Geometric rigidity

Pierre Pansu

Grenoble, june 2004

What is it all about ?

Classifying finite dimension linear representations of certain discrete groups.

Existing results apply mainly to lattices (i.e. finite covolume discrete subgroups) of semi-simple algebraic groups.

Prototype : $\Gamma = Sl(n,\mathbf{Z})$, $G = Sl(n,\mathbf{R})$.

One wonders wether methods might apply to larger classes of discrete subgroups.

Counting linear representations

Rough dimension count for Hom(Γ,G)/G. If Γ has g generators and r relators,

$$\dim(\mathrm{Hom}(\Gamma,G)/G) \approx \dim(G)(g-r-1).$$

Example. S compact orientable surface of genus γ. Then $\Gamma = \pi_1(S)$ has a presentation with 2g generators and 1 relator, thus $\dim(\mathrm{Hom}(\Gamma,G)/G) \approx \dim(G)(2\gamma - 2)$. This is sharp for G = PSl(2,**R**), PSl(2,**C**), but not for PU(2,1).

Compact 3-manifold groups

If $\Gamma = \pi_1(M^3)$, a Morse function with n_i critical points of index i, $n_0=n_3=1$, yields a presentation with $g=n_1$, $r=n_2$. Since $0 = \chi(M) = 1-n_1+n_2-1$, $g=r$.

Thurston. If $G = PSL(2,\mathbf{C})$, the dimension count can be refined to give $\dim(Hom(\Gamma,G)/G) = 0$.

This looks sharp, in view of Calabi-Weil infinitesimal rigidity. Higher dimensional lattices should be even more rigid.

Superrigidity of lattices

Margulis (1974). Let G, H be semi-simple algebraic groups over local fields, without compact factors. Assume G has real rank at least 2. Let Γ be an irreducible lattice in G.

Every homomorphism $\Gamma \rightarrow$ H whose image is unbounded and Zariski dense extends to G $\rightarrow$ H.

Dictionnary

Margulis (1974). Let G, H be semi-simple algebraic groups over local fields, without compact factors. Assume G has real rank at least 2. Let Γ be an irreducible lattice in G.

Every homomorphism $\Gamma \to H$ whose image is unbounded and Zariski dense extends to $G \to H$.

Let X, Y be finite dimensional symmetric spaces or buildings, without euclidean or compact factors. Assume X has rank at least 2. Let Γ be a discrete irreducible group of isometries of X such that

$$\mathrm{vol}(\Gamma \backslash X) < \infty.$$

Every reductive isometric action of Γ on Y leaves invariant either a point or a convex subset C of Y which is pluriisometric to a product of irreducible factors of X.

Rank 1 lattices

Corlette, **Gromov-Schoen** (1992). Preceding results extend to the case when X is a hyperbolic space over the quaternions or the octonions of dimension > 4.

Remark. They do not extend to the two other families of rank 1 symmetric spaces, real hyperbolic spaces $\mathbf{R}H^n$ or complex hyperbolic spaces $\mathbf{C}H^n$.

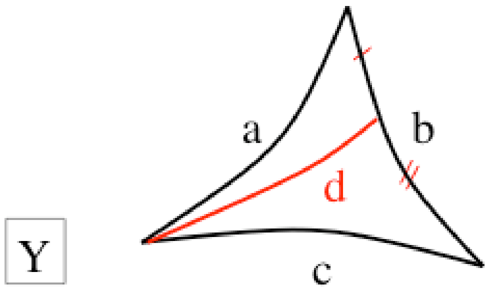
Arithmeticity

Corollary. Irreducible lattices in semi-simple Lie groups other than SO(n,1) = Isom($\mathbf{R}H^n$) and SU(n,1) = Isom($\mathbf{C}H^n$) are *arithmetic* i.e. obtained as integer matrices in a linear representation defined over **Q** of G (up to lifting in a product G × L, L compact, and up to commensurability).

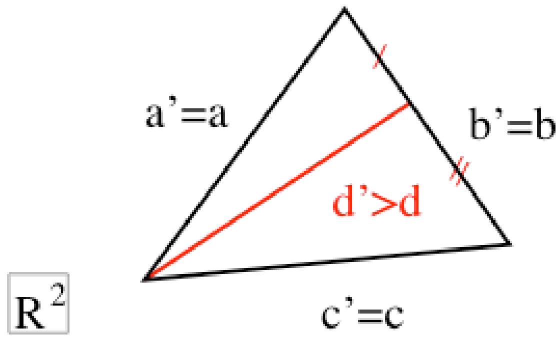
Since algebraic groups over **Q** are classified (**Tits** 1966), this can be viewed as a classification of commensurability classes of lattices.

CAT(0) spaces

Symmetric spaces and buildings are examples of CAT(0) metric spaces.



Definition. A geodesic space Y is CAT(0) if for every triangle, medians are shorter that in the euclidean comparison triangle.



For Riemannian manifolds, CAT(0) ⇔ simply connected, nonpositive curvature

Generalization

Question. Let Γ be a discrete finite covolume group of isometries of a metric space X. Find conditions on X so that every isometric action of Γ sur on a complete CAT(0) space Y has a fixed point or leaves a convex subset homothetic to X invariant.

This is the program of *geometric superrigidity*.

What is it good for ?

1. Understand why lattices of SU(n,1) are not superrigid.
2. Investigate infinite dimensional representations of lattices.
3. Study group actions on compact manifolds, via the associated action on an auxiliary CAT(0) space, like the space of measurable metrics.
4. Prove that certain groups are not linear.

