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DIFFERENTIABILITY OF VOLUME GROWTH [1]

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to the memory of Franco Tricerri

1 Introduction

Let (M,g) be a Riemannian manifold, and x a point in M. Let $B(x,r)$ denote the ball of radius r and center x. The *volume growth* is the function v on $\mathbf{R}$ defined by

$$v(r)=\begin{cases}0 \text{ if } r\leq 0;\\ \mathrm{Vol}\,B(x,r) \text{ otherwise.}\end{cases}$$

Which functions on $\mathbf{R}$ can be the volume growth of some metric on M ? It follows from [GP1] that any smooth function on $]0,+\infty[$ with positive derivative can be approximated (in the fine C^1 topology) by a volume growth. Therefore, the restrictions on v are only local.

In this paper, we are concerned with the differentiability of v when M has dimension 2. After a description of the generic picture, see Theorem 1, we prove that if the metric is real analytic, then the volume growth is semianalytic, see Theorem 2. Finally, we survey results from [GP2] and [GP3] giving lower bounds on the singularities for an arbitrary metric.

Section 2 contains precise statements. Proofs are to be found in sections 3 to 7. We need the fact that genericly the energy functional on the space of based loops has nondegenerate critical points. A proof of this variant of R. Abraham's Bumpy Metrics Theorem, [A], is included in an appendix.

2 Results

2.1 The generic picture

For a generic smooth metric on a 2-dimensional manifold, the cutlocus of a point x is a smoothly embedded graph. Each point $y\in Cutlocus(x)$ is joined to x by as many minimizing geodesics as there are arcs of $Cutlocus(x)$ emanating from y, never more than 3. In particular, an endpoint y of $Cutlocus(x)$ is joined to x by a unique minimizing geodesic along which it is conjugate to x.

The distance to x restricted to $Cutlocus(x)$ has three types of singular values

1. the set R_1 of distances to x of endpoints;
2. the set R_2 of critical values of the distance to x restricted to the interiors of edges of $Cutlocus(x)$;
3. the set R_3 of distances to x of branch points of $Cutlocus(x)$.

Generically, the sets R_1, R_2 and R_3 are disjoint and locally finite.

If $r>0$, the derivative $v'(r)$ is the length of the geodesic sphere $\partial B(x,r)$. The geodesic sphere is a subset of the wave front $W(r)$, image of the circle of radius r in T_xM by the

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exponential map. The length of the wave front is a smooth function of r, but parts of it brutally collapse sometimes.

If r_0 is not a critical value, then the geodesic sphere $\partial B(x, r_0)$ is smooth except at points of $Cutlocus(x)$, where it is transverse to $Cutlocus(x)$. Therefore v' is smooth in a neigborhood of r_0.

If $r_0 \in R_2$, two smooth branches of $W(r)$ touch quadraticly at some point $y \in Cutlocus(x)$. Either y is a local minimum of the distance to x restricted to the interior of an edge of $Cutlocus(x)$. In that case, the length of $\partial B(x,r)$ is smooth up to r_0 and decreases like $-\sqrt{r-r_0}$ for $r > r_0$. Or y is a local maximum, and $v'(r)$ decreases like $\sqrt{r_0 - r}$ for $r < r_0$, and is smooth for $r > r_0$.

If $r_0 \in R_3$, three smooth branches of the wave front $\partial B(x, r_0)$ meet transversally at some point $y \in Cutlocus(x)$. Near y, for $r < r_0$, $\partial B(x,r)$ is a tiny triangle that collapses to a point. Therefore the length of $\partial B(x,r)$ is smooth on both sides of r_0 but its right derivative is strictly larger than its left derivative.

If $r_0 \in R_1$, the wave front $W(r_0)$ has a singularity of the form $y^4 = x^3$. The length of $\partial B(x,r)$ is smooth up to r_0 and decreases like $-(r-r_0)^{3/2}$ for $r > r_0$.

