Submanifolds and differential forms in Carnot groups

After M. Gromov and M. Rumin

Carnot manifolds

Definition. Let M be a manifold, $H \subset TM$ a subbundle. Assume that iterated Lie brackets of sections of H generate TM. Call these data a Carnot manifold.

Choose euclidean metrics on fibers of H. Minimizing lengths of horizontal curves defines a Carnot metric.

Problem. How far can Carnot manifolds be from Riemannian metrics.

Example : Heisenberg group

Heis^3 = 3×3 unipotent real matrices,
H = kernel of left-invariant
1-form dz-ydx .

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

Choose left-invariant metric
dx^2+dy^2. Then dilations $\delta_\varepsilon(x,y,z) \longrightarrow (\varepsilon x, \varepsilon y, \varepsilon^2 z)$
multiply distance by ε.

$\Longrightarrow$ Hausdorff dimension = 4.

Carnot groups

Definition. A Carnot group is a Lie group G equipped with one-parameter group of automorphisms δ_ε such that

$V^1=\{v\in Lie(G) \mid \delta_\varepsilon v=\varepsilon v\}$ generates $Lie(G)$.

Take H = left-translated V^1. Then left-invariant Carnot metrics are δ_ε-homogeneous.

Define $V^{i+1}=[V^i, V^1]$. Then

Hausdorff dimension = $\sum i \, dim(V^i)$.

Equiregular Carnot manifolds

Definition. Given $H \subset TM$ and $x \in M$, let $H^i(x)$ = subspace of T_xM generated by values at x of i-th order iterated brackets of sections of H. Say H is <u>equiregular</u> if $dim(H^i)$ is constant.

Example. $M=\mathbb{R}^3$, H generated by ∂_x, $\partial_y + x^2\partial_z$, is not equiregular.

Tangent cones of equiregular Carnot manifolds

Theorem (Nagel-Stein-Wainger, Mitchell). An equiregular Carnot manifold is asymptotic to a Carnot group G_x at each point $x \in M$. In particular, Hausdorff dimension = $\sum i\, (dim(H^i) - dim(H^{i-1}))$.

Example. In dimension 3, equiregular ⇔ contact. Then $G_x = Heis^3$. In dimension 5, if $codim(H)=1$, equiregular ⇔ contact (then $G_x = Heis^5$) or $H = ker(\alpha)$ with $d\alpha$ of rank 2 (then $G_x = Heis^5 \times \mathbb{R}^2$).

BiLipschitz equivalence

Theorem. Two Carnot groups are biLipschitz homeomorphic (resp. quasiconformally homeomorphic) if and only if they are isomorphic.

Theorem (Mostow-Margulis). If $f{:}M \rightarrow M'$ is a quasiconformal homeomorphism, then for all $x \in M$, $G'_{f(x)}$ is isomorphic to G_x .

Hölder equivalence

Theorem (Rashevsky, Chow,..). A Carnot manifold is α-Hölder-homeomorphic to a Riemannian manifold, $\alpha \geqslant 1/r$ if $H^r = TM$.

Remark. A Carnot manifold of dimension n and Haudorff dimension d is not α-Hölder-homeomorphic to a Riemannian metric if $\alpha > n/d$.

Question. What is the best α , $\alpha(M,H)$?

Example. $1/2 \leqslant \alpha(Heis^3) \leqslant 3/4$.

The case of $Heis^3$ (1/2)

Theorem. $\alpha(Heis^3) \leqslant 2/3$. i.e. $Heis^3$ is not α-Hölder homeomorphic to $\mathbb{R}^3$ for $\alpha > 2/3$.

Lemma. Every topological surface in $Heis^3$ has Hausdorff dimension $\geqslant 3$.

Theorem follows : $f \in C^\alpha : \mathbb{R}^3 \to Heis^3 \Rightarrow$

$$3 \leqslant dim_{Hau} f(S) \leqslant dim_{Hau} S / \alpha = 2/\alpha .$$

The case of $Heis^3$ (2/2)

Proof of Lemma.

1. topdim(S) $\geqslant 2 \Rightarrow \exists$ continuous curve c such that every neighboring curve intersects S.
2. Can take c smooth, embedded and horizontal.
3. Insert c in a smooth submersion $p: Heis^3 \to \mathbb{R}^2$ with horizontal fibers.
4. Tube generated by ε-ball has volume $\leqslant C\tau\,\varepsilon^3$.
5. Cover S with ε_j-balls. Corresponding tubes cover a fixed open set.
6. $\sum_j \varepsilon_j^3 \geqslant Vol(\cup \text{Tubes})/C$ is bounded away from 0. Therefore $dim_{Hau} S \geqslant 3$.

