1 Appendix : Isoperimetric profiles of wedges

1.1 Itai Benjamini's problem

Definition. Let M be a Riemannian manifold with boundary. Define the *isoperimetric profile* of M as the largest function $I : (0, vol(M)) \to \mathbf{R}_+$ such that for every compact set $S \subset M$,

$$area((\partial S) \setminus (\partial M)) \geq I(vol(S)).$$

Classical example. When M is a Euclidean ball of radius R,

$$I(v) \geq \text{const}.R^{-1} \min\{v, vol(M) - v\}.$$

Question. Let $f : \mathbf{R}_+ \to \mathbf{R}_+$ be nondecreasing. Assume f grows less than linearly. Consider the wedge in R^3,

$$W_f = \{(x, y, z) \,|\, x > 0,\, 0 < z < f(x)\}.$$

View W_f as a manifold with boundary. What is the isoperimetric profile of W_f ?

Guess. Since, at large scales, W_f looks like a thickened half-plane, minimizers should look like thickened disks.

1.2 The result

Let $C_R = W_f \cap \{x^2 + y^2 < R^2\}$ be the thickened disk of radius R in W_f. Then

$$\frac{1}{3}R^2 f(R) < vol(C_R) < \frac{\pi}{2}R^2 f(R)$$

and

$$2Rf(R) < area(\partial C_R) < \pi R f(R).$$

Furthermore,

$$\lim_{R\to+\infty} \frac{vol(C_R)}{\frac{\pi}{2}R^2 f(R)} = 1, \quad \lim_{R\to+\infty} \frac{area(\partial C_R)}{\pi R f(R)} = 1.$$

Notation. Denote by $R(v)$ the solution of $\frac{\pi}{2}R^2 f(R) = v$. Set $J(v) = \pi R(v) f(R(v))$. By definition of the isoperimetric profile I of the wedge,

$$area(\partial C_R) \geq I(vol(C_R)).$$

This implies

$$\limsup_{v \to +\infty} \frac{I(v)}{J(v)} \leq 1.$$

Conjecture. *The isoperimetric profile I of the wedge satisfies*

$$\lim_{v \to +\infty} \frac{I(v)}{J(v)} = 1.$$

In this appendix, only a weaker statement is proven. Fortunately, it is sufficient for Itai's needs.

Theorem 1 *Let $f : \mathbf{R}_+ \to \mathbf{R}_+$ be nondecreasing. Assume $x \mapsto f(2x)/f(x)$ and $x \mapsto f(x)/x$ are bounded near $+\infty$. Then for v large enough, the isoperimetric profile I of the wedge $W_f = \{(x, y, z) \,|\, x > 0,\, 0 < z < f(x)\}$ satisfies*

$$I(v) \geq const.\, J(v).$$

Remark. A concave function satisfies the assumptions of Theorem 1.

The proof of Theorem 1 occupies the rest of the appendix. The main trick consists in cutting the wedge into portions Q_j which are biLipschitz equivalent to *beams*, i.e. products of rectangles with lines. By a reflection principle, isoperimetry of beams reduces to isoperimetry of flat manifolds of the form torus$\times$line, which is known. The edge of the wedge requires a specific treatment.

Remark. If f increases faster than linearly, then one expects the isoperimetric profile of W_f to be $I(v) \sim v^{2/3}$. It is so, for instance, if f has its derivative bounded below, since then W_f is biLipschitz to a half-space.

1.3 Profiles for flat manifolds

Notation. $M_{L,h} = \mathbf{R}/L\mathbf{Z} \times \mathbf{R} \times \mathbf{R}/h\mathbf{Z}$. We use the following result due to L. Hauswirth, J. Pérez, P. Romon, and A. Ros, [HPRR] (see also [RR]).