Non linear groups

Remark. There exist finitely presented groups without non trivial finite dimensional linear representations (e.g. infinite *simple* groups).

Question. Does a *generic* finitely presented group admit non trivial finite dimensional represented groups ?

Random groups

Claim (Gromov 2002). Let Γ be a random group modelled on a graph of large girth. With probability tending to 1 as girth tends to infinity, for all n, every homomorphism $\Gamma \to Gl(n,\mathbf{R})$ has a finite image.

Question. Find an elementary proof of this.

Geometric superrigidity ⇒ finiteness of representations

Let $\Gamma \subset Gl(n,\mathbf{C})$ be a superrigid group.

1. Choose a $\bar{\mathbf{Q}}$-point h of $Hom(\Gamma, Gl(n,\mathbf{C}))$.
2. $Y = \mathbf{C} \Rightarrow h(\Gamma) \subset Sl(n,\mathbf{C})$, in fact $Sl(n,\bar{\mathbf{Q}})$.
3. Γ finitely generated $\Rightarrow h(\Gamma) \subset Sl(n,\mathbf{F})$, $\mathbf{F}$ a finite extension of $\mathbf{Q}$. Extension of scalars yields $h' : \Gamma \to Sl(N,\mathbf{Q})$.
4. $Y =$ building of $Sl(N,\mathbf{Q}_p) \Rightarrow h'(\Gamma') \subset Sl(N,\mathbf{Z})$, for Γ' of finite index in Γ.
5. $Y = Sl(N,\mathbf{C})/SU(N) \Rightarrow h'(\Gamma) \subset SU(N)$.
6. $h'(\Gamma') \subset SU(N) \cap Sl(N,\mathbf{Z})$ is finite.

Results discussed in the sequel

- Affine actions (Bochner, Matsushima, Garland, Zuk).
- Actions on CAT(0) manifolds (Eells-Sampson, Corlette, Jost-Yau, Mok-Siu-Yeung, Wang).
- Actions on certain CAT(0) spaces (Gromov-Schoen, Wang, Gromov, Iseki-Nayatani).
- Irreducible lattices in products (Monod).
- Commensurators (Margulis, Monod).

Contents

1. Harmonic maps
- Eells-Sampson and Ricci > 0
- Affine actions and vanishing theorems
- A combinatorial vanishing theorem
- More general CAT(0) targets

2. Induction
- Irreducible lattices in products
- Commensurator superrigidity

Equivariant maps

Notation. M compact manifold, $\Gamma = \pi_1(M)$, N universal covering of M. h isometric action of Γ on Y.

Goal. Prove that h has a fixed point.

Notation. $L^{[2]}(M,h)$ = space of equivariant Lipschitz maps $f:N \to Y$, equipped with L^2 distance. Nonempty if Y is contractible.

Remark. h has a fixed point $\Leftrightarrow$ $L^{[2]}(M,h)$ contains a constant map.

Energy

Definition. Let $f \in L^{[2]}(M,h)$. Its *energy* is

$$E(f) = 1/2 \int_M |df|^2 .$$

Critical points of E are called *harmonic maps*.

Remark. Y is CAT(0) $\Rightarrow L^{[2]}(M,h)$ is CAT(0) and E is convex. If nonempty, the set of critical points of E is convex, closed and E achieves its absolute minimum on it.

Strategy

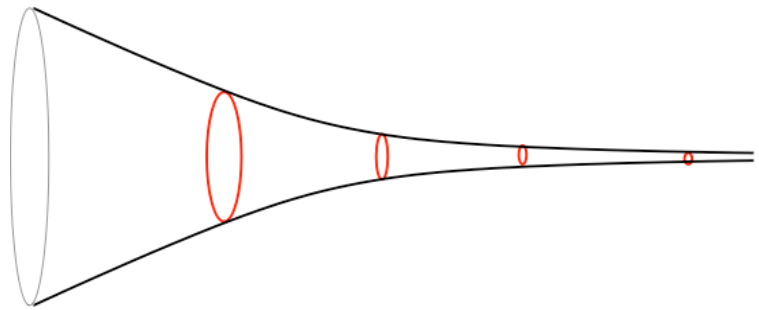
Strategy (Eells-Sampson 1964, Yau late '70). Under a suitable assumption on M and h,

- show that E achieves its minimum at some map f,
- show that f is constant.

Example. $M=S^1$, $Y=\mathbf{R}H^2$,

$h : z \to z+1$, inf $E = 0$

is not achieved.



Eells-Sampson and Ricci > 0

1. Harmonic maps
- Eells-Sampson and Ricci > 0
- Affine actions and vanishing theorems
- A simplicial vanishing theorem
- More general CAT(0) targets

2. Induction
- Irreducible lattices in products
- Commensurator superrigidity

Eells-Sampson's fixed point theorem

Eells-Sampson (1964). Let Y be a CAT(0) manifold. Assume $h(\Gamma)$ is discrete cocompact.

Then E achieves its minimum at some f.

If M has Ricci > 0, then f is constant.

If Ricci $\geqslant 0$, then $h(\Gamma)$ leaves invariant a convex d-flat and M fibres over a d-torus.

Actions on a line

Goal. Prove the second part of Eells-Sampson's theorem in case $Y = \mathbf{R}$.

Remark. If $f : N \to \mathbf{R}$ is equivariant, $\alpha = df$ is a 1-form on M.

