat{\psi}(r\xi)|^2\, \frac{dr}{r}.$$

Observe that in general this measure is not equidistributed on the sphere, thus the above theorem cannot be generalized to complex measures.Also notice that, because of the condition on the integral of ψ, $u_\epsilon \geq 0$ implies $\psi = 0$.

To cover the case of nonnegative sequences, set $v_\epsilon = |\log \epsilon|^{-1/2} u_\epsilon$. Then the MDM of (v_ϵ) is

$$\mu' = \delta(x) \otimes (2\pi)^{-d} |\hat{\psi}(0)|^2\, d\sigma(\xi),$$

which is consistent with the theorem.

5. Applications to non–elliptic equations.

The aim of Delort's theorem was to study the oscillations of a sequence of the bounded sequence (ψ_n) in H^1 defined by

$$\Delta\psi_n = \omega_n,$$

and it is easy to relate MDM for ψ_n to MDM for ω_n. Actually we obtain informations of quite different kind when dealing with a non–elliptic equation $Pu_n = \omega_n$ .This is the purpose of the following result.

Theorem 2. *Let P be a differential operator of order m on an open set Ω in $\mathbf{R}^d$. Assume that the principal symbol p of P satisfies, for every $x \in \Omega$,*

$$\int_{S^{d-1}} \frac{d\sigma(\xi)}{|p(x,\xi)|^2} = +\infty.$$

Let (u_n) be a bounded sequence of $H^{m-d/2}_{loc}(\Omega)$ such that $Pu_n \geq 0$ (or more generally $Pu_n = \omega_n$, where the sequence (ω_n) satisfies the assumptions of theorem 1). Then the sequence (Pu_n) is relatively compact in $H^{-d/2}_{loc}(\Omega)$.

Proof. By theorem 1, the MDM of (Pu_n) is $\nu_x \otimes \sigma_\xi$, and, by symbolic calculus, it is also $|p|^2\mu$, where μ is a MDM for (u_n). Hence, in the space of positive measures on the Borel σ–algebra of $\Omega \times S^{d-1}$, we have

$$\mu_x, \xi \geq \frac{1}{|p(x,\xi)|^2} \nu_x \otimes \sigma_\xi .$$

Since $\mu(K \times S^{d-1}) < +\infty$ for every compact subset K of Ω, we conclude, in view of the condition on p and by Fubini's theorem, that $\nu = 0$, which completes the proof.

Let us end these remarks by listing some applications of theorem 2, which can be seen as "one–sided compensated compactness results."

i) In $\mathbf{R}^2_{t,x}$, if u_n is bounded in H^1_{loc} satisfies $(\partial_t^2 - \partial_x^2)u_n \geq 0$, then one can pass to the limit in $|\partial_t u_n|^2 - |\partial_x u_n|^2$.

ii) In $\mathbf{R}^2_{t,x}$, if u_n is bounded in L^2_{loc} and satisfies $\partial_t u_n \geq 0, \partial_x u_n \geq 0$, then u_n belongs to a compact subset of L^2_{loc}.

iii) In $\mathbf{R}^4_{t,x}$, if u_n is bounded in L^2_{loc} and satisfies $(\partial_t^2 - \Delta_x)u_n \geq 0$, then one can pass to the limit in $|R_t u_n|^2 - |R_x u_n|^2$, where R denotes the Riesz transform.

References.

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