Chapter VIII

Compactness

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In this chapter, we will be concerned with the space of J-holomorphic maps of compact Riemann surfaces (S, J_S) into a fixed compact almost complex manifold (V, J). Even if an upper bound on area is imposed, this space is not compact in general.

Example. — The conics $x \mapsto (x, \varepsilon/x)$, $\mathbf{CP}^1 \to \mathbf{CP}^2$ all have the same area. As ε tends to 0, they converge smoothly, except at $x = 0$.

The graph of a dilation $x \mapsto (x, \varepsilon x)$, $\mathbf{CP}^1 \to \mathbf{CP}^1 \times \mathbf{CP}^1$ converges smoothly, as ε tends to 0, except at $x = \infty$.

In both cases, the images $f_\varepsilon(S)$ converge, as subsets in the range V, to a union of two holomorphic curves.

In his paper [3], M. Gromov states a compactness property for sets of holomorphic curves, which lies between the convergence of images (as subsets in the range) and the convergence of maps (parameter included): given a sequence $f_j : (S, J_S) \to (V, J)$ of holomorphic maps with bounded areas, there exists a subsequence that converges smoothly away from a finite set of points, at which "bubbles" develop.

This kind of result, which is classical in analytic geometry (E. Bishop's compactness theorem for analytic submanifolds in Kähler manifolds [1]), appeared more recently in analysis on manifolds. In this context, the bubbling off phenomenon was first discovered by J. Sacks and K. Uhlenbeck in their work on harmonic maps of a Riemann surface to a Riemannian manifold [6]. Since then, it has shown up in other variational problems where a noncompact symmetry group arises (see the report by J.P. Bourguignon [2]).

In these notes, which follow [3] closely, a proof of Gromov's compactness theorem for closed holomorphic curves is given. Holomorphic curves with boundary are covered only in an easy special case.

The first step in the proof is the compactness of "cusp-curves", i.e., convergence up to a change of parameter. In the second step, convergence of parametrised curves is obtained as a consequence of the convergence of graphs in $S \times V$.

There are other approaches to compactness theorems, due to T. Parker and J. Wolfson [5] and Rugang Ye [7].

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1. Riemann surfaces with nodes

Let us view a holomorphic curve in a fixed almost complex manifold V as the following set of data:

- an oriented differentiable 2-manifold S;
- a conformal or complex structure J_S on S;
- a holomorphic map $f : S \to V$.

In a degenerating family of holomorphic curves, it may happen that the complex structures J_S themselves degenerate. Thus a first step in understanding holomorphic curves is to understand noncompactness in the moduli space of complex structures on surfaces. This is a classical subject, for which a good reference is [4].

As we have seen in the above examples, topology can change in a degenerating family of holomorphic curves. It turns out a similar phenomenon is already present in the moduli space of complex structures on a surface. It is easy to visualise in terms of metrics of constant negative curvature.

1.1. Compact surfaces with constant curvature

Let S be a compact, orientable surface of genus greater than 1. In 2 dimensions, Riemannian metrics and complex structures are interrelated objects. Given a Riemannian metric and an orientation, rotation by 90^0 in each tangent space is well-defined and defines an almost-complex structure, which is automatically integrable (see chapter II for references). Two Riemannian metrics define the same complex structure if and only if they are conformal. Conversely, in a conformal class of Riemannian metrics, the curvature -1 condition singles out a unique metric. This is the content of the *uniformisation theorem*.

As a consequence, there is a 1-1 correspondence between complex structures on S and Riemannian metrics on S with curvature -1 (this correspondence breaks down for spheres and tori).

Noncompactness in the space of metrics with curvature -1 is easy to understand. The key notion is that of *injectivity radius* (see chapter III). Locally, any metric of curvature -1 is isometric to the hyperbolic plane (i.e., the disc with its hyperbolic metric). The injectivity radius at a point p is the largest r such that the geodesic disc centred at p is isometric to a geodesic disc of the hyperbolic plane. The injectivity radius of the Riemannian surface is the infimum of the injectivity radii of points of S.

PROPOSITION 1.1.1. — *For every $\varepsilon > 0$, the space of Riemannian metrics on S with injectivity radius greater than ε is compact.*

Indeed, the Gauss-Bonnet formula implies that all metrics with curvature -1 on S have the same area. This means that S can be covered by a number of hyperbolic discs of radius ε that depends only on ε. The gluing data then vary in a compact set. □

Here, S_j converges to S means that there exist diffeomorphisms $\varphi_j : S_j \to S$ which pull back the metric of S_j to metrics on S which converge uniformly in C^∞-topology.

1.2. The modulus of an annulus

It is useful to translate the compactness criterion of proposition 1.1.1 into purely conformal terms. A small injectivity radius means that there exists a short closed geodesic γ, which is not null homotopic. Then γ has a large embedded tubular neighbourhood. In an orientable surface, such a neighbourhood is an incompressible annulus with large modulus.