Proposition 1 ([HPRR]). *If $L \geq \frac{9\pi}{16}h$, the isoperimetric profile of $M_{L,h}$ is achieved by spheres, cylinders or strips bounded by parallel planes. In other words, for every $S \subset M_{L,h}$,*

$$\begin{aligned} vol(S) \leq \frac{4\pi}{81}h^3 \quad &\Rightarrow \quad area(\partial S) \geq (36\pi)^{1/3}\, vol(S)^{2/3}, \\ \frac{4\pi}{81}h^3 \leq vol(S) \leq \frac{1}{\pi}L^2 h \quad &\Rightarrow \quad area(\partial S) \geq (4\pi)^{1/2} h^{1/2}\, vol(S)^{1/2}, \\ vol(S) \geq \frac{1}{\pi}L^2 h \quad &\Rightarrow \quad area(\partial S) \geq 2Lh. \end{aligned}$$

Corollary 2 *Assume $L \geq h$. For $S \subset M_{L,h}$,*

$$\begin{aligned} vol(S) \leq \frac{1}{32\pi}\, h^3 \quad &\Rightarrow \quad area(\partial S) \geq \frac{1}{4}\, vol(S)^{2/3}. \\ vol(S) \leq \frac{1}{2\pi}\, L^2 h \quad &\Rightarrow \quad area(\partial S) \geq \frac{1}{2}\, L^{-1} vol(S). \end{aligned}$$

Proof. First, assume that $L \geq \frac{9\pi}{16}h$. If $vol(S) \geq \frac{4\pi}{81}h^3$, then

$$\begin{aligned} area(\partial S) &\geq (4\pi)^{1/2} h^{1/2}\, vol(S)^{1/2} \\ &= (4\pi)^{1/2} (\frac{L^2 h}{vol(S)})^{1/2} L^{-1} vol(S) \\ &\geq 2\, L^{-1} vol(S). \end{aligned}$$

Otherwise, if $\frac{4\pi}{81}h^3 \leq vol(S) \leq \frac{1}{\pi}L^2 h$,

$$\begin{aligned} area(\partial S) &\geq (36\pi)^{1/3}\, vol(S)^{2/3} \\ &= (36\pi)^{1/3} (\frac{L^3}{vol(S)})^{1/3}\, L^{-1} vol(S) \\ &\geq (288\pi)^{1/3} (\frac{h^3}{vol(S)})^{1/3}\, L^{-1} vol(S) \\ &\geq 18\, L^{-1} vol(S). \end{aligned}$$

In both cases, $area(\partial S) \geq 2\, L^{-1} vol(S)$.

If $h \leq L \leq \frac{9\pi}{16}h$, use the affine 2-biLipschitz diffeomorphism of $M_{L,h}$ to $M_{L',h}$ with $L' = 2L \geq \frac{9\pi}{16}h$. This sends S to S' with $vol(S') \leq 8\, vol(S)$.

If $vol(S) \leq \frac{1}{32\pi}\, h^3$ then $vol(S') \leq \frac{4\pi}{81}\, h^3$, thus $area(\partial S') \geq (36\pi)^{1/3} vol(S')^{2/3}$ and finally $area(\partial S) \geq \frac{1}{4}\, vol(S)^{2/3}$.

If $vol(S) \leq \frac{1}{2\pi}\, L^2 h$ then $vol(S') \leq \frac{1}{\pi}\, L'^2 h$, thus $area(\partial S') \geq 2\, L'^{-1} vol(S')$ and finally $area(\partial S) \geq \frac{1}{2}\, L^{-1} vol(S)$.$\diamond$

1.4 Selecting a scale R

Let f be a nondecreasing function on $\mathbf{R}_+$. Assume that $f(2x)/f(x) \leq \lambda$ for x large enough. Assume too that $f(x)/x$ is bounded near $+\infty$. Up to multiplying f by a constant (which only changes the constant in the final result), one can assume that $f(x)/x \leq 1$ for x large enough.

Given $v > 0$, let R be defined by $R^2 f(R) = 16\pi\lambda^3 v$. Fix v large enough so that $f(R)/R \leq 1$ and $f(2R)/f(R) \leq \lambda$.

