

DENSITY OF ZARISKI DENSITY FOR SURFACE GROUPS

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ABSTRACT. We show that a surface group contained in a reductive real algebraic group can be deformed to become Zariski dense, unless its Zariski closure acts transitively on a Hermitian symmetric space of tube type. This is a kind of converse to a rigidity result of Burger, Iozzi and Wienhard.

RÉSUMÉ. On montre qu'un groupe de surface contenu dans un groupe algébrique réel réductif peut être déformé pour devenir Zariski dense, sauf si son adhérence de Zariski agit transitivement sur un espace symétrique hermitien de type tube. C'est une réciproque partielle d'un théorème de rigidité dû à Burger, Iozzi et Wienhard.

1. INTRODUCTION

We are interested in the dimension of representation varieties. If G is a Lie group and Γ a finitely generated group, the corresponding representation variety is the space $Hom(\Gamma, G)$ of homomorphisms $\Gamma \rightarrow G$. This space has a *virtual dimension* $vdim_\phi(Hom(\Gamma, G))$ at each representation ϕ , which will be defined in 2.1. For instance, if G is a semisimple real algebraic group and Γ is the fundamental group of a closed surface of Euler characteristic $\chi(\Gamma)$, $vdim_\phi(Hom(\Gamma, G)) = (1 - \chi(\Gamma))\dim(G)$.

In general, we expect $Hom(\Gamma, G)$ to be a stratified space with open strata of dimension $vdim_\phi(Hom(\Gamma, G))$ and lower dimensional strata. Examples (see below) show that there may exist open strata of lower dimension. This suggests the following definitions.

Definition 1.1. *Let Γ be the fundamental group of a closed surface. Let G be a real Lie group so that $Hom(\Gamma, G)$ has a natural structure of an analytic space. Say a homomorphism $\phi : \Gamma \rightarrow G$ is*

- smooth if $Hom(\Gamma, G)$ is a smooth analytic manifold near ϕ and its dimension equals $vdim(Hom(\Gamma, G))$ (for this, it is necessary and sufficient that the dimension of the Zariski tangent space of $Hom(\Gamma, G)$ at ϕ be equal to its virtual dimension);
- fully flexible if smooth homomorphisms are dense in a neighborhood of ϕ in $Hom(\Gamma, G)$;

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- partially rigid if there is a neighborhood of ϕ in $\text{Hom}(\Gamma, G)$ which contains no smooth homomorphisms.

The sets \mathcal{S} of smooth homomorphisms, \mathcal{F} of fully flexible homomorphisms and \mathcal{R} of partially rigid homomorphisms are open in $\text{Hom}(\Gamma, G)$. By definition, \mathcal{F} is the interior of the closure of \mathcal{S} , and \mathcal{R} is the complement of the closure of \mathcal{S} . Roughly speaking, \mathcal{F} is the union of open strata of dimension $\text{vdim}(\text{Hom}(\Gamma, G))$, and \mathcal{R} is the union of open strata of lower dimensions. Figure 1 at the end of section 3 will illustrate our terminology.

Question 1. *When do there exist homomorphisms $\Gamma \rightarrow G$ which are partially rigid? fully flexible? neither partially rigid nor fully flexible?*

This question is related to Zariski density. Indeed (see Corollary 2.4), if G is connected algebraic and if $\text{genus}(\Gamma) \geq \dim(G)^2$, the set \mathcal{Z} of homomorphisms $\Gamma \rightarrow G$ whose image is Zariski dense is a dense subset of \mathcal{S} . Therefore, under the genus restriction, “smooth” can be replaced by “Zariski dense” in the definitions of \mathcal{F} and \mathcal{R} . In particular, partially rigid homomorphisms are those which cannot be slightly perturbed to become Zariski dense.

1.1. A long rigidity story. We collect examples of rigid homomorphisms.

1.1.1. Toledo invariants. Let X be a Hermitian symmetric space, with Kähler form Ω (the metric is normalized so that the minimal sectional curvature equals -1). Let Σ be a closed surface of negative Euler characteristic, let $\Gamma = \pi_1(\Sigma)$ act isometrically on X . Pick a smooth equivariant map $\tilde{f} : \tilde{\Sigma} \rightarrow X$.

Definition 1.2. *Define the Toledo invariant of the action $\rho : \Gamma \rightarrow \text{Isom}(X)$ by*

$$T_\rho = \frac{1}{2\pi} \int_\Sigma \tilde{f}^* \Omega.$$

Then the invariant is independent of the choice of \tilde{f} and

- (1) T_ρ depends continuously on ρ .
- (2) One can choose $\ell_X \in \mathbb{Q}$ positive and maximal among the possible values such that $T_\rho \in \ell_X \mathbb{Z}$.
- (3) $|T_\rho| \leq |\chi(\Sigma)| \text{rank}(X)$.

1.1.2. Rigidity in rank 1.

Example 1.3. *When $X = H_{\mathbb{C}}^1$ is the unit disk, inequality $|T_\rho| \leq |\chi(\Sigma)|$ is due to J. Milnor, [26]. Furthermore $\ell_X = 1$, T takes all integer values between $-|\chi(\Sigma)|$ and $|\chi(\Sigma)|$.*

Theorem 1.4. (W. Goldman, [11]). *Let $X = H_{\mathbb{C}}^1$. The level sets of T coincide with the connected components of $\text{Hom}(\Gamma, \text{PU}(1, 1))$. Furthermore $|T_\rho| = |\chi(\Sigma)|$ if and only if $\rho(\Gamma)$ is discrete and cocompact in $\text{PU}(1, 1) = \text{Isom}(H_{\mathbb{C}}^1)$.*

Note that all components of $\text{Hom}(\Gamma, \text{PU}(1, 1))$ have the same dimension $3(1 - \chi(\Sigma))$.

Theorem 1.5. (D. Toledo, 1979, 1989, [33]). *Let $X = H_{\mathbb{C}}^n$ have rank 1. Then $|T_{\rho}| \leq |\chi(\Sigma)|$. Furthermore, $|T_{\rho}| = |\chi(\Sigma)|$ if and only if $\rho(\Gamma)$ stabilizes a complex geodesic $H_{\mathbb{C}}^1$ in X and acts cocompactly on it.*

It follows that, for $n \geq 2$, different components of $\text{Hom}(\Gamma, \text{PU}(n, 1))$ can have different dimensions. Actions with maximal Toledo invariant are partially rigid. They form an open subset of $\text{Hom}(\Gamma, \text{PU}(n, 1))$ of dimension $(1 - \chi(\Sigma))\dim(P(U(1, 1) \times U(n - 1)))$. Actions with vanishing Toledo invariant are fully flexible [2], they form an open subset of $\text{Hom}(\Gamma, \text{PU}(n, 1))$ of dimension $(1 - \chi(\Sigma))\dim(\text{PU}(n, 1))$.

1.1.3. Higher rank.

Definition 1.6. *Actions ρ such that $|T_{\rho}| = |\chi(\Sigma)|\text{rank}(X)$ are called maximal representations.*

Example 1.7. *Pick cocompact actions ρ_1, \dots, ρ_r of Γ on $H_{\mathbb{C}}^1$. Then the direct sum representation on the polydisk $(H_{\mathbb{C}}^1)^r$ is maximal. When the polydisk is embedded in a larger symmetric space of rank r , it remains maximal. It follows that all Hermitian symmetric spaces admit maximal representations.*

Proposition 1.8. (Burger, Iozzi, Wienhard, [7]). *In case X is Siegel's upper half space (i.e. $\text{Isom}(X) = \text{Sp}(n, \mathbb{R})$), some of these actions can be bent to become Zariski dense.*

But this may fail for other Hermitian symmetric spaces.

Theorem 1.9. (L. Hernández Lamóneda, [16], S. Bradlow, O. García-Prada, P. Gothen, [3]). *Maximal reductive representations of Γ to $\text{PU}(p, q)$, $p \leq q$, can be conjugated into $P(U(p, p) \times U(q - p))$.*

1.1.4. *Tube type.* The key to these different behaviours of different Hermitian symmetric spaces seems to lie in the classical notion of tube type domains.

Definition 1.10. *Say a Hermitian symmetric space is of tube type if it can be realized as a domain in \mathbb{C}^n of the form $\mathbb{R}^n + iC$ where $C \subset \mathbb{R}^n$ is a proper open cone.*

Example 1.11. *The Grassmannian $\mathcal{D}_{p,q}$, $p \leq q$, with isometry group $\text{PU}(p, q)$ is of tube type only if $p = q$.*

Products of tube type spaces are of tube type, so polydisks are of tube type.

Remark 1.12. *All maximal tube type subsymmetric spaces in a Hermitian symmetric space are conjugate. For instance, the maximal tube type subsymmetric space in $\mathcal{D}_{p,q}$ is $\mathcal{D}_{p,p}$.*

Theorem 1.13. (Burger, Iozzi, Wienhard, [7]). *Let Γ be a closed surface group and X a Hermitian symmetric space. Every maximal representation $\Gamma \rightarrow \text{Isom}(X)$ stabilizes a tube type subsymmetric space Y . Conversely, for every tube type Hermitian symmetric space X , $\text{Isom}(X)$ admits Zariski dense maximal surface subgroups.*

In particular, maximal representations of surface groups in non tube type Hermitian symmetric spaces are partially rigid.

Example 1.14. *In case X is the n -ball $\mathcal{D}_{1,n}$ (resp. $\mathcal{D}_{p,q}$), one recovers Toledo's (resp. Bradlow et al.) results.*

1.2. New flexibility results. Here are answers to Question 1.

First, there always exist fully flexible homomorphisms. Here, a real algebraic group means the group of real points of a linear algebraic group defined over \mathbb{R} .

Theorem 1. *Let G be a connected reductive real algebraic group and let Γ be the fundamental group of a closed surface of genus greater than 1. Then the trivial homomorphism $\Gamma \rightarrow G$ can be deformed into fully flexible homomorphisms.*

Second, “neither fully flexible nor partially rigid” does not show up if genus is large enough.

Theorem 2. *Let G be a connected reductive real algebraic group and let Γ be the fundamental group of a closed surface of genus larger than $2\dim(G)^2$. Then homomorphisms $\Gamma \rightarrow G$ are either fully flexible or partially rigid.*

In other words, the sets \mathcal{F} of fully flexible and \mathcal{R} of partially rigid homomorphisms are unions of connected components of $\text{Hom}(\Gamma, G)$. To show that a component is fully flexible, it suffices to find one single Zariski dense representation in it (in fact, a representation whose centralizer contains the center of G as a finite index subgroup will do).

Remark 1.15. *It is likely that the restriction on genus, in this result as well as in forthcoming ones, is irrelevant. Nevertheless, our arguments dwell heavily on it.*

Theorem 2 leads to a refined question.

Question 2. *Which surface groups in real reductive algebraic groups are fully flexible?*

As far as the flexibility of a homomorphism ϕ is concerned, a key role is played by the center of the centralizer of the image of ϕ . It splits the Lie algebra of G into real root spaces $\mathfrak{g}_{\lambda, \mathbb{R}}$ which carry natural symplectic structures. In particular, to each pure imaginary root λ , there corresponds a symplectic representation ρ_λ on $\mathfrak{g}_{\lambda, \mathbb{R}}$ and a Toledo invariant T_λ .

Definition 1.16. Let $\mathfrak{t} \subset \mathfrak{g}$ be a torus, centralized by a homomorphism $\phi : \Gamma \rightarrow G$. Among the roots of the adjoint action of \mathfrak{t} on \mathfrak{g} , let P be the subset of pure imaginary roots λ such that ρ_λ is a maximal representation with $T_\lambda > 0$. Say \mathfrak{t} is balanced with respect to ϕ if 0 belongs to the interior of the sum of the convex hull of the imaginary parts of elements of P and the linear span of the real and imaginary parts of roots not in $\pm P$.

Here is a necessary and sufficient condition for full flexibility, for surface groups of sufficiently large genus.

Theorem 3. Let G be a semisimple real algebraic group. Let Γ be the fundamental group of a closed surface of genus larger than $2\dim(G)^2$. Let $\phi : \Gamma \rightarrow G$ be a homomorphism with reductive Zariski closure. Then ϕ is fully flexible if and only if \mathfrak{c} , the center of the centralizer of $\phi(\Gamma)$ in \mathfrak{g} , is balanced with respect to ϕ .

Using a result by M. Burger, A. Iozzi and A. Wienhard leads to the following consequence.

Corollary 4. Let G be a reductive real algebraic group. Let Γ be the fundamental group of a closed surface of genus larger than $2\dim(G)^2$. Assume there exists a non fully flexible homomorphism $\phi : \Gamma \rightarrow G$. Then the Zariski closure of $\phi(\Gamma)$ admits a transitive isometric action on a tube type Hermitian symmetric space, and the action of Γ on this space is a maximal representation.

It is not that easy to decide whether the balance condition is satisfied or not by a given homomorphism $\phi : \Gamma \rightarrow G$. For rank 1 simple groups, it is possible, and a complete answer to Question 2 for large genus can be given¹. Up to the genus restriction, this is a converse to Toledo's rigidity theorem 1.5.

Corollary 5. Let G be a rank one almost simple Lie group. Let Γ be the fundamental group of a closed surface of genus larger than $2\dim(G)^2$. Let $\phi : \Gamma \rightarrow G$ be a homomorphism. Then ϕ is fully flexible, unless G is $SU(m, 1)$ and $\phi(\Gamma)$ is discrete, cocompact in a conjugate of $S(U(1, 1) \times U(m - 1)) \subset SU(m, 1)$.

Remark 1.17. *Higgs bundles.*

Character varieties of surface groups can be described in terms of Higgs bundles, leading to a purely algebro-geometric approach to Question 2. It gives detailed informations on the topology of character varieties (see [18], [19],[3]) in many cases. But, as far as we can see, the published results do not seem to contain a complete answer to Question 2. See [22], [29], [15] for related results. Our results are among the very few that apply to *all* linear semi-simple Lie groups.

¹An other class, classical simple groups will be treated in the companion paper [21]

1.3. Tools. Our main tools were already known and used by W. Goldman in his 1985 paper, [12]. He was able to reduce the local study of the representation variety $Hom(\Gamma, G)$ to cohomology calculations : near the conjugacy class of a reductive homomorphism ϕ , $Hom(\Gamma, G)$ identifies with its quadratic approximation, the set of cocycles in $Z^1(\Gamma, \mathfrak{g}_{Ad\phi})$ whose cup-square vanishes. Then, in an example, he was able to compute the cup-square map, thanks to an avatar of the index theorem due to W. Meyer, [25].

Part of the new input comes from M. Burger, A. Iozzi and A. Wienhard's theory of tight homomorphisms and maximal representations.

The first step in our investigation of flexibility deals with centers of centralizers. They split the Lie algebra \mathfrak{g} into symplectic root spaces $\mathfrak{g}_{\lambda, \mathbb{R}}$, the cup-square map splits accordingly. On each $H^1(\Gamma, \mathfrak{g}_{\lambda, \mathbb{R}})$, the cup-square map is *scalar*, i.e. of the form $Q_\lambda t_\lambda$ where $t_\lambda \in H^2(\Gamma, \mathfrak{g}_{\lambda, \mathbb{R}})$ is a vector, and Q_λ a scalar valued quadratic form whose signature can be interpreted as a Toledo invariant. The quadratic approximation has a dense set of smooth points (and full flexibility holds) unless sufficiently many of the Q_λ 's are definite, i.e. the representation of Γ on $\mathfrak{g}_{\lambda, \mathbb{R}}$ is maximal. Tube-type Hermitian domains then arise from Burger-Iozzi-Wienhard's results from [7]. It turns out that balancedness is necessary and sufficient simultaneously for existence and density of smooth points in the quadratic approximation, this is the origin of Theorem 2.

The second step is a direct construction of cohomology classes which shows that representations with semisimple centralizers are fully flexible. This is based on some linear algebra of symplectic vector spaces, and involves a characteristic class computation. More generally, one shows that the presence of a Levi factor in the centralizer does not affect the issue of full flexibility.

Further steps allow reduction from general representations to reductive ones, and from reductive ambient groups to semisimple ones.

1.4. Organization of the paper. Section 2 explains the dimension count that reduces density of Zariski density to smoothness of the representation variety. The short section 3 recalls W. Goldman's approach to rigidity. In section 4, the role of the center of the centralizer is unveiled. Levi factors of centralizers appear in section 5. The full flexibility of the trivial representation (Theorem 1) appears in subsection 5.1 and is completed in section 6.3. The flexibility statement in Theorem 3 is proven in subsection 5.4. Due to complications caused by finite centers, the converse (rigidity statement) requires lifting homomorphisms in central extensions (section 6). It also requires full flexibility of representations with abelian, or, more generally, amenable image (section 7), so the full proof of Theorem 3 is postponed until subsection 7.3. The last reduction, to cover non reductive homomorphisms, comes in section 8, where the proofs of Theorem 2 and Corollary 4 are completed. Section 9 deals with the rank one case and Corollary 5. Two

appendices collect results not easily accessible in the litterature, W. Meyer's signature formula and the structure of centralizers of reductive subgroups.

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2. DIMENSION CONSIDERATIONS

The point of this section is to prove that Zariski dense homomorphisms $\Gamma \rightarrow G$, Γ the fundamental group of a closed surface, G the group of real points of a reductive real algebraic group, form a dense subset of \mathcal{S} when the genus is high enough.

2.1. The virtual dimension $\text{vdim}(Hom(\Gamma, G))$. Let Γ be a finitely generated group. Fix once and for all a finite generating set $S \subset \Gamma$. Let \mathbb{F} denote the free group generated by S , and $N \subset \mathbb{F}$ the set of relations, i.e. the kernel of the tautological homomorphism $\mathbb{F} \rightarrow \Gamma$. Let G be a Lie group. Let $F : G^S \rightarrow G^N$ map $\phi \in G^S$, viewed as a homomorphism $\mathbb{F} \rightarrow G$, to the collection of all $\phi(n)$, $n \in N$. Then $Hom(\Gamma, G) = F^{-1}(\mathbf{e})$, where \mathbf{e} is the constant collection (e, e, \dots) . This provides $Hom(\Gamma, G)$ with the structure of a real analytic set.

At a point ϕ , the Zariski tangent space of $Hom(\Gamma, G)$ identifies with the space $Z^1(\Gamma, \mathfrak{g}_{Ad \circ \phi})$ of 1-cocycles with values in the Lie algebra \mathfrak{g} . We show that its dimension is bounded below by topological data independent of ϕ .

The dimension of the Zariski tangent space is

$$\begin{aligned} \dim(Z^1(\Gamma, \mathfrak{g}_{Ad \circ \phi})) &= \dim(H^1(\Gamma, \mathfrak{g}_{Ad \circ \phi})) + \dim(B^1(\Gamma, \mathfrak{g}_{Ad \circ \phi})) \\ &= \dim(H^1(\Gamma, \mathfrak{g}_{Ad \circ \phi})) + \dim(C^0(\Gamma, \mathfrak{g}_{Ad \circ \phi})) \\ &\quad - \dim(H^0(\Gamma, \mathfrak{g}_{Ad \circ \phi})). \end{aligned}$$

For Γ a closed surface group, and ρ a linear representation of Γ on a vector space V_ρ , elementary homological algebra gives

$$\dim(H^0(\Gamma, V_\rho)) - \dim(H^1(\Gamma, V_\rho)) + \dim(H^2(\Gamma, V_\rho)) = \chi(\Gamma) \dim(V_\rho).$$

For $\rho = Ad \circ \phi$, this yields

$$\dim(Z^1(\Gamma, \mathfrak{g}_{Ad \circ \phi})) = \dim(H^2(\Gamma, \mathfrak{g}_{Ad \circ \phi})) + (1 - \chi(\Gamma)) \dim(G).$$

If G is connected reductive with radical R (which coincides with its center), pick a semisimple Levi factor S . Then $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{t}$, and

$$H^2(\Gamma, \mathfrak{g}_{Ad \circ \phi}) = H^2(\Gamma, \mathfrak{s}_{Ad \circ \phi}) \oplus H^2(\Gamma, \mathbb{R}) \otimes \mathfrak{t}.$$

Therefore,

$$\dim(Z^1(\Gamma, \mathfrak{g}_{Ad \circ \phi})) \geq (1 - \chi(\Gamma)) \dim(G) + \dim(\mathfrak{t}),$$

which does not depend on ϕ . It is this number that we shall call the virtual dimension of $Hom(\Gamma, G)$.

