Rigidity and harmonic maps

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Introduction

- Easy examples
- Terminology
- Survey of results

Example

Let $\Gamma = T_{3,4,7} = \langle s, t, u \mid s^2, t^2, u^2, (st)^3, (tu)^4, (us)^7 \rangle$*. Every isometric action of* Γ *on hyperbolic plane either factors through a finite group or is the symmetry group of a tiling by* $(3, 4, 7)$*-triangles.*

Proof.

1. If either s, t, u, st, tu, us acts trivially, action factors through $\mathbb{Z}/2\mathbb{Z}$.
2. Otherwise, each of st, tu, us fixes a point.
3. It two of these points coincide, action fixes a point.
4. Every isometric action on the circle factors through $\mathbb{Z}/2\mathbb{Z}$.
5. s fixes P_{st} and P_{us}, thus (P_{st}, P_{tu}, P_{us}) is a $(3, 4, 7)$-triangle.

Remark

The argument extends to isometric actions on the 2-sphere or Euclidean plane. The conclusion is these cases is that every action factors through $\mathbb{Z}/2\mathbb{Z}$*.*

Example

Let $\Gamma = T_{3,4,7} = \langle s, t, u \mid s^2, t^2, y^2, (st)^3, (tu)^4, (us)^7 \rangle$. *Every non-trivial isometric action of* Γ *on hyperbolic 3-space either fixes a point or leaves invariant a totally geodesic plane.*

Proof.

1. If either s, t, u, st, tu, us acts trivially, action is trivial.
2. Each of st, tu, us fixes a line.
3. If two of these lines intersect, action fixes a point.
4. If two fixed lines are asymptotic, action fixes a point at infinity. Busemann function gives rise to action on the real line.
5. No action on the real line. Thus action on Euclidean plane, thus fixed point.
6. Fixed lines D_{st} and D_{us} have a common perpendicular Δ.
7. s fixes D_{st} and D_{us}, therefore these lines are coplanar.
8. The plane containing Δ, orthogonal to D_{st} and D_{us}, is invariant.

Remark

- *The argument seems to generalize to actions on more general nonpositively curved spaces.*
- *But nontrivial actions fixing a point exist in high dimensions.*

Example

$T_{3,4,\infty}$ admits a continuum of different actions on hyperbolic plane. However, every discrete faithful action admitting a finite area fundamental domain is the symmetry group of a tiling by $(3,4,\infty)$-triangles.

Proof.

1. Action depends on one real parameter, the distance x between the fixed points P_{st} and P_{tu}. Let x_0 be the value of x for the standard action, preserving a tiling.
2. If $x < x_0$, *us* has a fixed point. The action is either nondiscrete (case *us* has infinite order) or nonfaithful (case *us* has finite order).
3. Il $x > x_0$, fundamental domains have infinite area.

Examples

Tilings of hyperbolic plane by right-angled hexagons depend on continuous parameters. More generally, torsion free orientable surface groups of genus g depend on $6g-6$ parameters.

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Terminology

Superrigidity. (Temporary definition). Let Γ be a group acting isometricly on a space X. Say (Γ, X) is superrigid for a class of space $\mathcal{Y}$ if every isometric action of Γ on a space $Y \in \mathcal{Y}$

- either fixes a point,
- or leaves invariant a copy of X (homotheticly) embedded in Y.

Mostow rigidity. Let Γ be a group acting discretely and faithfully on a space X with a finite volume fundamental domain. Say (Γ, X) is Mostow rigid if there is only one such action, up to isometry.

Local rigidity. Let Γ be a finitely generated group acting on a space X. Say (Γ, X) is infinitesimally rigid if every nearby action is conjugate to original action.

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Rigidity in the 60's

circa 1958 : A. Selberg proves infinitesimal rigidity of cocompact lattices of $Sl(n, \mathbb{R})$, $n \geq 3$.

1960 : E. Calabi proves infinitesimal rigidity of cocompact lattices of hyperbolic space. E. Calabi-E. Vesentini extend it to symmetric domains in $\mathbb{C}^n$.

1962 : A. Weil proves infinitesimal rigidity of uniform irreducible lattices in semisimple Lie groups non isogenic to $Sl(2, \mathbb{R})$. Nonuniform case (and $Sl(2, \mathbb{C})$, non uniform case). H. Garland proves infinitesimal rigidity of nonuniform irreducible lattices in semisimple Lie groups non isogenic to $Sl(2, \mathbb{R})$ or $Sl(2, \mathbb{C})$ in 1967.

1967 : G.D. Mostow proves Mostow rigidity of lattices in hyperbolic spaces.

1970 : Generalization to other symmetric spaces.

Margulis' superrigidity theorem

Theorem

(G.A. Margulis (1974) *Let G, H be semisimple algebraic groups over local fields, without compact factors. Assume that the real rank of G is ≥ 2. Let Γ be an irreducible lattice in G.*
Every homomorphism $\Gamma \to H$ with unbounded and Zariski dense image extends to a homomorphism $G \to H$.

In other words, finite dimensional linear representations of Γ come from G (except unitary ones).

What is real rank ?
According to Cartan, every semisimple algebraic group over an Archimedean local field ($\mathbb{R}$ or $\mathbb{C}$) is (isogenous to) the isometry group of a symmetric space. Its rank is the maximal dimension of a flat totally geodesic subspace.
According to F. Bruhat and J. Tits, a Euclidean building is attached to every semi-simple algebraic group over a non Archimedean local field ($\mathbb{Q}_p$, $\mathbb{F}_q((t))$ and their finite extensions). The rank is the dimension of the building.