Let $D\alpha$ = covariant derivative, $\delta\alpha = tr(D\alpha)$, $d\alpha = A(D\alpha)$, where A skew-symmetrizes 2-tensors. Then

$$\forall f,\ d\alpha = 0.$$

$$f \text{ harmonic} \Leftrightarrow \delta\alpha = 0.$$

Bochner's formula

Bochner (1946). For T a 2-tensor, let

$$Q_B(T) = |T|^2 - |A(T)|^2 - tr(T)^2.$$

Then for every 1–form α on M,

$$\int_M Q_B(D\alpha) = -\int_M Ricci(\alpha,\alpha).$$

Remark. Q_B is positive definite on the subspace S^2_0 of tracefree symmetric 2-tensors.

$$f : M \to \mathbf{R} \text{ harmonic} \Leftrightarrow D\alpha \in S^2_0 .$$

Thus Ricci $> 0 \Rightarrow \alpha = 0$, f constant. □

Eells-Sampson's formula

Let Y be a manifold, $f : N \to Y$ equivariant. Then df is a 1-form with values in the vector bundle f^*TY. Again,

$$ddf = 0 \text{ and } f \text{ harmonic} \Leftrightarrow \delta df = 0.$$

Eells-Sampson (1964). Let $F \to M$ be an orthogonal vector bundle with metric connection. For all F-valued 1–forms α on M,

$$\int_M Q_B(D\alpha) = -\int_M \mathrm{Ricci}(\alpha,\alpha) + \mathrm{tr}(\alpha^* R^F).$$

If Y is CAT(0) and $F = f^*TY$, then $R^F \leqslant 0$. Thus Ricci > 0 $\Rightarrow \alpha = 0$, f constant. Ricci $\geqslant 0 \Rightarrow$ f totally geodesic, curvature of Y vanishes on the image of f. □

Affine actions and vanishing theorems

1. Harmonic maps
- Eells-Sampson and Ricci > 0.
- Affine actions and vanishing theorems
- A simplicial vanishing theorem
- More general CAT(0) manifolds and spaces

2. Induction
- Irreducible lattices in products
- Commensurator superrigidity

Affine actions

Definition. Γ has property FH is every affine isometric action Γ on a Hilbert space has a fixed point.

Goal. Prove that most lattices have property FH.

Guichardet. FH implies Kazhdan's property (T) (for countable groups).

Matsushima's formula

A *curvature tensor* is a 4-tensor such that

$Q_{1234} = -Q_{2134} = Q_{3412}$ and $Q_{1234}+Q_{2314}+Q_{3124} = 0$.

Matsushima (1962), revisited by **Mok-Siu-Yeung** (1993). Let Q be a *parallel* curvature tensor on M. It defines a quadratic form on 2-tensors. Then, for all vector bundle valued 1-forms α on M,

$$\int_M Q(D\alpha) = 1/2 \int_M (\langle Q, \alpha^*R^F\rangle + \langle Q(\alpha),R(\alpha)\rangle).$$

Parallel curvature tensors

Examples.

- Q_B corresponds to the curvature tensor I of $\mathbf{R}H^n$, which can be transplanted on any Riemannian manifold. Therefore Bochner $\subset$ Matsushima.
- M is locally symmetric $\Leftrightarrow$ its curvature tensor R est parallel. In this case, let $R^{\perp}$ be the component of I orthogonal to R. If N is irreducible, and α is an equivariant Hilbert space valued 1-form, Matsushima's formula reads

$$\int_M R^{\perp}(D\alpha) = 0.$$

Positivity

Calabi, **Vesentini**, **Borel** (1960), **Matsushima**, **Kaneyuki-Nagano** (1962). If N is distinct from $\mathbf{R}H^n$ and $\mathbf{C}H^n$, the quadratic form $R^{\perp}$ is positive definite on $S^2{}_0$.

Corollary. Every affine isometric action of a uniform lattice which is neither real or complex hyperbolic has a fixed point (property FH). In particular, these lattices have Kazhdan's property (T).

Avoiding harmonic maps

Proof. Positivity $\Rightarrow \exists\, C_1, C_2 > 0$ such that

$$R^{\perp}(T) + C_1(|A(T)|^2 + tr(T)^2) \geqslant C_2|T|^2 .$$

For all equivariant $\mathcal{H}$ -valued 1-forms α,

Matsushima + Bochner $\Rightarrow$

$$\|\alpha\|^2 \leqslant C_3\|D\alpha\|^2 \leqslant C_4(\|d\alpha\|^2 + \|\delta\alpha\|^2),$$

Thus $\delta|_{ker(d)}$ is an isomorphism. So is its adjoint d. Let $f : N \rightarrow \mathcal{H}$ be equivariant. Then