DEFINITION 1.2.1. — *Let A be an annulus equipped with a smooth complex structure. Then A is either conformal to the punctured Euclidean plane, the punctured disc or a unique cylinder $S^1 \times [0, L]$ where the S^1 factor has unit length. The number L is called the* modulus *of A. Here is an alternative definition.*

$$\frac{1}{L} = \inf \int_A |du|^2 \tag{1.2.2}$$

over smooth functions u on A which take the value 0 on one boundary component and 1 on the other.

Conversely, if a compact Riemann surface S contains an incompressible annulus A with large modulus, then the conformal metric with curvature -1 has a small injectivity radius. A compactness criterion for sets of compact Riemann surfaces follows, where the following topology is used: the sequence (S_j, J_{S_j}) converges to (S, J_S) if there exist diffeomorphisms $\varphi_j : S \to S_j$ such that the pushed forward complex structures $(\varphi_j)_*(J_{S_j})$ on S converge uniformly and smoothly to J_S.

PROPOSITION 1.2.3. — *Fix a number L. Consider all compact Riemann surfaces S of fixed genus greater than 1, with the following property: the modulus of every incompressible annulus in S is less than L. This space is compact.*

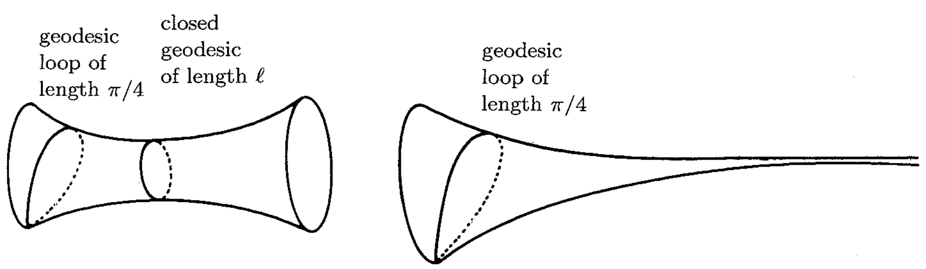
1.3. Degeneration

Given a complete Riemannian surface S with curvature -1, and $\varepsilon > 0$, the ε-*thick* part of S is the set of points where the injectivity radius is greater than ε. Note that the number of connected components of the ε-thick part of S is bounded in terms of the genus and ε.

DEFINITION 1.3.1. — *A sequence of complete surfaces S_j with curvature -1 converges to S if for every $\varepsilon > 0$, the ε-thick part of S_j converges to the ε-thick part of S.*

An obvious extension of Proposition 1.1.1 says that every sequence of complete surfaces with curvature -1 and uniformly bounded area has a convergent subsequence (in the $\mathcal{C}^\infty$-topology). The next point is to describe how area and topology can change in the limit.

The part of a compact surface with curvature -1 where the injectivity radius is small is also easy to describe. The model is the quotient of the hyperbolic plane by an isometry which translates a geodesic γ by a distance $\ell < \pi/4$. In this surface, the injectivity radius is an increasing function of the distance from γ. Let us *denote* by $A(\ell)$ the $\pi/8$-thin part of this surface, i.e., the set of points where the injectivity radius is less than $\pi/8$. When ℓ tends to 0, the annulus $A(\ell)$ splits into two isometric parts called *standard cusps*, and denoted by C. The standard cusp C is a complete Riemannian surface with boundary, conformal to a punctured disc. It can be viewed as the quotient of a *horodisc*, an open subset in the hyperbolic plane, by the parabolic rotation which translates boundary points by a distance $\pi/8$. Observe that the area of the ε-thin part of $A(\ell)$ tends to 0 as ε tends to 0 uniformly in ℓ. As a consequence, in the convergence $A(\ell) \to C \cup C$, the areas converge. Furthermore, as ℓ varies from 0 to $\pi/8$, the area of $A(\ell)$ varies between two positive constants.



the annulus $A(\ell)$ the cusp C

Figure 10

The next proposition describes the decomposition of the compact oriented constant curvature surface S into thick and thin parts.

PROPOSITION 1.3.2 (Thick and thin decomposition). — *The set of points in S where the injectivity radius is less than $\pi/8$ is a disjoint union of annuli, each of them being isometric to one of the $A(\ell)$.*

Thus, if a sequence S_j of surfaces with curvature -1 degenerates but converges to S in the sense of definition 1.3.1, a finite (bounded) number of disjoint closed geodesics have lengths that tend to zero, and S is diffeomorphic to the complement of the union of these curves. Furthermore, S contains two isometrically embedded

copies of the standard cusp C for each removed curve. As a consequence, S is conformal to a disjoint union Σ of compact Riemann surfaces with points removed (exactly 2 for each removed curve). The total genus of these surfaces is no more than the genus of the S_j. For j large enough, S_j is obtained from Σ by performing a series of connected sums, thus to reconstruct the manifold S_j, its is sufficient to remember which pairs of punctures have to be glued together.