Margulis' theorem, geometric version

Theorem

Let X, Y be finite dimensional symmetric spaces or buildings, without Euclidean or compact factors. Assume that X has rank ≥ 2. Let Γ be a discrete irreducible group of isometries of X such that $Vol(\Gamma \setminus X) < +\infty$.
Every reductive isometric action of Γ on Y either fixes a point or preserves a convex subspace of Y which is pluriisometric to a product of factors of X.

Meaning of terms.
When X splits nontrivially into a Riemannian product, a group of isometries of X is *irreducible* if no finite index subgroup preserves the decomposition. When the metric on each factor is multiplied by some constant, one gets a space which is *pluriisometric* to X. *Reductive* will be defined later.

Updated terminology.
Say a pair $\Gamma \subset G$ is superrigid with respect to a class $\mathcal{Y}$ of metric spaces if every isometric action of Γ on a space $Y \in \mathcal{Y}$ preserves a subset $Y' \subset Y$ on which the action extends to an isometric action of G.

Definition

Let Γ be group acting on a space X. Say Γ is a lattice if the action is faithful, discrete, and admits finite volume fundamental domains. If there exists a compact fundamental domain, one says that Γ is uniform.

Definition

Let G be a semisimple Lie group. A lattice $\Gamma \subset G$ is arithmetic if Γ is obtained by the following operations.

1. *Find an algebraic Q-group $\mathbf{H}$ defined over $\mathbb{Q}$ whose group of real points is isomorphic to $L \times G$ for some compact group L.*
2. *Take integer matrices in $\mathbf{H}$.*
3. *Project down to G.*
4. *Choose Γ commensurable to obtained group.*

(For nonuniform lattices, $L = \{1\}$ always works).

Examples

$Sl(n, \mathbb{Z}) \subset Sl(n, \mathbb{R})$. $T_{3,4,7}$ is arithmetic in a nonobvious manner.

Results

1959 : A. Selberg proves local rigidity for certain lattices, shows that this implies algebraicity, and conjectures some kind of arithmeticity.

1959 : I. Piatetskii-Shapiro conjectures that arithmeticity holds for nonuniform lattices.

1968 : A. Selberg stresses the role played by unipotent elements and conjectures that they always exist for nonuniform lattices.

1971 : G. Margulis announces arithmeticity of nonuniform lattices. The proof relies on existence of unipotents, rationality of maximal unipotent subgroups, compatibility of the rational structures of intersecting maximal unipotent subgroups.

Proposition

Superrigidity with respect to symmetric spaces and buildings implies arithmeticity.

Remark

There is a generalization involving local fields : the characterization of S-arithmetic lattices.

1990 : K. Corlette proves superrigidity with respect to the class of symmetric spaces for lattices in rank one symmetric spaces $\mathbb{H}H^n$, $n \geq 2$ and $\mathbb{O}H^2$.

1994 : M. Gromov and R. Schoen prove superrigidity with respect to the class of Euclidean buildings for lattices in rank one symmetric spaces $\mathbb{H}H^n$, $n \geq 2$ and $\mathbb{O}H^2$.

Examples

Striking superrigidity failures : bending of real hyperbolic manifolds with mirror symmetry, branched and blow-down holomorphic maps between complex hyperbolic surfaces.

Corollary

Irreducible lattices in semisimple Lie groups are arithmetic, except possibly for $SO(n,1)$, $SU(n,1)$. *In other words, lattices in most semisimple Lie groups are classified.*

Examples

Examples of nonarithmetic lattices are known to exist in $SO(n,1)$ *for all* $n \geq 2$, *and in* $SU(n,1)$ *for* $n = 2$, *3.*

Further superrigidity results

Very restricted targets.

circa 1978 : W. Thurston proves that maps of nonzero degree between real hyperbolic manifolds are covering maps.

1989 : D. Toledo proves that surface groups in $PU(n, 1)$ with maximum Toledo invariant preserve a totally geodesic complex line.

An arithmeticity result in complex hyperbolic geometry

2006 : B. Klingler; S.K. Yeung and Prasad : Classification of fake complex projective planes, i.e. of compact complex hyperbolic surfaces with $b_1 = 0$, $b_2 = 1$.

More general targets.

1990 : Corlette's result implies superrigidity (of certain rank one lattices) with respect to the class of Riemannian manifolds with nonpositive curvature operator.

1993 : N. Mok, Y. Siu and S.K. Yeung; J. Jost and S.T. Yau : All rank ≥ 2 cocompact lattices are superrigid with respect to the class of nonpositively curved Riemannian manifolds. All quaternionic and octonionic lattices are superrigid with respect to the class of Riemannian manifolds with nonpositive complex curvature.

Further superrigidity results

More general domains and targets.

2004 : N. Monod proves that superrigidity holds for cocompact irreducible lattices in arbitrary products of locally compact groups, with respect to the class of $CAT(0)$ spaces.

2006 : T. Gelander, A. Karlsson and G.A. Margulis extend this, by a different method, to a class of spaces with convex distance function.

2006 : I think I can prove, using ideas of M.T. Wang, M. Gromov, H. Izeki and S. Nayatani, superrigidity for lattices of rank 2 Euclidean buildings, with respect to the class of $CAT(0)$ spaces.

Plan of forthcoming talks

I will explain briefly why superrigidity implies arithmeticity. Then I will concentrate on the harmonic map method.

1. Isometric actions on the real line
2. Local rigidity
3. Matsushima's formula
4. Kazhdan's property (T)
5. Harmonic map proof of Margulis superrigidity, after Mok, Siu and Yeung
6. Garland's formula
7. Fixed points for isometric actions on $CAT(0)$ spaces
8. $\tilde{A}_2$-buildings

Superrigidity implies algebraicity

Let $\Gamma \subset G \subset Gl(N, \mathbb{C})$ be a superrigid lattice, where G is a $\mathbb{Q}$-group.