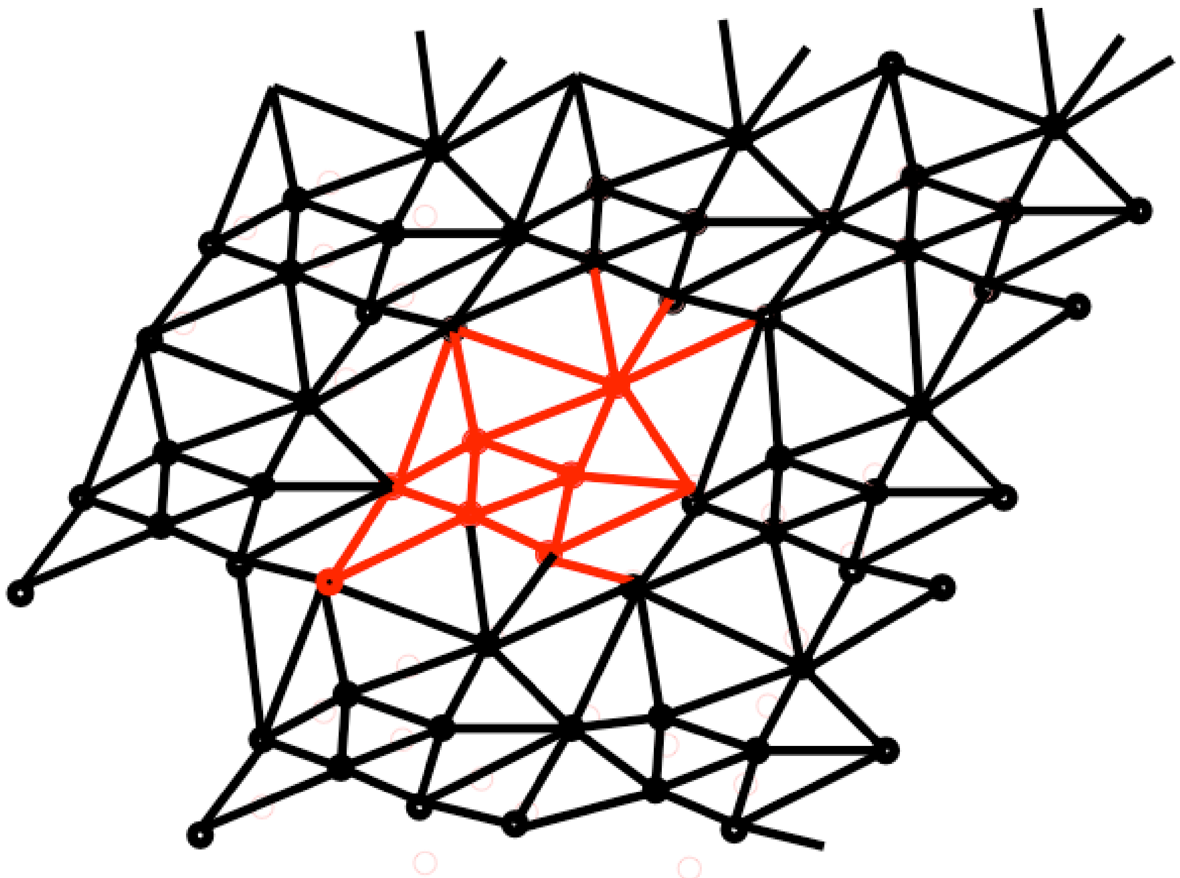
$g = f - d^{-1}df$ is equivariant and constant. □

A combinatorial vanishing theorem

1. Harmonic maps
- Eells-Sampson and Ricci > 0
- Affine actions and vanishing theorems
- A combinatorial vanishing theorem
- More general CAT(0) targets

2. Induction
- Irreducible lattices in products
- Commensurator superrigidity

Combinatorial equivariant maps



From manifolds to simplicial complexes

Goal. A combinatorial analogue of Matsushima's formula.

Weights. A *weight* on a complex C is a positive function m on simplices of C, such that for each k-simplex σ, $m(\sigma)$ is the sum of $m(\tau)$ over all (k+1)-simplices τ containing σ.

Example. Let m = 1 for all 2-simplices and propagate to 1- and 0-simplices.

Energy. Given a weighted complex (C,m) and a map g' : C→ Y, $E(g) = 1/2 \sum_{c,c'} m(c,c')\, d(g(c),g(c'))^2$.

Combinatorial harmonic maps

Definition. Let Z be a finite simplicial complex with universal covering X, $\Gamma = \pi_1(Z)$, h an isometric action of Γ on Y. Let $L^{[2]}(Z,h)$ be the set of equivariant maps of the set of vertices of X to Y.

$$E(f) = 1/2 \sum_{z,z'} m(z,z')d(f(x),f(x'))^2.$$

Critical points of E are *harmonic maps*.

Interpretation. Z is network of springs of strength m. E is potential energy, harmonic map is equilibrium configuration.

Notations in case of affine action

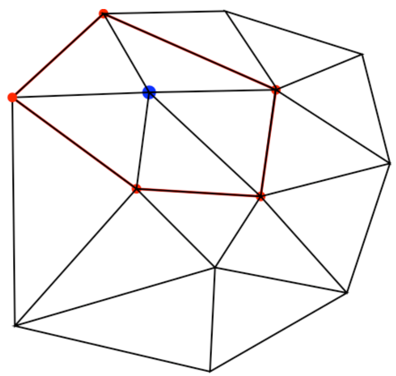
Barycenter. Let (C,m) be a finite weighted graph.
For $g : C \to \mathcal{H}$, let

$$\mathrm{bar}(g) = \sum_c m(c)g(c) / \sum_c m(c).$$

Energy. For $g : C \to \mathcal{H}$, let

$$E(g) = 1/2 \sum_{c,c'} m(c,c') \left|g(c)-g(c')\right|^2.$$

Links. Link(c) inherits weight m(c, •).



Combinatorial harmonic maps : affine case

Definition. Let Z be a finite simplicial complex with universal covering X, $\Gamma = \pi_1(Z)$, h an affine isometric action of Γ on a Hilbert space $\mathcal{H}$. Let $L^{[2]}(Z,h)$ be the Hilbert space of equivariant maps f of the set of vertices of X to $\mathcal{H}$. Let

$$E(f) = 1/2 \sum_{z,z'} m(z,z') \left|f(x)-f(x')\right|^2.$$

Harmonic map equation.

$$f \text{ is harmonic} \Leftrightarrow \forall x,\ f(x) = \mathrm{bar}(f_{|\mathrm{link}(x)}).$$

Bottom of spectrum

Definition. (C,m) weighted graph, $g : C \to R$. The *Rayleigh quotient* of g is $RQ(g) = E(g)/\|g - bar(g)\|^2$. Then $\lambda(C) = \inf_g RQ(g)$ is the *bottom of the spectrum* of the discrete Laplacian on C.