Results to be covered

a. Two proofs of isoperimetric inequality.
b. An existence result for horizontal submanifolds.
c. A Carnot version of de Rham theorem.
d. Applications to Hölder equivalence.

Contents

Isoperimetric inequality

Theorem. Let K be a compact subset in an equiregular Carnot manifold of Haudorff dimension d. There exist constants c and C such that for every domain D in K,

$$\mathcal{H}^d(D) \leqslant c \Rightarrow \mathcal{H}^d(D) \leqslant C\ \mathcal{H}^{d-1}(\partial D)^{d/d-1}.$$

Corollary. $\alpha(M) \leqslant (n-1)/(d-1)$.

Proof of Carnot isoperimetric inequality (1/4)

Flow tube estimates. Let X be a smooth horizontal vector field, B an ε-ball, T the tube swept by B in time τ under the flow of X. Then

$$\mathcal{H}^d(T) \leqslant \mathrm{const}(X,K)\ \tau/\varepsilon\ \mathcal{H}^d(B).$$

For the tube $T(S)$ swept by a hypersurface S,

$$\mathcal{H}^d(T(S)) \leqslant \mathrm{const}(X,K)\ \tau\ \mathcal{H}^{d-1}(S).$$

Choose smooth horizontal vector fields $X_1,\ldots,X_k$ such that the « iterated orbit » of any point $m \in K$ under them in time τ contains $B(m, \tau)$ and is contained in $B(m,\lambda\tau)$, $\lambda = \mathrm{const}(K)$.

Proof of Carnot isoperimetric inequality (2/4)

Local isoperimetric inequality. For every ball B of radius $R \leqslant \mathrm{const}(K)$, such that $B' = \lambda B \subset K$, and for every subset $D \subset K$ with $\mathcal{H}^d(D) \leqslant \mathcal{H}^d(B)/2$,

$$\mathcal{H}^d(D \cap B) \leqslant \mathrm{const}(K)\, R\, \mathcal{H}^{d-1}(\partial D \cap B').$$

Indeed, one of the fields X_j moves a proportion const(K) of the measure of $D \cap B$ outside $D \cap B$ in time $\tau = 2R$. Thus the X_j-tube in time $2R$ of $\partial D \cap B'$ contains const$(K)\mathcal{H}^d(D \cap B)$.

Proof of Carnot isoperimetric inequality (3/4)

Covering lemma. If $\mathcal{H}^d(D) \leqslant$ const(K), there exists a collection of disjoint balls B_j such that

- D is covered by concentric balls $2B_j$.
- $\mathcal{H}^d(D \cap \lambda^{-1} B_j) \geqslant \lambda^d \mathcal{H}^d(\lambda^{-2} B_j)/2$.
- $\mathcal{H}^d(D \cap B_j) \leqslant \lambda^d \mathcal{H}^d(\lambda^{-1} B_j)/2$.

Indeed, given $m \in D$, let $B(m)$ be the last of the balls $B(m,\lambda^{-i})$ to satisfy $\mathcal{H}^d(D \cap B) \leqslant \lambda^d \mathcal{H}^d(\lambda^{-1} B)/2$.

Then let B_0 be the largest $B(m)$, B_1 the largest which is disjoint from B_0, …

Proof of Carnot isoperimetric inequality (4/4)

End of proof. Local isoperimetric ineq. in $\lambda^{-1}B_j \Rightarrow$

$$\mathcal{H}^d(D\cap\lambda^{-1}B_j) \leqslant \mathrm{const}(K)\, R_j\, \mathcal{H}^{d-1}(\partial D\cap B_j).$$

Since

$$\mathcal{H}^d(D\cap \lambda^{-1}B_j) \geqslant \lambda^d\, \mathcal{H}^d(\lambda^{-2}B_j)/2 \geqslant \mathrm{const}(K)\, R_j^{\,d},$$

one gets

$$\mathcal{H}^d(D\cap \lambda^{-1}B_j) \leqslant \mathrm{const}(K)\, \mathcal{H}^{d-1}(\partial D\cap B_j)^{d/d-1}.$$

Finally,

$$\mathcal{H}^d(D\cap 2B_j) \leqslant \mathcal{H}^d(2B_j) \leqslant \mathrm{const}(K)\mathcal{H}^d(B_j)$$

$$\leqslant \mathrm{const}(K)\mathcal{H}^d(D\cap\lambda^{-1}B_j).$$

So one can sum up and estimate $\mathcal{H}^d(D)$.