It should be clear by now that a description of degeneration in families of Riemannian surfaces with curvature -1 should include disconnected or non compact surfaces with finite area as well.

1.4. Noncompact surfaces with constant curvature

In general, a complete Riemannian surface with curvature -1 and finite area contains finitely many disjoint isometric copies of the standard cusp, whose union has a compact complement. As a consequence, it is conformal to the complement of a finite subset in a compact surface equipped with a smooth complex structure. Conversely, given a compact Riemann surface Σ and a finite subset $F \subset \Sigma$, the uniformisation theorem applies provided the Euler characteristic of $\Sigma \setminus F$ is negative: there exists a unique complete conformal metric on $\Sigma \setminus F$ with curvature -1 and finite area.

As a consequence, given a compact oriented surface Σ, there is a 1-1 correspondence between the data of a complex structure on Σ together with a finite subset $F \subset \Sigma$ and complete Riemannian metrics on $\Sigma \setminus F$ with curvature -1 (this correspondence now includes spheres with at least three punctures and tori with at least one puncture).

Let us call a *Riemann surface with nodes* the data of a disjoint union of compact surfaces equipped with complex structures, a finite set of marked points called *nodes*, and an equivalence relation identifying certain marked points in pairs. Its genus is defined to be the sum of the genus of all components.

There is a natural topology on the set of Riemann surfaces with nodes. One says that Σ_j converges to Σ if Σ_j is almost obtained from Σ by a connected sum at pairs of identified marked points. This means that there is a disjoint collection A_j of annuli in Σ_j of large modulus, a map φ_j from Σ_j to Σ which is a diffeomorphism away from A_j and collapses each annulus in A_j to a node of Σ, such that the pushed forward complex structures $(\varphi_j)_* J_{\Sigma_j}$ converge on compact subsets of $\Sigma \setminus \{\text{marked points}\}$ to J_Σ.

By construction, uniformisation yields a homeomorphism between the space of Riemann surfaces with nodes and the space of Riemannian surfaces with nodes with curvature -1 and finite area.

PROPOSITION 1.4.1. — *The space of Riemann surfaces with nodes of genus less than g is compact*

2. Cusp-curves

2.1. Definitions

DEFINITION 2.1.1. — *A* cusp-curve *in an almost complex manifold* (V, J) *is a* J*-holomorphic map from a Riemann surface with nodes to* V*, i.e., the data of a disjoint union of compact surfaces* Σ_ℓ *equipped with complex structures, a finite set of nodes, and an equivalence relation identifying certain nodes in pairs, together with a* J*-holomorphic map* $f : \cup_\ell \Sigma_\ell \to V$ *compatible with the identifications.*

DEFINITION 2.1.2 (C^k-topology on cusp-curves). — *Let* $f : \cup_\ell \Sigma_\ell \to V$ *be a cusp-curve. Given* $\varepsilon > 0$*, a Hermitian metric* ν *on* Σ_ℓ *and a neighbourhood* U *of the nodes, a neighbourhood of the cusp-curve* f *is defined as follows. It is the set of cusp-curves* $\tilde f : \cup_\ell \tilde\Sigma_\ell \to V$ *such that there exists a continuous map*

$$\sigma : \cup_\ell \tilde\Sigma_\ell \to \cup_\ell \Sigma_\ell$$

with the following properties: σ *is a diffeomorphism except above the nodes; the pull-back of a node is an annulus of modulus* $\geq 1/\varepsilon$ *or a node;*

$$\left\| f - \tilde f \circ \sigma^{-1} \right\|_{C^k} < \varepsilon$$

away from U*, where the metric* ν *on* Σ_ℓ *and a fixed metric* μ *on* V *are used when measuring norms;*

$$\left\| J_\Sigma - \sigma_*^{-1} \tilde J_{\tilde\Sigma} \right\|_{C^k} < \varepsilon$$

away from U*, and*

$$\left| \operatorname{area}(f) - \operatorname{area}(\tilde f) \right| < \varepsilon.$$

By construction, the examples in the introduction illustrate the convergence of a sequence of holomorphic spheres to the join of two holomorphic spheres.

Figure 11

Since constant maps are holomorphic, the number of distinct curves in the image $f(\Sigma_\ell) \subset V$ can decrease in the limit. The above topology is non Hausdorff, but this should not be taken too seriously. It is the space of non parametrised curves which is Hausdorff.