Example. If C is a circle subdivised in 6 edges with equal weights, $\lambda(C) = 1/2$. A random regular graph of large size and degree has $\lambda(C)$ close to 1.

Intuition. $\lambda(C)$ large means C is strongly interconnected.

Garland's formula

Garland (1972), **Borel** (1973). h affine isometric action of Γ on Hilbert space $\mathcal{H}$, f : $X \rightarrow \mathcal{H}$ equivariant.

For $x \in X$, let $ED(f,x) = 1/2 \| f_{|link(x)} - f(x) \|^2$, so that $E(f) = \sum_z ED(f,x)$.

If f is harmonic, then

$$E(f) = 2 \sum_z RQ(f_{|link(x)}) \, ED(f,x).$$

Corollary (**Zuk** 1996). If for all z, $\lambda(link(z)) > 1/2$, then Γ has property FH.

Example. Sufficiently thick euclidean buildings, certain hyperbolic buildings.

Proof of Garland's formula

$$E(f) = 1/2 \sum_{z, z'} m(z,z') \left|f(x')-f(x'')\right|^2$$
$$= 1/2 \sum_{z, z',z''} m(z,z',z'') \left|f(x')-f(x'')\right|^2$$
$$= \sum_{z} E(f_{|link(x)}).$$

For each $x \in X$,

$$E(f_{|link(x)}) = RQ(f_{|link(x)}) \left\| f_{|link(x)} - bar(f_{|link(x)}) \right\|^2$$
$$= RQ(f_{|link(x)}) \left\| f_{|link(x)} - f(x) \right\|^2$$
$$=2\, RQ(f_{|link(x)})\, ED(f,x). \ \square$$

Property FH for random groups

Zuk (2003). Let Γ be a random group, i.e. given by a presentation $<S,R>$ where R consists of words of length L. Assume

$$\#R = (\#S)^{dL} \text{ where } d > 1/3.$$

With probability tending to 1 as L tends to infinity, every affine isometric action of Γ on a Hilbert space has a fixed point.

Remark. $d < 1/2 \Rightarrow \Gamma$ is nonelementary hyperbolic (**Gromov** 1993).

Why density > 1/3 ?

Gromov, **Zuk**. Let S’ = all words of length < L/3. Then Γ = $<S',R'>$ where R’ consists of random words of length 3. The Cayley complex of this new presentation is simplicial, its links are approximately random regular graphs of size #S’=(#S)$^{L/3}$. Their degree is

$$\#R/\#S' = (\#S)^{L(d-(1/3))},$$

which tends to infinity with L if d > 1/3.

Unclear wether bound 1/3 is sharp. **Ollivier** and **Wise** (2004) show that random groups in density < 1/8 are not FH.

More general CAT(0) targets

1. Harmonic maps
- Eells-Sampson and Ricci > 0
- Affine actions and vanishing theorems
- A simplicial vanishing theorem
- More general CAT(0) targets

2. Induction
- Irreducible lattices in products
- Commensurator superrigidity

Harmonic maps from symmetric spaces to manifolds

Mok-Siu-Yeung, Jost-Yau (1993). Every equivariant harmonic map of a higher rank compact irreducible locally symmetric space to a CAT(0) manifold is totally geodesic and pluriisometric.

Corlette (1990), **ibid**. Same conclusion for quotients of $\mathbf{H}H^n$ and $\mathbf{O}H^2$, under a stronger negativity assumption on the range (satisfied by symmetric spaces) : nonpositive complex sectional curvature.

Harmonic map proof of Margulis superrigidity

Proof. M compact higher rank locally symmetric space. Then there exists on M a parallel curvature tensor Q such that

- $Q \gg 0$ on S^2_0,
- $\langle Q,T\rangle \leqslant 0$ for all curvature tensors with nonpositive sectional curvature.

Matsushima ⇒ if $f : N \to Y$ is equivariant harmonic,

$0 \leqslant \int_M Q(Ddf) = 1/2 \int_M \langle Q,R\rangle |df|^2 + \langle Q, f^*R^Y\rangle \leqslant 0.$

⇒ f totally geodesic.□

Construction of a parallel curvature tensor

M higher rank locally symmetric space. $\mathcal{F}(x)$ = set of 2-flats F through x. p : $T_xM \to T_xF$ orthogonal projection. Pull back Bochner's curvature tensor on F and average, i.e. set

$$Q_F = p^*Q_{B,F} \text{ and } Q = \int_{\mathcal{F}(x)} Q_F \, dF.$$

Then Q is K-invariant, thus parallel.

$$\langle Q,S\rangle = \int_{\mathcal{F}(x)} S(F)\, dF$$

is an average of sectional curvatures. Therefore

$\langle Q,S\rangle \leqslant 0$ if S has nonpositive sectional curvature.

Harmonic maps from symmetric spaces to buildings

Gromov-Schoen (1992). Every equivariant harmonic map of a compact irreducible locally symmetric space (not covered by $\mathbf{R}H^n$ or $\mathbf{C}H^n$) to a Euclidean building is constant.