Sobolev meets Poincaré

Isoperimetric inequality is equivalent to Sobolev inequality for compactly supported u,

$$\|u\|_{d/d-1} \leqslant \text{const. } \|du\|_{1} .$$

Local isoperimetric inequality is equivalent to (weak) (1,1)-Poincaré inequality, for arbitrary u defined on a ball λB of radius R,

$$\mathrm{Inf}_{c\in\mathbb{R}} \int_{B} |u-c| \leqslant \text{const. } R \int_{\lambda B} |du| .$$

Carnot case : replace du with $d^H u = du_{|H}$, the <u>horizontal</u> <u>differential</u>.

Proof of Isoperimetric ⇔ Sobolev

1. Isoperimetric ⇔ Sobolev for characteristic functions 1_D of domains D.
2. Every nonnegative compactly supported function u is a sum of characteristic functions, $u = \int_0^\infty 1_{\{u>t\}} dt$.
3. Coarea formula

$$\int_0^\infty \mathcal{H}^d(\{u=t\})\, dt = \|d^H u\|_1.$$

Proof of Local Isoperimetric ⇔ Poincaré

1. Up to replacing u with u-c, $\mathcal{H}^d(\{u>0\}\cap B)$, $\mathcal{H}^d(\{u<0\}\cap B) \leqslant \mathcal{H}^d(B)/2$.
2. $u=u_+-u_-$ where $u_+=max\{u,0\}$.
3. $\int_B u_+ = \int_B \int_0^\infty 1_{\{u>t\}}\, dt = \int_0^\infty \mathcal{H}^d(\{u>t\} \cap B)\, dt$.
4. Local isoperimetric inequality implies $\int_B u_+ \leqslant$ const. $\int_0^\infty \mathcal{H}^d(\{u=t\} \cap \lambda B)\, dt$.
5. Coarea $\Rightarrow \int_B |u| \leqslant$ const. $\int_{\lambda B} |d^H u|$.

Hausdorff dimension of higher codimensional submanifolds

1. Hausdorff dimension of hypersurfaces and the isoperimetric inequality
2. Hausdorff dimension of higher codimensional submanifolds
3. From submanifolds to differential forms

Topological dimension

Theorem (Alexandrov). A subset $V \subset M^n$ has topological dimension $n\text{-}k$ $\Leftrightarrow$ there are $k\text{-}1$-cycles in $M \backslash V$ which do not bound chains of small diameter in $M \backslash V$.

Corollary. If topdim(V) $\geqslant n\text{-}k$, there exists a k-dimensional polyhedron P and a continuous map $f\colon P \to M$ such that any f' C^0-close to f hits V. Call this f <u>transverse</u> to V.

Folded maps

Definition. P polyhedron. A map $f{:}P\to M$ is folded if P is covered with subpolyhedra P_j such that f is a smooth immersion on simplices of P_j and a homeomorphism of P_j onto a smooth submanifold with boundary.

Terminology. Say a Carnot manifold is k-rich if given a C^0 map $f{:}P\to M$, where $dim(P)=k$, there exists a nearby horizontal folded map $F{:}P\times\mathbb{R}^q\to M$ which is an immersion on P_j and a submersion on $P_j\times\mathbb{R}^q$.

k-wealth⇒lower bound on Hausdorff dimension

Lemma. Assume M^n is k-rich. Then for every n-k-dimensional subset $V \subset M$,

$$dim_{Hau}(V)\text{-}dim(V) \geqslant dim_{Hau}(M)\text{-}dim(M).$$

It follows that $\alpha(M) \leqslant (n-k)/(d-k)$.

Indeed, given $F: P \times \mathbb{R}^q \to M$ which is close to a transversal to V, pick $\mathbb{R}^{n-k} \subset \mathbb{R}^q$ on which F is a diffeo. F defines τ-tubes. Tubes generated by ε-balls have volume $\leqslant \tau^k \varepsilon^{d-k}$. Cover V with ε_j-balls. Then $\sum_j \varepsilon_j^{d-k} \geqslant \tau^{-k} Vol(\cup \text{Tubes})$ is bounded away from 0.

Results

Theorem (Gromov). Let *dim* $M=n$. Assume $h\text{-}k \geqslant (n\text{-}h)k$. Then a generic h-dimensional distribution H on M is k-rich. A contact structure on M^{2m+1} is k-rich for all $k \leqslant m$.

Proof.

1. Linear algebra : existence of regular isotropic subspaces in H.
2. Analysis : microflexibility of regular horizontal submanifolds (Nash).
3. Topology : local to global (Smale).

Isotropic subspaces

Notation. Let $H=ker(\vartheta)$, where ϑ is $\mathbb{R}^{n-h}$-valued. If $V\subset M$ is horizontal, then $d\vartheta_{|TV} = 0$.