This leads to the following theorem. We use the notation

$$r_+ = \begin{cases} 0 \text{ if } r \le 0; \\ r \text{ if } r > 0. \end{cases}$$

Theorem 1 *Let g be a generic smooth metric on a 2-dimensional manifold M. Then the volume growth function is smooth away from the locally finite set $\Sigma = \{0\} \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. In a neighborhood of a point r_0 of Σ, the derivative v' admits a Puiseux expansion of the form*

(o) $v'(r) = g(r) - h((r-r_0)_+)$ *if* $r_0 = 0$;

(i) $v'(r) = g(r) + (r-r_0)h((r-r_0)_+^{1/2})$ *if* $r_0 \in \Sigma_1$;

(ii) $v'(r) = g(r) + h((r-r_0)_+^{1/2})$ *or* $v'(r) = g(r) - h((r_0-r)_+^{1/2})$ *if* $r_0 \in \Sigma_2$;

(iii) $v'(r) = g(r) + h((r_0-r)_+)$ *if* $r_0 \in \Sigma_3$;

where g and h are smooth, $h(0) = 0$ and $h'(0) < 0$.

2.2 The real analytic case

Definition 1 *A set K in $\mathbf{R}^n$ is semianalytic if, locally in $\mathbf{R}^n$, it is a finite union of subsets K_i defined by finitely many analytic equations and inequations.*

Let I be an interval in $\mathbf{R}$. A function $\phi : I \to \mathbf{R}$ is semianalytic if its graph $\{(x, \phi(x)) \mid x \in I\}$ is a semianalytic subset of $\mathbf{R}^2$.

Here is a concrete characterization of semianalytic functions (see Corollary 3). A function ϕ on $\mathbf{R}_+$ is semianalytic if and only if there exists a sequence $0 \le x_0 < \cdots < x_k < \cdots$ such that

i) ϕ is analytic away from the x_k;

ii) on each side of x_k, ϕ admits a Puiseux expansion of the form $\phi(x) = h(|x - x_k|^{1/p})$ where p is an integer and h is analytic on a neighborhood of 0 in $\mathbf{R}$.

Theorem 2 *Let (M, g) be a real analytic complete Riemannian manifold of dimension 2. For each x in M, the volume of balls centered at x and its derivative are semianalytic functions of the radius.*

2.3 Results about arbitrary smooth metrics

For the sake of completeness, we quote two results concerning general smooth metrics. The idea is that the singularities of the volume growth observed in the generic case can get only worse for an arbitrary metric.

Theorem 3 ([GP2]) *Let (M, g) be a smooth 2-dimensional Riemannian manifold, and $x \in M$. Let ℓ be the injectivity radius at x. Let λ be the length of the shortest homotopicly nontrivial loop at x. Let L be the largest distance of a point of M to x. Then*

1. *at $\frac{\lambda}{2}$, the volume growth v is not of class $C^{1+\alpha}$ for any $\alpha > \frac{1}{2}$;*
2. *at 0 and at L, v is not of class C^2;*
3. *at ℓ, v is not of class $C^{2+\alpha}$ for any $\alpha > \frac{1}{2}$.*

Furthermore, the number of singular radii is bounded below by the topology of M. Fix a point $x \in M$ and a generic metric ds^2. Say that $y \in M$ is a *critical point* if either

(o) $y = x$ or

(i) $y \in Cutlocus(x)$ is a local minimum of the restriction of the distance to x to the interior of an edge of $Cutlocus(x)$;

(ii) $y \in Cutlocus(x)$ is a branch point of $Cutlocus(x)$ or a local maximum of the restriction of the distance to x to the interior of an edge of $Cutlocus(x)$.

In case (o) (resp. (i), resp (ii)), say that y has index 0 (resp. 1, resp. 2). It is easy to approximate the distance function by a smooth proper Morse function with the same critical points and the same indices. Morse inequalities then provide topological lower bounds for $\#R_2$ and $\#R_2 + \#R_3$.

For slightly less generic metrics, the distance to x can take equal values at different critical points, and now one gets weaker estimates. Let $S(M)$ (resp. $s(M)$) denote the minimum number of critical *values* (resp. of index 1) of a smooth proper function with nondegenerate critical points on M. For such less generic metrics, one merely gets $\#R_2 + \#R_3 \ge S(M)$ and $\#R_2 \ge s(M)$.