Proof. Such a map is smooth (i.e. locally factors through an isometric embedding of a euclidean space) away from a set of codimension 2. Thus integration by parts applies.□

Simplicial maps to CAT(0) spaces

L^2 distance. Given maps g, g' : C→ Y,

$$d(g,g')^2 = \sum_c m(c)\, d(g(c),g'(c))^2.$$

Barycenter. The *barycenter* bar(g) of g is the point of Y which minimizes

$$y \to d(g,y)^2 = \sum_c m(c)\, d(g(c),y)^2.$$

Harmonic map equation.

$$f \text{ is harmonic} \Leftrightarrow \forall x,\ f(x) = \mathrm{bar}(f_{|\mathrm{link}(x)}).$$

Bottom of spectrum. Let $\lambda(C,Y)$ denote the inf of $RQ(g)=E(g)/d(g,\mathrm{bar}(g))^2$ over all maps g : C→ Y.

Tangent cones

Let Y be CAT(0). Then Y admits at each point y a tangent cone $T_y Y$ which is again a CAT(0) metric space.

Wang. $\lambda(C,Y) \geqslant \inf_{y \in Y} \lambda(C,T_y Y)$. In particular, if Y is a manifold or a tree,

$$\lambda(C,Y) \geqslant \lambda(C,\mathbf{R}) = \lambda(C).$$

Nonlinear Garland formula

Wang (2000). Let $ED(f,x) = d(f_{|link(x)}, f(x))^2$. If f is harmonic,

$$E(f) = 2\sum_z RQ(f_{|link(x)})\, ED(f,x).$$

In particular, if for each z and y,

$$\lambda(link(z), T_y Y) > 1/2,$$

every equivariant harmonic map $f : X \to Y$ is constant.

Corollary (**Gromov** 2003, **Iseki**-**Nayatani** 2004). Every isometric action of Γ on Y has a fixed point.

Nonlinear heat flow

Proof. **Mayer** (1998) constructs a gradient flow for every continuous convex function on a complete CAT(0) space. This applies to E on $L^{[2]}(Z,h)$, yielding $t \rightarrow f_t$. If all $x \in X$ and $y \in Y$ satisfy $\lambda(\mathrm{link}(x), T_y Y) \geqslant \lambda > 1/2$, Garland's formula implies

$$\partial E(f_t)/\partial t \leqslant - 4 \lambda E(f_t).$$

Since

$$| \partial f_t/\partial t |^2 = - \partial E(f_t)/\partial t,$$

one concludes that f_t subconverges to a constant map. □

Comparing simplicial curvatures

Iseki-Nayatani (2004) attach to every metric space Y a number $\delta(Y)$ such that for every weighted graph C,

$$\lambda(C,Y) \geqslant (1-\delta(Y))\, \lambda(C).$$

$\delta(Y) = 0$ if Y is a tree.

$\delta(Y) \leqslant 0.41$ if Y is the building of $PGl(3,\mathbf{Q}_2)$.

Corollary. Random groups in density >1/3, lattices in $PGl(3,\mathbf{Q}_p)$ for large p, must have a fixed point when acting on this specific building.

Izeki-Nayatani's invariant

Consider probability measures μ on Y and distance decreasing maps $\phi : Y \to \mathcal{H}$ such that

$$\forall y \in Y, |\phi(y)| = d(y, bar(\mu)).$$

Define

$$\delta(Y) = \sup_{\mu} \inf_{\phi} |\textstyle\int_Y d\mu|^2 / \textstyle\int_Y |\phi|^2 \, d\mu \quad .$$

Induction

1. Harmonic maps
- Eells-Sampson and Ricci > 0
- Affine actions and vanishing theorems
- A simplicial vanishing theorem
- More general CAT(0) targets

2. Induction
- Irreducible lattices in products
- Commensurator superrigidity

Induction and Kazhdan's property (T)

Kazhdan (1968). If Γ is a lattice in G, then

G has property (T) $\Rightarrow$ Γ has property (T).

Proof. π unitary representation of Γ. Then

π has almost Γ-invariant vectors $\Leftrightarrow$

$L^2(\Gamma\backslash G,\pi)$ has almost G-invariant vectors. □

Inducing isometric actions

Inducing a Γ–action h on Y means considering the G-action (by precomposition with right translations) on

$$L^2(\Gamma\backslash G,h)=\{\text{equivariant } L^2 \text{ maps } G\to Y\},$$

where L^2 means $g\to d(f(g),y_0)$ is L^2 on a fundamental domain (this requires a condition on h if Γ is not uniform).

Lemma. Γ lattice in G.

$\exists$D G-invariant closed convex subset in $L^{[2]}(\Gamma\backslash G,Y)$ such that $\forall$f, f'$\in$ D, d(f(g),f'(g)) is independant on the choice of g$\in$G.

$\Rightarrow$ $\exists$C Γ-invariant closed convex subset in Y on which the Γ–action extends to a G-action.

Evanescence

Definition (Monod). A continuous isometric G-action on Y is *evanescent* if there exist an unbounded $T \subset Y$ such that for every compact set $Q \subset G$, the Q-orbits of elements of T have diameters bounded in terms of Q only.

Remark. An affine action of a σ-compact group is evanescent $\Leftrightarrow$ the associated linear representation has almost invariant vectors.

Induction and evanescence

Proposition. Let G be locally compact second countable. Let Γ be a uniform lattice in G, h a non evanescent isometric action on a CAT(0) space Y. Then the induced G-action on $L^{[2]}(\Gamma\backslash G,h)$ is non evanescent.

Irreducible lattices in products

1. Harmonic maps
- Eells-Sampson and Ricci > 0
- Affine actions and vanishing theorems
- A simplicial vanishing theorem
- More general CAT(0) targets

2. Induction
- Irreducible lattices in products
- Commensurator superrigidity

Irreducible lattices

Definition. Let G_i be locally compact second countable groups. A lattice $\Gamma \subset G = G_1 \times \ldots \times G_n$ is *irreducible* if its projections to factors are dense.