Cusp-curves with boundary. — Let T be a compact complex manifold with boundary of dimension 1 (i.e., it has an atlas of holomorphic charts onto open subsets of $\mathbf{C}$ or of a closed half plane). Its *double* is a compact Riemann surface S with a natural antiholomorphic involution τ which exchanges T and $S \setminus T$ while fixing the boundary ∂T. If $f : T \to V$ is a continuous map, holomorphic in the interior of T, it is convenient to extend f to S by

$$f = f \circ \tau.$$

DEFINITION 2.1.3. — *A* cusp-curve with boundary *in* (V, J)*, is the data of finitely many compact Riemann surfaces* S_ℓ *obtained by doubling surfaces with boundary* T_ℓ *and a continuous map* $f : \cup_\ell S_\ell \to V$*, holomorphic in the interior of each* T_ℓ*, and such that* $f \circ \tau = f$*. A finite set of "nodes" is given, together with identifications in pairs compatible with* τ *and* f*.*

The topology is the same as for closed surfaces. As in the case of closed surfaces, in a convergent sequence of cusp-curves, a finite number of simple closed curves, but also, of simple arcs with endpoints on ∂T_ℓ, may collapse to a node.

2.2. Compactness theorems

THEOREM 2.2.1 (Compactness for closed cusp-curves). — *Let* V *be a closed Riemannian manifold. Let* J_j *be a convergent (in* $C^{k+\alpha}$*) sequence of almost complex structures on* V*, and* $f_j : S \to V$ *a sequence of* J_j*-holomorphic curves with bounded areas. There exists a subsequence which converges (in* $C^{k+1+\alpha}$*) to a cusp-curve* $f : \cup_\ell \Sigma_\ell \to V$*, where, topologically,* $\cup_\ell \Sigma_\ell$ *is obtained from* S *by collapsing a finite number of disjoint simple closed curves. In particular, the genus cannot increase in the limit*

$$\sum_\ell g(\Sigma_\ell) \leq g(S).$$

THEOREM 2.2.2 (Compactness for cusp-curves with boundary). — *Let* V *be a closed Riemannian manifold,* W *a real analytic submanifold of* V*. Let* J_j *be a convergent (in* $C^{k+\alpha}$*) sequence of almost complex structures on* V *which are integrable in a neighbourhood of* W*, and for which* W *is totally real. For every sequence* $f_j : (T, \partial T) \to (V, W)$ *of* J_j*-holomorphic curves with boundary in* W*, and bounded areas, has a* $C^{k+1+\alpha}$ *convergent subsequence.*

3. Proof of the compactness theorem 2.2.1

3.1. Scheme of the proof

Constant curvature -1 metrics μ_j on S will be chosen so that the maps f_j become uniformly Lipschitz. Gromov's Schwarz Lemma provides us with a Lipschitz bound for maps of large hyperbolic discs immersed in S into small balls of V. By removing a controlled number of points on S and choosing for μ_j the conformal metric with cusps at these points, one can ensure that each (non removed) point of S is the centre of such an immersed disc. One can then choose convergent subsequences of metrics and maps. In the limit, a holomorphic map of finite area of a punctured surface is obtained. It needs be extended across the punctures.

The main tools, Gromov's Schwarz lemma and the removable singularity theorem, are needed in the following form (see chapter VII).

PROPOSITION 3.1.1. — *Let (V,μ) be a compact Riemannian manifold. Let J be an almost complex structure of class $C^{k+\alpha}$ on V. There exist constants ε_0 (depending only on the C^0 norm of μ and on the C^α norm of J) and C (depending only on the C^0 norm of μ and on the $C^{k+\alpha}$ norm of J) such that every J-holomorphic map of the unit disc to an ε_0-ball of V has its derivatives up to order $k+1+\alpha$ near the origin bounded by C.*

PROPOSITION 3.1.2. — *Let (V,μ) be a compact Riemannian manifold. Let J be an almost complex structure of class $C^{1+\alpha}$ on V. Every J-holomorphic map $f : D^* \to V$ of finite area extends to a J-holomorphic map of the disc to V.*

A coarse form of monotonicity will also be needed. The constant ε_0 in this statement is the one in proposition 3.1.1.

PROPOSITION 3.1.3. — *Let (V,μ) be a compact Riemannian manifold. Let J be an almost complex structure of class C^α on V. Let $x \in V$. If S is a J-holomorphic curve in V with boundary contained in the sphere $\partial B(x,\varepsilon_0)$, then*

$$\text{area}\,(S) \geq \varepsilon_0^2.$$

3.2. Choice of metric on the domain

Let $f : S \to V$ be a holomorphic map. In this paragraph, a metric with curvature -1 will be constructed on S (perhaps with finitely many points removed) in such a way that the image by f of every unit geodesic disc in this metric has diameter less than ε_0, the constant that enters the Schwarz lemma 3.1.1.

Removal of a net. — In V, let us choose a maximal system of disjoint ε_0-balls with centres on $f(S)$. Let F be the set of centres. According to the monotonicity property 3.1.3, $f(S)$ leaves a definite quantum of area inside each of these balls, thus the number of points in F is bounded by a constant N that depends only on the area and ε_0.

Since F is maximal, every point of $f(S)$ lies at a distance at most $2\varepsilon_0$ from a point of F. Next we want to bound the diameter of discs contained in $f(S) \setminus F$. The trick is to construct annuli with large modulus.