Here is a table plotting the values of S and s for 2-dimensional manifolds.

manifold	s	S
$\mathbf{R}^2$	0	1
S^2	0	2
non compact, finite type, $\ne \mathbf{R}^2$	1	2
compact, $\ne S^2$	1	3
infinite type	$+\infty$	$+\infty$

It turns out that $S(M)$ and $s(M)$ give sharp bounds for the number of singularities of the volume growth for *arbitrary* metrics, except for the real projective plane. Given a metric ds^2 on M ands a point $x \in M$, let $\Sigma'(g)$ denote the set of values of r such that v is not of class $C^{1+\alpha}$ for any $\alpha > \frac{1}{2}$ in any neighborhood of r. Let $\Sigma''(g)$ denote the set of values of r such that v is not of class C^2 in any neighborhood of r.

Theorem 4 **([GP3])** *Let M be a 2-dimensional manifold, and $x \in M$. For any metric ds^2 on M, one has*

$$\#\Sigma'(g) \geq S(M), \quad \#\Sigma''(g) \geq s(M)$$

if $M \neq \mathbf{R}P^2$. If $M = \mathbf{R}P^2$, then

$$\#\Sigma'(g) \geq 2, \quad \#\Sigma''(g) \geq 1$$

These estimates are sharp, equality is achieved by certain constant curvature metrics.

The metrics on $\mathbf{R}P^2$ such that $\#\Sigma'(g) = 2$ and $\#\Sigma''(g) = 1$ form a very small and explicite family. For such a metric, all geodesics through x come back to x after the same time λ, the volume growth is not C^1 at λ. All other metrics have $\#\Sigma'(g) \geq 3 = S(\mathbf{R}P^2)$ and $\#\Sigma''(g) \geq 1 = s(\mathbf{R}P^2)$ and this is again sharp.

3 Facts about semianalytic functions

We shall need a few standard properties of semianalytic functions.

Lemma 2 *Let $\phi : [0,1] \to \mathbf{R}$ be a semianalytic function. There exists a subdivision $0 = x_0 < x_1 < \cdots < x_k = 1$ of $[0,1]$ such that over each interval $[x_i, x_{i+1}]$, the graph of ϕ is contained in an embedded analytic arc.*

Proof. Let K denote the graph of ϕ. By assumption definition, near one of its points (x_0, y_0) the set K is a finite union of subsets K_i defined by finitely many equations $f_{i_j}(x,y) = 0$ and inequalities $g_{i_j}(x,y) < 0$ where the functions f_{i_j} and g_{i_j} are analytic in a neighborhood of (x_0, y_0). We can assume that the sets $\{f_{i_j} = 0\}$ et $\{g_{i_j} = 0\}$ contain (x_0, y_0) and are all distinct near (x_0, y_0). Then they have only finitely many intersection points. Therefore, we can assume that they are disjoint outside (x_0, y_0). As a consequence, each K_i has at most one equation. Since the projection $\pi : (x,y) \mapsto x$ is injective on K, each K_i is defined by exactly one equation f_i and a number of inequalities. Near (x_0, y_0), the analytic set $\{f_i = 0\}$ consists of finitely many arcs intersecting at (x_0, y_0). The inequalities $g_{i_j} < 0$ do not cut these arcs, thus they merely suppress certain arcs. As a consequence, the set K_i itself consists of finitely many arcs K_{i_ℓ} which meet in (x_0, y_0). The projection $\pi(K_{i_\ell})$ contains a half neighborhood of x_0. Since π is injective on K, only two cases can occur.

- either there is only one set K_i consisting of two arcs which project onto two opposite half neighborhoods of x_0 ;
- or there are exactly two sets K_i, each of them an arc which projects onto a half neighborhood of x_0.

In both cases, on some interval $]x_0 - \epsilon, x_0]$ (resp. on $[x_0, x_0 + \epsilon[$), ϕ parametrizes an analytic set. Only finitely many neighborhoods of the $]x_0 - \epsilon, x_0 + \epsilon[$ are needed to cover $[0,1]$. ∎

Proposition 3 *A function ϕ on $[0,1]$ is semianalytic if and only if there exists a sequence $0 = x_0 < \cdots < x_k = 1$ such that*

i) ϕ is analytic away from the x_k;

ii) on each side of x_k, ϕ admits a Puiseux expansion of the form $\phi(x) = h(|x - x_k|^{1/p})$ where p is an integer and h is analytic on a neighborhood of 0 in $\mathbf{R}$.