Definition. Let $m\in M$. A subspace $S\subset H_m$ is <u>isotropic</u> if $d\vartheta_{|S} = 0$.

Examples. 1-dimensional subspaces are always isotropic. If H is a contact structure on M^{2m+1}, isotropic subspaces have dimension $\leqslant m$.

Linearizing horizontality

Goal. Apply inverse function theorem to horizontal submanifold equation.

Write $E(V)= \vartheta_{|TV}$. Then

$$V \text{ is horizontal} \Leftrightarrow E(V)=0.$$

X vectorfield along $V\subset M$. Then

$$D_V E(X)=(d(\iota_X\vartheta)+\iota_X(d\vartheta))_{|TV} .$$

Regular subspaces

Definition. Assume $H=ker(\vartheta)$, where ϑ is $\mathbb{R}^{n-h}$ -valued. Say $S \subset H_m$ is <u>regular</u> if

$H_m \to Hom(S, \mathbb{R}^{n-h})$, $X \to \iota_X(d\vartheta))_{|S}$ is onto.

Examples. In contact manifolds, all horizontal subspaces are regular.

All 1-dimensional subspaces are regular $\Leftrightarrow$ H satisfies the strong bracket generating hypothesis. This is very rare.

Generic case

Proposition. If $S \subset H_m$ is isotropic and regular, then $h\text{-}k \geqslant (n\text{-}h)k$, where $h = dim\ H$, $k = dim\ S$.

Conversely, if $h\text{-}k \geqslant (n\text{-}h)k$, a generic h-dimensional distribution admits regular isotropic k-planes, away from a small subset.

Indeed, regular isotropic k-planes are the smooth points of the variety of isotropic k-planes. Their existence is a Zariski open condition on a 2-form ω. Assumption allows to construct at least one such 2-form. The map $\vartheta \to d\vartheta_{|ker(\vartheta)}$ is transverse to the set of bad ω.

Algebraic inverses

Proposition. If $TV \subset H$ is regular, $D_V E$ admits an algebraic right inverse.

Indeed, if $M_m : T_m^*V \to H_m$ is a right inverse of $X \to \iota_X(d\vartheta))_{|TV}$, $\beta \to M(\beta)$, $\Omega^1(V) \to C^\infty(H)$ is a right inverse of $D_V E$.

Remark. For $f{:}V \to \mathbb{R}^q$, the first order linear operator $L(f) = Af + \sum_i B_i \partial_i f \in \mathbb{R}^{q'}$ is algebraicly invertible for generic A and B_i if $q \gg q'$.

Indeed, to solve $L(f) = g$, it suffices to solve $B_i f = 0$ and $(A - \sum_i \partial_i B_i) f = g$.

Nash implicit function theorem

Theorem (Nash). Let F, G be bundles over V. Assume $E:C^\infty(F) \to C^\infty(G)$ is a differential operator whose linearization $D_f E$ admits a differential right inverse M_f, which is defined for f in a subset A of $C^\infty(F)$ defined by an open differential relation. Let s be large enough.

Then for each $f \in A$, there exists a right inverse E_f^{-1} of E, defined on a C^s-neighborhood of $E(f)$ in $C^\infty(G)$. Furthermore, E_f^{-1} depends smoothly on parameters, and is local : $E_f^{-1}(g)(v)$ depends on $g_{|B(v,1)}$ only.

Approximate solutions.

Corollary. Any germ f_0 that solves

$$E(f_0)(m) = o(|m-m_0|^s)$$

can be deformed to a true local solution f_1 : $E(f_1)=0$.

Indeed, choose $g \in C^\infty(G)$ such that $g= -E(f_0)$ near m_0, but g is C^s-small. Set $f_t = E_f^{-1}(E(f_0)+tg)$.

In other words, it suffices to construct solutions up to order s ($s=2$ is enough for the horizontal manifold problem). This implies local existence.

Microflexibility (1/2)

Definition. Say an equation is <u>(micro)flexible</u> if given compact sets $K' \subset K \subset V$, a solution f defined on a neighborhood of K, and a deformation f_t, $t \in [0,1]$, of its restriction to K', the deformation extends to a neighborhood of K (for a while, i.e. for $t \in [0,\varepsilon]$). It should also work for families f_p parametrized by a polyhedron P.

Example. Inequations are always microflexible.

Microflexibility (2/2)

Corollary. If for $f \in A$, $D_f E$ admits a differential right inverse, then $A \cap \{E=0\}$ is microflexible.