1. *Every homomorphism* $\Gamma \to \mathbb{C}^*$ *is trivial on some finite index subgroup.*
Since translation actions on $\mathbb{C}$ have fixed points, every homomorphism $\Gamma \to \mathbb{C}$ is trivial. In particular, the abelianization of Γ has no free part. A homomorphism $\Gamma \to \mathbb{C}^*$ factors through the finite torsion part of the abelianization.

2. $\Gamma \subset Sl(n, \mathbb{C})$.
Use $det : \Gamma \to \mathbb{C}^*$.

3. *Up to conjugacy,* $\Gamma \subset Sl(n, \bar{\mathbb{Q}})$.
Indeed, $Hom(\Gamma, Sl(n, \mathbb{C}))$ is an affine algebraic variety defined over $\mathbb{Q}$. By assumption, it has at least one complex point. According to the Nullstellensatz, it also has a nearby $\bar{\mathbb{Q}}$-point, this is a representation by matrices with entries in $\bar{\mathbb{Q}}$. By superrigidity, the corresponding homomorphism extends to an inner automorphism of $Sl(n, \mathbb{C})$.

4. *There exists a number field* F *such that* $\Gamma \subset Sl(n, F)$.
Indeed, matrix coefficients belong to a finitely generated subfield of $\bar{\mathbb{Q}}$, i.e. a finite extension of $\mathbb{Q}$.

Superrigidity implies arithmeticity

5. *Up to enlarging matrices,* $\Gamma \subset Sl(N, \mathbb{Q})$*.*
Indeed, F can be realized as a ring of matrices (express its multiplication rule in a $\mathbb{Q}$-basis).

6. $G \subset Sl(N)$ *is defined over* $\mathbb{Q}$*.*
Indeed, it is the Zariski closure of a subset of $Sl(N, \mathbb{Q})$.

7. *Let* p *be a prime. Then, up to finite index,* p *does not divide denominators of matrix coefficients of elements of* Γ*.*
Indeed, the image of $\Gamma \to Sl(N, \mathbb{Q}) \to Sl(N, \mathbb{Q}_p)$ fixes a vertex in the building associated to $Sl(N, \mathbb{Q}_p)$. Vertices correspond to norms on $\mathbb{Q}_p^N$. Vectors of norm ≥ 1 form a subgroup commensurable to $\mathbb{Z}_p^N$. Therefore a finite index subgroup of Γ fixes $\mathbb{Z}_p^N$, i.e. maps to $Sl(N, \mathbb{Z}_p)$.

8. *Up to finite index,* $\Gamma \subset Sl(N, \mathbb{Z})$*.*
Indeed, only finitely many primes occur in the denominators of matrix coefficients. Up to finite index, no prime divides any denominator, thus entries are integers.

9. Γ *is commensurable to* $G_{\mathbb{Z}}$*.*
Indeed, up to finite index, $\Gamma \subset G_{\mathbb{Z}}$. Since $[G_{\mathbb{Z}} : \Gamma]\, vol(G_{\mathbb{Z}} \setminus G) = vol(\Gamma \setminus G) < +\infty$, Γ has finite index in $G_{\mathbb{Z}}$.

A subgroup of index 2 acts by translation.

Lemma

A translation action of Γ *on* $\mathbb{R} \Leftrightarrow$ *a class in* $H^1(\Gamma, \mathbb{R})$.
Class vanishes $\Leftrightarrow$ *action has a fixed point.*

De Rham proof

Assume Γ acts freely on manifold X. Pick a Γ-equivariant map $f : X \to \mathbb{R}$. Then df induces a closed 1-form on $\Gamma \setminus X$. This form is exact $\Leftrightarrow$ f is invariant $\Leftrightarrow$ $f(X)$ is pointwise fixed by Γ.

Therefore superrigidity of isometric actions of Γ on the real line $\Leftrightarrow$ vanishing of $H^1(\Gamma, \mathbb{R})$.

Theorem

(S. Bochner, circa 1940). Let $\Gamma = \pi_1(M)$ *where* M *is compact Riemannian with Ricci* > 0*. Then* $H^1(\Gamma, \mathbb{R}) = 0$*.*

Proof. Define *energy* of 1-forms as $\frac{1}{2}$ of squared L^2-norm,

$$E(\alpha) = \frac{1}{2} \int_M |\alpha|^2.$$

Hodge : every cohomology class contains an energy minimizing 1-form α, which satisfy $d\alpha = d^*\alpha = 0$. This is equivalent to $\Delta\alpha = 0$, where $\Delta = dd^* + d^*d$ is the *Laplacian*. Bochner's formula

$$\int_M |D\alpha|^2 - |d\alpha|^2 - |d^*\alpha|^2 = -\int_M Ricci(\alpha).$$

Remark

Useless, since non flat locally symmetric spaces have negative Ricci curvature.

Proposition

(A. Weil, 1959). For lattices in Lie groups, local rigidity follows from infinitesimal rigidity, i.e. vanishing of $H^1(\Gamma, Lie(G)_{Ad})$.

Proof. The Zariski tangent space of the variety $Hom(\Gamma, G)$ is equal to the space of 1-cocycles $Z^1(\Gamma, Lie(G)_{Ad})$. The tangent space at $\rho \in Hom(\Gamma, G)$ of the G-orbit $G\rho$ is the space of 1-coboundaries $B^1(\Gamma, Lie(G)_{Ad\circ\rho})$.
Vanishing of $H^1(\Gamma, Lie(G)_{Ad})$ implies that the G-action $G \times Hom(\Gamma, G) \to Hom(\Gamma, G)$ has constant rank near id, i.e. $Hom(\Gamma, G)$ is smooth and consists of exactly one orbit near id.
In other words, every representation which is close enough to the original one it conjugate to it.