Indeed, if K is an analytic subset of $\mathbf{R}^2$ which at $(x_k, \phi(x_k))$ does not contain a vertical line segment, then K can be locally parametrized by $]-\epsilon, \epsilon[\to \mathbf{R}^2$, $t \mapsto (x_k + t^p, h(t))$ (see [L2], page 172). If K contains all points $(x, \phi(x))$ for $x \in x_k, x_k + \epsilon'$, then $\phi(x) = h(\pm|x - x_k|^{1/p})$ on that interval.

Conversely, if ϕ is piecewise analytic with Puiseux expansions, its graph is clearly semianalytic. ∎

Lemma 4 *Let $\phi : [0,1] \to \mathbf{R}$ be a nonconstant semianalytic function. Let Θ be an analytic function on $\mathbf{R}^2$. Then*

$$y \mapsto a(y) = \int_{\{x \mid \phi(x) > y\}} \Theta(x,y)\, dx$$

is semianalytic on $\mathbf{R}$.

Proof. According to Lemma 2, we can assume that the graph of ϕ is an analytic set. Near y_0 and on each side of it, the equation $\phi(x) = y$ has finitely many solutions $x = \alpha_j(y)$ which admit a Puiseux expansion ([L2] page 172)

$$\alpha_j(y) = h_j(|y - y_0|^{1/p_j})$$

where h_j is analytic on a neighborhood of 0 in $\mathbf{R}$. Clearly, α_j is semianalytic on each side of y_0.

Let us label the solutions $\psi_j(y)$ in increasing order, from $j = 1$ to $j = 2k$, unless $\phi(0) < y$, in which case one sets $\alpha_1(y) = 0$ or $\phi(1) < y$, in which case one sets $\alpha_{2k}(y) = 1$. Let $G(x,y) = \int_0^x \Theta(t,y)\, dt$. This is an analytic function. The expression

$$a(y) = \sum_{j=1}^{k} G(\alpha_{2j}(y), y) - G(\alpha_{2j-1}(y), y)$$

shows that a is semianalytic on each side of y_0. ∎

Lemma 5 *Let a be a differentiable function on an interval of $\mathbf{R}$. Assume its derivative a' is semianalytic. Then a is semianalytic.*

Proof. Near x_0, we integrate a Puiseux expansion

$$a'(x) = h(|x - x_0|^{1/p}).$$

Then

$$a(x) = a(x_0) + \int_{x_0}^{x} a'(z)\, dz = a(x_0) \pm \int_0^{|x-x_0|^{1/p}} pu^{p-1} h(u)\, du$$

is semianalytic on each side of x_0. ∎

4 Proof of theorem 2

We need a few facts from [GP1]. Let S^1 denote the unit circle in the tangent plane T_xM. For a direction $\theta \in S^1$, let

$$cut(\theta) \stackrel{\text{def}}{=} \sup\{r \mid \text{the geodesic } s \mapsto \exp(s\theta) \text{ is minimizing up to } r\}.$$

For a smooth metric, the function $r \to v(r) = \mathrm{Vol}\, B(x,r)$ is absolutely continuous. Its derivative v' is continuous away from a countable set. At such a point,

$$v'(r) = \int_{\{cut(\theta)>r\}} \Theta(\theta,r)d\theta$$

where $\Theta(\theta,r)$ is the density of the volume element in polar coordinates centered at x.

If the metric is real analytic, v' is continuous except in a very special case. If v' is discontinuous at r, then $cut(\theta) = r$ for all θ. In particular, there is at most one such r, which will not affect semianalyticity.

According to [B2], for a real analytic metric, the cutlocus $Cutlocus(x)$ in M is subanalytic (see [H]). It follows that the cutlocus in the tangent plane $K = \partial U$ where

$$U = \{(r,\theta) \mid r < cut(\theta)\}$$

is subanalytic as well. Indeed, by definition, subanalyticity is a local property, and images of subanalytic sets by proper analytic maps are subanalytic. For each $t > 0$, the subset $U_t = U \cap \{cut < t\}$ is open in K, and K_t is the projection on the first factor of $T_xM \times \mathbf{R}$ of the subanalytic set

$$U_t' = \{(r,\theta,s) \mid 0 \le r \le s < t \text{ and } \exp_x(s\theta) \in Cutlocus(x)\}.$$

This shows that U is subanalytic, and so is its boundary, [H].