Indeed, given solutions f on K and f_t on K', extend f_t arbitrarily to f'_t defined on K. For t small, one can set $e_t = E_{f'_t}^{-1}(0)$. Locality $\Rightarrow e_t = f_t$ near K'.

Remark. (Micro)flexibility means that restriction of solutions from K to K' is a fibration (submersion).

h-principle

Definition (Gromov). Given an equation of order *r*, there is a notion of nonholonomic solution, « *r*-jet of a solution ».

Example. For horizontal immersions $V \to (M,H)$, a nonholonomic solution is a continuous map $f\colon V \to M$ together with an isotropic injective linear map $T_m V \to H_{f(m)}$.

Say an equation satisfies the C^0 h-principle if every nonholonomic solution can be C^0-approximated by solutions (and also familywise).

h-principle as a homotopy theory

h-principle localizes near a compact subset *K*, and has a relative version for a pair *(K,K')*.

Proposition. h-principle for *K'* + h-principle for *(K,K')* ⇒ h-principle for *K*.

Theorem (Smale). h-principle for *K'* + h-principle for *K* + diff. invariance + flexibility ⇒ h-principle for *(K,K')*.

Microflexibility versus flexibility

Corollary. Flexibility + local existence ⇒ global existence (h-principle).

Theorem (Gromov). Microflexibility on $V=W\times\mathbb{R}$ ⇒ flexibility on W. Microflexibility implies h-principle for folded solutions.

Smooth horizontal immersions

Definition. $S \subset H$ is <u>superregular</u> if $S \subset S'$, S' is regular isotropic and dim S' = dim $S + 1$.

Theorem (Gromov). Let $dim\ M=n$. Assume $h\text{-}k \geqslant (n\text{-}h)k$. For a generic h-dimensional distribution H on M the h-principle holds for $k\text{-}1$-dimensional superregular horizontal immersions.

Theorem (Duchamp). In a contact manifold M^{2m+1} , the h-principle holds for k-dimensional horizontal immersions for all $k \leqslant m$.

Regularity and calculus of variations

Remark. The space of regular horizontal immersions $V \to (M,H)$ is a smooth manifold. Therefore, one can write Euler-Lagrange equations for the extremals of functionals on such immersions.

Example. Variational Hamiltonian Legendrian surfaces in S^5 .

Back to the Hölder equivalence problem

Let (M^n,H^h) be a Carnot manifold of Hausdorff dimension d. Then from existence of horizontal (folded) submanifolds, one gets the following upper bounds for $\alpha=\alpha(M,H)$, the best possible exponent for a Hölder homeomorphism $\mathbb{R}^n \to M$.

1. $\alpha \leqslant (n-1)/(d-1)$ in general.
2. $\alpha \leqslant (n-k)/(d-k)$ for generic H, if $h-k \geqslant (n-h)k$.
3. $\alpha \leqslant (m+1)/(m+2)$ for contact H, $n=2m+1$.

From submanifolds to differential forms

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Horizontal forms

Remark. Let $p:M\to\mathbb{R}^q$ be a submersion with horizontal fibers. Then p^*vol is a horizontal form, i.e. its wedge product with a form that vanishes on H is zero. Thus k-wealth implies abundance of horizontal n-k-forms.

Notation. Let Θ^* denote the ideal of forms that vanish on H and A^* its annihilator,

$$A^*=\{\eta|\eta\wedge\vartheta=0\ \forall\,\vartheta\in\Theta^*\}.$$

Elements of A^* are called <u>horizontal forms</u>.

Existence of horizontal forms (1/2)

Proposition. Every closed *n-1*-form is cohomologous to a horizontal form.

Proof (Heisenberg group case). Let ϑ be the contact form, ω a *n-1*-form. Since $d\vartheta_{|H}$ is symplectic, there is a unique *n-3*-form β on H such that $(d\vartheta)_{|H}\wedge\beta - \omega_{|H} = 0$. Extend β arbitrarily. Then $\omega + d(\vartheta\wedge\beta)$ is horizontal.

Existence of horizontal forms (2/2)

Proof (general case). Consider the operators $d^H : \Theta^1 \to \Omega^2/\Theta^2$ and $d_H : A^{n-2} \to \Omega^{n-1}/A^{n-1}$ induced by the exterior differential. Both are order zero, and adjoints of each other, since, for $\vartheta \in \Theta^*$ and $\eta \in A^*$,

$$(d_H\vartheta)\wedge\eta \pm \vartheta\wedge d^H\eta = d(\vartheta\wedge\eta) = 0.$$

Bracket generating $\Rightarrow$ d^H is injective, so d_H is surjective. Given a closed $n\text{-}1$-form ω, there exists $\eta \in A^{n-2}$ such that $d_H\eta = -\omega$ mod A^{n-1}, i.e. $\omega + d\eta$ is horizontal.