Note that $R \leq 16\lambda^{3/2} R(v)$. Indeed, if n is the smallest integer larger than $\log(64\lambda^3)/\log 4$, then $4^{n-1} \leq 64\lambda^3 \leq 4^n$,

$$\frac{\pi}{2}(2^{-n}R)^2 f(2^{-n}R) \leq \frac{\pi}{2}\, 4^{-n} R^2 f(R) \leq 16\pi\lambda^3 R^2 f(R) = v = \frac{\pi}{2} R(v)^2 f(R(v)),$$

which implies $2^{-n}R \leq R(v)$. By construction, $2^n \leq (256\lambda^3)^{1/2}$, thus $R \leq 16\lambda^{3/2} R(v)$.

1.5 Cutting sets into 3 pieces

Let $S \subset W_f$ have volume v. The goal is to estimate $area(\partial S)$ by v/R. Let k be the largest integer such that $f(kR) \leq R$.

Split S into 3 parts,

$$A = S \cap \{0 < x < R\}, \quad B = S \cap \{R < x < (k+1)R\},$$

$$C = S \cap \{x > (k+1)R\}.$$

1.6 Estimate for $vol(A)$

Consider the projection along the first factor $proj(x, y, z) = (y, z)$. It decreases areas. Note that the restriction of $proj$ to ∂S has global degree zero.

At points of $\partial S \cap \partial W_f$, $proj$ has local degree 1. Therefore every point of $T = proj(\partial S)$ is the projection of at least one point of ∂S which does not belong to ∂W_f, $T = proj(\partial S \cap W_f)$. It follows that $area(T) \leq area(\partial S \cap W_f)$. Now $A \subset [0, R] \times T$, thus

$$vol(A) \leq R\, area(T) \leq R\, area((\partial S) \cap W_f).$$

1.7 Chopping and reflecting

Chop B and C into pieces

$$P_j = S \cap \{jR \leq x \leq (j+1)R\},$$

so that

$$B = \bigcup_{j \leq k} P_j, \quad C = \bigcup_{j \geq k+1} P_j.$$

P_j is contained in a portion $Q_j = W_f \cap \{jR \leq x \leq (j+1)R\}$ which is biLipschitz to a beam $B_{R,f(jR)} = \{jR \leq x \leq (j+1)R,\, 0 \leq z \leq f(jR)\}$, where the Lipschitz constant is at most $f((j+1)R)/f(jR) \leq \lambda$. Reflect P_j in the hyperplanes $\{x = jR\}$, $\{z = 0\}$, and map it to the quotient manifold M_{L_j,h_j} with $L_j = 2R$, $h_j = 2f(jR)$. The obtained set $S_j \subset M_{L,h}$ has

$$4\lambda^{-3} \leq \frac{vol(S_j)}{vol(P_j)} \leq 4\lambda^3, \quad 4\lambda^{-2} \leq \frac{area(\partial S_j)}{area(\partial P_j \setminus \partial Q_j)} \leq 4\lambda^2.$$

In particular, for all j, $vol(S_j) \leq 4\lambda^3 vol(S)$.

1.8 Estimate for $vol(B)$

For $j \leq k$, $vol(S_j) \leq \frac{1}{4\pi} R^2 f(R) \leq \frac{1}{2\pi} L_j^2 h_j$ and $h_j \leq L_j$. Corollary 2 gives

$$area(\partial S_j) \geq \frac{1}{2}\,(2R)^{-1} vol(S_j).$$

It follows that

$$\begin{aligned} vol(P_j) &\leq \frac{1}{4}\lambda^3 vol(S_j) \\ &\leq 2\lambda^3 R\, area(\partial S_j) \\ &\leq 8\lambda^5 R\, area(\partial P_j \setminus \partial Q_j), \end{aligned}$$

and finally

$$vol(B) \leq 8\lambda^5 R\, area((\partial S) \cap W_f).$$

Combining estimates for A and B leads to

$$\begin{aligned} vol(A) + vol(B) &\leq (8\lambda^5 + 1) R\, area((\partial S) \cap W_f) \\ &\leq 9\lambda^5\, R\, area((\partial S) \cap W_f). \end{aligned}$$