Definition 2.1. Let G be a reductive real algebraic group with radical R . Let Γ be a closed surface group. Define the virtual dimension of $\text{Hom}(\Gamma, G)$ as

$$\text{vdim}(\text{Hom}(\Gamma, G)) = (1 - \chi(\Gamma))\dim(G) + \dim(R).$$

Say a homomorphism ϕ is smooth (short for a smooth point of $\text{Hom}(\Gamma, G)$) of dimension $\text{vdim}(\text{Hom}(\Gamma, G))$ if

$$\dim(Z^1(\Gamma, \mathfrak{g}_{\text{Ad}\circ\phi})) = \text{vdim}(\text{Hom}(\Gamma, G)).$$

Let \mathcal{S} denote the set of smooth points of $\text{Hom}(\Gamma, G)$.

This fits with [17], Definition 1.5, or [4], Definition 3.3.

Since smoothness is defined in terms of the Zariski tangent space only, it is invariant under analytic isomorphisms.

By semi-continuity, in a neighborhood of a point $\phi \in \text{Hom}(\Gamma, G)$ which is smooth in the sense of Definition 2.1, $\dim(Z^1(\Gamma, \mathfrak{g}_{\text{Ad}\circ\phi}))$ is constant, so $\text{Hom}(\Gamma, G)$ is a smooth analytic manifold of dimension $\text{vdim}(\text{Hom}(\Gamma, G))$.

2.2. Zariski dense homomorphisms belong to \mathcal{S} . Let Γ be a surface group and ρ a linear representation of Γ . Poincaré duality [31] gives an isomorphism of $H^2(\Gamma, V_\rho)$ to $H^0(\Gamma, V_\rho^*)$, where ρ^* denotes the contragredient representation. Since $H^0(\Gamma, V_\rho)$ counts Γ -invariant vectors in V , $H^2(\Gamma, V_\rho)$ counts Γ -invariant vectors in V^* .

If $\phi : \Gamma \rightarrow G$ is a homomorphism to a semisimple group G , invariant vectors in the adjoint representation $\rho = \text{Ad} \circ \phi$ form the Lie algebra of the centralizer $Z_G(\phi(\Gamma))$ of $\phi(\Gamma)$ in G . Since the adjoint representation preserves a nondegenerate quadratic form, the Killing form, it is isomorphic to its contragredient. Therefore $H^2(\Gamma, \mathfrak{g}_{\text{Ad}\circ\phi})$ is isomorphic to $H^0(\Gamma, \mathfrak{g}_{\text{Ad}\circ\phi})$.

Lemma 2.2. Let G be semisimple, let Γ be a surface group. Let $\phi : \Gamma \rightarrow G$ be a homomorphism such that $\phi(\Gamma)$ has a discrete centralizer. Then ϕ belongs to the set \mathcal{S} of smooth homomorphisms (see Definitions 1.1 and 2.1). In particular, Zariski dense homomorphisms belong to \mathcal{S} .

Proof: If $\phi(\Gamma)$ has a discrete centralizer, then $H^2(\Gamma, \mathfrak{g}_{\text{Ad}\circ\phi}) = H^0(\Gamma, \mathfrak{g}_{\text{Ad}\circ\phi}) = 0$, which implies that ϕ is a smooth point of $\text{Hom}(\Gamma, G)$. ■

When G is merely reductive, there is an analogue of Lemma 2.2, Proposition 6.3. It relies on properties of central extensions which are postponed until section 6.

2.3. A crude upper bound on $\dim(\text{Hom}(\Gamma, G))$. Let Γ be a surface group and ρ a linear representation of Γ . Then $\dim(H^0(\Gamma, V_\rho)) \leq \dim(V_\rho)$, $\dim(H^0(\Gamma, V_\rho)) \leq \dim(V_\rho^*)$, thus

$$\dim(H^1(\Gamma, V_\rho)) \leq (2 - \chi(\Gamma))\dim(V_\rho).$$

and

$$(1) \quad \dim(Z^1(\Gamma, V_\rho)) \leq (3 - \chi(\Gamma))\dim(V_\rho).$$

Lemma 2.3. *Let G be a Lie group. Let Γ be a finitely generated group. Then, in a neighborhood of the conjugacy class of a homomorphism $\phi : \Gamma \rightarrow G$,*

$$\dim(\text{Hom}(\Gamma, G)) \leq (3 - \chi(\Gamma))\dim(G).$$

Proof: Since one can locally embed $\text{Hom}(\Gamma, G)$ in its Zariski tangent space, its dimension is at most $\dim(Z^1(\Gamma, \mathfrak{g}_{\text{ad}\circ\phi})) \leq (3 - \chi(\Gamma))\dim(G)$, by inequality (1). ■

2.4. Zariski dense homomorphisms are dense in \mathcal{S} .

Corollary 2.4. *Let G be a connected reductive real algebraic group of dimension > 2 . Let Γ be the fundamental group of a closed surface of genus g . Assume that $g \geq \dim(G)^2$. Then the set of non Zariski dense homomorphisms $\Gamma \rightarrow G$ has dimension $< -\chi(\Gamma)\dim(G)$. In particular, Zariski dense representations form a dense subset of \mathcal{S} .*

Proof: Connected proper subgroups of G are determined by their Lie algebras, which are linear subspaces in \mathfrak{g} . Algebraic subgroups of G have finitely many connected components. Therefore proper algebraic subgroups of G come in countably many families, each of which has dimension less than $\dim(G)^2$. The variety of pairs (H, ϕ) where $H \subset G$ is a proper algebraic subgroup and $\phi : \Gamma \rightarrow H$ is a homomorphism has dimension at most $\dim(G)^2 + (3 - \chi(\Gamma))(\dim(G) - 1)$. If $g \geq \dim(G)^2$,

$$\begin{aligned} & -\chi(\Gamma)\dim(G) - (\dim(G)^2 + (3 - \chi(\Gamma))(\dim(G) - 1)) \\ &= 2g - 2 - \dim(G)^2 - 3\dim(G) + 3 \\ &\geq \dim(G)^2 - 3\dim(G) + 1 > 0. \end{aligned}$$

Since \mathcal{S} is a smooth $\text{vdim}(\text{Hom}(\Gamma, G))$ -dimensional manifold, and

$$\text{vdim}(\text{Hom}(\Gamma, G)) \geq -\chi(\Gamma)\dim(G),$$

a subset of smaller dimension has empty interior in it. ■

Remark 2.5. *The connectedness assumption in Corollary 2.4 is necessary.*

Indeed, if $G' \subset G$ is a proper open subgroup, and genus is large enough, homomorphisms $\Gamma \rightarrow G'$ form a proper open and closed subset of $\text{Hom}(\Gamma, G)$, there are smooth ones but none of them is Zariski dense.

This motivates the following generalization of Corollary 2.4.

Proposition 2.6. *Let G be a reductive real algebraic group of dimension > 2 . Let Γ be the fundamental group of a closed surface of genus g . Assume that $g \geq \dim(G)^2$. Let $\phi : \Gamma \rightarrow G$ be a smooth homomorphism. Let G' denote the smallest open subgroup of G containing $\phi(\Gamma)$. Then homomorphisms whose Zariski closure is G' are dense in a neighborhood of ϕ in $\text{Hom}(\Gamma, G)$.*

Proof: G' is well defined since G has only finitely many connected components. As $\text{Hom}(\Gamma, G')$ is open in $\text{Hom}(\Gamma, G)$, ϕ is smooth as a homomorphism $\Gamma \rightarrow G'$, so we can assume that $G = G'$. There are finitely many

proper open subgroups G'' in G . Each $\text{Hom}(\Gamma, G'')$ is closed in $\text{Hom}(\Gamma, G)$, so is their union. Thus a neighborhood \mathcal{V} of ϕ in $\text{Hom}(\Gamma, G)$ consists of homomorphisms which are not contained in any proper open subgroup of G . As observed in the proof of Corollary 2.4, in \mathcal{V} , homomorphisms which are contained in proper closed subgroups of empty interior form a set of lower dimension. Thus Zariski dense homomorphisms are dense in \mathcal{V} . ■

Example 2.7. *Let Γ be the fundamental group of a closed surface of genus g . Let $\phi : \Gamma \rightarrow \text{Sl}(n, \mathbb{R})$ be an irreducible representation. Then ϕ is smooth and thus fully flexible. If furthermore $g \geq n^4$, ϕ can be deformed to become Zariski dense.*

Proof: An element of the centralizer of $\phi(\Gamma)$ is an invertible linear map which intertwines the representation ϕ with itself. According to Schur's lemma, it has to be proportional to identity. In other words, the centralizer of $\phi(\Gamma)$ in $\text{Sl}(n, \mathbb{R})$ is finite. Thus ϕ belongs to the open set \mathcal{S} . In particular, ϕ is fully flexible. If $g \geq n^4$, Corollary 2.4 applies, and Zariski dense homomorphisms are dense in a neighborhood of ϕ . ■

3. SECOND ORDER CALCULATIONS

The second order integrability condition for infinitesimal deformations at ϕ of representations of a group Γ in a Lie group G can be expressed in terms of the *cup-product*, a symmetric bilinear map

$$[\cdot \smile \cdot] : H^1(\Gamma, \mathfrak{g}_{\text{Ad}\circ\phi}) \rightarrow H^2(\Gamma, \mathfrak{g}_{\text{Ad}\circ\phi}).$$

For $u \in Z^1(\Gamma, \mathfrak{g}_{\text{Ad}\circ\phi})$,

$$[u \smile u](\alpha, \beta) = [u(\alpha), \text{Ad} \circ \phi(\alpha)u(\beta)].$$

It is well-known, [27], that for a representation ϕ from Γ to a reductive group G , if there exists a smooth path ϕ_t in $\text{Hom}(\Gamma, G)$ which is tangent to $u \in Z^1(\Gamma, \mathfrak{g}_{\text{Ad}\circ\phi})$, then $[u \smile u] = 0$. According to W. Goldman (Theorem 3 in [12], [13]), for surface groups, this necessary condition is also sufficient. Here is a more general statement, due to W. Goldman, J. Millson and C. Simpson, following an idea of P. Deligne.

Definition 3.1. *Let Γ be a finitely generated group, let $\phi : \Gamma \rightarrow G$ be a homomorphism to a reductive Lie group G . The quadratic model for the representation space near ϕ is*

$$\mathcal{Q}_\phi = \{u \in Z^1(\Gamma, \mathfrak{g}_{\text{Ad}\circ\phi}) \mid [u \smile u] = 0 \text{ in } H^2(\Gamma, \mathfrak{g}_{\text{Ad}\circ\phi})\}.$$

Theorem 3.2. (W. Goldman and J. Millson, [14], Theorem 1, for variations of Hodge structures, C. Simpson, [30], Corollary 2.4, for general reductive homomorphisms). *Let G be a semi-simple Lie group. Let Γ be the fundamental group of a compact Kähler manifold. If the Zariski closure of $\phi(\Gamma)$ is reductive, the analytic germ of the algebraic variety $\text{Hom}(\Gamma, G)$ at ϕ is equivalent to the analytic germ of \mathcal{Q}_ϕ at 0.*

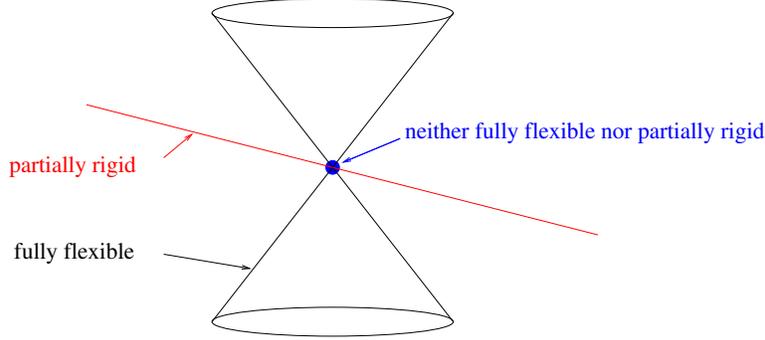


FIGURE 1. Partially rigid and fully flexible points

Under the analytic isomorphism of Theorem 3.2, smooth points of $\text{Hom}(\Gamma, G)$ are mapped to smooth points of the quadratic model, i.e. points $v \in \mathcal{Q}_\phi$ where the map $u \mapsto [u \smile v]$, $Z^1(\Gamma, \mathfrak{g}_{\text{Ad} \circ \phi}) \rightarrow H^2(\Gamma, \mathfrak{g}_{\text{Ad} \circ \phi})$, is a submersion.

Remark 3.3. *Theorem 3.2 implies that locally, $\text{Hom}(\Gamma, G)$ can be defined by homogeneous quadratic equations. This is not sufficient to imply Theorem 2.*

Indeed, consider the following system of 2 homogeneous quadratic equations on \mathbb{R}^4 :

$$\begin{cases} x^2 + y^2 - tz - t^2 = 0, \\ x^2 + y^2 + z^2 - t^2 = 0. \end{cases}$$

The solution set \mathcal{Q} is the union of the line $L = \{x = 0, y = 0, z = -t\}$ (which is 1-dimensional) and of the quadratic cone $C = \{x^2 + y^2 - t^2 = 0, z = 0\}$ (which is 2-dimensional). Using the terminology of Definitions 1.1, 2.1, the smooth points of \mathcal{Q} , as well as the fully flexible ones, are the points of $C \setminus \{0\}$, the partially rigid points of \mathcal{Q} are the points of $L \setminus \{0\}$, and 0 is neither fully flexible nor partially rigid.

4. TORAL CENTRALIZERS

4.1. Toledo invariants attached to a torus.

Proposition 4.1. *Let G be a semisimple real algebraic group. Let \mathfrak{t} be a (not necessarily maximal) torus in \mathfrak{g} . After complexification the \mathfrak{t} -module \mathfrak{g} splits as*

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{g}_\lambda$$

where $\Lambda \subset \mathfrak{t}^* \otimes \mathbb{C}$ is a finite set of roots. For a nonzero root $\lambda \in \Lambda$, the real vector space

$$\mathfrak{g}_{\lambda, \mathbb{R}} = \mathfrak{g} \cap ((\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}) + (\mathfrak{g}_{\bar{\lambda}} \oplus \mathfrak{g}_{-\bar{\lambda}})).$$

carries a natural complex valued alternating 2-form Ω_λ . If $t_\lambda \in \mathfrak{t} \otimes \mathbb{C}$ denotes the complex root vector defined by $\lambda(t) = t_\lambda \cdot t$ (complexified Killing inner product), then, for $X, Y \in \mathfrak{g}_{\lambda, \mathbb{R}}$,

$$[X, Y]^{\mathfrak{t}} = \Re(\Omega_\lambda(X, Y)t_\lambda),$$

(Killing orthogonal projection on \mathfrak{t}). Also, $\Omega_{-\lambda} = -\Omega_\lambda$, $\Omega_{\bar{\lambda}} = \overline{\Omega_\lambda}$.

If $\lambda \neq 0$ is real, so are t_λ and Ω_λ . Furthermore Ω_λ is a symplectic structure on $\mathfrak{g}_{\lambda, \mathbb{R}}$.

If $\lambda \neq 0$ is pure imaginary, so are t_λ and Ω_λ . The imaginary part $\Im(\Omega_\lambda)$ is a symplectic structure on $\mathfrak{g}_{\lambda, \mathbb{R}}$.

If λ is neither real nor pure imaginary, $\mathfrak{g}_{\lambda, \mathbb{R}}$ carries a natural complex structure J_λ and Ω_λ is \mathbb{C} -bilinear with respect to J_λ . It defines a complex symplectic structure on $(\mathfrak{g}_{\lambda, \mathbb{R}}, J_\lambda)$.

Proof: By assumption, after complexification of the Lie algebra and accordingly the Killing form, \mathfrak{g} splits into root spaces

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Lambda} \mathfrak{g}_\lambda,$$

where $\Lambda \subset \mathfrak{t}^* \otimes \mathbb{C}$ is a finite set of roots, and $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \otimes \mathbb{C} \mid \forall t \in \mathfrak{t}, ad_t(X) = \lambda(t)X\}$. For $t \in \mathfrak{t}$, consider the alternating form on \mathfrak{g} defined by

$$\Omega_t(X, Y) = t \cdot [X, Y].$$

Since the Killing form is bi-invariant, for any $X, Y \in \mathfrak{g}$, $ad_X(Y) \cdot t + Y \cdot ad_X(t) = 0$, thus

$$\Omega_t(X, Y) = ad_t(X) \cdot Y$$

for all $t \in \mathfrak{t}$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, the subspaces \mathfrak{g}_α and \mathfrak{g}_β are orthogonal with respect to the Killing form and to Ω_t unless $\alpha + \beta = 0$. Indeed, let $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$ so that $\alpha + \beta \neq 0$ and $t \in \mathfrak{t}$ so that $(\alpha + \beta)(t) \neq 0$. From $[X, Y] \cdot t = -\beta(t)X \cdot Y = \alpha(t)X \cdot Y$ we get $(\alpha + \beta)(t)X \cdot Y = 0$ for any X, Y . Hence \mathfrak{g}_α and \mathfrak{g}_β are orthogonal. Similarly for Ω_t . Since ad_t 's are skewsymmetric with respect to the Killing form, all nonzero roots occur in pairs $(\lambda, -\lambda)$. For nonzero $\lambda \in \Lambda$, set $\mathfrak{g}_{\pm\lambda} = \mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}$. Distinct $\mathfrak{g}_{\pm\lambda}$'s are Killing and Ω_t -orthogonal. It follows that the Killing form restricted to each $\mathfrak{g}_{\pm\lambda}$ and to \mathfrak{g}_0 is nondegenerate. Furthermore, Ω_t restricted to each $\mathfrak{g}_{\pm\lambda}$ such that $\lambda(t) \neq 0$ is nondegenerate, and the nullspace of Ω_t equals the sum of \mathfrak{g}_λ such that $\lambda(t) = 0$.