Second proof of isoperimetric inequality

Goal. $\mathcal{H}^d(D) \leqslant C\, \mathcal{H}^{d-1}(\partial D)^{d/d-1}$ (Carnot group case).

1. Choose horizontal representative of generator of $H^{n-1}(G\backslash\{e\}/\langle\delta_2\rangle,\mathbb{Z})$, lift it to $G\backslash\{e\}$. Get horizontal form ω_e such that $|\omega_e(g)| \leqslant$ const.$|g|^{1-d}$. Left translate it at each $g\in G$.
2. $\mathcal{H}^d(D)= \int_D(\int_{\partial D} \omega_g)\, dg \leqslant \int_{D\times\partial D} d(g,g')^{1-d}\, dgdg'$ $= (\int_{\partial D} (\int_D d(g,g')^{1-d}\, dg)dg'$.
3. If B is ball centered at g' with $\mathcal{H}^d(D)=\mathcal{H}^d(B)$,
$$\int_D d(g,g')^{1-d}\, dg \leqslant \int_B d(g,g')^{1-d}\, dg = \mathcal{H}^d(D)^{1/d}.$$

More horizontal forms, contact case

Notation. Let $I^* = \{\alpha\wedge\vartheta + \beta\wedge d\vartheta\}$ denote the differential ideal generated by forms that vanish on H and J^* its annihilator,

$$J^* = \{\eta \,|\, \eta\wedge\vartheta = 0 \;\forall\, \vartheta \in I^*\}.$$

Theorem (Rumin). M^{2m+1} contact manifold.

There exists a second order differential operator $D: \Omega^m/I^m \to J^{m+1}$ such that the complex

$$0 \to \Omega^1/I^1 \to \ldots \to \Omega^m/I^m \to J^{m+1} \to \ldots \to J^{2m+1} \to 0$$

is homotopy equivalent to the de Rham complex.

Rumin's second order operator

Proof (case *Heis*^{3}). ω a 1-form mod I^1. Let β be the unique function such that $(d\omega + \beta\, d\vartheta)_{|H} = 0$. Set $D\omega = d(\omega + \beta \wedge \vartheta)$. Then d^H D = D $d_H = 0$.

Locally, $D\omega = 0 \Leftrightarrow \omega + \beta\vartheta$ is exact $\Leftrightarrow \omega$ is d_H-exact.

Also, locally, a 2-form η is closed $\Leftrightarrow \eta = d(\omega + \beta d\vartheta)$ in which case $(\omega + \beta\vartheta)_{|H} = 0$, thus $\eta = D\omega$.

Corollary. There exist small open subsets in M with closed non exact horizontal q-forms for all $q \geqslant m+1$.

Weights of differential forms

Definition. Say a q-form ω on a Carnot manifold has <u>weight</u> $\geqslant w$ if its vanishes on all q-vectors of $H_{i1} \otimes \ldots \otimes H_{iq}$ whenever $i_1 + \ldots + i_q < w$.

Examples. On a Carnot group, the dual basis of V^i has weight i, and weight adds up under wedge product.

A q-form ω on a Carnot group has weight $\geqslant w \Leftrightarrow$ the L^∞ norm $\| \delta_\varepsilon{}^* \omega \|_\infty \leqslant$ const. ε^w.

A q-form is horizontal $\Leftrightarrow$ it has weight $\geqslant d-n+q$.

Goal. Show that the minimal weight of forms needed to represent a cohomology class is a Hölder covariant.

Alexander-Spanier cochains (1/4)

Definition (Alexander-Spanier). A straight q-cochain is a function on $q+1$-tuples of points of diameter $< \delta$. Set $|c|_\varepsilon$ = sup of c on $q+1$-tuples of points of diameter $< \varepsilon$.

Properties. If δ is small enough, straight cochains compute cohomology. There are straight chains too, dual to cochains.

Alexander-Spanier cochains (2/4)

Proposition. On a Riemannian manifold, straight cocycles representing nonzero classes of degree q satisfy $|c|_{\varepsilon} \geqslant$ const. ε^{q} .

Indeed, fix a cycle *c'* such that *c(c')* is nonzero. Subdivide *c'* into const. ε^{-q} simplices of diameter $\leqslant \varepsilon$. Then $c(c') \leqslant$ const. $\varepsilon^{-q} |c|_{\varepsilon}$.