A classical theorem of S. Lojasiewicz ([L1], see for example [KJZ]) asserts that every subanalytic set in the plane is semianalytic. Thus Lemma 4 applies and v' is semianalytic, then Lemma 5 applies and v is semianalytic. ∎

Remark 6 *Consider the following semialgebraic subset in* $\mathbf{R}^3$

$$A = \{(x,y,z) \in \mathbf{R}^3 \mid 0 \le x \le 1,\ y \ge 0,\ x^3 - y(x^2+z^2) \ge 0\}$$

and its horizontal slices $A_r = A \cap \{z = r\}$. *The function*

$$\mathrm{area}(A_r) = \frac{1}{2}(1 - r^2\log(1+r^2) + r^2\log(r^2))$$

is not semianalytic (I thank K. Kurdyka for this example). Therefore, it is likely that Theorem 2 does not extend to higher dimensional manifolds.

5 Facts about generic cut loci

Fix a 2-dimensional manifold M and a point x in M. According to M. Buchner [B1], [B3], there is a residual subset $\mathcal{M}$ of the space of smooth complete riemannian metrics on M such that for every metric in $\mathcal{M}$, the following holds.

The cutlocus $Cutlocus(x)$ is a smoothly embedded finite graph. Each vertex belongs either to 3 or to one edge. Vertices belonging to only one edge are called *endpoints*. Vertices belonging to 3 edges are called *triple points*. Interior points on edges are called *double points*.

Each endpoint is joined to x by a unique minimizing geodesic, along which it is conjugate to x. This geodesic is tangent and opposite to the cutlocus at y. The only points of the cutlocus which are conjugate to x along a minimizing geodesic are endpoints.

Each double point is joined to x by 2 minimizing geodesics which are transverse to the cutlocus.

Each triple point y is joined to x by 3 minimizing geodesics. These geodesics are neither tangent nor orthogonal to the cutlocus at y, and the pairwise angles they make at y are all less than π.

6 Differentiability of the *cut* function

In this section, we give sufficient conditions on a metric in order that the *cut* function be smooth on its domain of definition (an open subset of S^1) and have nondegenerate critical points, and only finitely of them many below any level. The Puiseux expansions described in theorem 1 then follow from Lemme 1 in [GP2].

The cutlocus in the tangent plane is the set

$$K = \{(r,\theta) \mid cut(\theta) = r\}.$$

It is part of the inverse image of the cutlocus $Cutlocus(x) \subset M$. Assume the metric is in $\mathcal{M}$. According to results in the previous section, S^1 splits into four subsets,

1. B_1 is the finite set consisting of inverse images of the endpoints of $Cutlocus(x)$;
2. B_2 is the inverse image of the interior of the 1-skeleton of $Cutlocus(x)$;
3. B_3 is the finite set consisting of inverse images of triple points of $Cutlocus(x)$;
4. B_∞ is the open subset where $cut = +\infty$.

6.1 Triple points

Lemma 7 *Assume that the Riemannian metric $g \in \mathcal{M}$. Let $\theta_0 \in B_3$. The cut function is smooth on both sides of θ_0, its left and right derivatives at θ_0 are nonzero.*

Indeed, since $\frac{\partial}{\partial r}\exp_x(cut(\theta_0),\theta_0)$ is transverse to the cutlocus, the 3 edges of $Cutlocus(x)$ pull back to 3 arcs transverse to $\frac{\partial}{\partial r}$ and pairwise transverse. Then *cut* is the minimum of 3 smooth functions which take equal values only at θ_0. Thus *cut* is piecewise smooth. The derivatives $cut'(\theta_0)$ are the cosines of certain angles of minimizing geodesics with arcs of the cutlocus, which were assumed not to vanish. ∎

6.2 Double points

If $\theta_0 \in B_2$, near $(cut(\theta_0),\theta_0)$, K consists of a smooth arc transverse to $\frac{\partial}{\partial r}$, thus *cut* is smooth near θ_0. Each critical point θ_0 of *cut* in B_2 corresponds to a geodesic loop of length $2cut(\theta_0)$. Therefore, for each t, the subset of critical points of cut in B_2 with $cut \leq t$ is compact.