Theorem

(A. Weil, 1960). For uniform lattices in semisimple Lie groups others than $Sl(2,\mathbb{R})$, $H^1(\Gamma, Lie(G)_{Ad}) = 0$.

Proof. De Rham : $H^1(\Gamma, Lie(G)_{Ad})$ is the cohomology of E-valued 1-forms on $M = \Gamma \setminus G/K$, where E is a vector bundle equipped with a flat connection ∇. Equip E with a natural metric. Then $\nabla = D + S$ where D is a metric connection, which has nonzero curvature R^D.

The vector-valued Bochner formula

$$\int_M |D\alpha|^2 - |d^D\alpha|^2 - |(d^D)^*\alpha|^2 = -\int_M Ricci(\alpha) + tr(\alpha^* R^D),$$

contains an extra term, $tr(\alpha^* R^D)$, which compensates for Ricci curvature (and for $|d^D\alpha|^2 - |(d^D)^*\alpha|^2$, which do not vanish), except when $G = Sl(2,\mathbb{R})$.

Y. Matsushima could directly integrate by parts the curvature term in Weil's formula. The formula he got is simple since the curvature tensor of a locally symmetric space is parallel. More generally, call a *curvature tensor* a tensor which shares all the symmetries of the curvature tensor of a Riemannian manifold. Such a tensor Q acts on 1-forms $Q(\alpha)$ and on 2-tensors $\dot{Q}(\tau)$.

Theorem

(Y. Matsushima, 1962). Let M be a compact Riemannian manifold, with curvature tensor R. Let E be a vectorbundle on M with a metric and a metric connection D. Let α be an E-valued 1-form. Let Q be a parallel curvature tensor field on M. Then

$$\int_M \langle \dot{Q} D\alpha, D\alpha \rangle = \frac{1}{2} \int_M \Big(\langle Q, \alpha^* R^D \rangle + \langle Q(\alpha), R(\alpha) \rangle \Big) .$$

Most Riemannian manifolds admit only one parallel curvature tensor field, the curvature tensor I of the unit sphere. The choice $Q = I$ gives Bochner's formula.

Matsushima's formula

Most symmetric spaces admit exactly 2 independant curvature tensor fields, I and R. The choice $Q = R_\perp$, projection of I ont the line orthogonal to R, gives *Matsushima's formula*

$$\int_M \langle \dot{R}_\perp D\alpha, D\alpha \rangle = 0$$

Corollary

Let Γ be a uniform lattice in a semisimple Lie group without factors isogenic to $SO(n,1)$ or $SU(n,1)$. Then $H^1(\Gamma, \mathbb{R}) = 0$.

Proof.
Under these assumptions, $\dot{R}_\perp$ is positive definite on traceless symmetric 2-tensors *(E. Calabi, E.Vesentini, A. Borel, Y. Matsushima, S. Kaneyuki, T. Nagano).* Thus $\int_M \dot{R}_\perp D\alpha, D\alpha\rangle = 0$ for harmonic α implies $D\alpha = 0$.
Parallel 1-forms exist only if M/K split an Euclidean de Rham factor. Therefore $\alpha = 0$.

1968 : D. Kazhdan proves that $H^1(\Gamma, \mathbb{R}) = 0$ for all higher rank lattices as follows.

Definition

Say a locally compact group G has property (T) if the trivial representation of G is isolated in the space of unitary representations of G.

Theorem

(D. Kazhdan, B. Kostant, A. Guichardet, P. Delorme).

1. *If Γ is a lattice in a locally compact group G, then Γ has property (T) if and only if G has it.*
2. *A semisimple Lie group G has property (T) if and only if no simple factor of G is isogenous to $SO(n, 1)$ or $SU(n, 1)$.*
3. *Γ has property (T) if and only if $H^1(\Gamma, \pi) = 0$ for all unitary representations π of Γ.*
4. *Γ has property (T) if and only if every affine isometric action of Γ on a Hilbert space has a fixed point.*

Remark

Property (T) implies that Γ is finitely generated, this was Kazhdan's motivation.

Proof. $H^1(\Gamma,\pi)$ can be computed using L^2 E-valued differential 1-forms α on $M=\Gamma\setminus G/K$, for some flat hermitian vectorbundle E on M.

Positivity of $\dot{R}_\perp$ on traceless symmetric 2-tensors implies a pointwise inequality

$$|D\alpha|^2 \leq \mathrm{const.}\left(|d^D\alpha|^2+|(d^D)^*\alpha|^2+\langle \dot{R}_\perp D\alpha, D\alpha\rangle\right).$$

Bochner's formula $\int_M \left(|D\alpha|^2-|d^D\alpha|^2_{L^2}-|(d^D)^*\alpha|^2\right)=-\int_M Ricci(\alpha)$ and negativity of Ricci curvature give

$$\int_M |\alpha|^2 \leq \mathrm{const.}\int_M \left(|D\alpha|^2-|d^D\alpha|^2_{L^2}-|(d^D)^*\alpha|^2\right).$$

Combining these inequalities with Matsushima's formula $\int_M\langle \dot{R}_\perp D\alpha, D\alpha\rangle=0$ yields

$$\parallel \alpha \parallel_{L^2_1} \leq \mathrm{const.}\left(\parallel d^D\alpha \parallel^2_{L^2}+\parallel (d^D)^*\alpha \parallel^2_{L^2}\right).$$

This implies that $(d^D)^*$ on closed 1-forms is invertible, thus its adjoint d^D from 0-forms to closed 1-forms is invertible too.