Alexander-Spanier cochains (3/4)

Proposition. Let ω be a closed form of weight $\geqslant w$ on some open set U. Then $\forall\varepsilon$, $[\omega]$ can be represented by a straight cocycle c_ε such that

$$|c_\varepsilon|_\varepsilon \leqslant \text{const.}\ \varepsilon^w \ .$$

Indeed, assume M is a Carnot group. Equip it with an invariant Riemannian metric g and its dilates $\delta_\varepsilon{}^*g$. Fill in straight simplices of unit size with geodesic singular simplices. For each straight chain σ, this gives a family σ_ε . Set $c_\varepsilon(\sigma)=\int_{\sigma\varepsilon}\omega$. Then $|c_\varepsilon| \leqslant \text{const.}\ \varepsilon^w$.

Alexander-Spanier cochains (4/4)

Corollary (Gromov). Define $W_q(M)$ as the largest w such that there exists an open set $U \subset M$ and a nonzero class in $H^q(U, \mathbb{R})$ which can be represented by a form of weight $\geqslant w$. If there exists a α-Hölder homeomorphism $\mathbb{R}^n \to M$, then $\alpha \leqslant q/W_q$.

Examples. For all Carnot manifolds, $W_{n-1}(M) \leqslant d-1$. For contact M^{2m+1}, Rumin's theorem applies with $q=m+1$, $W_q(M) \leqslant m+2$. One recovers the bound given by h-principle for horizontal m-manifolds.

Rumin's complex in general

Goal. Produce a retraction r of the de Rham complex onto a subcomplex consisting of forms of high weight.

Retraction means $r=1-dB-Bd$. Removing low weight components $\Leftrightarrow$ B inverts d on low weights.

Notation. $\Omega^{q,w}=\{q\text{-forms of weight} \geqslant w\}$.

Properties. $\Omega^{*,w} \wedge \Omega^{*,w'} \subset \Omega^{*,w+w'}$. $d\Omega^{*,w} \subset \Omega^{*,w}$.

$d_0 : \Omega^{*,w} / \Omega^{*,w+1} \to \Omega^{*,w} / \Omega^{*,w+1}$ is algebraic, $d_0 =$ Lie algebra differential on tangent cone.

Equihomological Carnot manifolds

Definition. Say an equiregular Carnot manifold is equihomological if dimensions of cohomology groups of tangent Lie algebras are constant.

A choice of complements

V^k of H^{k-1} in H^k,

F of $ker\ d_0$ in Λ^*T^*M and

E of $im\ d_0$ in $ker\ d_0$,

determines an inverse d_0^{-1}. Set $r=1-dd_0^{-1}-d_0^{-1}d$.

Rumin's complex

Theorem (Rumin). Assume M is equihomological. The iterates r^j stabilize to a projector p of Ω^*M, with image $\mathcal{E} = ker\ d_0^{-1} \cap ker(d_0^{-1}d)$ and kernel $\mathcal{F} = im\ d_0^{-1} + im(dd_0^{-1})$. Both are subcomplexes. p is a differential operator. Furthermore

$$\mathcal{E} = \{\eta \in E+F \mid d\eta \in E+F\}.$$

In particular, $\mathcal{E}^1$ identifies with Ω^1/Θ^1.

Corollary. Assume that, in degree q, $E+F \subset \Omega^{q,w}$. Then $W_q(M) \geqslant w$. It follows that $\alpha(M) \leqslant q/w$.

Graduation and duality

Let $\mathcal{G}$ be a Carnot Lie algebra, $\mathcal{G}=V^1\oplus \ldots \oplus V^r$. Let

$$\Lambda^{q,w} = \oplus_{i1+2i2+\ldots+rir=w} \Lambda^{i1} V^{1*} \otimes \ldots \otimes \Lambda^{ir} V^{r*}.$$

For adapted metric, use Hodge $*$: $\Lambda^{q,w} \to \Lambda^{n-q,d-w}$.

***Im d_0* criterion**. If $im\, d_0^{q-1} \supset \oplus_{w'<w} \Lambda^{q,w'}$ then $W_q(G) \geq w$.

***Ker d_0* criterion**. If $ker\, d_0^{q} \subset \oplus_{w'\leq w} \Lambda^{q,w'}$ then $W_{n-q}(G) \geq d-w$.

Examples

Degree *n-1*. $ker\ d_0^1 = V^{1*} = \Lambda^{1,1} \Rightarrow W_{n-1}(G) \geqslant d-1$.