Let λ be a nonzero root. Let

$$\mathfrak{g}_{\lambda, \mathbb{R}} := \mathfrak{g} \cap (\mathfrak{g}_{\pm\lambda} + \mathfrak{g}_{\pm\bar{\lambda}}) = \mathfrak{g} \cap (\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda} + \mathfrak{g}_{\bar{\lambda}} + \mathfrak{g}_{\bar{\lambda}}),$$

(the sum is direct only if λ is neither real nor pure imaginary). Let $X = X_\lambda + X_{-\lambda}, Y = Y_\lambda + Y_{-\lambda} \in \mathfrak{g}_{\pm\lambda}$. For $t \in \mathfrak{t}$,

$$ad_t(X) = \lambda(t)(X_\lambda - X_{-\lambda}),$$

$$\Omega_t(X, Y) = ad_t(X) \cdot Y = \lambda(t)(X_\lambda - X_{-\lambda}) \cdot (Y_\lambda + Y_{-\lambda})$$

is a complex multiple of the complex symplectic structure

$$\omega_\lambda(X, Y) = (X_\lambda - X_{-\lambda}) \cdot (Y_\lambda + Y_{-\lambda}).$$

If λ is neither real nor pure imaginary, the splitting $\mathfrak{g}_{\lambda, \mathbb{R}} \otimes \mathbb{C} = \mathfrak{g}_{\pm\lambda} \oplus \overline{\mathfrak{g}_{\pm\lambda}}$ defines a complex structure J_λ on $\mathfrak{g}_{\lambda, \mathbb{R}}$ (its complexification equals multiplication by i on $\mathfrak{g}_{\pm\lambda}$ and by $-i$ on $\mathfrak{g}_{\pm\bar{\lambda}}$). Then ω_λ defines a complex valued J_λ -complex bilinear alternating 2-form Ω_λ on $\mathfrak{g}_{\lambda, \mathbb{R}}$ as follows:

$$\Omega_\lambda(X + \bar{X}, Y + \bar{Y}) = 2\omega_\lambda(X, Y).$$

Let $t_\lambda \in \mathfrak{t} \otimes \mathbb{C}$ be the vector defined by

$$t_\lambda \cdot t = \lambda(t) \quad \text{for all } t \in \mathfrak{t} \otimes \mathbb{C}.$$

Then, for $X, Y \in \mathfrak{g}_{\pm\lambda}$,

$$t \cdot [X, Y] = \Omega_t(X, Y) = \lambda(t)\omega_\lambda(X, Y) = t \cdot (\omega_\lambda(X, Y)t_\lambda),$$

thus

$$[X, Y]^{t \otimes \mathbb{C}} = \omega_\lambda(X, Y)t_\lambda,$$

and, for real $X + \bar{X}, Y + \bar{Y} \in \mathfrak{g}_{\lambda, \mathbb{R}}$, since

$$t \cdot [X + \bar{X}, Y + \bar{Y}] = t \cdot [X, Y] + t \cdot [\bar{X}, \bar{Y}] = \lambda(t)\omega_\lambda(X, Y) + \bar{\lambda}(t)\omega_{\bar{\lambda}}(\bar{X}, \bar{Y}),$$

$$\begin{aligned} [X + \bar{X}, Y + \bar{Y}]^t &= ([X, Y] + [\bar{X}, \bar{Y}])^t \\ &= 2\Re(\omega_\lambda(X, Y)t_\lambda) \\ &= \Re(\Omega_\lambda(X + \bar{X}, Y + \bar{Y})t_\lambda). \end{aligned}$$

If $\lambda \in \Lambda$ is real, $\mathfrak{g}_\lambda = \mathfrak{g}_{\bar{\lambda}}$ and $X = X_\lambda + X_{-\lambda}, Y = Y_\lambda + Y_{-\lambda} \in \mathfrak{g}_{\pm\lambda}$. For $t \in \mathfrak{t}$,

$$ad_t(X) = \lambda(t)(X_\lambda - X_{-\lambda}),$$

$$\Omega_t(X, Y) = ad_t(X) \cdot Y = \lambda(t)(X_\lambda - X_{-\lambda}) \cdot (Y_\lambda + Y_{-\lambda})$$

$\Omega_\lambda = (X_\lambda - X_{-\lambda}) \cdot (Y_\lambda + Y_{-\lambda})$ is real and symplectic.

If $\lambda \in \Lambda$ is pure imaginary, then $\mathfrak{g}_{-\lambda} = \mathfrak{g}_{\bar{\lambda}}$. Since $t \cdot [X + \bar{X}, Y + \bar{Y}] = \lambda(t)\Omega_\lambda(X + \bar{X}, Y + \bar{Y})$ is real, Ω_λ is pure imaginary and $\Im(\Omega_\lambda)$ is symplectic where

$$\Omega_\lambda(X + \bar{X}, Y + \bar{Y}) = (X - \bar{X}) \cdot (Y + \bar{Y})$$

is pure imaginary. ■

Proposition 4.2. *Let G be a semisimple Lie group. Let Γ be a surface group. Let $\phi : \Gamma \rightarrow G$ be a homomorphism whose Zariski closure is reductive. Let \mathfrak{c} denote the center of the centralizer \mathfrak{z} of ϕ in \mathfrak{g} . Let $G^\mathfrak{c}$ denote the centralizer of \mathfrak{c} in G . Among the roots of the adjoint action of \mathfrak{c} on $\mathfrak{g} \otimes \mathbb{C}$, pick a subset $\Lambda' \subset \Lambda$ containing exactly one representative of each set $\{\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}\}$ so that*

$$\mathfrak{g} = \mathfrak{g}_{0, \mathbb{R}} \oplus \bigoplus_{\lambda \in \Lambda'} \mathfrak{g}_{\lambda, \mathbb{R}}$$

where $\mathfrak{g}_{0,\mathbb{R}} = \mathfrak{g}_0 \cap \mathfrak{g}$. Pick a G^c -invariant splitting $\mathfrak{z} = \mathfrak{c} \oplus \mathfrak{s}$ with \mathfrak{s} a semisimple Lie subalgebra. Since $H^2(\Gamma, \mathfrak{g}) = \mathfrak{z}$, the quadratic map $Q : H^1(\Gamma, \mathfrak{g}) \rightarrow H^2(\Gamma, \mathfrak{g})$ splits as $Q = Q^c + Q^s$. Then, on

$$H^1(\Gamma, \mathfrak{g}) = H^1(\Gamma, \mathfrak{g}_{0,\mathbb{R}}) \oplus \bigoplus_{\lambda \in \Lambda'} H^1(\Gamma, \mathfrak{g}_{\lambda,\mathbb{R}}),$$

the \mathfrak{c} -valued quadratic map Q^c vanishes on $H^1(\Gamma, \mathfrak{g}_{0,\mathbb{R}})$ and splits as a direct sum of nondegenerate scalar quadratic forms

$$Q^c = \bigoplus_{\lambda \in \Lambda'} \Re(Q_\lambda t_\lambda).$$

If λ is neither real nor pure imaginary, $\mathfrak{g}_{\lambda,\mathbb{R}}$ has a natural complex structure, Q_λ is a \mathbb{C} -bilinear form.

For real or pure imaginary λ , $\mathfrak{g}_{\lambda,\mathbb{R}}$ is a symplectic vector space, whence a symplectic representation ρ_λ of Γ , with Toledo invariant T_λ . Then the real valued quadratic form Q_λ (resp. $\Im(Q_\lambda)$) lives on a vector space $H^1(\Gamma, \mathfrak{g}_{\lambda,\mathbb{R}})$ whose dimension is equal to $-\chi(\Gamma)\dim(\mathfrak{g}_{\lambda,\mathbb{R}})$. The signature of Q_λ is equal to $4T_\lambda$.

If λ is real, $T_\lambda = 0$ automatically.

Assume that λ is pure imaginary and that the quadratic form $\Im(Q_\lambda)$ on $H^1(\Gamma, \mathfrak{g}_{\lambda,\mathbb{R}})$ is definite. Then $\mathfrak{g}_{\lambda,\mathbb{R}}$ carries a natural complex structure J_λ which is orthogonal with respect to the Killing form. The signature of the Killing form, viewed as a non-degenerate sesquilinear form on $(\mathfrak{g}_{\lambda,\mathbb{R}}, J_\lambda)$, vanishes. In other words, the representation ρ_λ lives in a proper subgroup $U(p_\lambda, p_\lambda) \subset Sp(\mathfrak{g}_{\lambda,\mathbb{R}}, \Omega_\lambda)$, $4p_\lambda = \dim_{\mathbb{R}}(\mathfrak{g}_{\lambda,\mathbb{R}})$.

Proof: Since ϕ is reductive, its centralizer is reductive (Corollary 11.2), and so its center \mathfrak{c} is a torus, and we can apply Proposition 4.1.

The splitting of cohomology follows from the corresponding splitting of \mathfrak{g} and of the \mathfrak{c} -component of the Lie bracket. For nonzero λ , $H^0(\Gamma, \mathfrak{g}_\lambda) = \{a \in \mathfrak{g}_\lambda \mid Ad \circ \phi(\gamma)a = a \text{ for all } \gamma \in \Gamma\}$ and such an a must belong to \mathfrak{g}_0 , and since $\mathfrak{g}_\lambda \cap \mathfrak{g}_0 = 0$, $H^0(\Gamma, \mathfrak{g}_\lambda) = 0$. Hence $\dim_{\mathbb{C}} H^1(\Gamma, \mathfrak{g}_\lambda) = -\chi(\Gamma)\dim_{\mathbb{C}}(\mathfrak{g}_\lambda) > 0$, thus no term vanishes. On $H^1(\Gamma, \mathfrak{g}_{\pm\lambda} + \mathfrak{g}_{\pm\bar{\lambda}})$, define the complex valued quadratic form Q_λ by

$$Q_\lambda = \Omega_\lambda(u \smile u).$$

From the proof of Proposition 4.1, \mathfrak{g}_α and \mathfrak{g}_β are orthogonal with respect to Ω_t , $t \in \mathfrak{c}$ for $\alpha + \beta \neq 0$ and Ω_t vanishes on \mathfrak{g}_0 , hence $\Omega_t(u_\lambda \smile u_\gamma) = 0$ for $\lambda + \gamma \neq 0$ and on \mathfrak{g}_0 . Then, for $u = u_0 + \sum_{\lambda \in \Lambda'} u_\lambda$, the \mathfrak{c} -valued quadratic form Q^c satisfies

$$\begin{aligned} Q^c(u) &= [u \smile u]^c \\ &= \sum_{\lambda \in \Lambda'} [u_\lambda \smile u_\lambda]^c \\ &= \sum_{\lambda \in \Lambda'} \Re(Q_\lambda t_\lambda). \end{aligned}$$

Note that \mathfrak{g}_λ is $Ad \circ \phi$ -invariant. Since for $t \in \mathfrak{c}$, $X \in \mathfrak{g}_\lambda$,

$$[t, Ad \circ \phi(\gamma)X] = [Ad \circ \phi(\gamma)t, Ad \circ \phi(\gamma)X] = Ad \circ \phi(\gamma)[t, X] = \lambda(t)(Ad \circ \phi(\gamma)X).$$

For real or pure imaginary λ , let $\rho_\lambda : \Gamma \rightarrow Sp(\mathfrak{g}_{\lambda, \mathbb{R}}, \Omega_\lambda)$ denote the symplectic linear representation defined by ϕ composed with the adjoint representation restricted to $\mathfrak{g}_{\lambda, \mathbb{R}}$. Let E_{ρ_λ} be the corresponding symplectic vectorbundle on Σ . From Meyer's signature formula (see Lemma 10.2), it follows that the signature of the quadratic form $Q_\lambda(u) = \int_\Sigma \Omega_\lambda(u \smile u)$ on $H^1(\Gamma, \mathfrak{g}_{\lambda, \mathbb{R}})$ is equal to $4c_1(E_{\rho_\lambda})$. Finally, Lemma 10.3 equates the first Chern class $c_1(E_\lambda)$ to the Toledo invariant T_λ .

If λ is neither real nor pure imaginary, the complex structure J_λ on $\mathfrak{g}_{\lambda, \mathbb{R}}$ defines a complex structure on $H^1(\Gamma, \mathfrak{g}_{\lambda, \mathbb{R}}) : u \mapsto J_\lambda \circ u$, with respect to which the complex valued quadratic form Q_λ is \mathbb{C} -bilinear.

If λ is real, ρ_λ fixes the splitting $\mathfrak{g}_{\lambda, \mathbb{R}} = (\mathfrak{g} \cap \mathfrak{g}_\lambda) \oplus (\mathfrak{g} \cap \mathfrak{g}_{-\lambda})$ as a sum of real Lagrangian subspaces. It follows from Lemma 10.3 that $T_\lambda = 0$.

Assume that λ is pure imaginary. The Killing orthogonal splitting

$$\mathfrak{g}_{\lambda, \mathbb{R}} \otimes \mathbb{C} = \mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda} = \mathfrak{g}_\lambda \oplus \bar{\mathfrak{g}}_\lambda$$

defines a Killing orthogonal complex structure, again denoted by J_λ , on $\mathfrak{g}_{\lambda, \mathbb{R}}$. It equals multiplication by i on \mathfrak{g}_λ and by $-i$ on $\mathfrak{g}_{-\lambda}$. If $Z \in \mathfrak{c}$ and $X \in \mathfrak{g}_\lambda$,

$$\begin{aligned} [Z, X + \bar{X}] &= \lambda(Z)X + \bar{\lambda}(Z)\bar{X} \\ &= \lambda(Z)(X - \bar{X}) \\ &= \Im m(\lambda(Z))(iX - i\bar{X}) \\ &= \Im m(\lambda(Z))J_\lambda(X + \bar{X}), \end{aligned}$$

so ad_Z is proportional to J_λ . By construction,

$$\Im m \Omega_\lambda(x + \bar{x}, y + \bar{y}) = -i(x - \bar{x}) \cdot (y + \bar{y}) = -J_\lambda(x + \bar{x}) \cdot (y + \bar{y}),$$

hence $-\Im m \Omega_\lambda(X, Y) = J_\lambda X \cdot Y$. Thus we can view the Killing form as a non-degenerate sesquilinear form on the complex vector space $(\mathfrak{g}_{\lambda, \mathbb{R}}, J_\lambda)$. Let $\sigma = p_\lambda - q_\lambda$ be its signature, with $p_\lambda + q_\lambda = \dim_{\mathbb{C}}(\mathfrak{g}_{\lambda, \mathbb{R}}, J_\lambda)$. Then the signature of the Killing form as a quadratic form is 2σ . We can view ρ_λ as a homomorphism to $U(p_\lambda, q_\lambda)$, or as an action on the Grassmannian $\mathcal{D}_{p_\lambda, q_\lambda}$.

Let T_λ denote the corresponding Toledo invariant. Maximality means that

$$4T_\lambda = -\chi(\Gamma)\dim_{\mathbb{R}}(\mathfrak{g}_{\lambda, \mathbb{R}}) = -\chi(\Gamma)(2p_\lambda + 2q_\lambda).$$

On the other hand, the Milnor-Wood inequality for actions on Hermitian symmetric spaces (due to J. Milnor, [26], V. Turaev, [34], A. Domic and D. Toledo, [9], J.-L. Clerc and B. Ørsted, [8]), states that

$$|T_\lambda| \leq |\chi(\Gamma)|\text{rank}(\mathcal{D}_{p_\lambda, q_\lambda}) = |\chi(\Gamma)|\min\{p_\lambda, q_\lambda\}.$$

Thus $2p_\lambda + 2q_\lambda \leq 4\min\{p_\lambda, q_\lambda\}$. This implies that $p_\lambda = q_\lambda$. In other words, the representation ρ_λ can be maximal only if it lives in the proper subgroup $U(p_\lambda, p_\lambda) \subset Sp(\mathfrak{g}_{\lambda, \mathbb{R}}, \Omega_\lambda)$. \blacksquare

Example 4.3. *Let G be a complex Lie group. Then no root λ is pure imaginary.*

Indeed, the centralizer of an arbitrary subset of G is a complex Lie subgroup, so its center \mathfrak{c} is a complex vector space and roots are \mathbb{C} -linear, hence they cannot be pure imaginary.

4.2. The central quadratic model.

Definition 4.4. *Let G be a semisimple Lie group. Let Γ be a surface group. Let $\phi : \Gamma \rightarrow G$ be a homomorphism whose Zariski closure is reductive. Let \mathfrak{c} denote the center of the centralizer of $\phi(\Gamma)$ in \mathfrak{g} . We call the space*

$$\mathcal{Q}_\phi^{\mathfrak{c}} = \{u \in Z^1(\Gamma, \mathfrak{g}) \mid [u \smile u]^{\mathfrak{c}} = 0 \text{ in } H^2(\Gamma, \mathfrak{c})\}$$

the central quadratic model for $\text{Hom}(\Gamma, G)$ near ϕ . Its smooth points are the points v where $u \mapsto [u \smile v]^{\mathfrak{c}}$ is a submersion. The projection of $\mathcal{Q}_\phi^{\mathfrak{c}}$ in $H^1(\Gamma, \mathfrak{g})$ is denoted by $\mathcal{Q}_\phi^{\mathfrak{c}}$.

By definition, the central quadratic model contains the Goldman-Millson-Simpson quadratic model, and coincides with it if the centralizer of $\phi(\Gamma)$ is abelian. A smooth point of the Goldman-Millson-Simpson quadratic model is smooth as a point of the central quadratic model as well.

Lemma 4.5. *Let G be a semisimple Lie group. Let Γ be a surface group. Let $\phi : \Gamma \rightarrow G$ be a reductive homomorphism. Let C denote the center of the centralizer of $\phi(\Gamma)$ in G . A real cohomology class $u = u_0 + \sum_{\lambda \in \Lambda'} u_\lambda \in H^1(\Gamma, \mathfrak{g})$ with $Q^{\mathfrak{c}}(u) = 0$, defines a smooth point of the central quadratic model if and only if the set of roots λ in Λ' such that $u_\lambda \neq 0$ spans $\mathfrak{c}^* \otimes \mathbb{C}$.*

Proof: The differential of $Q^{\mathfrak{c}}$ at u is

$$u' \mapsto \sum_{\lambda \in \Lambda'} \Re(d_{u_\lambda} Q_\lambda(u') t_\lambda).$$

The image of this linear map over \mathbb{R} is clearly contained in the linear span of the set of all real and imaginary parts of root vectors t_λ such that $u_\lambda \neq 0$. In fact, it is equal to it. Indeed, when $\lambda \neq 0$ and u_λ is non zero, the map

$$H^1(\Gamma, \mathfrak{g}_{\lambda, \mathbb{R}}) \rightarrow H^2(\Gamma, \mathfrak{c}), \quad u' \mapsto \Re(d_{u_\lambda} Q_\lambda(u') t_\lambda)$$

is surjective onto the linear span of $\Re(t_\lambda)$ and $\Im(t_\lambda)$. If λ is real or pure imaginary, this is because Q_λ (resp. $\Im(Q_\lambda)$) is nondegenerate as a real bilinear form. If λ is neither real nor pure imaginary, $H^1(\Gamma, \mathfrak{g}_{\lambda, \mathbb{R}})$ has a complex structure and Q_λ is nondegenerate as a \mathbb{C} -bilinear form. We conclude that $d_u Q^{\mathfrak{c}}$ is surjective if and only if

$$\mathfrak{c} = \text{span}_{\mathbb{R}} \left(\bigcup_{\lambda \neq 0, u_\lambda \neq 0} \{\Re(t_\lambda), \Im(t_\lambda)\} \right).$$

An equivalent formulation is

$$\mathfrak{c}^* \otimes \mathbb{C} = \text{span}_{\mathbb{C}}(\{\lambda \mid u_\lambda \neq 0\}).$$

■

Remark 4.6. *Note that under this condition, not only is \bar{Q}_ϕ^c smooth near u , but its quotient under the stabilizer of u in C , which is a local model for $\text{Hom}(\Gamma, G)/G$, is a smooth orbifold of the same dimension.*

Indeed, the action of C is locally free near u . For each root λ such that $u_\lambda \neq 0$, at least one of the (2 or 4) components (in $\mathfrak{g}_{\pm\lambda}$ and $\mathfrak{g}_{\pm\bar{\lambda}}$) of u_λ on root spaces does not vanish. Let us denote it by x_λ . If $t \in \mathfrak{c}$ satisfies $ad_t(u) = 0$, then $ad_t(x_\lambda) = 0$. This implies that one of $\lambda(t)$, $-\lambda(t)$, $\bar{\lambda}(t)$, $-\bar{\lambda}(t)$ vanishes. In either case, $\lambda(t) = 0$. Since this holds for all λ such that $u_\lambda \neq 0$, and these span $\mathfrak{c}^* \otimes \mathbb{C}$, $t = 0$.