1.9 Estimate for $vol(C)$

For $j \geq k+1$, $vol(S_j) \leq \frac{1}{4\pi} R^2 f(R) \leq \frac{1}{32\pi}(2R)^3 \leq \frac{1}{32\pi} \min\{h_j, L_j\}^3$. Corollary 2 gives

$$area(\partial S_j) \geq \frac{1}{4}\, vol(S_j)^{2/3}.$$

It follows that

$$\begin{aligned} vol(P_j)^{2/3} &\leq (\frac{1}{4})^{2/3} \lambda^2 vol(S_j)^{2/3} \\ &\leq 4^{1/3} \lambda^2\, area(\partial S_j) \\ &\leq 4^{-2/3} \lambda^4\, area(\partial P_j \setminus \partial Q_j), \end{aligned}$$

and finally

$$\begin{aligned} vol(C)^{2/3} &= (\sum_{j \geq k+1} vol(P_j))^{2/3} \\ &\leq \sum_{j \geq k+1} vol(P_j)^{2/3} \\ &\leq 4^{-2/3} \lambda^4 \sum_{j \geq k+1} area(\partial P_j \setminus \partial Q_j) \\ &\leq 4^{-2/3} \lambda^4\, area((\partial S) \cap W_f). \end{aligned}$$

1.10 Conclusion

Let $\tau = vol(C)$, $v = vol(S)$. We have shown that

$$area((\partial S) \cap W_f) \geq \frac{1}{2}(4^{2/3} \lambda^{-4} \tau^{2/3} + \frac{1}{9} \lambda^{-5} R^{-1} (v - \tau)).$$

The right hand side is a concave function of $\tau \in [0, v]$, which achieves its minimum either at 0 or at v. One concludes that

$$area((\partial S) \cap W_f) \geq \min\{2^{1/3}\lambda^{-4}v^{2/3}, \frac{1}{18}\lambda^{-5}R^{-1}v\}.$$

It turns out that the minimum is $\frac{1}{18}\lambda^{-5}R^{-1}v$. Therefore

$$\begin{aligned} area((\partial S) \cap W_f) &\geq \frac{1}{24}\lambda^{-5}R^{-1}v \\ &\geq \frac{1}{24}\lambda^{-5}\frac{1}{16}\lambda^{-3/2}R(v)^{-1}v \\ &= \frac{1}{776}\lambda^{-13/2}J(vol(S)). \end{aligned}$$

1.11 Remark

Only a coarser version, with unsharp constants, of the result of [HPRR] is needed. A coarser proof can be given, along the following lines.

For every Riemannian manifold with a cocompact isometry group, extremal domains exist, so that the profile never vanishes. Extremal domains with small enough volume look like small balls. In a flat manifold, they are exactly balls. This occurs at a scale of the order of the injectivity radius. This gives the first statement of Corollary 2, with an unsharp constant c,

$$vol(S) \leq c\,h^3 \quad \Rightarrow \quad area(\partial S) \geq (36\pi)^{1/3}\,vol(S)^{2/3}.$$

Up to rescaling, one can assume that $L = 1$. Up to biLipschitz equivalence, one can assume that $1/h$ is an integer. Let $M_{1,1} \to M_{L,h}$ be the $1/h$-fold cover of $M_{L,h}$. If $S \subset M_{L,h}$ has volume $c\,L^2h$, its inverse image $\tilde{S}$ in $M_{1,1}$ has volume $\leq c$, thus

$$\frac{L}{h}\,area(\partial S) = area(\partial \tilde{S}) \geq (36\pi)^{1/3}\ vol(\tilde{S})^{2/3} = (36\pi)^{1/3}\ vol(S)^{2/3}(\frac{L}{h})^{2/3}.$$

Since $vol(S) \leq c\,L^2h$, $\frac{h}{L} \geq vol(S)/cL^3$, $(\frac{h}{L})^{1/3}vol(S)^{2/3} \geq vol(S)/c^{1/3}L$, which leads to

$$area(\partial S) \geq \text{const.}\,L^{-1}vol(S),$$

as announced.$\diamond$

References

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