Contact. $\Lambda^q \mathcal{G}^* = \Lambda^q V^{1*} \oplus V^{2*} \otimes \Lambda^{q-1} V^{1*}$.

$$d_0(\eta + \vartheta \wedge \beta) = d\vartheta \wedge \beta + \vartheta \wedge 0.$$

d_0^q vanishes on $\Lambda^q V^{1*}$, is injective on $V^{2*} \otimes \Lambda^{q-1} V^{1*}$ if $q \geqslant m+1$. Thus $ker\ d_0^q = \Lambda^{q,q} \Rightarrow W_{n-q}(G) \geqslant d-q$.

Rank 2 distributions. $im\ d_0^1 \supset \Lambda^{2,2} \Rightarrow W_{n-2}(G) \geqslant 2$.

Genericly ($n>4$), $im\ d_0^1 \supset \Lambda^{2,2} \oplus \Lambda^{2,3} \Rightarrow W_{n-2}(G) \geqslant 3$.

Regular isotropic planes

Remark. If (G,V^1) admits a regular isotropic horizontal k-plane S, then $\ker d_0^k \subset \Lambda^{k,k}$. Therefore $W_k(G) \geqslant d-k$.

Indeed, if $\omega \in \ker d_0^k \cap \Lambda^{k,>k}$, then $\omega=\sum_{i=1}^{n-h} a_i \vartheta_i \wedge \eta_i$ where $\vartheta_i \in \Lambda^{1,>1}$. If $X \in V^1$, since S is isotropic, $(\iota_X d\omega)_{|S} = \sum_{i=1}^{n-h} a_i (\iota_X d\vartheta_i)_{|S} \wedge \eta_{i|S}$. Choose X such that all $(\iota_X d\vartheta_i)_{|S}$ vanish but one, which is $(\iota_X d\vartheta_{i0})_{|S}=*_S \eta_{i0|S}$. Conclude that $\omega=0$.

Quaternionic Heisenberg group

$\mathcal{G}=V^1 \oplus V^2$ where $V^1=\mathbb{H}^n$ and $V^2=\Im m\ \mathbb{H}$, $[u,v]= \Im m\langle u,v\rangle$. Then $Aut(\mathcal{G},V^1) \supset Sp(n)Sp(1)$ and $\Lambda^{2,*}=\Lambda^{2,2}\oplus\Lambda^{2,3}\oplus\Lambda^{2,4}$ is a decomposition into irreducible summands. Therefore $ker\ d_0{}^2 = \Lambda^{2,2}$, which implies $W_{n-2}(G) \geqslant d\text{-}2 = 4n+4$.

Remark. Isotropic subspaces S exist in each dimension $k \leqslant n$. They form a unique orbit $\mathbb{R}^k \subset \mathbb{R}^n \subset \mathbb{H}^n$, entirely regular. Therefore horizontal submanifolds of dimension $<n$ obey h-principle.

Rumin's retraction in the contact case (1/2)

Let $H = ker\ \vartheta$, where ϑ is a contact form, dim=$2m+1$.

Choose $V^2 = ker\ d\vartheta$. Then

$$d_0(\eta+\vartheta \wedge \beta) = d\vartheta \wedge \beta + \vartheta \wedge 0 = L\,\beta.$$

Choose, for $k \leqslant m$,

$$E^k = \{\eta \in \Omega^k \mid \vartheta \wedge L^{m-k+1}\,\eta = \vartheta \wedge L^{m-k} d\eta = 0\},$$

$$F^k = \{\vartheta \wedge \eta \mid \eta \in \Omega^{k-1}\},$$

and for $k \geqslant m+1$,

$$E^k = \{\eta = d\vartheta \wedge \beta \mid d\vartheta \wedge \beta = 0\},$$

$$F^k = \{\vartheta \wedge L^{m-k+1}\,\eta \mid \eta \in \Omega^*\}.$$

Rumin's retraction in the contact case (2/2)

q-forms on H can be uniquely written $\eta = \eta_0 + L\eta_1 + \ldots + L^{m'}\eta_{m'}$ where η_i are primitive and $m'=m$ or $m-1$ depending wether q is even or odd. Define

$L^{-1}\eta = \eta_1 + \ldots + L^{m'-1}\eta_{m'}$. Then, for $\omega = \eta + \vartheta \wedge \beta$,

$$p\omega = r\omega = \eta_0 - \vartheta \wedge (L^{-1}d\eta + \beta - \beta_m).$$

Therefore, $p^{-1} \circ d \circ p$ coincides with the second order operator D.

Conclusion

As far as the Hölder equivalence problem is concerned, the algebraic approach using differential forms seems to give better results than horizontal submanifolds : not all closed currents are laminated.

Possibility of improvement : produce retraction onto a subcomplex on which d_0 vanishes.

New (metric-analytic ?) idea needed for Hölder equivalence problem for $Heis^3$.

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