Proposition 4.7. *Let G be a semisimple Lie group. Let Γ be a surface group. Let $\phi : \Gamma \rightarrow G$ be a homomorphism whose Zariski closure is reductive. Let C denote the center of the centralizer of $\phi(\Gamma)$ in G . Let $P \subset \Lambda$ denote the subset of pure imaginary roots for which the quadratic form $\mathfrak{S}m(Q_\lambda)$ is positive definite. Denote by $N = \Lambda \setminus \pm P$ the set of roots which are neither in P nor opposites of elements of P . The following are equivalent.*

- (1) *The central quadratic model has at least one smooth point.*
- (2) *The central quadratic model has a dense set of smooth points.*
- (3) *The convex hull of $\mathfrak{S}m(P) + \text{span}(\Re(N) \cup \mathfrak{S}m(N))$ in \mathfrak{c}^* contains 0 in its interior.*

Proof: Let $P' \subset \Lambda$ be a set which contains for each root $\lambda \in N = \Lambda \setminus \pm P$ a representative of the pair $\{\pm\lambda\}$ or of the 4-tuple $\{\pm\lambda, \pm\bar{\lambda}\}$. Split $P' = R \cup I \cup M$ according to whether λ is real, pure imaginary or neither. When $\lambda \in R$, let $C_\lambda = t_\lambda$ and $q_\lambda = Q_\lambda$. When $\lambda \in P \cup I$, let $C_\lambda = -\mathfrak{S}m(t_\lambda)$ and $q_\lambda = \mathfrak{S}mQ_\lambda$. For $E \subset \Lambda$, denote by $C_E = \{C_\lambda \mid \lambda \in E\}$. With this notation, we get an expansion for Q^c in real terms only,

$$Q^c = \sum_{\lambda \in P \cup I \cup R} q_\lambda C_\lambda + \sum_{\lambda \in M} (\Re(Q_\lambda) \Re(t_\lambda) - \mathfrak{S}m(Q_\lambda) \mathfrak{S}m(t_\lambda))$$

Assume that \bar{Q}_ϕ^c contains at least one smooth point. Since this set is a cone, smooth points can be found arbitrarily close from 0. According to Lemma 4.5, this means that there exists a small nonzero $v = \sum_{\lambda \in \{0\} \cup P \cup P'} v_\lambda$, $v_\lambda \in H^1(\Gamma, \mathfrak{g}_{\lambda, \mathbb{R}})$, such that $Q^c(v) = 0$, and such that the set E of roots $\lambda \in \Lambda$ such that $v_\lambda \neq 0$ spans $\mathfrak{c}^* \otimes \mathbb{C}$. *A fortiori*, $(E \cap P) \cup P'$ spans $\mathfrak{c}^* \otimes \mathbb{C}$, i.e. $C_{E \cap P} \cup C_I \cup C_R \cup \Re(t_M) \cup \mathfrak{S}m(t_M)$ spans \mathfrak{c} . Write

$$0 = Q^c(v) = \sum_{\lambda \in E \cap P} q_\lambda(v_\lambda) C_\lambda + \sum_{\lambda \in I \cup R} q_\lambda(v_\lambda) C_\lambda + \sum_{\lambda \in M} \Re(Q_\lambda(v_\lambda) t_\lambda).$$

By construction, for $\lambda \in E \cap P$, $q_\lambda(v_\lambda) > 0$. Thus 0 belongs to the interior of the convex hull of the image of $C_{E \cap P}$ in the quotient vector space $\mathfrak{c}/\text{span}(C_{R \cup I} \cup \Re(t_M) \cup \mathfrak{S}m(t_M))$. In other words, 0 of \mathfrak{c} belongs to the interior of the sum of the convex hull of $C_{E \cap P}$ and the linear span of $C_{R \cup I} \cup \Re(t_M) \cup \mathfrak{S}m(t_M)$. This implies that $0 \in \text{interior}(\text{convex hull}(\mathfrak{S}m(P)) + \text{span}(\Re(N) \cup \mathfrak{S}m(N)))$.

Conversely, assume that 0 of \mathfrak{c}^* belongs to the interior of the sum of the convex hull of $\Im m(P)$ and the linear span of the real and imaginary parts of elements of N . In other words, there exists a subset $E \subset P \cup P'$, small positive numbers a_λ for $\lambda \in E \cap P$, small nonzero real numbers a_λ for $\lambda \in E \cap (R \cup I)$ and small nonzero complex numbers c_λ for $\lambda \in E \cap M$ such that

$$\sum_{\lambda \in (E \cap P) \cup (E \cap (R \cup I))} a_\lambda C_\lambda + \sum_{\lambda \in M \cap E} \Re(c_\lambda t_\lambda) = 0.$$

For above E , if $\lambda \notin E$, let $a_\lambda = 0$ or $c_\lambda = 0$. Note that

$$\text{span}_{\mathbb{R}}(C_{(E \cap P) \cup (E \cap (R \cup I))} \cup \Re(t_{E \cap M}) \cup \Im m(t_{E \cap M})) = \mathfrak{c},$$

thus $\text{span}_{\mathbb{C}}(E) = \mathfrak{c}^* \otimes \mathbb{C}$.

Let $u \in H^1(\Gamma, \mathfrak{g})$ satisfy $Q^\mathfrak{c}(u) = 0$, i.e., $u \in \bar{Q}_\phi^\mathfrak{c}$. We will find an arbitrary small v such that $u + v$ is a smooth point. This will show that smooth points are dense in $\bar{Q}_\phi^\mathfrak{c}$. For $\lambda \in P \cup R \cup I$, there exists a small $v_\lambda \in H^1(\Gamma, \mathfrak{g}_{\lambda, \mathbb{R}})$ such that $Q_\lambda(u_\lambda + v_\lambda) = Q_\lambda(u_\lambda) + a_\lambda \neq 0$. Indeed, for $\lambda \in E \cap P$, q_λ is positive definite, and takes all positive values. For $\lambda \in R \cup I$, q_λ is indefinite, and takes all real values. In the same manner, for $\lambda \in M$, Q_λ is \mathbb{C} -bilinear and nondegenerate, and takes all complex values, thus there exists a small v_λ such that $Q_\lambda(u_\lambda + v_\lambda) = Q_\lambda(u_\lambda) + c_\lambda$. For $\lambda \in P \setminus E$, take $v_\lambda = 0$. Let $v = \sum_{\lambda \in P \cup P'} v_\lambda$. Then

$$\begin{aligned} Q^\mathfrak{c}(u + v) &= \sum_{\lambda \in P \setminus E} q_\lambda(u_\lambda) C_\lambda + \sum_{\lambda \in (E \cap P) \cup (E \cap (R \cup I))} q_\lambda(u_\lambda + v_\lambda) C_\lambda \\ &\quad + \sum_{\lambda \in M} \Re(Q_\lambda(u_\lambda + v_\lambda) t_\lambda) \\ &= \sum_{\lambda \in P \setminus E} q_\lambda(u_\lambda) C_\lambda + \sum_{\lambda \in (E \cap P) \cup (E \cap (R \cup I))} (q_\lambda(u_\lambda) + a_\lambda) C_\lambda \\ &\quad + \sum_{\lambda \in M} \Re(Q_\lambda(u_\lambda) + c_\lambda) t_\lambda \\ &= Q^\mathfrak{c}(u) + \sum_{\lambda \in (E \cap P) \cup (E \cap (R \cup I))} a_\lambda C_\lambda + \sum_{\lambda \in M} \Re(c_\lambda t_\lambda) \\ &= 0. \end{aligned}$$

By construction for $\lambda \in E$, $Q_\lambda(u_\lambda + v_\lambda) = Q_\lambda(u_\lambda) + a_\lambda \neq 0$, hence the set of roots λ such that $(u + v)_\lambda \neq 0$ contains E , and it spans $\mathfrak{c}^* \otimes \mathbb{C}$. Thus, with Lemma 4.5, $u + v$ gives a smooth point of the central quadratic model (in cohomology). Hence smooth points are dense in the central quadratic model. \blacksquare

4.3. Flexibility for toral centralizers. Here comes the proof of Theorem 3 in case the centralizer is abelian.

Corollary 4.8. *Let G be a semisimple real algebraic group. Let Γ be the fundamental group of a surface of genus $\geq \dim(G)^2$. Let $\phi : \Gamma \rightarrow G$ be a reductive homomorphism. Assume that the centralizer \mathfrak{c} of $\phi(\Gamma)$ is abelian. Split \mathfrak{g} into real root spaces $\mathfrak{g}_{\lambda, \mathbb{R}}$ under \mathfrak{c} . When λ is pure imaginary, $\mathfrak{g}_{\lambda, \mathbb{R}}$ is a symplectic vector space, which gives rise to a Toledo invariant T_λ . For all nonzero pure imaginary roots λ , the following Milnor-Wood type inequality holds.*

$$(2) \quad 4|T_\lambda| \leq -\chi(\Gamma)\dim(\mathfrak{g}_{\lambda, \mathbb{R}}).$$

Denote by P denotes the set of nonzero pure imaginary roots $\lambda \in \Lambda$ such that $4T_\lambda = -\chi(\Gamma)\dim(\mathfrak{g}_{\lambda, \mathbb{R}})$. Denote by N the set of roots which are neither in P nor opposites of elements of P . Then ϕ is fully flexible if and only if

$$0 \in \text{interior}((\text{convex hull}(\mathfrak{Im}(P)) + \text{span}(\Re(N) \cup \mathfrak{Im}(N))).$$

Proof: Note that $H^0(\Gamma, \mathfrak{g}_{\lambda, \mathbb{R}}) = H^2(\Gamma, \mathfrak{g}_{\lambda, \mathbb{R}}) = 0$, hence $\dim H^1(\Gamma, \mathfrak{g}_{\lambda, \mathbb{R}}) = -\chi(\Gamma)\dim(\mathfrak{g}_{\lambda, \mathbb{R}})$. The signature of a quadratic form on a vector space is less than or equal to the dimension of this space, this gives inequality (2). Equality holds if and only if Q_λ is definite. Thus the set P of roots coincides with the set defined in Proposition 4.7. Proposition 4.7 yields the criterion for the quadratic model \tilde{Q}_ϕ to be smooth of dimension $-\chi(\Gamma)\dim(G)$ at a dense set of points. Theorem 3.2 implies that the same holds for $\text{Hom}(\Gamma, G)$ in a neighborhood of the conjugacy class of ϕ . ■

Example 4.9. *Let Γ be a cocompact lattice in $H = SU(1, 1)$. Embed $SU(1, 1)$ as a lower right diagonal block in $G = SU(2, 1)$. Then the obtained homomorphism $\Gamma \rightarrow SU(2, 1)$ is not fully flexible.*

In this case, treated originally by W. Goldman, [12], the centralizer \mathfrak{z} of H is the 1-dimensional subspace generated by $Z = \text{diag}(-2i, i, i)$. Under

$$\text{ad}_Z, \mathfrak{g} = \left\{ \begin{bmatrix} v_1 & v_2 & v_3 \\ -\bar{v}_2 & v_4 & v_5 \\ \bar{v}_3 & \bar{v}_5 & v_6 \end{bmatrix} \right\}, \text{ with } v_2, v_3, v_5 \in \mathbb{C} \text{ and } v_1, v_4, v_6 \in \mathfrak{Im}(\mathbb{C}),$$

splits as

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{3i, \mathbb{R}},$$

where $\mathfrak{g}_0 = \mathfrak{z} \oplus \mathfrak{su}(1, 1)$ and $\mathfrak{g}_{3i, \mathbb{R}} = \left\{ \begin{bmatrix} 0 & v_2 & v_3 \\ -\bar{v}_2 & 0 & 0 \\ \bar{v}_3 & 0 & 0 \end{bmatrix} \right\}$ is a 4-dimensional real

vector space isomorphic to \mathbb{C}^2 with Z acting by multiplication by $\text{diag}(3i, 3i)$ and $SU(1, 1)$ by its standard representation. The restriction of the Killing form to $\mathfrak{g}_{3i, \mathbb{R}}$ is a quadratic form q of signature $(1, 1)$ since $v_2 \in \mathfrak{t}$ and $v_3 \in \mathfrak{p}$ for the Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$.

Consider the flat bundle E_{3i} on Σ with fiber $\mathfrak{g}_{3i, \mathbb{R}}$ equipped with the symplectic form $\mathfrak{Im}\Omega_{3i}$. View the lifted bundle $\tilde{E}_{3i} = \tilde{\Sigma} \times \mathfrak{g}_{3i, \mathbb{R}}$ as a trivial bundle over $H_{\mathbb{C}}^1 = \tilde{\Sigma}$. Let $\tilde{\tau} \subset \tilde{E}_{3i}$ denote the tautological complex line bundle, i.e., at a point $p \in H_{\mathbb{C}}^1$ representing a negative line ℓ , the fiber of $\tilde{\tau}$

is ℓ . Let $\tilde{\tau}^\perp$ be its orthogonal complement. Both descend to bundles over Σ , and $E_{3i} = \tau \oplus \tau^\perp$. The quadratic form q is negative definite on τ and positive definite on τ^\perp . Let J be the (variable) complex structure on W defined by $J = i$ on τ and $J = -i$ on τ^\perp . Then for $B = \begin{pmatrix} b_- \\ b_+ \end{pmatrix} \in E_{3i} = \tau \oplus \tau^\perp$, $\Im m \Omega_{3i}(B, JB) = -q(b_-) + q(b_+)$ is positive definite. Thus J is tamed by $\Im m \Omega_{3i}$. As a complex plane bundle, (E_{3i}, J) is isomorphic to $E_{3i}^{1,0} = \{B \in E_{3i} \otimes \mathbb{C} \mid JB = iB\} = \tau \otimes \mathbb{C} \simeq \tau \oplus \tau$. Therefore $c_1(E_{3i}, \Omega_{3i}) = c_1(\tau \oplus \tau) = 2c_1(\tau)$. Recall that, as a homogeneous vector bundle, the tangent bundle of $H_{\mathbb{C}}^1$ is $TH_{\mathbb{C}}^1 = Hom(\tilde{\tau}, \tilde{\tau}^\perp) = \tilde{\tau}^* \otimes \tilde{\tau}^\perp$. Therefore, as bundles on Σ , $T\Sigma = \tau^* \otimes \tau^\perp$, $c_1(T\Sigma) = c_1(\tau^*) + c_1(\tau^\perp)$. Since $\tau \oplus \tau^\perp = E_{3i}$ is flat, $c_1(\tau^\perp) = -c_1(\tau)$, and $c_1(T\Sigma) = -2c_1(\tau)$. Therefore $c_1(E_{3i}, \Omega_{3i}) = -c_1(T\Sigma)$. As a number, $c_1(E_{3i}, \Omega_{3i}) = -\chi(\Gamma)$. So the Milnor-Wood inequality is an equality.

Example 4.10. Let $\phi_1, \phi_2 : \Gamma \rightarrow Sl(2, \mathbb{R})$ be two nonconjugate discrete cocompact homomorphisms. Use them to map Γ as a Zariski-dense subgroup of $Sl(2, \mathbb{R}) \times Sl(2, \mathbb{R})$. Then map $Sl(2, \mathbb{R}) \times Sl(2, \mathbb{R})$ to $Sl(4, \mathbb{R})$. The obtained homomorphism is fully flexible.

Indeed, the centralizer of $Sl(2, \mathbb{R}) \times Sl(2, \mathbb{R})$ in $Sl(4, \mathbb{R})$ is 1-dimensional, so the set \mathcal{P} of considered homomorphisms consists in a connected component of a single χ_T . \mathcal{P} contains in its closure the diagonal embedding studied in Example 5.8, which is fully flexible. Since full flexibility is an open condition, \mathcal{P} contains fully flexible elements, thus all of \mathcal{P} is fully flexible, according to Corollary ??.

5. LEVI FACTORS OF CENTRALIZERS

5.1. The trivial homomorphism. The simplest case of a homomorphism whose centralizer has a nontrivial Levi factor is the trivial homomorphism. This special case is of course classical. The method and some intermediate results will turn out to be useful later on.

Lemma 5.1. Let Σ be a closed surface with genus $g \geq 1$. Let $\Gamma = \pi_1(\Sigma)$ act trivially on a semisimple Lie algebra \mathfrak{g} . Let Q denote the quadratic map $H^1(\Gamma, \mathfrak{g}) \rightarrow H^2(\Gamma, \mathfrak{g})$, $u \mapsto [u \smile u]$. Let us view an element $u \in H^1(\Gamma, \mathfrak{g})$ as a linear map $H_1(\Gamma, \mathbb{R}) \rightarrow \mathfrak{g}$. Define $\text{span}(u)$ as the image of this map. If the centralizer of $\text{span}(u)$ in \mathfrak{g} is trivial, then

- (1) Q is a submersion at u ;
- (2) the G -action on $H^1(\Gamma, \mathfrak{g})$ is locally free at u .

Proof: 1. Denote by $(X, Y) \mapsto X \cdot Y$ the Killing form on \mathfrak{g} . Assume $Q : u \mapsto [u \smile u]$ is not a submersion at u_0 . This means that the linear map $v \mapsto [u_0 \smile v]$, $H^1(\Gamma, \mathfrak{g}) \rightarrow H^2(\Gamma, \mathfrak{g}) = \mathfrak{g}$, is not onto. Let $X \in \mathfrak{g}$ be orthogonal to its image. Let Y_1, \dots, Y_k be a basis of $\text{span}(u_0)$. Then

$$u_0 = \sum_{i=1}^k a_i \otimes Y_i,$$

where $a_i \in H^1(\Gamma, \mathbb{R})$ are linearly independent. By Poincaré duality, the pairing $(a, b) \mapsto a \smile b$ on $H^1(\Gamma, \mathbb{R})$ is nondegenerate. Therefore, there exist classes $b_i \in H^1(\Gamma, \mathbb{R})$ such that $a_i \smile b_j = \delta_{ij}$. For $Y \in \mathfrak{g}$, take $v_{i,Y} = b_i \otimes Y$. Then $[u_0 \smile v_{i,Y}] = [Y_i, Y]$,

$$0 = X \cdot [u_0 \smile v_{i,Y}] = X \cdot [Y_i, Y] = -[Y_i, X] \cdot Y.$$

This shows that for all $i = 1, \dots, k$, $[Y_i, X] = 0$, i.e. X centralizes the subalgebra generated by $\text{span}(u_0)$.