Example. The automorphism group of the product of two trees admits such lattices (**Burger**-**Mozes** 1997).

Superrigidity of irreducible lattices in products

Monod (2004). Let Γ be an irreducible uniform lattice in $G = G_1 \times \ldots \times G_n$. For every isometric and non evanescent action of Γ on a complete, separable CAT(0) space Y, there is either a fixed point or a closed convex Γ-invariant subset $C \subset Y$ which is equivariantly isometric to a product of minimal G_i-spaces.

The proof relies on the following splitting theorem for G actions.

Splitting of actions of products

Monod (2004). Let G_i be topological groups. Let $G = G_1 \times \ldots \times G_n$ act isometricly and continuously on a CAT(0) space Y. If the action is not evanescent, there is a closed convex G-invariant subset $D \subset Y$ which is equivariantly isometric to a product of G_i-spaces.

Encompasses results of **Schroeder** (1985), **Bridson** and **Jost-Yau** (1999) for locally compact targets.

Proof of the splitting theorem

Let n=2.

- There is a minimal closed convex G_i-invariant set D_i (weak compactness of bounded convex sets + non evanescence).
- The union D of the minimal closed convex G_1-invariant sets splits isometricly as $D_1 \times D_2$.
- The G-action on $D_1 \times D_2$ is the product action.

Proof of superrigidity

The splitting theorem applied to $L^{[2]}(\Gamma\backslash G,h)$ yields a G-invariant subset $D = D_1 \times D_2$. Let f, f' $\in D_1$ be joined by a geodesic σ. The slopes of σ (with respect to the infinite product structure of $L^{[2]}(\Gamma\backslash G,h)$) are G_2-invariant functions. By irreducibility, they are constant. This means that $g \rightarrow d(f(g),f'(g))$ is constant. The lemma applies. □

Challenge. Analyze isometric actions of higher rank semi-simple Lie groups and prove superrigidity.

Commensurator rigidity

1. Harmonic maps

- Eells-Sampson and Ricci > 0
- Affine actions and vanishing theorems
- A simplicial vanishing theorem
- More general CAT(0) targets

2. Induction

- Irreducible lattices in products
- Commensurator superrigidity

Commensurators and arithmeticity

Definition. Γ subgroup of G. Comm(Γ,G) is the group of $g \in G$ such that $g^{-1}\Gamma g \cap \Gamma$ has finite index in Γ.

Example. $\Gamma = Sl(n,\mathbf{Z})$, $G = Sl(n,\mathbf{R}) \Rightarrow Comm(\Gamma,G) = Sl(n,\mathbf{Q})$.

Typical of arithmetic lattices.

Margulis. G semi-simple Lie group with trivial center and no compact factors, Γ irreducible lattice in G. Then Γ is arithmetic $\Leftrightarrow$ Comm(Γ,G)is dense in G $\Leftrightarrow$ Γ has infinite index in Comm(Γ,G).

Proof of commensurator arithmeticity criterion

Recall the proof of arithmeticity for superrigid lattices. Let h be a $\overline{\mathbf{Q}}$-point of Hom(Γ,Sl(n,$\mathbf{C}$)). After extension of scalars, get $\Gamma \hookrightarrow$ Sl(N,$\mathbf{Q}$).

Then Comm(Γ,G) $\subset$ Sl(N,$\mathbf{Q}$) for G= Sl(N,$\mathbf{R}$).

Apply superrigidity to

$$\Gamma \hookrightarrow \mathrm{Comm}(\Gamma,G) \hookrightarrow \mathrm{Sl}(N,\mathbf{Q}) \hookrightarrow \mathrm{Sl}(N,\mathbf{Q}_p).$$

Get $\Gamma' \subset$ Sl(N,$\mathbf{Z}$), for Γ' of finite index in Γ.

Observe that superrigidity is used only for homomorphisms $\Gamma \to$ H which are restrictions of homomorphisms defined on Comm(Γ,G).

Commensurator superrigidity

Margulis. G, H semi-simple, Γ lattice in G. G without compact factors. Let Λ be a dense subgroup of G contained in Comm(Γ,G). Then any homomorphism h : Λ → H such that h(Λ) is unbounded and Zariski dense extends to G → H.

Remark. Margulis gave an (unpublished) harmonic map proof of this.

Geometric commensurator superrigidity

Monod (2004). Γ uniform lattice in G locally compact, σ-compact. Let Λ be a dense subgroup of G contained in Comm(Γ,G). Suppose Λ acts isometricly on a complete CAT(0) space Y which is nonevanescent and unbounded on Γ. After restriction to a non empty Γ-invariant closed convex subset, the Γ-action extends to G.

Case of the automorphism group of a tree treated by **Lebeau** (Y is a tree) and **Burger-Mozes** (Y CAT(-1), (1996)).

Proof of commensurator rigidity (1/2)

Let $Y' = L^{[2]}(\Gamma\backslash G,h)$.

1. Averaging over finite subsets A of $\Gamma\backslash\Lambda$. For $f \in Y'$ and $a \in \Lambda$, let $f_a : g \to h(a)^{-1}f(ag)$. The set $\{f_a ; a \in A\}$ is finite and permuted by Γ. Therefore its barycenter $F_A(f) \in Y'$. The map $F_A : Y' \to Y'$ is G-equivariant.
2. Non evanescence implies that the F_A-orbits are bounded. The fixed-point set D_A of F_A is non empty, convex and G-invariant. If f, f' $\in D_A$, then $g \to d(f(g),f'(g))$ is Λ_A-invariant, Λ_A = group generated by ΓA.
3. D_A splits isometricly as $C \times T_A$ where C is a minimal closed convex G-invariant set.