Harmonic map approach to superrigidity

Let X be a symmetric space, Y a Riemannian manifold. Let Γ be a lattice of X, acting isometricly on Y. If $f : X \to Y$ is an equivariant map, df is a d^D-closed f^*TN-valued 1-form. Half of its squared L^2 norm is called the *energy* of f,

$$E(f) = \frac{1}{2}\int_M |df|^2.$$

Critical points of the energy are called *harmonic maps*. They satisfy $(d^D)^* df = 0$. Note that totally geodesic maps, characterized by $Ddf = 0$, are harmonic. The issue of superrigidity is the converse : prove that

$$(d^D)^* df = 0 \Rightarrow Ddf = 0.$$

A vector valued form of Matsushima's formula reads

$$\int_{\Gamma\backslash X} \langle \dot{Q} Ddf, Ddf\rangle = \frac{1}{2}\int_{\Gamma\backslash X}\left(\langle Q, R\rangle |df|^2 + \langle Q, f^* R^Y\rangle\right).$$

Problem : find a parallel curvature tensorfield Q such that

1. $\langle Q, R\rangle = 0$;
2. $\dot{Q}$ is positive definite on traceless symmetric 2-tensors;
3. $\langle Q, T\rangle \leq 0$ under suitable assumptions on curvature tensor T.

Choice of parallel curvature tensorfield, after Mok, Siu and Yeung

Theorem

(Mok, Siu and Yeung, 1993). Let X be a symmetric space which is neither a real nor a complex hyperbolic space. There exists a parallel curvature tensorfield Q on X such that

1. $\langle Q, R\rangle = 0$;
2. $\dot{Q}$ *is positive definite on traceless symmetric 2-tensors;*
3. $\langle Q, T\rangle \leq 0$ *for all nonpositively curved curvature tensor T (resp. for all curvature tensors T with nonpositive complex curvature if $rank(X) = 1$).*

Proof. Choose vectors u, $v \in TX$ which are tangent to a flat totally geodesic plane. Let S be orthogonal projection on the line generated by $X \wedge Y$ in $\Lambda^2 TX$. Let Q be the average of $k_* S$ for $k \in K \subset G = Isom(X)$. Then Q is K-invariant, and thus parallel. For T a curvature tensor,

$$\langle Q, T\rangle = \int_K T(k(u), k(v), k(u), k(v))\, dk$$

is an average of sectional curvatures of T, so it is ≤ 0 if T is nonpositively curved. In rank one, take u, $v \in TX \otimes \mathbb{C}$ in $T^{1,0}Z$ for some complex hyperbolic plane $Z \subset X$. Then $\langle Q, T\rangle$ is an average of *complex curvatures* $\Re e\,(T(k(u), k(v), k(u), k(v)))$. Note that all symmetric spaces Y have nonpositive complex curvature.

Remark

As in the linear case, Matsushima's formula should be used to prove the existence of an equivariant harmonic map without any reductivity assumption.

Definition

Let C be a finite simplicial 2-complex. Put on each edge a weight equal to the number of faces that contain it, put on each vertex the total weight of edges containing it.

For a map $g : C \to Y$ sending vertices of C to a metric space Y, define the energy

$$E(g) = \sum_{edges\ e} m(e) d(g(orig(e)), g(end(e)))^2 = \frac{1}{2} \sum_{c} \sum_{c' \sim c} m(c,c') d(g(c), g(c'))^2.$$

Let X be a simplicial 2-complex with a cocompact action of a group Γ. If $f : X \to Y$ is equivariant, define

$$E(g) = \sum_{edges\ e\ of\ \Gamma\backslash X} m(e) d(f(orig(\tilde{e})), f(end(\tilde{e})))^2,$$

where $\tilde{e}$ denotes a lift of e to X. Say f is harmonic *if it minimizes energy among equivariant maps.*

Proposition

Let Y be CAT(0). Then an equivariant map $f : X \to Y$ is harmonic if and only if for each vertex x of X, $f(x)$ coincides with the barycenter *of $f_{|link(x)}$, i.e. the unique point of Y which minimizes the weighted sum of squares of distances to the images of the neighbours of X.*

Bottom of spectrum

Definition

(M.T. Wang, 1998). *Let C be a finite weighted graph. Let*

$$d(g, bar(g))^2 = \sum_{c \in C} m(c) d(g(c), bar(g))^2$$

denote the L^2 distance of map g to its barycenter. Define the Rayleigh quotient

$$RQ(g) = \frac{E(g)}{d(g, bar(g))^2}.$$

The bottom of spectrum *of C relative to Y is the infimum of Rayleigh quotients of nonconstant maps $C \to Y$,*

$$\lambda(C, Y) = \inf_{g : C \to Y} RQ(g).$$

Example

When $Y = \mathbb{R}$, the bottom of spectrum equals the smallest positive eigenvalue of the combinatorial Laplacian $\Delta g(c) = \sum_{neighbours\ c'\ of\ c} m(c, c')(g(c) - g(c'))$.

Garland's formula

It is a combinatorial analogue of Matsushima's formula, discovered by H. Garland in 1972, in order to prove vanishing of cohomology for, compact quotients of Euclidean buildings. A. Borel (1973) generalized it to arbitrary simplicial complexes. A. Zuk applied it to prove Kazhdan's property. The nonlinear version is due to M.T. Wang (1998).

Theorem

(H. Garland, 1972, M.T. Wang, 1998). *Let X be a simplicial complex, Γ a uniform lattice in X, acting isometricly on a metric space Y. Let $f : X \to Y$ be an equivariant map. For $x \in X$, denote by*

$$ED(f,x) = \frac{1}{2} d(f_{|link(x)}, f(x))^2,$$

(where links inherit weights from X). Then

$$E(f) = \sum_{x\in\Gamma\backslash X} ED(f,\tilde{x}).$$

If f is harmonic, then

$$E(f) = 2 \sum_{x\in\Gamma\backslash X} RQ(f_{|link(x)}) ED(f,x).$$

In particular, if, for all $x \in X$, $\lambda(link(x), Y) > \frac{1}{2}$, every equivariant harmonic map $X \to Y$ is constant.