2. The differential of the G -action on $H^1(\Gamma, \mathfrak{g})$ is $(Z, u) \mapsto ad_Z(u)$, for $Z \in \mathfrak{g}$. If u_0 is viewed as a linear map $H_1(\Gamma, \mathbb{R}) \rightarrow \mathfrak{g}$, then $ad_Z(u_0) = ad_Z \circ u_0$. This vanishes only if Z centralizes the image $\text{span}(u_0)$ of u_0 . ■

Corollary 5.2. *Let G be a semisimple Lie group. Let Γ be the fundamental group of a closed surface of genus greater than 1. The trivial homomorphism of Γ to G belongs to the closure of \mathcal{S} .*

Proof: Take two vectors X and Y in \mathfrak{g} whose centralizers intersect trivially (this is an open dense condition). Take non-colinear classes a and b in $H^1(\Gamma, \mathbb{R})$ such that $a \smile b = 0$. Then $v = a \otimes X + b \otimes Y$ satisfies $[v \smile v] = 2(a \smile b)[X, Y] = 0$. Since the centralizer of $\text{span}(v)$ is trivial, Lemma 5.1 ensures that the quadratic cone $\{u \mid [u \smile u] = 0\}$ is smooth at v . Furthermore, since X and Y do not have a common centralizer, the G -action is locally free near v . One concludes that the quadratic model $\mathcal{Q}_{triv} = \{u \mid [u \smile u] = 0\}$ is smooth near the equivalence class of v , which can be chosen arbitrarily close to the origin. ■

Remark 5.3. *This proves Theorem 1 for semisimple Lie groups. The generalization to reductive groups is postponed until subsection 6.3.*

Note that the preceding argument is not sufficient to prove full flexibility of the trivial representation itself, i.e. density of smooth homomorphisms in a neighborhood of the trivial homomorphism. Instead of constructing just one explicit class v , one needs to adapt v to a given solution u of $[u \smile u] = 0$. We start with some linear algebraic preliminaries.

Lemma 5.4. *Let (V, Ω) be a $2g$ -dimensional symplectic vector space. Let W be a n -dimensional vector space, and ω an alternating 2-form on W . Let E be a codimension q vectorsubspace in $\text{Hom}(W, V)$. Let f_0 be an element of $\text{Hom}(W, V)$. If $g \geq 2nq$, there exists a linear map $f : W \rightarrow V$ such that*

- (1) $f \in E$;
- (2) $f_0 + \epsilon f$ is injective for all nonzero ϵ ;
- (3) $f^* \Omega = \omega$;
- (4) the map $\Phi_f : E \rightarrow W^* \otimes W^*$ defined for $w, w' \in W$ by

$$\Phi_f(f')(w, w') = \Omega(f(w), f'(w'))$$

is onto.

Proof: First, we prove that there exists a symplectic $2n$ -dimensional subvector space $P \subset V$ such that $\text{Hom}(W, P) \subset E$.

Let \mathcal{G} be the Grassmannian of $2n$ -planes in V , and let \mathcal{H} denote the subset of symplectic planes, i.e. those on which Ω induces a symplectic structure. Let τ denote the tautological $2n$ -bundle on \mathcal{G} .

Let ℓ be a linear form on $\text{Hom}(W, V)$. Then, for each $P \in \mathcal{G}$, $\text{Hom}(W, P)$ is a subspace of $\text{Hom}(W, V)$, therefore ℓ defines a linear form on $\text{Hom}(W, P)$, i.e. an element of $\text{Hom}(P, W)$. In other words, ℓ defines a section of the bundle $\text{Hom}(\tau, W)$, which is isomorphic to the direct sum of n copies of τ^* . Since τ^* is isomorphic to τ , ℓ defines as well a section of τ^n , the direct sum of n copies of τ . Choosing q independent linear equations for E defines a section s of τ^{nq} .

Let us show that s must vanish somewhere on \mathcal{H} . For this, since $\tau|_{\mathcal{H}}^{nq}$ is oriented, it is sufficient to prove that the Euler class $e(\tau|_{\mathcal{H}}^{nq})$ does not vanish. Pick a complex structure J on V compatible with Ω and let \mathcal{C} denote the Grassmannian of complex n -dimensional J -complex subspaces in V . Since $\mathcal{C} \subset \mathcal{H}$, it suffices to show that $e(\tau|_{\mathcal{C}}^{nq})$ does not vanish. As a real vectorbundle, $\xi = \tau|_{\mathcal{C}}$ coincides with the tautological complex n -plane bundle $\tau_{\mathbb{C}}$ over \mathcal{C} . In other words, $\xi \otimes \mathbb{C} = \tau_{\mathbb{C}} \oplus \overline{\tau_{\mathbb{C}}}$. Thus

$$\begin{aligned} e(\xi)^2 &= e(\xi \oplus \xi) \\ &= e(\xi \otimes \mathbb{C}) \\ &= c_n(\tau_{\mathbb{C}})c_n(\overline{\tau_{\mathbb{C}}}) \\ &= (-1)^n c_n(\tau_{\mathbb{C}})^2, \end{aligned}$$

and

$$e(\tau|_{\mathcal{C}}^{2nq}) = e(\xi)^{2nq} = (-1)^{n^2q} c_n(\tau_{\mathbb{C}})^{2nq}.$$

Let \mathcal{C}_{∞} denote the direct limit of the Grassmannians \mathcal{C} as g tends to infinity. According to A. Borel, [5], the cohomology algebra of \mathcal{C}_{∞} is freely generated by the Chern classes of the tautological bundle $\tau_{\mathbb{C}, \infty}$. Thus $c_n(\tau_{\mathbb{C}, \infty})^{2nq} \neq 0$. It follows that $e(\tau|_{\mathcal{C}}^{nq})$ does not vanish for g large enough. In fact, $g \geq 2nq$ is sufficient, see [23], chapter 3, section 2. We conclude that for $g \geq 2nq$, there exists a J -complex n -plane $P \in V$ such that for all \mathbb{R} -linear maps $f : W \rightarrow P$, $f \in E$.

Let r denote the rank of ω . Since $r \leq n$, there exists an n -dimensional subspace $P' \subset P$, transverse to $\text{Im}(f_0) \cap P$, such that $\Omega|_{P'}$ has rank r . Let $f_1 : W \rightarrow P'$ be a bijective linear map. Then the pulled-back form $f_1^* \Omega$ has rank r . There exists a bijective linear map $h : W \rightarrow W$ such that $h^*(f_1^* \Omega) = \omega$. Then $f = f_1 \circ h$ satisfies $f^* \Omega = \omega$ and f belongs to E . The map f is injective, $\text{Im}(f)$ and $\text{Im}(f_0)$ are transverse, thus $f_0 + \epsilon f$ is injective for all nonzero ϵ .

Let ϕ denote the restriction of Φ_f to $\text{Hom}(W, P) \subset E$. Then

$$\begin{aligned} \ker(\phi) &= \{f' \in \text{Hom}(W, P) \mid \forall w, w' \in W, \Omega(f(w), f'(w')) = 0\} \\ &= \{f' \in \text{Hom}(W, P) \mid \forall w, w' \in W, f'(w') \perp P'\} \\ &= \text{Hom}(W, P'^{\perp}). \end{aligned}$$

In particular, $\dim(\ker(\phi)) = n^2$, $\dim(\text{im}(\phi)) = 2n^2 - n^2 = n^2$, i.e. ϕ (and thus Φ_f) is onto. \blacksquare

Lemma 5.5. *Let Σ be a closed surface with genus $g \geq 1$. Let $\Gamma = \pi_1(\Sigma)$ act trivially on a semisimple Lie algebra \mathfrak{g} . Let Q denote the quadratic map $H^1(\Gamma, \mathfrak{g}) \rightarrow H^2(\Gamma, \mathfrak{g})$, $u \mapsto [u \smile u]$. Let $L : H^1(\Gamma, \mathfrak{g}) \rightarrow H^2(\Gamma, \mathfrak{g})$ be an arbitrary linear map. Let v_0 be an arbitrary element of $H^1(\Gamma, \mathfrak{g})$. Let $\mathcal{U}_{L, v_0} \subset H^1(\Gamma, \mathfrak{g})$ denote the subset of elements v such that*

- (1) $L(v) = 0$;
- (2) $v_0 + \epsilon v$ has full span for all nonzero ϵ ;
- (3) the map

$$H^1(\Gamma, \mathfrak{g}) \rightarrow H^2(\Gamma, \mathfrak{g}), \quad u' \mapsto L(u') + [v \smile u']$$

is onto.

If $g \geq 2\dim(\mathfrak{g})^2$, then $Q : \mathcal{U}_{L, v_0} \rightarrow H^2(\Gamma, \mathfrak{g})$ is onto.

Proof: Let e_1, \dots, e_{2g} be a basis of $H^1(\Gamma, \mathbb{R})$, Z_1, \dots, Z_n a basis of \mathfrak{g} . If $u = \sum_{i=1}^{2g} \sum_{j=1}^n x_{ij} e_i \otimes Z_j$, $u' = \sum_{i=1}^{2g} \sum_{j=1}^n x'_{ij} e_i \otimes Z_j$, then $[u \smile u'] = \sum_{i,j,k,l} x_{ik} x'_{jl} [Z_k, Z_l] = \sum_{k,\ell=1}^n y_{k\ell} [Z_k, Z_\ell]$, where the matrices $X = (x_{ij})_{i=1,\dots,2g, j=1,\dots,n}$, $Y = (y_{k\ell})_{k,\ell=1,\dots,n}$ and $A = (e_i \smile e_j)_{i,j=1,\dots,2g}$ are related by

$$Y = X^\top A X'.$$

In other words, the cup-product on $H^1(\Gamma, \mathbb{R})$ induces a bilinear map $(u, u') \rightarrow u \smile u'$, $H^1(\Gamma, \mathbb{R}) \otimes \mathfrak{g} \rightarrow H^2(\Gamma, \mathbb{R}) \otimes \mathfrak{g} \otimes \mathfrak{g}$, whose image by the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ equals $[u \smile u'] \in H^2(\Gamma, \mathfrak{g})$. Note that $u \smile u \in H^2(\Gamma, \mathbb{R}) \otimes \Lambda^2 \mathfrak{g}$.

Let $V = H^1(\Gamma, \mathbb{R})$ be equipped with the symplectic form Ω defined by evaluating the cup product on the fundamental class, and $W = \mathfrak{g}^*$. If we identify $H^1(\Gamma, \mathfrak{g})$ with $\text{Hom}(H_1(\Gamma, \mathbb{R}), \mathfrak{g}) = \text{Hom}(V^*, \mathfrak{g}) = \mathfrak{g} \otimes V = \text{Hom}(W, V)$, one can identify $u \in H^1(\Gamma, \mathfrak{g})$ with a map $f_u \in \text{Hom}(W, V)$. Note that the correspondence is given, for $w \in \mathfrak{g}^*, \gamma \in \Gamma$, by

$$f_u(w)(\gamma) = w(u(\gamma)).$$

Since the cup product in $H^1(\Gamma, \mathbb{R})$ corresponds to the symplectic form Ω via above identification, using the notation of Lemma 5.4, $\Phi_{f_u}(f_{u'})(w, w') = \Omega(f_u w, f_{u'} w')$, and hence $u \smile u$ identifies with $f_u^* \Omega$ and $u \smile u'$ with $\Phi_{f_u}(f_{u'})$. We define a subspace E of $\text{Hom}(W, V) = H^1(\Gamma, \mathfrak{g})$ by

$$f_u \in E \Leftrightarrow L(u) = 0.$$

As an element f_0 of $\text{Hom}(W, V)$, pick f_{v_0} .

Since \mathfrak{g} is semi-simple, every $Z \in \mathfrak{g} = H^2(\Gamma, \mathfrak{g})$ is the image under the Lie bracket of some $\omega \in \Lambda^2 \mathfrak{g}$. Lemma 5.4 provides us with an element $v \in H^1(\Gamma, \mathfrak{g})$ such that $L(v) = 0$ and $v \smile v = \omega$, whence $Q(v) = Z$. Lemma 5.4 asserts that $\Phi_{f_v} : E \rightarrow W^* \otimes W^*$, is onto. This translates into surjectivity of

$$\ker(L) \rightarrow H^2(\Gamma, \mathbb{R}) \otimes \mathfrak{g} \otimes \mathfrak{g}, \quad u' \mapsto v \smile u'.$$

Again, this implies that the map $u' \mapsto [v \smile u']$, $\ker(L) \rightarrow H^2(\Gamma, \mathfrak{g})$ is onto. A fortiori,

$$H^1(\Gamma, \mathfrak{g}) \rightarrow H^2(\Gamma, \mathfrak{g}), \quad u' \mapsto L(u') + [v \smile u']$$

is onto. Note that $f_{v_0} + \epsilon f_v$ being injective implies that $v_0 + \epsilon v$ has full span. Indeed $((f_{v_0} + \epsilon f_v)(w))(\gamma) = w((v_0 + \epsilon v)(\gamma))$, hence if $(f_{v_0} + \epsilon f_v)(w) = 0$, then $w((v_0 + \epsilon v)(\gamma)) = 0$ for all $\gamma \in \Gamma$, so $v_0 + \epsilon v$ must have full span to have $w = 0$.

Therefore $v \in \mathcal{U}_{L, v_0}$. We conclude that the image $Q(\mathcal{U}_{L, v_0})$ contains all of $H^2(\Gamma, \mathfrak{g})$. ■

Lemma 5.5 leads to the full flexibility of the trivial representation. It is also the key to the proof of Theorem 3 in the next subsection.

Corollary 5.6. *Let G be a semi-simple real algebraic group. Let Σ be a compact surface with genus $\geq \dim(G)^2$. Then the trivial homomorphism $\Gamma \rightarrow G$ is fully flexible.*

Proof: Let $u_0 \in H^1(\Gamma, \mathfrak{g})$ be such that $[u_0 \smile u_0] = 0$. We need a $v \in H^1(\Gamma, \mathfrak{g})$ such that $u_\epsilon = u_0 + \epsilon v$ has full span and satisfies $[u_\epsilon \smile u_\epsilon] = 0$. We achieve this by requiring that $[u_0 \smile v] = 0$ and $[v \smile v] = 0$. Therefore we set $L(v) = [u_0 \smile v]$ and $v_0 = u_0$. Lemma 5.5 yields a cohomology class v such that $[u_0 \smile v] = 0$, $[v \smile v] = 0$ and $u_0 + \epsilon v$ has full span for all nonzero ϵ . According to Lemma 5.1, the conjugacy class of $u_0 + \epsilon v$ is a smooth point of the quadratic model \bar{Q}_{triv} . With Theorem 3.2 and Corollary 2.4, this shows that Zariski dense representations are dense in a neighborhood of the trivial representation. ■

5.2. Semisimple centralizers. Here comes the proof of an other special case of Theorem 3: when the centralizer is semisimple.

Proposition 5.7. *Let G be a semi-simple real algebraic group. Let Σ be a compact surface with genus $\geq 2\dim(G)^2$ and $\phi : \pi_1(\Sigma) = \Gamma \rightarrow G$ be a homomorphism with reductive Zariski closure. Assume that its centralizer \mathfrak{z} is semi-simple. Then ϕ is fully flexible.*

Proof: Let H denote the Zariski closure of $\phi(\Gamma)$ and $Z = Z_G(H)$ its centralizer. Since HZ is reductive, there exists an HZ -invariant splitting

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}'.$$

Let $u_0 \in H^1(\Gamma, \mathfrak{g})$ be such that $[u_0 \smile u_0] = 0$. Write $u_0 = v_0 + w_0$ with $v_0 \in H^1(\Gamma, \mathfrak{z})$ and $w_0 \in H^1(\Gamma, \mathfrak{z}')$. For $v \in H^1(\Gamma, \mathfrak{z})$, let $L(v) = [u_0 \smile v] \in H^2(\Gamma, \mathfrak{z}) = H^2(\Gamma, \mathfrak{g})$.

Note that Γ acts trivially on \mathfrak{z} . According to Lemma 5.5, there exists $v \in H^1(\Gamma, \mathfrak{z})$ such that $L(v) = 0$ and $[v \smile v] = 0$. Then $u = u_0 + v$ satisfies $Q(u) = 0$. The map

$$u' \mapsto [u \smile u'] = L(u') + [v \smile u'], \quad H^1(\Gamma, \mathfrak{z}) \rightarrow H^2(\Gamma, \mathfrak{z}) = H^2(\Gamma, \mathfrak{g})$$

is onto, thus the cup-product map on $H^1(\Gamma, \mathfrak{g})$ is a submersion at u .

The \mathfrak{z} -component of u is $v_0 + v$. Since $v_0 + v$ has full span, its stabilizer under \mathfrak{z} is trivial. It follows that the $Z = Z_G(\phi(\Gamma))$ -action on $H^1(\Gamma, \mathfrak{g})$ is locally free at u .

Since v can be chosen to be arbitrarily small, one concludes that there are smooth points of the quadratic model $\bar{Q}_\phi = \{[u \smile u] = 0\}$ in all neighborhoods of the class of u . With Theorem 3.2, this shows that smooth points are dense in a neighborhood of ϕ in $\text{Hom}(\Gamma, G)$. \blacksquare

Example 5.8. *Map Γ to a discrete cocompact subgroup of $Sl(2, \mathbb{R})$, then map $Sl(2, \mathbb{R})$ diagonally to $Sl(2, \mathbb{R}) \times Sl(2, \mathbb{R})$, and finally realize $Sl(2, \mathbb{R}) \times Sl(2, \mathbb{R})$ as block-diagonal matrices in $Sl(4, \mathbb{R})$. The obtained homomorphism is fully flexible.*

Indeed, the centralizer of the diagonal $Sl(2, \mathbb{R})$ in $Sl(4, \mathbb{R})$ is isomorphic to $Sl(2, \mathbb{R})$.

5.3. Reductive centralizers. We show that occurrence of a nontrivial Levi factor in a centralizer can only bring extra flexibility, without destroying the flexibility allowed by the center of the centralizer.

Proposition 5.9. *Let G be a semisimple real algebraic group. Let Γ be the fundamental group of a closed surface Σ of genus $\geq 2\dim(G)^2$. Let $\phi : \Gamma \rightarrow G$ be a homomorphism with reductive Zariski closure. Assume that the central local quadratic model \bar{Q}_ϕ^c for $\text{Hom}(\Gamma, G)$ near ϕ has at least one smooth point. Then the full quadratic model \bar{Q}_ϕ has a dense set of smooth points.*

Proof: Let H denote the Zariski closure of $\phi(\Gamma)$ and $Z = Z_G(H)$ its centralizer. Since HZ is reductive, there exists an HZ -invariant splitting

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}'.$$

The Lie algebra of Z splits as $\mathfrak{z} = \mathfrak{s} \oplus \mathfrak{c}$ where \mathfrak{s} is semi-simple and \mathfrak{c} is the center of \mathfrak{z} . Let $u \in H^1(\Gamma, \mathfrak{g})$ be such that $Q(u) = 0$.

Assume that \bar{Q}_ϕ^c has at least one smooth point. There exist arbitrarily small $v \in H^1(\Gamma, \mathfrak{g})$ such that $[(u+v) \smile (u+v)]^c = 0$. Furthermore, one can assume that

- the linearization of Q^c at $u+v$, $u' \mapsto [(u+v) \smile u']^c$, is onto ;
- for $Z \in \mathfrak{c}$, $ad_Z(u+v) = 0$ implies $Z = 0$.