Annuli. — Let A be an annulus contained in $f(S) \setminus F$, whose boundary components have length at most ε_0. Then the diameter of A is less than $12\varepsilon_0$. Indeed, every point of A lies at a distance at most $2\varepsilon_0$ of the boundary (by construction of F), and, for the same reason, the boundary components lie at most $4\varepsilon_0$ apart.

If a Riemannian annulus has a large modulus, then a slightly smaller isotopic annulus has a short boundary. Indeed, let $\Phi : S^1 \times [0,L] \to A$ be a conformal mapping. Denote by

$$S_t^1 = \Phi(S^1 \times \{t\}).$$

Then

$$\begin{aligned} \text{area}\,(A) &\geq \int_0^L \left(\int_{S^1\times\{t\}} |\Phi'|^2 \right) dt \\ &\geq \int_0^L \left(\int_{S^1\times\{t\}} |\Phi'| \right)^2 dt \\ &\geq \int_0^L \text{length}\,(S_t^1)^2\, dt. \end{aligned}$$

Thus there exists a t such that

$$\text{length}\,(S_t^1) \leq (\text{area}\,(A)/L)^{1/2}.$$

Splitting A into three *adjacent* annuli (i.e., they share boundary components like plumbing fixtures), one finds short curves in the extreme annuli. We sum up the discussion in a

LEMMA 3.2.1. — *Let A_0, A, A_1 be adjacent annuli in $S \setminus f^{-1}(F)$. If the moduli of A_0 et A_1 are larger than L, a constant which depends only on the area of $f(S)$ and on ε_0, then*

$$\text{diameter}\, f(A) \leq 12\varepsilon_0.$$

Bound on the diameter of discs in the μ^*-metric. — We choose to give $S \setminus f^{-1}(F)$ the unique complete conformal metric μ^* with curvature -1.

LEMMA 3.2.2. — *There exists a constant ρ depending only on ε_0 and on the area of $f(S)$, such that every geodesic disc of radius ρ in the metric μ^* is contained in an annulus A admitting adjacent annuli with moduli greater than L as in lemma 3.2.1.*

Proof. — Fix $r = \min\{\pi/8, \exp(-2L)\}$ and set $\rho = r^2$. The thick and thin decomposition of surfaces with constant curvature, (proposition 1.1.1 and § 1.4) tells us what a geodesic disc of radius r looks like. Either it is isometric to a hyperbolic geodesic disc (when the injectivity radius is larger than r, the thick case), or it is contained in a tube around a closed geodesic $A(\ell)$ or in a standard cusp C (when the injectivity radius is less than r, the thin case).

- The thick case: Let x be a point of $S \setminus f^{-1}(F)$ where the injectivity radius is larger than $\pi/8$. Choose a point $y \in \partial B(x, 2r)$. The shells

$$\begin{aligned} A_0 &= B(y, r^2) - y \\ A &= B(y, 3r^2) - B(y, r^2) \\ A_1 &= B(y, r) - B(y, 3r^2) \end{aligned}$$

are topological annuli. Also $B(x,\rho) \subset A$, modulus $(A_0) = +\infty$ and

$$\text{modulus}\,(A_1) \approx \frac{1}{2}\log\frac{1}{r} \geq L$$

as required.

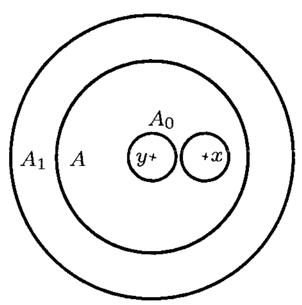


Figure 12

- The thin case: Then x is either close to a closed geodesic or in a cusp.

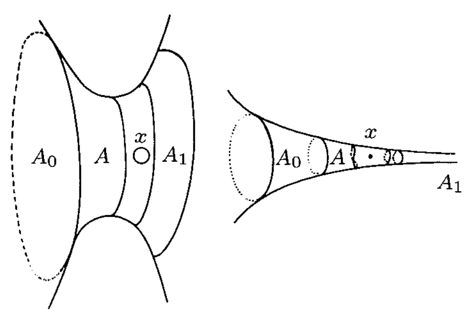


Figure 13

In both cases, the function

$$u = d(\cdot, \text{geodesic}) - d(x, \text{geodesic})$$

or

$$u = d(\cdot, \text{cusp}) - d(x, \text{cusp})$$

is smooth on $B(x, \pi/8)$, and $|du| = 1$. The sets

$$\begin{aligned} A_0 &= u^{-1}[-\pi/8, -r^2] \\ A &= u^{-1}[-r^2, +r^2] \supset B(x,\rho) \\ A_1 &= u^{-1}[+r^2, \pi/8]. \end{aligned}$$

are topological annuli. Using formula (1.2.2) one gets

$$\text{modulus}\,(A_i) \geq \frac{(\pi/8)^2}{r\,\text{sh}\,(\pi/8)}$$

of the order of $1/r$, which is much larger than L. □

Conclusion. — For all $x \in S \setminus f^{-1}(F)$, one has

$$|f'|^{\mu}_{\mu^*} \leq \text{const}\,(V, \mu, \|J\|_{C^\alpha}\,,\ \text{area})\,.$$