Proof of commensurator rigidity (2/2)

4. Non evanescence implies that $\cup_A T_A$ is bounded. Weak compactness implies that the T_A have a non empty intersection. If f, f' $\in D = \cap_A D_A$, then $g \rightarrow d(f(g),f'(g))$ is Λ-invariant, thus constant, and the lemma applies.□

References (1)

N. A'Campo, M. Burger, *Réseaux arithmétiques et commensurateur d'après G.A. Margulis*, Invent. Math. **116** (1994), 1-25.

A. Borel, *On the curvature tensor of the hermitian symmetric manifolds*, Ann. Math. **71** (1960), 508-521.

A. Borel, *Cohomologie de certains groupes discrets et laplacien p-adique (d'après H. Garland),* Séminaire Bourbaki (1973-74), exposé 437, 12-35, Lecture Notes in Math. **431**, Springer, Berlin (1975).

M. Bridson, A. Haefliger, *Metric spaces of nonpositive curvature*, Grundlehren **319**, Springer, Berlin (1999).

M. Burger, S. Mozes, *Lattices in products of trees*, Publ. Math. IHES **92** (2000), 113-150.

M. Burger, S. Mozes, *CAT(-1) spaces, divergence groups and their commensurators*, J. Amer. Math. Soc. **9** (1996).

E. Calabi, E. Vesentini, *On compact locally symmetric Kähler manifolds,* Ann. Math. **71** (1960), 472-507.

K. Corlette, *Archimedean superrigidity and hyperbolic geometry*, Ann. Math. **135** (1992), 165-182.

References (2)

P. De la Harpe, A. Valette, *La propriété (T) de Kazhdan pour les groupes localement compacts*, Astérisque **175**, Soc. Math. de France (1989).

J. Eells, J. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **85** (1964), 109-160.

H. Garland, *p-adic curvature and the cohomology of discrete subgroups of p-adic groups*, Ann. Math. **97** (1973), 375-423.

M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory vol. 2 (Sussex, 1991), 1-295, London Math. Soc. Lecture Notes Ser. **182**, Cambridge Univ. Press, Cambridge, 1993.

M. Gromov, *Random walks on random groups*, GAFA **13** (2003), 73-148.

M. Gromov, P. Pansu, Rigidity of lattices : an introduction, Geometric topology : recent developments, Montecatini Terme 1990, 39-137, Lecture Notes **1504**, Springer, Berlin, 1991.

M. Gromov, R. Schoen, *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one*, Publ. Math. IHES, **76** (1992), 165-246.

H. Izeki, S. Nayatani, *Combinatorial harmonic maps and discrete-group actions on Hadamard spaces*, preprint (2004).

References (3)

E. Lebeau, *Applications harmoniques entre graphes finis et un théorème de superrigidité*, Ann. Inst. Fourier **46** (1996), 1183-1203.

J. Jost, S.T. Yau, *Harmonic maps and superrigidity*, Differential geometry, partial differential equations on manifolds, Proc. Symp. Pure Math**. 54-I** (1993), 245-280.

J. Jost, S.T. Yau, *Harmonic maps and rigidity theorems for spaces of nonpositive curvature*, Commun. Anal. Geom. **7** (1999), 681-694.

G.A. Margulis, *Arithmeticity and finite dimensional representations of uniform lattices*, Funkc. Anal. I Prilozh. **8** (1974), 258-259.

G.A. Margulis, *Discrete subgroups of semi-simple Lie groups*, Ergebn. der Math. **17**, Springer, Berlin (1990).

G.A. Margulis, *Superrigidity for commensurability subgroups and generalized harmonic maps*, preprint.

Y. Matsushima, *On the first Betti number of compact quotient spaces of higher dimensional symmetric spaces*, Ann. Math. **75** (1962), 312-330.

U. Mayer, *Gradient flows of nonpositively curved metric spaces and harmonic maps*, Commun. Anal. Geom. **6** (1998), 199-253.

N. Mok, Y.T. Siu, S.K. Yeung, *Geometric superrigidity*, Invent. Math. **113** (1993), 57-83.

Références (4)

N. Monod, *Superrigidity for irreducible lattices and geometric splitting*, preprint (2004).

P. Pansu, *Sous-groupes discrets des groupes de Lie : rigidité, arithméticité*, Sém. Bourbaki, exposé 778, (1993-94), 69-105, Astérisque **105**, Soc. Math. de France (1995).

P. Pansu, *Formules de Matsushima, de Garland et propriété (T) pour des groupes agissant sur des espaces symétriques ou des immeubles,* Bull. Soc. Math. De France **126** (1998), 107-139.

V. Schroeder, *A splitting theorem for spaces of nonpositive curvature*. Invent. Math. **79** (1985), 323-327.

W. Thurston, *Geometry and topology of 3-manifolds*, Lecture notes, Princeton (1979).

J. Tits, *Classification of algebraic groups*, Algebraic groups and discrete subgroups, Proc. Symp. Pure Math. Amer. Math. Soc. **9** (1966), 33-62.

M.T. Wang, *Generalized harmonic maps and representations of discrete groups*, Commun. Anal. Geom. **8** (2000), 545-563.

A. Zuk, *La propriété (T) de Kazhdan pour les groupes agissant sur les polyèdres*, C. R. Acad. Sci. Paris **323** (1996), 453-458.

A. Zuk, *Property (T) and Kazhdan constants for discrete groups*, GAFA **13** (2003), 643-670.