The arguments in the proof of Proposition 5.7 take care of the \mathfrak{s} -component $Q^{\mathfrak{s}}$ by taking a linear map $L : H^1(\Gamma, \mathfrak{s}) \rightarrow H^2(\Gamma, \mathfrak{s})$, $w \mapsto [(u+v)^{\mathfrak{s}} \smile w]$ and $(u+v)^{\mathfrak{s}}$ as an arbitrarily given element. Lemma 5.5 provides us with a small $w \in H^1(\Gamma, \mathfrak{s})$ such that $[(u+v)^{\mathfrak{s}} \smile w] = 0$ and $[w \smile w] = -[(u+v) \smile (u+v)]^{\mathfrak{s}}$. Furthermore, one can assume that $u^{\mathfrak{s}} + v^{\mathfrak{s}} + w$ has full span in \mathfrak{s} , and that the map

$$H^1(\Gamma, \mathfrak{s}) \rightarrow H^2(\Gamma, \mathfrak{s}), \quad u' \mapsto [(u^{\mathfrak{s}} + v^{\mathfrak{s}}) \smile u'] + [w \smile u']$$

is onto. Then

$$\begin{aligned} [(u+v+w) \smile (u+v+w)] &= [(u+v+w) \smile (u+v+w)]^{\mathfrak{c}} \\ &\quad + [(u+v+w) \smile (u+v+w)]^{\mathfrak{s}} \\ &= 2[(u+v) \smile w]^{\mathfrak{c}} + [(u+v) \smile (u+v)]^{\mathfrak{s}} + [w \smile w] \\ &= 2[((u+v)^{\mathfrak{b}} + (u+v)^{\mathfrak{d}'}) \smile w]^{\mathfrak{c}} = 2[(u+v)^{\mathfrak{b}} \smile w]^{\mathfrak{c}} \\ &= 2[((u+v)^{\mathfrak{c}} + (u+v)^{\mathfrak{s}}) \smile w]^{\mathfrak{c}} = 2[(u+v)^{\mathfrak{s}} \smile w]^{\mathfrak{c}} = 0. \end{aligned}$$

Also $u' \in H^1(\Gamma, \mathfrak{s}) \mapsto [(u+v+w) \smile u']^{\mathfrak{s}} = [(u+v)^{\mathfrak{s}} \smile u'] + [w \smile u']$ is onto to $H^2(\Gamma, \mathfrak{s})$ and $u' \in H^1(\Gamma, \mathfrak{c}) \mapsto [(u+v+w) \smile u']^{\mathfrak{c}} = [(u+v) \smile u']^{\mathfrak{c}}$ is onto to $H^2(\Gamma, \mathfrak{c})$. This yields $Q(u+v+w) = 0$, with Q being a submersion at $u+v+w$.

Therefore $Q^{-1}(0)$ is smooth of dimension $-\chi(\Gamma)\dim(G)$ near $u+v+w$. Since v , and consequently w , could be chosen arbitrarily small, this proves that \widehat{Q}_ϕ has a dense set of smooth points. \blacksquare

5.4. Proof of Theorem 3, flexibility part. Now we prove the following: *Let G be a semisimple real algebraic group. Let Γ be the fundamental group of a closed surface of genus $\geq 2\dim(G)^2$. Let $\phi : \Gamma \rightarrow G$ be a homomorphism with reductive Zariski closure H . Assume that \mathfrak{c} , the center of the centralizer of $\phi(\Gamma)$, is balanced with respect to ϕ . Then ϕ is fully flexible.*

Proof: Proposition 4.7 provides a dense set of smooth points for the central quadratic model. Proposition 5.9 asserts that the full quadratic model has the same property. Theorem 3.2 transfers this property to a neighborhood of ϕ in $\text{Hom}(\Gamma, G)$. Thus ϕ is fully flexible. \blacksquare

6. CENTRAL EXTENSIONS

In this section, we examine how smoothness lifts via central extensions. This is motivated by a technical difficulty in the proof of the remainder of Theorem 3 (necessary condition for full flexibility), see subsection 7.3. As a by-product, this allows to generalize our results from semisimple to reductive ambient groups.

6.1. Lifting homomorphisms.

Lemma 6.1. *Let $1 \rightarrow D \rightarrow G \xrightarrow{\pi} Q \rightarrow 1$ be a central extension of Lie groups such that D is discrete. Let Γ be a finitely generated group. Let $\pi \circ : Hom(\Gamma, G) \rightarrow Hom(\Gamma, Q)$, $\phi \mapsto \pi \circ \phi$. Then $\pi \circ$ is a local isomorphism of real analytic sets. In particular, it induces a local diffeomorphism $\mathcal{S}_G \rightarrow \mathcal{S}_Q$ (which needs not be onto).*

Proof: Let us use the notation of subsection 2.1. Let $\langle S|N \rangle$ be a presentation of Γ . By noetherianity, near a point $\phi \in Hom(\Gamma, G)$ (resp. $\pi \circ \phi \in Hom(\Gamma, Q)$), a finite set $N' \subset N$ of relations suffice to define the real analytic subset $Hom(\Gamma, G)$ (resp. $Hom(\Gamma, Q)$). There is a commutative diagram

$$\begin{array}{ccc} G^S & \xrightarrow{F_G} & G^{N'} \\ \pi^S \downarrow & & \pi^{N'} \downarrow \\ Q^S & \xrightarrow{F_Q} & Q^{N'} \end{array}$$

in which the vertical arrows are local analytic isomorphisms. This shows that $\pi \circ : Hom(\Gamma, G) \rightarrow Hom(\Gamma, Q)$ is a local analytic isomorphism. In particular, it maps smooth points to smooth points and conversely, and is a smooth diffeomorphism on the subset of smooth points. \blacksquare

Corollary 6.2. *Let G be a connected, reductive real algebraic group with radical R and $\pi : G \rightarrow G/R$. Let Γ be a closed surface group of negative Euler characteristic. Let $\pi \circ : Hom(\Gamma, G) \rightarrow Hom(\Gamma, G/R)$, $\phi \mapsto \pi \circ \phi$. Then $\pi \circ$ is an open map, its image is a union of connected components of $Hom(\Gamma, G/R)$. Furthermore, smooth homomorphisms $\Gamma \rightarrow G$ are exactly lifts of smooth homomorphisms $\Gamma \rightarrow G/R$, i.e. $\mathcal{S}_G = (\pi \circ)^{-1}(\mathcal{S}_{G/R})$.*

Proof: Since G is connected, reductive algebraic, \mathfrak{r} , the Lie algebra of R , is the center of \mathfrak{g} . Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be a Levi decomposition of \mathfrak{g} , with \mathfrak{s} semisimple. Then $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$, thus the Lie algebra $\mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}]$ of $D = R \cap [G, G]$ is trivial, and D is discrete.

Let $R' = R/D$ and $G' = G/D$. Then $R' \cap [G', G'] = \{1\}$. Therefore the homomorphism $[G', G'] \rightarrow G'/R' = G/R$ is injective. Its image has the same Lie algebra as G/R , it is an open subgroup of G/R , thus it is equal to G/R , since G/R is connected. In other words, $[G', G']$ is isomorphic to G/R .

Since R' is central in G' , the map $R' \times [G', G'] \rightarrow G'$, $(r, h) \mapsto rh$, is a homomorphism, it is injective as $R' \cap [G', G'] = \{1\}$, it is surjective as $[G', G']$ maps onto G'/R' , thus

$$G' \simeq R' \times [G', G'] \simeq R' \times G/R.$$

The map $\pi : G \rightarrow G/R$ factors as $\pi'' \circ \pi'$ where $\pi' : G \rightarrow G'$ and $\pi'' : G' \rightarrow G/R$. Since D is discrete and π' is a central extension, Lemma 6.1 applies, $\pi' \circ$ is locally homeomorphic, it maps smooth points of $Hom(\Gamma, G)$ to smooth points of $Hom(\Gamma, G')$ (and vice-versa) and is a smooth local diffeomorphism there.

Since $G' \simeq R' \times G/R$, $\text{Hom}(\Gamma, G') = \text{Hom}(\Gamma, R') \times \text{Hom}(\Gamma, G/R)$ and $\pi'' \circ$ equals the projection onto the second factor. Since $\text{Hom}(\Gamma, R')$ is smooth, $\pi'' \circ$ maps smooth points to smooth points and vice-versa. We conclude that $\mathcal{S}_G = (\pi \circ)^{-1}(\mathcal{S}_{G/R})$. ■

6.2. Virtual dimension for reductive groups. Here, as announced in subsection 2.2, we justify the Definition 2.1 of virtual dimensions for representation varieties of reductive algebraic groups.

Proposition 6.3. *Let G be a connected reductive real algebraic group with radical R . Let Γ be a closed surface group. Let $\phi : \Gamma \rightarrow G$ be a homomorphism such that the identity component of the centralizer of $\phi(\Gamma)$ is equal to R . Then ϕ belongs to the set \mathcal{S} of smooth homomorphisms (see Definitions 1.1 and 2.1). In particular, Zariski dense homomorphisms belong to \mathcal{S} .*

Proof: Let $\pi : G \rightarrow G/R$. Under the assumption, the centralizer of $\pi \circ \phi(\Gamma)$ in G/R is discrete, so $\pi \circ \phi$ is a smooth point of $\text{Hom}(\Gamma, G/R)$. According to Corollary 6.2, ϕ is a smooth point of $\text{Hom}(\Gamma, G)$ as well. ■

6.3. Proof of Theorem 1. Here we complete the proof that if G is a connected reductive real algebraic group and Γ the fundamental group of a closed surface of genus > 1 , then the trivial homomorphism $\Gamma \rightarrow G$ can be deformed into fully flexible homomorphisms.

Proof: Let R denote the radical of G . According to Corollary 5.2, the trivial homomorphism belongs to the closure of $\mathcal{S}_{G/R}$ in $\text{Hom}(\Gamma, G/R)$. Corollary 6.2 shows that its trivial lift to G belongs to the closure of \mathcal{S}_G in $\text{Hom}(\Gamma, G)$. ■

6.4. Flexibility versus rigidity for reductive representations.

Proposition 6.4. *Let G be a connected, reductive, real algebraic group. Let Γ be the fundamental group of a closed surface of genus $\geq 2\dim(G)^2$. Reductive homomorphisms $\Gamma \rightarrow G$ are either partially rigid or fully flexible.*

Proof: Let \mathcal{F} denote the set of fully flexible homomorphisms $\Gamma \rightarrow G$. Recall that \mathcal{S} denote the set of homomorphisms which are smooth points of $\text{Hom}(\Gamma, G)$. By definition, \mathcal{F} is open in $\text{Hom}(\Gamma, G)$ and contains \mathcal{S} as a dense subset. So \mathcal{F} is contained in the closure of \mathcal{S} . Conversely, let ϕ be a reductive homomorphism which belongs to the closure of \mathcal{S} . Let R denote the radical of G and $\pi : G \rightarrow G/R$. Then $\pi \circ \phi$ is reductive again. Corollary 6.2 tells that $\text{Hom}(\Gamma, G/R)$ admits smooth points in every neighborhood of $\pi \circ \phi$. Since G/R is semisimple, Theorem 3.2 applies. The Goldman-Millson-Simpson quadratic model at $\pi \circ \phi$ has at least one smooth point. This is also valid for the central quadratic model $\mathcal{Q}_{\pi \circ \phi}^c$. Proposition 5.9 implies that the full quadratic model $\mathcal{Q}_{\pi \circ \phi}$ has a dense set of smooth points. And by Theorem 3.2 again, smooth points are dense in a neighborhood of $\pi \circ \phi$ in $\text{Hom}(\Gamma, G/R)$. By Corollary 6.2, the same holds in a neighborhood of ϕ in $\text{Hom}(\Gamma, G)$, i.e. ϕ is fully flexible. This shows that $\text{RedHom}(\Gamma, G) \cap \overline{\mathcal{S}} \subset \mathcal{F}$

where $RedHom(\Gamma, G)$ is the set of reductive homomorphisms, which implies that $RedHom(\Gamma, G) \cap \mathcal{F} = RedHom(\Gamma, G) \cap \overline{\mathcal{S}}$ is closed in $RedHom(\Gamma, G)$. Hence $RedHom(\Gamma, G) \cap \mathcal{F}$ is both open and closed in $RedHom(\Gamma, G)$, its complement consists of partially rigid homomorphisms only. ■

7. AMENABLE REPRESENTATIONS

Again, this section is motivated by a technical point in the proof of the remainder of Theorem 3. It will be proved that homomorphisms with reductive amenable Zariski closure are fully flexible. The generalization to nonreductive representations is treated in the next section.

7.1. Bounded homomorphisms.

Proposition 7.1. *Let G be a connected semisimple real algebraic group. Let Γ be the fundamental group of a closed surface of genus $\geq 2\dim(G)^2$. Then every bounded homomorphism $\phi : \Gamma \rightarrow G$ is fully flexible.*

Proof: For any pure imaginary root λ , consider $\rho_\lambda : \Gamma \rightarrow Sp(\mathfrak{g}_{\lambda, \mathbb{R}}, \Omega_\lambda)$, the symplectic linear representation defined by ϕ composed with the adjoint representation restricted to $\mathfrak{g}_{\lambda, \mathbb{R}}$. Since the representation into $Sp(\mathfrak{g}_{\lambda, \mathbb{R}}, \Omega_\lambda)$ factors through a compact group, one can construct an equivariant constant map. Hence the integration of the pull back of the Kähler form is zero. So there is no pure imaginary root with maximal Toledo invariant and the balanced condition is automatically satisfied. ■

7.2. Reductive amenable homomorphisms.

Lemma 7.2. *Let G be a real algebraic group. Let $\phi : \Gamma \rightarrow G$ be a homomorphism whose Zariski closure is reductive and amenable. Then the connected component of ϕ in the space $RedHom(\Gamma, G)$ of reductive homomorphisms $\Gamma \rightarrow G$ contains a bounded homomorphism.*

Proof: Let H denote the Zariski closure of $\phi(\Gamma)$. Since H is reductive, $H = SR$ where R , the radical, is connected and central, and S is semisimple. The connected abelian algebraic group R is isomorphic to a direct product $T \times V$ where V is a vectorspace and T is compact ([6], Proposition 8.15). $R \cap S$ is the center of S , thus is finite and contained in T . Therefore $TS \cap V = \{1\}$, and H is isomorphic to $TS \times V$. Since H is amenable, S is compact, so is TS . For $(u, v) \in TS \times V$, let $\delta_t(u, v) = (u, tv)$. If $t \neq 0$, this is an algebraic automorphism of H , thus the Zariski closure of $\phi_t = \delta_t \circ \phi$ is $\delta_t(H) = H$. The Zariski closure of ϕ_0 is TS . In all cases, ϕ_t is reductive, so $t \mapsto \phi_t$ connects ϕ to the bounded homomorphism ϕ_0 in $RedHom(TS \times V) \subset RedHom(\Gamma, G)$. ■

Corollary 7.3. *Let G be a connected reductive real algebraic group. Let Γ be the fundamental group of a closed surface of genus $\geq 2\dim(G)^2$. Homomorphisms whose Zariski closure is reductive and amenable are fully flexible.*

Proof: Let $\phi : \Gamma \rightarrow G$ be a homomorphism whose Zariski closure is reductive and amenable. Lemma 7.2 provides us with a bounded homomorphism ϕ_0 in the connected component of ϕ in $\text{RedHom}(\Gamma, G)$. According to Proposition 7.1, ϕ_0 is fully flexible. Proposition 6.4 asserts that the set $\mathcal{F} \cap \text{RedHom}(\Gamma, G)$ of fully flexible reductive homomorphisms is open and closed, thus ϕ is fully flexible. \blacksquare

This partial result will be generalized to arbitrary amenable homomorphisms in Corollary 8.4.

7.3. Proof of Theorem 3, rigidity part. Here we complete the proof of Theorem 3, i.e. we prove that: *Let G be a semisimple real algebraic group. Let Γ be the fundamental group of a closed surface Σ of genus $\geq 2\dim(G)^2$. Let $\phi : \Gamma \rightarrow G$ be a homomorphism whose Zariski closure H is reductive. Assume that ϕ is fully flexible. Then the center of the centralizer of $\phi(\Gamma)$ is balanced with respect to ϕ .* (The converse has been proven in subsection 5.4).

Proof: Let Z denote the centralizer of H , Z^0 the identity component of Z and \mathfrak{c} the center of \mathfrak{z} . We construct a small deformation ϕ' of ϕ whose centralizer in \mathfrak{g} is exactly \mathfrak{c} . It suffices that ϕ' be Zariski dense in HZ^0 . Note that

$$\mathfrak{c} = Z_{\mathfrak{z}}(\mathfrak{z}) = \mathfrak{z} \cap Z_{\mathfrak{g}}(\mathfrak{z}) = Z_{\mathfrak{g}}(H) \cap Z_{\mathfrak{g}}(Z^0) = Z_{\mathfrak{g}}(HZ^0).$$

Pick a Levi factor L in H . Since the radical R of H is central and connected ([6], 11.21), it is contained in Z^0 , so that $HZ^0 = LZ^0$.

For clarity, let us treat first a simple case. Assume that $L \cap Z = \{1\}$ and L and Z are connected. Then HZ is isomorphic to $L \times Z$. Let $pr_Z : LZ \rightarrow Z$ and $pr_L : LZ \rightarrow L$ denote the projections onto the factors. Since $\phi(\Gamma)$ is Zariski dense in H , $pr_L \circ \phi(\Gamma)$ is Zariski dense in L , and thus smooth as a homomorphism $\Gamma \rightarrow L$. The homomorphism $pr_Z \circ \phi$ has an abelian image. Corollary 7.3 implies that $pr_Z \circ \phi$ is fully flexible in Z , so it can be slightly deformed into a smooth homomorphism $\psi : \Gamma \rightarrow Z$. Since $\text{Hom}(\Gamma, LZ) = \text{Hom}(\Gamma, L) \times \text{Hom}(\Gamma, Z)$, it follows that $(pr_L \circ \phi, \psi)$ is a smooth homomorphism $\Gamma \rightarrow LZ$, which is close to $(pr_L \circ \phi, pr_Z \circ \phi) = \phi$. According to Corollary 2.4, there is a Zariski dense $\phi' : \Gamma \rightarrow LZ$ close to $(pr_L \circ \phi, \psi)$ and thus to ϕ .

In general, $F = L \cap Z^0$ is a finite group contained in the center of both L and Z^0 . The quotient LZ^0/F is isomorphic to the direct product $(L/F) \times (Z^0/F)$. Let $pr_L : LZ^0 \rightarrow LZ^0/F \rightarrow L/F$ and $pr_Z : LZ^0 \rightarrow LZ^0/F \rightarrow Z^0/F$ denote composed projections. The image of the homomorphism $pr_Z \circ \phi : \Gamma \rightarrow Z^0/F$ is Zariski dense in $pr_Z(R)$, thus is reductive and abelian. According to Corollary 7.3, there exists a smooth homomorphism $\psi : \Gamma \rightarrow Z^0/F$ nearby $pr_Z \circ \phi$. The image of $pr_L \circ \phi$ is Zariski dense in $L/F = H/R$. Then $pr_L \circ \phi$ and ψ are smooth points of $\text{Hom}(\Gamma, L/F)$ and $\text{Hom}(\Gamma, Z^0/F)$ respectively, thus $\bar{\phi} = (pr_L \circ \phi, \psi)$ is a smooth point of the product $\text{Hom}(\Gamma, L/F) \times \text{Hom}(\Gamma, Z^0/F) = \text{Hom}(\Gamma, L/F \times Z^0/F)$. Now LZ^0

is a finite central extension of LZ^0/F , thus, thanks to Lemma 6.1, $\bar{\phi}$ lifts to a smooth homomorphism $\tilde{\phi} : \Gamma \rightarrow LZ^0$ close to ϕ . The smallest open subgroup of LZ^0 containing $\phi(\Gamma)$ contains Z^0 (which is connected) and H , so this is $HZ^0 = LZ^0$. Since LZ^0 has only finitely many open subgroups, LZ^0 is still the smallest open subgroup of LZ^0 containing $\tilde{\phi}(\Gamma)$. According to Proposition 2.6, there is a Zariski dense $\phi' : \Gamma \rightarrow LZ^0$ nearby $\tilde{\phi}$ and so nearby ϕ . As we saw earlier, the centralizer of ϕ' equals \mathfrak{c} .