Indeed, if Φ denotes the conformal immersion of the unit Euclidean disc onto the μ^* geodesic ball $B(x,\rho)$, then

$$|f'|^{\mu}_{\mu^*}(x) \leq \text{const} \cdot |(f \circ \Phi)'(0)|^{\mu}_{m_0}$$

where the constant only depends on the radius ρ. Since the image of $f \circ \Phi$ has a small diameter, the Schwarz lemma 3.1.1 applies and a bound on the derivatives of $f \circ \Phi$ follows.

Notice that this bound does not depend on the injectivity radius of the metric μ^*.

3.3. End of proof

Convergence of metrics. — Again, we are given a sequence $f_j : S_j \to (V, \mu)$ of holomorphic curves with bounded area and genus. On each of them, a finite set F_j is removed and a metric μ_j^* is chosen in order that the maps f_j are uniformly Lipschitz.

Since the genus of S_j and the number of points in F_j are bounded, the area of μ_j^* stays bounded, and the compactness criterion 1.3.1 for surfaces with curvature -1 applies. Up to taking a subsequence, the metrics μ_j^* can be viewed as smoothly convergent metrics on larger and larger subsets of a fixed surface Σ which is diffeomorphic to the complement of finitely many "nodes" in a compact surface $\bar{\Sigma}$. The maps f_j are then uniformly Lipschitz maps on these subsets. A subsequence converges uniformly to a holomorphic map f defined on Σ. On compact subsets of Σ, the convergence is as smooth as the almost complex structure on V, thanks to the estimate on higher derivatives included in the Schwarz lemma and to elliptic regularity.

Convergence of areas. — According to 1.3.1, the metrics $\mu_j^\star$ converge smoothly away from sets of smaller and smaller area. Since the f_j are uniformly Lipschitz, their Jacobians are bounded and so

$$\text{area}\,(f(\Sigma)) = \lim_{j\to\infty} \text{area}(f_j(S_j)).$$

Bubbles intersect. — According to the removable singularity theorem 3.1.2, the map f extends across the nodes. It remains to prove that f is compatible with the identifications, i.e., it takes equal values at nodes that arise from the degeneration of the same annulus $A(\ell)$. As in the proof of the removable singularity theorem, one shows that if A_ε is the ε-thin part of $A(\ell)$, then the diameter of $f(A_\varepsilon)$ tends to 0 with ε. If not, the holomorphic annulus $f(A_\varepsilon)$ which has a short boundary and small area (Lipschitz estimate) would intersect a large ball, contradicting monotonicity. This ends the proof of theorem 2.2.1.

Remark. — The convergence of areas and monotonicity imply that the images $f_j(S_j)$ converge as subsets of V.

3.4. Compactness for cusp-curves with boundary

Under our strong assumption (the totally real data W is real analytic), only minor changes are needed to adapt the proof. The idea is that in the proof of the estimate for the first derivative of a conformal map

$$f : (S, \mu^\star) \to (V, \mu)$$

only *intrinsic* properties of the metric $\mu_{|f(S)}$ are used, like isoperimetric inequalities. In this respect, maps which are a mixture of holomorphic and anti-holomorphic are just as good as holomorphic maps.

LEMMA 3.4.1. — *Let W be a real analytic totally real submanifold in (V, J). Assume that J is integrable in a neighbourhood of W. Then there exists a unique anti-holomorphic involution τ_V defined in a neighbourhood of W whose fixed point set is W.*

It is sufficient to produce local holomorphic charts of V that take W to $\mathbf{R}^m \subset \mathbf{C}^m$. We can assume that $V = \mathbf{C}^m$. Write $\mathbf{C}^m = \mathbf{R}^m \oplus i\mathbf{R}^m$. Locally, W is the graph of a real analytic map φ of $\mathbf{R}^m$ to $\mathbf{R}^m$, i.e.,

$$W = \{x + i\varphi(x) \in \mathbf{C}^m;\ x \in \mathbf{R}^m\}.$$

The map φ extends holomorphically to a neighbourhood of $\mathbf{R}^m$ in $\mathbf{C}^m$. The map

$$z \mapsto z + i\varphi(z)$$

is a local diffeomorphism, as follows from the assumption $T_wW \cap iT_wW = \emptyset$. □

LEMMA 3.4.2. — *Assume the triple (V, J, W) satisfies the hypotheses of the previous lemma. Choose on V a Hermitian metric μ which is invariant under the involution τ_V. Let $f : S \to V$ be a cusp-curve with boundary (remember this means that $f \circ \tau = f$) such that $f(\partial T) \subset W$. Then the pull-back metric $f^\star\mu$ sur S is smooth and satisfies the isoperimetric inequalities and monotonicity holds.*

Remark. — These isoperimetric inequalities are the standard ones involved in the statement of the Schwarz lemma in chapter VII.