Proof of Garland's formula

$$
\begin{aligned}
E(f) &= \frac{1}{2}\sum_{x\in\Gamma\backslash X}\sum_{x'\sim x} m(x,x')d(f(x),f(x'))^2 \\
&= \sum_{x\in\Gamma\backslash X}\frac{1}{2}\sum_{x'\in link(x)} m(x,x')d(f(x),f(x'))^2 \\
&= \sum_{x\in\Gamma\backslash X} ED(f,x).
\end{aligned}
$$

$$
\begin{aligned}
E(f) &= \sum_{\text{edges } (x',x'') \text{ of } \Gamma\backslash X} m(x',x'')d(f(x'),f(x''))^2 \\
&= \sum_{\text{faces } (x,x',x'') \text{ of } \Gamma\backslash X} m(x,x',x'')d(f(x'),f(x''))^2 \\
&= \sum_{x\in\Gamma\backslash X} E(f_{|link(x)}).
\end{aligned}
$$

If f is harmonic, for each $x \in X$,

$$
\begin{aligned}
E(f_{|link(x)}) &= RQ(f_{|link(x)})d(f_{|link(x)}, bar(f_{|link(x)}))^2 \\
&= RQ(f_{|link(x)})d(f_{|link(x)}, f(x))^2 \\
&= 2RQ(f_{|link(x)})ED(f,x).
\end{aligned}
$$

Corollary

(A. Zuk, 1996). *Let X be a simplicial complex, Γ a uniform lattice in X. Assume that for all $x \in X$, $\lambda(link(x), \mathbb{R}) \geq \lambda > \frac{1}{2}$. Then Γ has Kazhdan's property (T).*

Lemma

Let Γ act isometricly on a Hilbert space $\mathcal{H}$, let $f : X \to \mathcal{H}$ be equivariant. Then

$$(2\lambda - 1)E(f) \leq \lambda \| \Delta f \|^2.$$

Proof. In $\ell^2(link(x))$, $f_{|link(x)} - bar(f_{|link(x)})$ is orthogonal to the constant function $\Delta f(x) = f(x) - bar(f_{|link(x)})$,

$$\begin{aligned} \| f_{|link(x)} - f(x) \|^2 &= \| f_{|link(x)} - bar(f_{|link(x)}) \|^2 + \| f(x) - bar(f_{|link(x)}) \|^2 \\ &= 2ED(f, x) + m(x)|\Delta f(x)|^2. \end{aligned}$$

$$\begin{aligned} E(f_{|link(x)}) &= RQ(f_{|link(x)}) d(f_{|link(x)}, bar(f_{|link(x)}))^2 \\ &= 2RQ(f_{|link(x)}) ED(f, x) + RQ(f_{|link(x)}) m(x)|\Delta f(x)|^2 \\ &\geq 2\lambda ED(f, x) + \lambda m(x)|\Delta f(x)|^2. \end{aligned}$$

Summing gives

$$E(f) \geq 2\lambda E(f) + \lambda \| \Delta f \|^2.$$

Fixed point property, linear case

Proposition

(Following H. Izeki and S. Nayatani, 2004). *Let X be a simplicial complex, Γ a uniform lattice in X, acting isometricly on a Hilbert space $\mathcal{H}$. Assume that for all $x \in X$, $\lambda(link(x), \mathbb{R}) \geq \lambda > \frac{1}{2}$. Then Γ fixes a point in $\mathcal{H}$.*

Proof. Start with arbitrary equivariant map $f : X \to \mathcal{H}$. Solve heat equation $\frac{\partial f_t}{\partial t} = -\Delta f_t$. Smooth short time solutions always exist, they satisfy

$$\begin{aligned} \frac{\partial}{\partial t} E(f_t) &= -\| \Delta f_t \|^2 \\ &\leq -\frac{2\lambda - 1}{\lambda} E(f_t), \end{aligned}$$

which shows exponential decay of energy. Since

$$\| \frac{\partial f_t}{\partial t} \|^2 = -\| \Delta f_t \|^2 = -\frac{\partial}{\partial t} E(f_t),$$

$$\int_0^{+\infty} \| \frac{\partial f_t}{\partial t} \|^2 E(f_t)^{-1/2} \, dt = 2E(f_0)^{1/2}$$

is *a priori* bounded, therefore f_t converges in L^2. This suffices to show existence of a sublimit f_∞ with vanishing energy, i.e. a constant equivariant map, i.e. a fixed point.

Observe that every group presentation can be modified, by adding generators, so that all relators have length 3.

Consider presentations on m fixed generators and $(2m-1)^{3d}$ relators chosen independently at random among the $(2m-1)^3$ possibilities. We are interested in properties which are satisfied with overwhelming probability as m tends to infinity. Such a property is said to be *satisfied by a random group in density d*.

Theorem

(M. Gromov, 1993). *Random groups in density $< \frac{1}{2}$ are infinite and hyperbolic.*
(A. Zuk, 2003). *Random groups in density $> \frac{1}{3}$ have Kazhdan's property (T).*

Proof. (very rough idea)
The Cayley complex has links which look like random graphs. Such graphs (M. Broder and E. Shamir, 1987) have bottom of spectra which tend to 1 as m tends to infinity.

Definition

Let Y be geodesic and CAT(0). If s, s' are geodesics emanating from $y \in Y$, let

$$d(s,s') = \lim_{t \to 0} \frac{d(s(t), s(t'))}{t},$$

(nondecreasing limit). Identify s and s' if $d(s,s') = 0$. This gives a metric space, denoted by $T_y Y$, with a distance nonincreasing map $\pi_y : Y \to T_y Y$.