There remains to make sure that critical points of cut in B_2 are nondegenerate.

Fix a critical point $\theta_0 \in B_2$. Let $r_0 = cut(\theta_0)$ and $y = \exp_x(r_0, \theta_0)$. Let $s \mapsto y(s)$ be the arclength parametrization of $Cutlocus(x)$ near y. By assumption, for s small, $y(s)$ is joined to x by exactly two minimizing geodesics $\gamma_{0,s}$ and $\gamma_{1,s}$ with initial speeds $\theta_{0,s}$ and $\theta_{1,s}$ and length $r(s)$. Since $y(s)$ is not conjugate to x along these geodesics, the map $s \mapsto \theta_{0,s}$ is a diffeomorphism at 0, and cut has a nondegenerate critical point at θ_0 if and only if r_0 has a nondegenerate critical point at 0.

Lemma 8 *Let y be a degenerate critical point of the restriction of the distance to x to an open arc of $Cutlocus(x)$. Then the two minimizing geodesics from y to x form a geodesic loop γ at x and x is conjugate to itself along γ.*

Proof. Consider the family of broken geodesics

$$\gamma_s(t) = \begin{cases} \gamma_{0,s}(tr(s)) \text{ if } t \in [0,1]; \\ \gamma_{1,s}(r(s)(2-t)) \text{ if } t \in [1,2]. \end{cases}$$

Its length is $2r(s)$. Denote as usual $T = \frac{\partial \gamma_s}{\partial t}$ and $V = \frac{\partial \gamma_s}{\partial s}$. At $t = r_0$, these vectorfields and their derivatives are possibly discontinuous, but have left and right limits $T(r_0^-)$ and $T(r_0^+)$, etc...

The vectorfield V is continuous since $V(r_0^-) = V(r_0^+) = y'(s)$. The first variation formula implies that

$$\langle V(r_0), T(r_0^+) - T(r_0^-) \rangle = 2r'(0) = 0$$

therefore $T(r_0^-) = T(r_0^+)$, thus γ_0 is a geodesic loop, smooth at r_0. Away from r_0, V is a Jacobi field. We next show that V is differentiable at r_0. This follows from the second variation formula, [GKM], section 4.

$$\begin{aligned} 2r''(0) &= \langle D_T V, V \rangle (r_0^-) - \langle D_T V, V \rangle (0) + \langle D_T V, V \rangle (2r_0) - \langle D_T V, V \rangle (r_0^+) \\ &+ \langle D_V V, T \rangle (r_0^-) - \langle D_V V, T \rangle (0) + \langle D_V V, T \rangle (2r_0) - \langle D_V V, T \rangle (r_0^+) \\ &= \langle D_T V, V \rangle (r_0^-) - \langle D_T V, V \rangle (r_0^+). \end{aligned}$$

Since $r''(0) = 0$ and $\langle D_T V, T \rangle (r_0^-) = \langle D_T V, T \rangle (r_0^+) = 0$, this implies that $D_T V(r_0^-) = D_T V(r_0^+)$. We conclude that V is a smooth nonzero Jacobi field along γ_0 which vanishes at endpoints. Therefore x is conjugate to itself along γ. ∎

Proposition 9 *There is a residual set $\mathcal{M}'$ of metrics for which x is a regular value of $\exp_x : T_x M \to M$. If a Riemannian metric $g \in \mathcal{M} \cap \mathcal{M}'$, then all critical points of cut in B_2 are nondegenerate.*

Proof. The first sentence is Theorem 5 of the appendix. For $g \in \mathcal{M} \cap \mathcal{M}'$, x is not conjugate to itself along geodesic loops, the critical points of the distance to x on edges of $Cutlocus(x)$ are nondegenerate, and points in B_2 are nondegenerate critical points of cut. ∎