The proof follows from the *Schwarz reflection principle.*

THEOREM 3.4.3 (Reflection principle). — *Let T be a Riemann surface with boundary, let τ be the natural antiholomorphic involution on the doubled surface S. Let $f : (T, \partial T) \to (V, W)$ be holomorphic. The formula*

$$\tilde f(x) = \tau_W \circ f \circ \tau(x)$$

defines a holomorphic extension of f to a neighbourhood of T in S.

The identity $f^\star\mu = \tilde f^\star\mu$ proves lemma 3.4.2. □

Choice of a metric on the domain. — Keep the construction τ-invariant. Arrange things so that F does not intersect W. Since the metric $\mu^\star$ is τ-invariant, the boundary ∂T is totally geodesic.

Convergence of metrics on the domain. — Observe that in the limit, the geodesics which collapse are of three types:

- simple closed curves;
- simple arcs joining two boundary points
- boundary components.

The presence of nodes on ∂T causes no extra difficulty.

Non compact ranges. — All of these results persist when, while W is compact, V has a *coercivity* property that prevents holomorphic curves with boundary on W from escaping to infinity. In the case when $W = \mathbf{C}^m$, monotonicity, which holds globally, implies coercivity.

4. Convergence of parametrised curves

Finally, we are concerned with maps $f_j : S \to V$ where the complex structure on S is fixed. In this case, the "bubbles", i.e., the extra components in a limiting cusp-curve, are always spheres. This remains true if the complex structure is allowed to vary in a compact subset of the moduli space of S.

4.1. Graphs

The trick is to apply cusp-curve compactness to the graphs $\mathrm{gr}(f_j) : S \to S \times V$ or more generally to sections of bundles $X \to S$.

THEOREM 4.1.1. — *Let X be a compact manifold and $\pi : X \to S$ a fibration over a Riemann surface S. Let J_j be a $C^{k+\alpha}$-convergent sequence of almost complex structures on X, such that for each j the fibration $\pi : (X, J_j) \to S$ is holomorphic. Let $f_j : S \to X$ be a sequence of sections with bounded area. Then there exist a finite subset $\Gamma \subset S$ and a J_∞-holomorphic section $f_\infty : S \to X$ such that*

(i) a subsequence of the f_j converges in $C^{k+1+\alpha}$ to f_∞ away from Γ;

(ii) if $\gamma \in \Gamma$ and $f_j(\gamma)$ do not converge to $f_\infty(\gamma)$, then the fiber X_γ contains a non trivial rational curve

$$\varphi_\gamma : S^2 \to X_\gamma$$

which passes through $f_\infty(\gamma)$;

(iii) for large j, the homotopy class of f_j in $[S, X]$ is the same as

$$f_\infty + \sum_\gamma \varphi_\gamma$$

(care has to be taken if S is not a sphere, but if S is a sphere, the formula holds in $\pi_2(X)$).

Proof. — Let $F : \cup_\ell \Sigma_\ell \to X$ be a limiting cusp-cuve for some subsequence of f_j. Let $\tilde{F} = \pi \circ F$. Since $\tilde{F}$ is holomorphic, the domain splits as

$$\cup_\ell \Sigma^\ell = \Sigma_1 \cup \Sigma_2$$

where $\tilde{F}$ is constant on each component of Σ_1, and $\tilde{F}$ is a ramified covering on each component of Σ_2. This implies that

$$\mathrm{genus}\,(\Sigma_2) \geq \mathrm{genus}\,(S).$$

According to theorem 2.2.1,

$$\mathrm{genus}\,(\Sigma_1 \cup \Sigma_2) \leq \mathrm{genus}\,(S).$$

Thus the surface Σ_1, whose genus is 0, is a union of spheres.

Let x be a point of S which is neither in $\Gamma = \tilde{F}(\Sigma_1)$ nor a branch point of $\tilde{F}$. One can assume that the $f_j(x)$ converge. As the $f_j(S) \cap X_x$ converge in the Hausdorff sense to $F(\cup_\ell \Sigma^\ell) \cap X_x$, one concludes that $F(\cup_\ell \Sigma^\ell)$ intersects X_x in a single point. As a consequence, the ramified covering $F : \Sigma_2 \to S$ is an isomorphism, and $F(\Sigma_2)$ is the image of a section f_∞. As the $f_j(S)$ Hausdorff converge to the image of a section, the maps f_j converge uniformly to f_∞ in a neighbourhood of x. Since all of the f_j send this neighbourhood into a small ball of X, the Schwarz lemma applies, and the convergence is in fact $C^{k+1+\alpha}$. □