Examples

If Y is a Riemannian manifold, $T_y Y$ is a finite dimensional Euclidean space. If Y is a finite tree, $T_y Y$ is a bunch of half lines with common endpoint.

A fixed point theorem

Theorem

(H. Izeki and S. Nayatani, 2004). *Let X be a simplicial complex, Γ a uniform lattice in X, acting isometricly on Y. Assume Y is complete, geodesic, CAT(0). Assume that for all $x \in X$, for all $y \in Y$, $\lambda(link(x), T_y Y) > \frac{1}{2}$. Then Γ fixes a point in Y.*

Proof. Define $(-\Delta f)(x)$ as a point in $T_{f(x)}Y$.

Prove a Garland inequality

$$(2\lambda - 1)E(f) \leq \lambda \| -\Delta f \|^2.$$

Use U. Mayer's gradient flow for convex functions on CAT(0) spaces.

Show exponential decay of energy.

Conclude that flow is defined for any time and converges to an ℓ^2 map of vanishing energy.

Theorem

(M.T. Wang, 1998). *Let C be a finite weighted graph. Let Y be a geodesic CAT(0) metric space. Then*

$$\lambda(C, Y) = \inf_{y \in Y} \lambda(C, T_y Y).$$

Proof.

Given $g : C \to Y$, let $y = bar(g)$. Let $g' = \pi_y \circ g$ where π_y is the projection from Y to $T_y Y$. Since π_y is distance nonincreasing, $E(g') \leq E(g)$.

It turns out that $bar(g') = y$ is the vertex of the cone. Indeed, at y, the distance in Y to a point z osculates the distance in $T_y Y$ to $\pi_y(z)$. So do $d(g, \cdot)^2$ and $d(g', \cdot)^2$. Since the first achieves a minimum at y, so does the second at the vertex of the cone. It follows that $d(g', bar(g'))^2 = d(g, bar(g))^2$, thus $RQ(g') \leq RQ(g)$, and $\lambda(C, T_y Y) \leq \lambda(C, Y)$.

Reverse inequality holds since each $T_y Y$ is a limit of rescaled copies of Y.

Corollary

If Y is a nonpositively curved manifold, then for every finite weighted graph C, $\lambda(C, Y) = \lambda(C, \mathbb{R})$.

Corollary

Les X be a simplicial complex, Γ a uniform lattice in X. Assume that at all vertices x, $\lambda(link(x), \mathbb{R}) > \frac{1}{2}$. Then Γ is superrigid with respect to nonpositively curved Riemannian manifolds : every isometric action of Γ on such a space has a fixed point.

Corollary

Random groups in density $> \frac{1}{3}$ are superrigid with respect to nonpositively curved Riemannian manifolds.

Proposition

(H. Izeki and S. Nayatani, 2004). *If Y is a tree, then for every finite weighted graph C, $\lambda(C,Y)=\lambda(C,\mathbb{R})$.*

Proof. Since we deal with finitely many points at a time, we can assume first that Y is a finite tree, and then replace it by a bunch of half-lines. One can assume that the given map has its barycenter at the vertex. Without changing the barycenter nor increasing energy, one can assume that the given map $g : C \to Y$ sends a point $a_i > 0$ in each branch, with weight 1.
The barycenter assumption translates into

$$\forall i, \quad a_i \leq \sum_{j\neq i} a_j,$$

a necessary and sufficient condition for the numbers a_i to be the length of the sides of a planar Euclidean polygon.
Let $e_i \in \mathbb{R}^2$ be unit vectors parallel to the sides, so that $\sum_i a_i e_i = 0$. Let $g' : C \to \mathbb{R}^2$ be the map which sends a point c mapped by g to the i-th branch to $d(g(c), bar(g))e_i$. Then $bar(g')=0$, $E(g')\leq E(g)$, $d(g', bar(g'))^2 = d(g, bar(g))^2$, thus $RQ(g')\leq RQ(g)$, $\lambda(C,Y)\geq\lambda(C,\mathbb{R}^2)=\lambda(C,\mathbb{R})$.

Definition

(H. Izeki and S. Nayatani, 2004). *Let Y be a geodesic CAT(0) space. Given a finite weighted subset $Z \in Y$ (sum of weights $= 1$), let $\phi : Z \to \mathcal{H}$ be a 1-Lipschitz map to Hilbert space such that for all $z \in Z$, $|\phi(z)| = d(z, bar(Z))$. Define*

$$\delta(Z) = \inf_{\phi} \frac{|bar(\phi)|^2}{\| \phi \|^2}.$$

The IN invariant of Y is $\delta(Y) = \sup_{Z \subset Y} \delta(Z) \in [0, 1]$.

Lemma

Let Y be a geodesic CAT(0) space, let C be a finite weighted graph. Then

$$\lambda(C, Y) \geq (1 - \delta(Y))\lambda(C, \mathbb{R}).$$

Proof. Given $g : C \to Y$, let $Z = g(C)$. Choose optimal ϕ for Z. Pythagore gives $d(\phi, bar(\phi))^2 = \| \phi \|^2 - |bar(\phi)|^2 = (1 - \delta(Z))\| \phi \|^2 = (1 - \delta(Z))d(g, bar(g))^2$.

$$\lambda(C, \mathbb{R}) \leq RQ(\phi \circ g) = \frac{E(\phi \circ g)}{d(\phi \circ g, bar(\phi \circ g))^2} \leq \frac{E(g)}{d(\phi, bar(\phi))^2} = \frac{1}{1 - \delta(Z)} RQ(g).$$