6.3 Endpoints

In this paragraph, we fix a Riemannian metric g in $\mathcal{M}$ and an endpoint y of the cutlocus $Cutlocus(x)$ of x. We can write $y = \exp_x(r_0, \theta_0)$ where $\theta_0 \in B_1$. For calculations, we use Fermi coordinates (X, Y) around y. This means that $Y(r, \theta)$ is the (signed) distance of the point $\exp_x(r, \theta)$ to the minimal geodesic from x to y, and $X(r, \theta)$ is the arclength (with origin at y) of the orthogonal projection of $\exp_x(r, \theta)$ onto this geodesic. In particular, $X(r, \theta_0) = r - r_0$ and $Y(r, \theta_0) = 0$ for all r. In Fermi coordinates, the Riemannian metric reads

$$g = dY^2 + e(X, Y) dX^2$$

where $e(X, 0) = 1$ and $\frac{\partial e}{\partial Y}(X, 0) = 0$.

Lemma 10 *Assume that the Riemannian metric $g \in \mathcal{M}$. Let $y = \exp_x(r_0, \theta_0)$ be an endpoint of $Cutlocus(x)$. Then*

$$\frac{\partial \Theta}{\partial r}(r_0, \theta_0) \neq 0 \quad \textit{and} \quad \frac{\partial \Theta}{\partial \theta}(r_0, \theta_0) = 0.$$

Proof. The function $r \mapsto \Theta(r, \theta_0)$ is a nonzero solution of the Jacobi equation

$$\frac{\partial^2 \Theta}{\partial r^2} + R\Theta = 0.$$

This is a linear second order differential equation. Since $\Theta(r_0, \theta_0) = 0$ (y is a conjugate point), the derivative $\frac{\partial \Theta}{\partial r}(r_0, \theta_0) \neq 0$. As a consequence, the set $Z = \{(r, \theta) \mid \Theta(r, \theta) = 0\}$ is a submanifold near θ_0, the equation $\Theta(r, \theta) = 0$ has a unique smooth solution $r = conj(\theta)$.

We show that $conj$ achieves a local minimum at θ_0. Since $\frac{\partial X}{\partial r}(r, \theta_0) = 1$ for all r, then for θ close to θ_0, the function $r \mapsto X(r, \theta)$ is increasing on $[0, conj(\theta)]$. It is a classical fact that $cut \leq conj$ (see for example [GKM] page 143). By assumption, the cutlocus near y is an arc opposite to the geodesic segment $\{Y = 0, \ X \leq 0\}$. Therefore $X \geq 0$ on the cutlocus, and we conclude that

$$X(conj(\theta), \theta) \geq X(cut(\theta), \theta) \geq 0 = X(r_0, \theta_0).$$

The first variation formula gives $\frac{\partial X}{\partial \theta}(r_0, \theta_0) = 0$. Expanding the derivative $0 = \frac{\partial}{\partial \theta} X(conj(\theta), \theta)$ leads to $\dfrac{\partial conj}{\partial \theta}(\theta_0) = 0$. Differentiating the identity $\Theta(conj(\theta), \theta) = 0$ gives $\frac{\partial \Theta}{\partial \theta}(r_0, \theta_0) = 0$. ∎

Lemma 11 *Assume that the Riemannian metric $g \in \mathcal{M}$. Let $V \subset M$ be a submanifold which is tangent to $Cutlocus(x)$ at y. Then $\exp^{-1}(V)$ is near (r_0, θ_0) the union of two smooth transversal submanifolds, one of which is tangent to the circle $\{r = r_0\}$ at (r_0, θ_0).*

Proof. Let V be given as a graph $Y = \psi(X)$ near y, with $\psi(0) = \psi'(0) = 0$. Let $\kappa = \psi''(0)$ be the curvature of V at y. We show that the function

$$h(r, \theta) = Y(r, \theta) - \psi(X(r, \theta))$$

has a nondegenerate critical point at (r_0, θ_0) with Hessian $-\kappa\, dr^2 - \frac{\partial \Theta}{\partial r}(r_0, \theta_0)\, dr\, d\theta$. Morse Lemma then applies.