4.2. The corresponding statement for curves with boundary

THEOREM 4.2.1. — *Let $\pi : X \to T$ be a fibration over a Riemann surface with boundary T. Let W be a submanifold of $\pi^{-1}(\partial T)$. Let J_j be a $C^{k+\alpha}$-convergent sequence of almost complex structures on X. Assume that for each j, $\pi : (X, J_j) \to S$ is holomophic, that W is totally real with respect to J_j, and that J_j is integrable in a neighbourhood of W. Let $f_j : (T, \partial T) \to (X, W)$ be a sequence of J_j-holomorphic sections of π, with bounded areas. Then there exists a finite subset Γ in T and a J_∞-holomorphic section $f_\infty : T \to X$ such that*

(i) a subsequence of the f_j $C^{k+1+\alpha}$-converges to f_∞ away from Γ;

(ii) if $\gamma \in \Gamma$ and $f_j(\gamma)$ does not converge to $f_\infty(\gamma)$, then the fiber X_γ contains a non trivial rational curve $\varphi_\gamma : S^2 \to X_\gamma$ (resp. a holomorphic disc $\varphi_\gamma : (D, \partial D) \to (X_\gamma, W_\gamma)$ if $\gamma \in \partial T$) which passes through $f_\infty(\gamma)$;

(iii) for large j, the homotopy class of f_j in $[(T, \partial T, (X, W))]$ is the same as

$$f_\infty + \sum_\gamma \varphi_\gamma$$

(care has to be taken if T is not a disc, but if T is a disc, the formula holds in $\pi_2(X, W)$).

4.3. Simple homotopy classes

Under certain extra assumptions which forbid bubbles, a compactness theorem for parametrised curves will follow. Here are two typical assumptions.

- There are no nontrivial rational curves in the fibres. This appears in the proof that a compact embedded Lagrangian submanifold in $\mathbf{C}^m$ cannot be exact (one argues by contradiction).
- The homotopy class of f_j in $[S, X]$ is simple and, in the case when $S = S^2$, normalise the parametrisation in three points.

DEFINITION 4.3.1. — *Let J be an almost complex structure on V. A homotopy class*

$$\beta \in \pi_2(V)$$

is J-simple if in any decomposition $\beta = \sum_j f_j$ where $f_j : S^2 \to V$ is J-holomorphic, at most one of the f_j is nonconstant.

Example. — Let $V = S^2 \times V_1$, equipped with an almost complex structure J tamed by $\omega = \omega_1 \oplus \omega_2$. Let β be the homotopy class of the first factor. Assume that, for every J-holomorphic curve c in V_1

$$\int_c \omega_2 \quad \text{is an integral multiple of} \quad \int_\beta \omega_1.$$

then the class β is J-simple.

Corollary 4.3.2. — *Let $\beta \in \pi_2(V)$ be a J-simple class. Fix three distinct points s_1, s_2, s_3 in S^2. For all A, all $\delta > 0$, there exists an $\varepsilon > 0$ such that the set of J-holomorphic maps $f : S^2 \to V$ which satisfy*

$$\begin{aligned} \operatorname{area}(f) &\leq A, \\ \operatorname{dist}(f(s_j), f(s_k)) &\geq \delta, \\ \left\|\overline{\partial} f\right\|_{C^\alpha} &\leq \varepsilon \end{aligned}$$

is compact.

The operator $\overline{\partial} f$ is explained in chapter V. It takes its values in the space of sections of a certain bundle $\overline{X}$. Every such section g determines an almost complex structure J_g on $X = S^2 \times V$, and the equation $\overline{\partial} f = g$ means that the graph of f is J_g-holomorphic. Furthermore, as g tends to 0, the almost complex structure J_g tends to the product structure on X.

Proof. — We prove the corollary by contradiction. Let f_j be a noncompact sequence of J_{g_j}-holomorphic curves with bounded areas, which satisfy the normalisation condition in the statement, and where the g_j tends to 0. Let $F : \cup_\ell \Sigma^\ell \to X$ be a limiting cusp-curve of the graphs of the f_j (it is holomorphic with respect to the product almost complex structure on X). Since β is J-simple, at most one of the maps

$$\Sigma^\ell \xrightarrow{F} X \longrightarrow V$$

is nonconstant. If the limiting section f_∞ is non constant in V, then there are no bubbles and compactness holds. Otherwise f_∞ is constant and there is exactly one bubble over some point $\gamma \in S^2$. Away from γ, the f_j converge uniformly to a constant map but this contradicts the assumption $\operatorname{dist}(f(s_j), f(s_k)) \geq \delta$. □

Example. — Fix three disjoint submanifolds Σ_1, Σ_2, Σ_3 in $S^2 \times V$ and require that $f(s_i) \in \Sigma_i$. In this case the conclusion is that compactness holds for J_g-holomorphic curves with g small enough (this is theorem 2.3.C of [3]).

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