Examples of values of IN invariant

Examples

1. *Hilbert spaces have* $\delta = 0$*, by definition.*
2. *For all* Y, $\delta(Y) = \inf_{y \in Y} \delta(T_y Y)$*. Therefore nonpositively curved manifolds have* $\delta(Y) = 0$*.*
3. *Trees have* $\delta = 0$*.*
4. δ *is continuous under ultralimits. Therefore (non proper) Euclidean buildings which are asymptotic cones of symmetric spaces have* $\delta(Y) = 0$*.*
5. *For all* Y *and probability measure spaces* Ω, $\delta(L^2(\Omega, Y)) \leq \delta(Y)$*.*
6. $\delta(Y_1 \times Y_2) \leq \max\{\delta(Y_1), \delta(Y_2)\}$*. Therefore, products of the above have* $\delta = 0$*.*
7. *The Euclidean building of* $Sl(3, \mathbb{Q}_p)$ *has* $\delta \geq \frac{(\sqrt{p}-1)^2}{2(p-\sqrt{p}+1)}$ *(equality conjectured).*
8. *The Euclidean building of* $Sl(3, \mathbb{Q}_2)$ *has* $\delta < \frac{1}{2}$*.*

The fixed point property $F\mathcal{Y}_{\leq\delta_0}$

Definition

Fix $\delta_0 \in [0,1]$. Say a group Γ has property $F\mathcal{Y}_{\leq\delta_0}$ if every isometric action of Γ on a geodesic CAT(0) space Y with $\delta(Y) \leq \delta_0$ has a fixed point.

Proposition

Let $\delta_0 < \frac{1}{2}$. Let X be a simplicial complex, Γ a uniform lattice in X. Assume that for all $x \in X$, $\lambda(link(x), \mathbb{R}) > \frac{1}{2(1-\delta_0)}$. Then Γ has property $F\mathcal{Y}_{\leq\delta_0}$.

Theorem

(A. Zuk, 2003, H. Izeki, T. Kondo and S. Nayatani, 2006). *If $\delta_0 < \frac{1}{2}$, random groups in density $> \frac{1}{3}$ have asymptoticly property $F\mathcal{Y}_{\leq\delta_0}$.*

Theorem

(T. Kondo, 2006). *In the space of marked groups, $F\mathcal{Y}_{\leq\delta_0}$ is an open condition. Furthermore, $F\mathcal{Y}_{<1/2}$ is dense.*

Definition

(H. Bass, 1980). *Say a group* Γ *has* finite representation type *if for all* n*, every homomorphism* $\Gamma \to Gl(n, \mathbb{C})$ *factors through a finite group.*

Theorem

(T. Kondo, 2006). *In the space of marked groups, there is a dense* G_δ *of groups which have property* $F\mathcal{Y}_{<1/2}$ *and finite representation type.*

Proposition

If a group has a fixed point in all its isometric actions on symmetric spaces and classical Euclidean buildings of type $\tilde{A}_n$ *(call this FIS), then it has finite representation type.*

Remark

$F\mathcal{Y}_{\leq \delta_0}$ *does not imply FIS. Indeed,* δ *tends to* $1/2$ *for classical buildings of type* $\tilde{A}_2$*, as* p *tends to infinity.*

1. Isometric actions on the real line
2. Local rigidity
3. Matsushima's formula
4. Kazhdan's property (T)
5. Harmonic map proof of Margulis superrigidity, after Mok, Siu and Yeung
6. Garland's formula
7. Fixed points for isometric actions on $CAT(0)$ spaces
8. $\tilde{A}_2$-buildings

Theorem

(M. Gromov, 2001). *Let C_k denote the k-cycle. Then, for every CAT(0) space Y,*

$$\lambda(C_k, Y) = \lambda(C_k, \mathbb{R}) = \frac{1}{2}|1 - e^{2i\pi/k}|^2.$$

In particular, $\lambda(C_6, Y) = \frac{1}{2}$.

Proof. Introduce

$$F(g) = \frac{1}{2\sum m(c)} \sum_{c,\, c' \in C} m(c)m(c')d(g(c), g(c'))^2.$$

Then $d(g, bar(g))^2 \leq F(g)$ with equality when Y is a Hilbert space.

Given $g : C_k \to Y$, extend g to a geodesic polygon, then to a ruled disk $f : D \to Y$. Since D has nonpositive curvature, there exists an embedding $g' : D \to \mathbb{R}^2$ which is isometric on the boundary and does not decrease other distances (Yu. Reshetnyak, 1968). Thus $E(g') = E(g)$ and

$$d(g', bar(g'))^2 = F(g') \geq F(g) \geq d(g, bar(g))^2,$$

thus $RQ(g') \leq RQ(g)$.

Definition

Let C be a weighted graph, Y a metric space. Define

$$\lambda^{Gro}(C,Y)=\inf_{g:C\to Y} RQ^{Gro}(g) \quad \textit{where} \quad RQ^{Gro}(g)=\frac{E(g)}{F(g)}.$$

If Y is geodesic CAT(0), $\lambda^{Gro}(C,Y)\leq\lambda(C,Y)$, with equality when Y is a Hilbert space.

Proposition

Let C be the incidence graph of a finite projective plane. Let Y be an arbitrary geodesic CAT(0) space. Then

$$\lambda^{Gro}(C,Y)=RQ^{Gro}(\iota),$$

where $\iota : C\to I$ is the embedding of C in the cone over C, for instance, as the link of a vertex in a Euclidean building of type $\tilde{A}_2$.

Proof. In the incidence graph of a finite projective plane, the number of 6-cycles containing two given vertices depends only on their distance. Sum up Gromov's estimate on F for all 6-cycles.

Unfortunately, $RQ^{Gro}(\iota)<\frac{1}{2}$. Note that $RQ(\iota)=\frac{1}{2}$.