Since ϕ is fully flexible, so is ϕ' . Corollary 4.8 implies that \mathfrak{c} is balanced with respect to ϕ' . Since ϕ' is close to ϕ , both homomorphisms give rise to the same Toledo invariants, \mathfrak{c} is balanced with respect to ϕ as well. ■

8. NON REDUCTIVE REPRESENTATIONS

The fact that $Hom(\Gamma, G)/G$ may be non Hausdorff greatly helps. It will allow us to continuously deform by conjugation non reductive representations to reductive ones.

8.1. Closures of G -orbits in $Hom(\Gamma, G)$.

Lemma 8.1. *Let G be a semisimple real algebraic group, $H \subset G$ a non reductive subgroup of G . There exists a one parameter subgroup $t \mapsto g_t$ of G such that the restriction of Ad_{g_t} to H converges as t tends to $+\infty$ to a homomorphism $T : H \rightarrow I$, where $I = Z_G(\{g_t\})$ is a product $I = G'K'A'$, where G' is semisimple of non compact type, K' is compact, A' is a real split torus, G' commutes with $K'A'$ and K' normalizes A' .*

Proof: We rely on section 4.4 of [10]. Let X denote the symmetric space of G , and let $X(\infty)$ denotes its visual boundary, equipped with Tits' metric. By definition, the unipotent radical U of H is non trivial. According to Proposition 4.4.3, page 312 of [10], U has fixed points on $X(\infty)$, all contained in an open Tits ball of radius π . So does its normalizer (Proposition 4.4.4 of [10]), so H has a fixed point x in $X(\infty)$. Next we use section 2.17 of [10]. Pick an origin o in X . The unit vector u at o pointing towards x corresponds to an element, still denoted by u , of the \mathfrak{p} part of the Cartan decomposition of \mathfrak{g} at o . According to Proposition 2.17.5, page 104 of [10], for every $g \in H$, the limit

$$T_x(g) = \lim_{t \rightarrow +\infty} e^{-tu} g e^{tu}$$

exists, and defines a group homomorphism $T_x : H \rightarrow I = Z_G(u)$. Furthermore, $I = K_x A_x$ where $K_x = I \cap K$ is compact and $A_x = \exp(Z_{\mathfrak{g}}(u) \cap \mathfrak{p})$, so I is stable under the Cartan involution corresponding to o . It follows that the orbit Y of o under I is a totally geodesic subspace of X . Let F denote the Euclidean factor of the de Rham decomposition $Y = Y' \times F$ of Y . The isometry group G' of Y' being generated by geodesic symmetries, embeds in G . For the same reason, the group A' of translations of F embeds in G . G' and A' are contained in I . Since $G' \times A'$ acts transitively on $Y = Y' \times F$, it is cocompact in I . The kernel of the isometric action of I on Y' is a subgroup

of $\text{Isom}(F)$ which contains A' , i.e. a semi-direct product $K' \ltimes A'$ with K' compact. Therefore I is isomorphic to $G' \times (K' \ltimes A')$. ■

The next lemma is a variant of Lemma 8.1, tailored for an induction procedure.

Lemma 8.2. *Let G be a semisimple real algebraic group, let G_1 and G_2 be commuting subgroups of G , such that G_1 is semisimple of non compact type, and G_2 contains no unipotent elements. Let $H \subset G_1G_2$ be a non reductive subgroup of G_1G_2 . There exists a one parameter subgroup $t \mapsto g_t$ of G_1 such that the restriction of Ad_{g_t} to H converges as t tends to $+\infty$ to a homomorphism $T : H \rightarrow G'_1G'_2$, where G'_1 is a proper subgroup of G_1 and is semisimple of non compact type, G'_2 commutes with G'_1 and contains no unipotent elements.*

Proof: Let U be the unipotent radical of H . Its projection to G_2 is unipotent, thus trivial, so $U \subset H \cap G_1$. In particular, the projection H_1 of H to G_1 is not reductive. Lemma 8.1 provides the needed one-parameter group $t \mapsto g_t$. Ad_{g_t} being trivial on G_2 , its restriction to $H_1G_2 \supset H$ converges to a homomorphism $T : H_1G_2 \rightarrow Z_{G_1}(\{g_t\})G_2$. Furthermore, $Z_{G_1}(\{g_t\}) = G'_1K'A'$ where G'_1 is a proper subgroup of G_1 and is semisimple of non compact type, and $K'A'$ commutes with G'_1 and contains no unipotent elements. Since $K'A'$ commutes with G_2 , $G'_2 = K'A'G_2$ does not contain any unipotent elements either. ■

Proposition 8.3. *Let G be a semisimple real algebraic group. Let Γ be a finitely generated group. Let G act on $\text{Hom}(\Gamma, G)$ by conjugation. Then every orbit closure contains at least one reductive homomorphism. More precisely, given a homomorphism $\phi : \Gamma \rightarrow G$ with Zariski closure H , there exists a homomorphism $T : H \rightarrow G$ such that $T \circ \phi$ is reductive and belongs to the closure of the G -orbit of ϕ .*

Proof: Let H be the Zariski closure of $\phi(\Gamma)$. If H is not reductive, apply Lemma 8.2 to $G_1 = G$ and trivial G_2 , and get $T_1 : H \rightarrow G'_1G'_2$. If $T_1(H)$ is not reductive, replace $\phi = \phi_0$ with $\phi_1 = T_1 \circ \phi$ and apply Lemma 8.2 again, and so on. When the procedure stops, we get a reductive homomorphism $\psi = \phi_k = T_k \circ \dots \circ T_1 \circ \phi$. By construction, each ϕ_j belongs to the closure of the G -orbit of ϕ_{j-1} . Thus ψ belongs to the closure of the G -orbit of ϕ . ■

Corollary 8.4. *Let G be a connected semisimple real algebraic group. Let Γ be the fundamental group of a closed surface of genus $\geq 2\dim(G)^2$. Homomorphisms whose Zariski closure is amenable are fully flexible.*

Proof: This is a combination of Proposition 8.3, openness of full flexibility and Corollary 7.3. Indeed, images of amenable groups are amenable. ■

8.2. Proof of Theorem 2. *Let G be a connected reductive real algebraic group. Let Γ be the fundamental group of a closed surface Σ of genus $\geq 2\dim(G)^2$. Homomorphisms $\Gamma \rightarrow G$ are either fully flexible or partially rigid.*

Proof: Assume first that G is semisimple. The sets \mathcal{F} of fully flexible homomorphisms and \mathcal{R} of partially rigid homomorphisms are both open and G -invariant. The complement of $\mathcal{F} \cup \mathcal{R}$ in $\text{Hom}(\Gamma, G)$ is closed and G -invariant. Assume that it is non empty. According to Proposition 8.3, the closure of the orbit of any of its elements contains a reductive homomorphism, which, according to Proposition 6.4, belongs to $\mathcal{F} \cup \mathcal{R}$, contradiction. We conclude that $\text{Hom}(\Gamma, G) = \mathcal{F} \cup \mathcal{R}$.

If G is merely reductive, let R denote its radical, $\pi : G \rightarrow G/R$ and denote by $\pi \circ : \text{Hom}(\Gamma, G) \rightarrow \text{Hom}(\Gamma, G/R)$ the induced map. From Corollary 6.2, we know that smooth points correspond under $\pi \circ$, i.e. $\mathcal{S}_G = (\pi \circ)^{-1} \mathcal{S}_{G/R}$. Since $\pi \circ$ is an open map, it commutes with closures and interiors, so $\mathcal{F}_G = (\pi \circ)^{-1} \mathcal{F}_{G/R}$ and $\mathcal{R}_G = (\pi \circ)^{-1} \mathcal{R}_{G/R}$. Since G/R is semisimple, we know that $\text{Hom}(\Gamma, G/R) = \mathcal{F}_{G/R} \cup \mathcal{R}_{G/R}$. It follows that $\text{Hom}(\Gamma, G) = \mathcal{F}_G \cup \mathcal{R}_G$. \blacksquare

8.3. Proof of Corollary 4. The following result is a special case of Theorem 5 of [7].

Theorem 8.5. *Let Γ be surface group, $\phi : \Gamma \rightarrow G$ a homomorphism. Let G act symplectically on (V, Ω) . Let $\rho : \Gamma \rightarrow \text{Sp}(V, \Omega)$ be a homomorphism. Assume equality holds in the Milnor-Wood inequality, i.e.*

$$|T_\rho| = -\chi(\Gamma) \dim(V).$$

Then the Zariski closure of $\rho(\Gamma)$ in $\text{Sp}(V, \Omega)$ is reductive, the symmetric space associated to its semisimple Levi factor is Hermitian of tube type.

Using this theorem, we can prove Corollary 4, i.e.: *Let G be a connected reductive real algebraic group. Let Γ be the fundamental group of a closed surface of genus $\geq 2 \dim(G)^2$. Let $\phi : \Gamma \rightarrow G$ be a homomorphism with Zariski closure H . If ϕ is not fully flexible, then H acts transitively on a tube type Hermitian symmetric space, producing a maximal representation of Γ .*

Proof: Assume first that G is semisimple and H is reductive. The centralizer \mathfrak{z} of H in \mathfrak{g} is reductive (Corollary 11.2). If ϕ is not fully flexible, then \mathfrak{z} cannot be semisimple (Proposition 5.7) so it has a nontrivial center \mathfrak{c} . Furthermore (sufficient condition for full flexibility in Theorem 3, proved in subsection 5.4) there is at least one nonzero root λ of \mathfrak{c} on \mathfrak{g} for which

$$(3) \quad 4|c_1(E_\lambda)| = -\chi(\Gamma) \text{rank}(E_\lambda).$$

Here $\mathfrak{g}_{\lambda, \mathbb{R}}$ is the real root space corresponding to λ , Ω_λ the symplectic form on it, E_λ the symplectic bundle on Σ induced by an equivariant map $\tilde{\Sigma} \rightarrow X(\mathfrak{g}_{\lambda, \mathbb{R}}, \Omega_\lambda)$. Since H centralizes \mathfrak{c} , it preserves the splitting of \mathfrak{g} and acts symplectically on $\mathfrak{g}_{\lambda, \mathbb{R}}$. Its image I in $\text{Sp}(\mathfrak{g}_{\lambda, \mathbb{R}}, \Omega_\lambda)$ coincides with the Zariski closure of the image of Γ in $\text{Sp}(\mathfrak{g}_{\lambda, \mathbb{R}}, \Omega_\lambda)$, which, according to Theorem 8.5, is reductive with a Levi factor which is Hermitian of tube type. I is a quotient of H , thus its Levi factor is a product of simple factors of a semisimple Levi factor of H . Thus at least one of them gives rise to a Hermitian symmetric

space of tube type, with a maximal representation of Γ , thanks to equation (3).

If H is not reductive, Proposition 8.3 yields a homomorphism $T : H \rightarrow G$ such that $\phi' = T \circ \phi$ is a limit of conjugates of ϕ and has a reductive Zariski closure H' . If ϕ is not fully flexible, neither is ϕ' , so one of the simple factors of the semisimple Levi factor of H' has a tube type Hermitian symmetric space. H , which surjects onto H' , acts transitively on this symmetric space.

In general, let G be connected reductive with radical R , and $\pi : G \rightarrow G/R$. Then G/R is semisimple. According to Corollary 6.2, the projected homomorphism $\pi \circ \phi$ is not fully flexible. Thus $\pi(H)$ acts transitively on a tube type Hermitian symmetric space, so does H . ■

9. SURFACE GROUPS IN REAL RANK 1 SIMPLE LIE GROUPS

Corollary 5 is proven here.

9.1. Consequences of general results. Let G be a simple Lie group of real rank 1. Let Γ be the fundamental group of a closed surface of large genus $\geq 2\dim(G)^2$. Let $\phi : \Gamma \rightarrow G$ be a homomorphism.

If the Zariski closure of $\phi(\Gamma)$ is not reductive, then it is contained in a proper parabolic subgroup. In rank 1, there is only one, the minimal parabolic subgroup AN , which is amenable. Thus Corollary 7.3 applies, and ϕ is fully flexible.

Assume that the Zariski closure H of $\phi(\Gamma)$ is reductive nonamenable. According to Lemma 11.1, the centralizer of a Levi factor of H is compact, so is the centralizer C of H .

Assume ϕ is not fully flexible in G . Let S denote the product of noncompact factors of a Levi factor of H . The symmetric space Y of S is totally geodesic in X , it has rank 1, so S has only one factor. According to Corollary 4, Y has to be Hermitian of tube type. It must be a complex hyperbolic space. Only the complex hyperbolic line is of tube type. It follows that S is locally isomorphic to $SL(2, \mathbb{R})$. Totally geodesic 2-planes in $X = H_{\mathbb{H}}^m$ are of the following 2 types

- (1) \mathbb{C} -line, i.e. subsets of \mathbb{F} -lines, $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$.
- (2) \mathbb{R} -planes, spanned by orthonormal tangent vectors a, b with $b \perp \mathbb{F}a$.

In each case, we shall compute the center of the centralizer. It turns out to always be 0 or 1-dimensional. A trivial center implies full flexibility (Proposition 5.7). According to Theorem 3, when the center of the centralizer is 1-dimensional, a necessary and sufficient condition for full flexibility is that the direct sum of symplectic representations ρ_λ be non maximal.

In the computation of the Toledo invariant of ρ_λ , only the S -component of ϕ plays a role. Indeed, since $Z_G(S)$ is compact, one can always choose an equivariant map $\tilde{\Sigma} \rightarrow Sp(\mathfrak{g}_{\lambda, \mathbb{R}})$ which factorizes via the symmetric space of S .

9.2. Notation. For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, let $O(m, p, \mathbb{F})$ denote respectively $O(m, p)$, $U(m, p)$, $Sp(m, p)$, F_4^{-20} with a corresponding notation for Lie algebras.

For instance, a nonamenable Lie group H that stabilizes a real plane contains $S = SO(2, 1, \mathbb{R}) \subset SO(2, 1, \mathbb{F}) \subset SO(m, 1, \mathbb{F})$. If H stabilizes a complex line, it contains $S = SU(1, 1) = SO(1, 1, \mathbb{C}) \subset SO(1, 1, \mathbb{F}) \subset SO(m, 1, \mathbb{F})$.

The following lemma will be useful.

Lemma 9.1. *For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, for $0 < p < m$,*

$$\mathfrak{o}(m, q, \mathbb{F}) = \mathfrak{o}(m - p, \mathbb{F}) \oplus \mathfrak{o}(p, q, \mathbb{F}) \oplus \text{Hom}_{\mathbb{F}}(\mathbb{F}^{m-p}, \mathbb{F}^{p+q}).$$

Proof: Let $e = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. Then a $(m - p) \oplus (p + q)$ -block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with entries in \mathbb{F} belongs to $\mathfrak{o}(m, q, \mathbb{F})$ if and only if $A \in \mathfrak{o}(m - p, \mathbb{F})$, $D \in \mathfrak{o}(p, q, \mathbb{F})$ and $B = -C^*e = -\bar{C}^\top e$. An element $M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in O(m - p, \mathbb{F}) \times O(p, q, \mathbb{F})$ acts on $N = \begin{pmatrix} 0 & -C^*e \\ C & 0 \end{pmatrix}$ by

$$\text{Ad}_M(N) = \begin{pmatrix} 0 & -A^{-1}C^*eD \\ D^{-1}CA & 0 \end{pmatrix},$$

justifying the notation for the complement of $\mathfrak{o}(m - p, \mathbb{F}) \oplus \mathfrak{o}(p, q, \mathbb{F})$ in $\mathfrak{o}(m, 1, \mathbb{F})$. For future use, we record the following formula. The Lie bracket of two elements $\begin{pmatrix} 0 & -B^*e \\ B & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -C^*e \\ C & 0 \end{pmatrix}$ of $\text{Hom}_{\mathbb{F}}(\mathbb{F}^{m-p}, \mathbb{F}^{p+q})$ equals $\begin{pmatrix} C^*eB - B^*eC & 0 \\ 0 & CB^*e - BC^*e \end{pmatrix}$. \blacksquare

9.3. Homomorphisms that stabilize an \mathbb{R} -plane. Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} . The centralizer of $SO(2, 1, \mathbb{R})$ in $O(2, 1, \mathbb{F})$ is the group of multiples of identity by units of \mathbb{F} , $O(1, \mathbb{F})$. The centralizer of $SO(2, 1, \mathbb{R})$ in $O(m, 1, \mathbb{F})$ stabilizes a \mathbb{F} -plane, thus it is equal to $O(1, \mathbb{F}) \times O(m - 2, \mathbb{F})$.

Case $\mathbb{F} = \mathbb{O}$. Recall that the maximal compact subgroup K of $G = F_4^{-20}$ is isomorphic to $Spin(9)$, and the action of K on the tangent space T to G/K is the 16-dimensional spin representation of $Spin(9)$. Let v, w be orthonormal vectors in T with $w \in (\mathbb{O}v)^\perp$. The fixator of v in this representation equals $M = Spin(7)$, T splits as $T = \mathbb{R}v \oplus \mathfrak{S}m(\mathbb{O})v \oplus (\mathbb{O}v)^\perp$, on which M acts respectively by the trivial, standard $SO(7)$ and spin representations. The fixator of v and w equals the fixator of w in the spin representation of $Spin(7)$, i.e. G_2 .

To sum up, the centralizer Z of $S = SO(2, 1)$ equals

- $O(m - 2)$ if $\mathbb{F} = \mathbb{R}$,
- $S(U(1) \times U(m - 2))$ if $\mathbb{F} = \mathbb{C}$,
- $Sp(1) \times Sp(m - 2)$ if $\mathbb{F} = \mathbb{H}$,
- G_2 if $\mathbb{F} = \mathbb{O}$.

Then, the center of \mathfrak{z} is

- $\mathfrak{so}(2)$ if $\mathbb{F} = \mathbb{R}$ and $m = 4$,
- trivial if $\mathbb{F} = \mathbb{R}$ and $m \neq 4$,
- trivial if $\mathbb{F} = \mathbb{C}$ and $m = 2$,
- $\mathfrak{u}(1)$ (the center of $S(U(1) \times U(m-2))$), if $\mathbb{F} = \mathbb{C}$, $m \geq 3$,
- trivial if $\mathbb{F} = \mathbb{H}$,
- trivial if $\mathbb{F} = \mathbb{O}$.

Thus, if genus is large enough, ϕ is fully flexible, except possibly in $SO(4,1)$ or in $SU(m,1)$, $m \geq 3$. It has been known for a long time that flexibility holds in these cases (these homomorphisms can be deformed by bending, see [32], [1]), but we include a proof along the lines of Theorem 3, for completeness' sake.

9.3.1. 4-dimensional real hyperbolic space. The splitting of $\mathfrak{so}(4,1)$ in root spaces according to the centralizer $\mathfrak{so}(2)$ is given by Lemma 9.1. There is only one space $\mathfrak{g}_{\lambda, \mathbb{R}}$, it is $Hom(\mathbb{R}^2, \mathbb{R}^3)$. Represent elements $C \in Hom(\mathbb{R}^2, \mathbb{R}^3)$ as pairs of vectors (c_1, c_2) of \mathbb{R}^3 . Then the $\mathfrak{so}(2)$ -component of the Lie bracket $[B, C]$ is $((b_1, b_2), (c_1, c_2)) \mapsto c_1 \cdot b_2 - c_2 \cdot b_1$ (here, \cdot denotes the $SO(2,1)$ -invariant symmetric bilinear form on \mathbb{R}^3). This is a symplectic structure. The subspaces $R = \{c_2 = 0\}$ and $R^* = \{c_1 = 0\}$ are Lagrangian and $SO(2,1)$ -invariant. Lemma 10.3 implies that the Toledo invariant attached to this symplectic structure vanishes. Therefore full flexibility holds.

9.3.2. Real planes in complex hyperbolic space. Again, Lemma 9.1 provides us with the only real root space of the centralizer, $Hom_{\mathbb{C}}(\mathbb{C}^{m-2}, \mathbb{C}^3)$. Let

$$Z = \begin{pmatrix} \frac{3i}{m+1} I_{m-2} & 0 \\ 0 & \frac{(2-m)i}{m+1} I_3 \end{pmatrix}$$

be a basis of the centralizer. Then ad_Z is nothing but the obvious complex structure on $Hom(\mathbb{C}^{m-2}, \mathbb{C}^3)$. Then $(B, C) \mapsto \Omega_Z(ad_Z B, C)$ is the direct sum of $m-2$ copies of the $SU(2,1)$ -invariant Hermitian form on \mathbb{C}^3 . Let $R = Hom_{\mathbb{R}}(\mathbb{R}^{m-2}, \mathbb{R}^3) \otimes \mathbb{C} \subset Hom_{\mathbb{C}}(\mathbb{C}^{m-2}, \mathbb{C}^3)$ denote the subspace of real matrices, and $R^* = iR$. These are two Lagrangian, $SO(2,1)$ -invariant subspaces. Lemma 10.3 implies that the Toledo invariant attached to this symplectic structure vanishes.

9.4. Homomorphisms that stabilize a \mathbb{C} -line. Let $\mathbb{F} = \mathbb{C}$ or \mathbb{H} . The centralizer of $SU(1,1)$ in $SO(1,1, \mathbb{H}) = Sp(1,1)$ equals $U(1)$ (the centralizer of $SU(1,1)$ in $U(1,1)$). The centralizer of $SU(1,1)$ in $SO(m,1, \mathbb{F})$ fixes a \mathbb{C} -line pointwise, thus it stabilizes the \mathbb{F} -line that contains it. Thus it is equal to $U(m-1)$ (complex case), $U(1) \times Sp(m-1)$ (quaternionic case).

Case $\mathbb{F} = \mathbb{O}$. Let T denote the tangent space to octonionic hyperbolic plane. Let v, w be orthonormal vectors in T with $w \in \mathbb{O}v$. The fixator of v in the isotropy representation equals $Spin(7)$, T splits as $T = \mathbb{R}v \oplus \mathfrak{S}m(\mathbb{O})v \oplus (\mathbb{O}v)^\perp$, on which M acts respectively by the trivial, standard $SO(7)$ and spin representations. The fixator of v and w equals the fixator of w in the

standard representation of $SO(7)$, i.e. a double cover of $SO(6)$, $Spin(6)$. Therefore the centralizer of $SU(1, 1)$ in F_4^{-20} is isomorphic to $Spin(6)$.

To sum up, the centralizer Z of $S = SU(1, 1)$ is

- (1) $S(U(1) \times U(m-1))$ if $\mathbb{F} = \mathbb{C}$,
- (2) $U(1) \times Sp(m-1)$ if $\mathbb{F} = \mathbb{H}$,
- (3) $Spin(6)$ if $\mathbb{F} = \mathbb{O}$.

Then, the center of \mathfrak{z} is

- (1) $\mathfrak{u}(1)$ (the center of $S(U(1, 1) \times U(m-1))$) if $\mathbb{F} = \mathbb{C}$,
- (2) $\mathfrak{u}(1)$ (the center of $U(1, 1) \subset Sp(1, 1) \subset Sp(m, 1)$) if $\mathbb{F} = \mathbb{H}$,
- (3) trivial if $\mathbb{F} = \mathbb{O}$.

9.4.1. *Octonionic case.* A trivial center implies full flexibility.

9.4.2. *Quaternionic case.* Choose $Z = \begin{pmatrix} 0 & 0 \\ 0 & iI_2 \end{pmatrix}$ as a basis of the centralizer. Once more, Lemma 9.1 provides us with the unique real root space, $Hom(\mathbb{F}^{m-1}, \mathbb{F}^2)$, on which ad_Z acts by multiplication by i , i.e. as the obvious complex structure.

We continue with $\mathbb{F} = \mathbb{H}$. The Killing form on $\mathfrak{sp}(m, 1)$ is proportional to $M \mapsto \Re(\text{Trace}_{\mathbb{H}}(M^2))$. For $X = \begin{pmatrix} 0 & -B^*e \\ B & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & -C^*e \\ C & 0 \end{pmatrix}$, up to a constant,

$$\begin{aligned} \Omega_Z(X, Y) &= Z \cdot [X, Y] \\ &= \Re(\text{Trace}_{\mathbb{H}}(Z[X, Y])) \\ &= \Re(i \text{Trace}_{\mathbb{H}}(CB^*e - BC^*e)) \\ &= 2\Re(i \text{Trace}_{\mathbb{H}}(CeB^*)). \end{aligned}$$

This alternating form is, in an $Sp(1, 1)$ -invariant manner, the direct sum of $m-1$ copies of a symplectic structure Ω associated with a \mathbb{C} -Hermitian form on \mathbb{H}^2 (it turns out to be the $Sp(1, 1)$ -invariant \mathbb{H} -Hermitian form). Therefore we need only compute the Toledo invariant in case $m=2$. In this case, $Sp(1, 1) = SO(4, 1)$, $SU(1, 1)$ is a diagonal $SO(2, 1)$. The calculation, done in paragraph 9.3.1, shows that the Toledo invariant vanishes. Flexibility follows. This example, studied in [20], is the starting point of the present work.

9.4.3. *Complex case.* Now $\mathbb{F} = \mathbb{C}$. Since the Killing form on $\mathfrak{su}(m, 1)$ is proportional to $M \mapsto \Re(\text{Trace}_{\mathbb{C}}(M^2))$, for $X = \begin{pmatrix} 0 & -B^*e \\ B & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & -C^*e \\ C & 0 \end{pmatrix}$,

$$\begin{aligned} \Omega_Z(X, Y) &= Z \cdot [X, Y] = \Re(\text{Trace}_{\mathbb{C}}(Z[X, Y])) \\ &= \Re(i \text{Trace}_{\mathbb{C}}(CB^*e - BC^*e)) = 2\Re(i \text{Trace}_{\mathbb{C}}(B^*eC)). \end{aligned}$$

This alternating form is, in an $SU(1, 1)$ -invariant manner, the direct sum of $m - 1$ copies of the symplectic structure Ω associated with the $SU(1, 1)$ -invariant Hermitian form on \mathbb{C}^2 . Therefore the corresponding Toledo invariant is equal to $m - 1$ times the Toledo invariant computed in Example 4.9. Thus it is maximal if and only if the projection ϕ' of ϕ to the Levi factor $S = SU(1, 1)$ of its Zariski closure is maximal. According to Goldman's theorem 1.4, this happens if and only if ϕ' is discrete and cocompact. We conclude that ϕ is partially rigid if and only if $\phi(\Gamma)$ is discrete and cocompact in $S(U(1, 1) \times U(m - 1))$.

9.5. Proof of Corollary 5. It follows from Theorem 3 and the case by case study of the preceding subsections.

Remark 9.2. *Real rank one allows many shortcuts in the proof of Theorem 3.*

10. APPENDIX 1 : COMPUTING THE CUP-PRODUCT

10.1. W. Meyer's signature formula. Given a linear form ℓ on the second cohomology, the cup-product composed with ℓ is a real valued quadratic form. Its signature will be given by the following lemma, which is a special case of W. Meyer's signature formula, [25]).

Definition 10.1. *The first Chern class of a symplectic vector bundle (E, Ω) is defined as follows : pick a smooth complex structure J on the fibres which is tamed by Ω , i.e. such that the quadratic form $e \mapsto \Omega(e, Je)$ is positive definite. Up to homotopy, J is unique. Therefore the first Chern class of the complex vector bundle (E, J) does not depend on the choice of J .*

Lemma 10.2. ([25]). *Let (E, Ω) be a flat symplectic vector bundle over a closed oriented 2-manifold Σ . The quadratic form $\kappa(a, b) = \int_{\Sigma} \Omega(a \smile b)$ on $H^1(\Sigma, E)$ is nondegenerate of signature $4c_1(E, \Omega)$.*

Proof: Since [25] is not so easily available, we include a short proof of the special case we need. See also [28].

In general, if E is a flat vector bundle over a closed n -manifold X , and Ω is a nondegenerate parallel bilinear form on E , Poincaré duality ensures that the pairing $(a, b) \rightarrow \int_X \Omega(a \smile b)$ is nondegenerate on $H^k(X, E) \otimes H^{n-k}(X, E)$. This can be seen as follows. Pick a smooth Euclidean structure G on the fibres of E . View G as a section of $Hom(E, E^*)$ since an inner product on a vector space determines an isomorphism to its dual. This defines an L^2 norm $\int_X \langle G\alpha, \alpha \rangle vol_X$ on E -valued forms on X . With respect to this Hilbert space structure, the adjoint of the exterior differential is

$$d^* = \pm G^{-1} \star d \star G,$$

where in this equation, d denotes exterior differentiation of E -valued forms, and \star denotes the Hodge \star operator, trivially extended from Λ^*T^*X to

$\Lambda^*T^*X \otimes E$. View Ω as a parallel section of $Hom(E, E^*)$. Then Ω commutes with exterior differentiation. Therefore

$$d^* = \pm G^{-1} \Omega \star d \star \Omega^{-1} G = \pm \star_{\Omega} d \star_{\Omega},$$

where \star_{Ω} maps harmonic forms to harmonic forms, and satisfies

$$\Omega(\alpha \wedge \star_{\Omega} \alpha) = |\alpha|^2 vol_X$$

pointwise. The integrated version

$$\int_X \Omega(\alpha \wedge \star_{\Omega} \alpha) = \|\alpha\|_{L^2}^2$$

implies that κ is nondegenerate on harmonic forms, and thus on cohomology.

Assume $X = \Sigma$ is 2-dimensional and Ω is skew-symmetric. Let J be a smooth complex structure on the fibres of E tamed by Ω . Choose $G(e) = \Omega(e, Je)$. Then

$$|\alpha(v)|_G^2 vol(v, \star v) = \Omega(\alpha(v), (\star_{\Omega} \alpha)(v)) = \Omega(\alpha(v), J\alpha(v)) vol(v, \star v).$$

Hence $\star_{\Omega} = \star \otimes J$ and it is an involution on E -valued 1-forms. Let $\pi_+ : \Lambda^1 T^* \Sigma \otimes E \rightarrow \Lambda^+$ denote projection onto the +1-eigenspace of \star_{Ω} , and $d^+ = \pi_+ \circ d$. If an E -valued 0-form e satisfies $d^- e = 0$, then $de = d^+ e + d^- e = d^+ e = \star_{\Omega} d^+ e = \star_{\Omega} de$ is harmonic (for $(\star_{\Omega} d \star_{\Omega} d + d \star_{\Omega} d \star_{\Omega})(de) = 0$) and exact, thus vanishes. In other words, $\ker d^- = \ker d \simeq H^0(\Sigma, E)$. Let α be an antiselfdual E -valued 1-form, i.e., $\star_{\Omega} \alpha = -\alpha$ and e an E -valued 0-form. Then $\langle \alpha, d^- e \rangle_{L^2} = \langle \alpha, de \rangle_{L^2}$. For

$$\langle \alpha, de \rangle = \langle -\star_{\Omega} \alpha, d^+ e + d^- e \rangle = \langle -\alpha, \star_{\Omega} d^+ e + \star_{\Omega} d^- e \rangle = \langle -\alpha, d^+ e - d^- e \rangle,$$

which implies $\langle \alpha, d^+ e \rangle = 0$.

Thus if α is orthogonal to the image of d^- , then $\alpha = -\star_{\Omega} \alpha$ is coclosed (for $\langle \alpha, de \rangle = \langle \delta \alpha, e \rangle = 0$ for all e , and hence $\delta \alpha = 0$) and also closed (for $d\alpha = -d \star_{\Omega} \alpha = -\star_{\Omega} \star_{\Omega} d \star_{\Omega} \alpha = -\star_{\Omega} \delta \alpha = 0$), thus harmonic. In other words, the cokernel of d^- is the space \mathcal{H}^- of antiselfdual harmonic 1-forms. Therefore $\text{Index}(d^-) = \dim \ker - \dim \text{coker} = \dim H^0(\Sigma, E) - \dim \mathcal{H}^-$. Since d^- is the $\bar{\partial}$ operator of the complex vector bundle (E, J) ,

$$\text{Index}(d^-) = \chi(\Sigma) \text{rank}_{\mathbb{C}}(E, J) + 2c_1(E, J),$$

(see for instance [24]). Similarly,

$$\begin{aligned} \text{Index}(d^+) &= \chi(\Sigma) \text{rank}_{\mathbb{C}}(E, -J) + 2c_1(E, -J) \\ &= \chi(\Sigma) \text{rank}_{\mathbb{C}}(E, J) - 2c_1(E, J), \end{aligned}$$

and finally, $H^1 = \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ and

$$\kappa(\alpha, \alpha) = \int_{\Sigma} \Omega(\alpha \wedge \star_{\Omega} \alpha) = \|\alpha\|^2$$

for $\alpha \in \mathcal{H}^+$, hence positive definite, and similarly negative definite on \mathcal{H}^- . Then

$$\begin{aligned} \text{Signature}(\kappa) &= \dim \mathcal{H}^+ - \dim \mathcal{H}^- \\ &= \text{Index}(d^-) - \text{Index}(d^+) \\ &= 4c_1(E, J). \end{aligned}$$

■

10.2. Link with Toledo invariants.

Lemma 10.3. *Let $\Gamma = \pi_1(\Sigma)$ be a surface group. For a symplectic linear representation $\rho : \Gamma \rightarrow Sp(V, \Omega)$,*

- *the Toledo invariant equals the first Chern class of the flat symplectic vectorbundle E_ρ associated to ρ .*
- *If $\rho(\Gamma)$ is contained in the subgroup that fixes a pair of complementary Lagrangian subspaces, then the Toledo invariant vanishes.*

Proof: Let $X(V, \Omega)$ denote the space of complex structures on V tamed by Ω and compatible with Ω . $Sp(V, \Omega)$ acts by conjugation on $X(V, \Omega)$ transitively. Hence the metric on $T_J X(V, \Omega)$ is determined by the killing form of $\mathfrak{sp}(V, \Omega)$, which is proportional to $\text{tr} A^2$. Tautologically, the trivial bundle $X(V, \Omega) \times V$ comes equipped with an $Sp(V, \Omega)$ -invariant complex structure J . The trivial connection D on it is not compatible with the complex structure, but $\nabla = D - DJ$ is. This is an $Sp(V, \Omega)$ -invariant connection. Let $F = \frac{1}{2\pi} \text{trace}(R^\nabla)$ denote the first Chern form obtained from its curvature. This is an $Sp(V, \Omega)$ -invariant 2-form on $X(V, \Omega)$, in fact, the Kähler form of the symmetric metric on the Hermitian symmetric space $X(V, \Omega)$ normalized so that the minimal sectional curvature equals -1 . Therefore, if $f : \tilde{\Sigma} \rightarrow X(V, \Omega)$ is a smooth ρ -equivariant map, the induced 2-form f^*F on Σ is a representative of the first Chern class of the flat symplectic vectorbundle E_ρ . Integrating it on Σ yields

$$T_\rho = c_1(E_\rho).$$

Assume that ρ fixes two complementary Lagrangian subspaces. In other words, $V = R \oplus R^*$ is isomorphic to the sum of a real vector space R and its dual, with the tautological symplectic structure $\Omega((q, p), (q', p')) = \langle p|q' \rangle - \langle p'|q \rangle$. $\rho(\Gamma)$ falls into the subgroup $H \subset Sp(V, \Omega)$ that preserves the splitting. This subgroup is isomorphic to the full linear group $Gl(R)$. Fix a Euclidean structure on R , whence an isomorphism $e : R \rightarrow R^*$. On V , consider the complex structure $J = \begin{pmatrix} 0 & e^{-1} \\ -e & 0 \end{pmatrix}$. The tangent space of $X(V, \Omega)$ at J consists in symmetric endomorphisms A which anticommute with J . For if A_t is a curve in $X(V, \Omega)$ starting from J , the differentiation gives $\dot{A}J + J\dot{A} = 0$. The complex structure on this space is $\mathcal{J}A = JA$. The Euclidean structure on this space is proportional to $\text{Trace}(A^2)$. The tangent space W to the orbit HJ is the subspace of symmetric endomorphisms

A which preserve the splitting $V = R \oplus R^*$. The elements $B = JA$ of $\mathcal{J}(W)$ exchange the summands. Therefore the products AB exchange the summands, $\text{Trace}(AB) = 0$, i.e. W is orthogonal to $\mathcal{J}(W)$. One concludes that the Kähler form F of $X(V, \Omega)$ vanishes on the orbit HJ . If $f : \tilde{\Sigma} \rightarrow X(V, \Omega)$ is a smooth ρ -equivariant map, the induced 2-form f^*F vanishes, thus $T_\rho = 0$. ■

11. APPENDIX 2 : STRUCTURE OF CENTRALIZERS

Lemma 11.1. *Let G be a semisimple real algebraic group. Let $H \subset G$ be a semisimple subgroup with centralizer \mathfrak{z} . Then there exists an Iwasawa decomposition $G = KAN$ of G such that \mathfrak{z} is a direct sum of Lie algebras $\mathfrak{s}' \oplus \mathfrak{k}' \oplus \mathfrak{a}'$ where \mathfrak{s}' is semisimple of noncompact type, $\mathfrak{k}' \subset \mathfrak{k}$ centralizes \mathfrak{a}' and $\mathfrak{a}' \subset \mathfrak{a}$.*

Proof: Since H is semisimple, at least one of its orbits on $X = G/K$ is totally geodesic. All totally geodesic orbits of H are pairwise equidistant. The union of these orbits is a totally geodesic subspace Y of $X = G/K$, isometric to a Riemannian product $Y = W \times V \times F$, where H acts on W as an open subgroup of the full isometry group of W and trivially on $V \times F$, V is a symmetric space of noncompact type and F is flat. Note that the isometry group S' of V canonically embeds in G , since it is generated by geodesic symmetries. Since it commutes with H , its Lie algebra \mathfrak{s}' has to be contained in \mathfrak{z} . $\exp \mathfrak{z}$ leaves Y invariant, thus it maps to $\text{Isom}(Y)$ with a compact kernel K' . Since $K' = \text{Fix}(Y)$ fixes V , it commutes with S' . Up to finite index, $\text{Isom}(Y)$ is a product $H \times A'$ where A' is the group of translations along F . Since A' commutes with H , its Lie algebra $\mathfrak{a}' \subset \mathfrak{z}$. Furthermore, A' commutes with K' and S . Fix points $w \in W$, $v \in V$ and $f \in F$, view (w, v, f) as a point in G and F as a flat in G passing through this point. Pick a maximal flat containing it. This defines an Iwasawa decomposition of G such that $K' \subset K$ and $A' \subset A$. We conclude that $\mathfrak{z} = \mathfrak{s}' \oplus \mathfrak{k}' \oplus \mathfrak{a}'$, where \mathfrak{k}' denotes the Lie algebra of K' . ■

Corollary 11.2. *Let G be a semisimple real algebraic group. Then centralizers of reductive subgroups of G are reductive.*

Proof: Let $H \subset G$ be reductive with centralizer Z . Write $H = SR$ where S is semisimple and R is the radical of H , a torus. Let Z' be the centralizer of S . Then $R \subset Z'$ is a subtorus of Z' , $Z \subset Z'$ and Z is the centralizer of R in Z' . According to Corollary 2, section 13.17 of [6], in connected, affine algebraic groups, centralizers of subtori are reductive. Thus Z is reductive. ■

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