Since $X(r,\theta) = r - r_0 + O((r-r_0)^2 + (\theta-\theta_0)^2)$, $\psi(X) = \frac{\kappa}{2}(r-r_0)^2 + O((r-r_0)^2 + (\theta-\theta_0)^2)$. Since $\frac{\partial X}{\partial \theta}(r_0,\theta_0) = 0$, differentiating the identity

$$\Theta = e(X,Y)(\frac{\partial X}{\partial r}\frac{\partial Y}{\partial \theta} - \frac{\partial X}{\partial \theta}\frac{\partial Y}{\partial r})$$

gives

$$\frac{\partial^2 Y}{\partial r \partial \theta}(r_0,\theta_0) = \frac{\partial \Theta}{\partial r}(r_0,\theta_0) \neq 0, \quad \frac{\partial^2 Y}{\partial \theta^2}(r_0,\theta_0) = \frac{\partial \Theta}{\partial \theta}(r_0,\theta_0) = 0,$$

thanks to Lemma 10. Finally, since $Y(r,\theta_0) = 0$ for all r, $\frac{\partial^2 Y}{\partial r^2}(r_0,\theta_0) = 0$. This gives the announced expression for the Hessian of h. ∎

Lemma 12 *Assume that the Riemannian metric $g \in \mathcal{M}$. Let $\theta_0 \in B_1$. Then the function cut is smooth near θ_0, $cut'(\theta_0) = 0$ and*

$$cut''(\theta_0) = -\frac{1}{3}(\frac{\partial^2 \Theta}{\partial \theta^2} / \frac{\partial \Theta}{\partial r})(cut(\theta_0),\theta_0).$$

Proof. By assumption, in Fermi coordinates the cut locus is given near $y = \exp_x(r_0,\theta_0)$ by $Cutlocus(x) = \{Y = \psi(X) \mid X \geq 0\}$ (here again $r_0 = cut(\theta_0)$). Furthermore, X vanishes on $Cutlocus(x)$ only at y. Since $\frac{\partial X}{\partial r} = 1$ and $\frac{\partial X}{\partial \theta} = 0$, the inequation $X(r,\theta) \leq 0$ is solved by $r \leq \phi(\theta)$ where $\phi(\theta_0) = cut(\theta_0)$ and $\phi'(tz) = 0$. Furthermore, $\phi < cut$ away from θ_0. Therefore the condition $X \geq 0$ cuts exactly one of the 4 arcs of the Morse critical level set $h^{-1}(0)$. Thus exactly the 2 arcs tangent to the circle $r = cut(\theta_0)$ contribute to the function *cut*. We conclude that *cut* is smooth with $cut'(\theta_0) = 0$.

Differentiating the identity $h(cut(\theta),\theta) = 0$ three times gives

$$cut''(\theta_0) = -\frac{1}{6}(\frac{\partial^3 h}{\partial \theta^3} / \frac{\partial^2 h}{\partial r \partial \theta})(r_0,\theta_0).$$

Next we express the derivatives of h in terms of derivatives of Θ. First some preparation. The equation

$$(*) \qquad 0 = \langle \frac{\partial}{\partial r}\exp_x, \frac{\partial}{\partial \theta}\exp_x \rangle = \frac{\partial X}{\partial r}\frac{\partial X}{\partial \theta} + e(X,Y)\frac{\partial Y}{\partial r}\frac{\partial Y}{\partial \theta}$$

gives $\frac{\partial X}{\partial \theta}(r,\theta_0) = 0$ for all r, and therefore $\frac{\partial^2 X}{\partial r \partial \theta}(r,\theta_0) = 0$. With $\psi'(0) = 0$, this implies $\frac{\partial^2 \psi(X)}{\partial r \partial \theta}(r_0,\theta_0) = 0$. From the proof of Lemma 11, we take $\frac{\partial^2 Y}{\partial r \partial \theta}(r_0,\theta_0) = \frac{\partial \Theta}{\partial r}(r_0,\theta_0)$ and find

$$\frac{\partial^2 h}{\partial r \partial \theta}(r_0,\theta_0) = \frac{\partial \Theta}{\partial r}(r_0,\theta_0).$$

Differentiating (*) with respect to θ yields

$$\frac{\partial^2 X}{\partial \theta^2}(r_0,\theta_0) = 0.$$

and thus $X(r_0,\theta) = O((\theta-\theta_0)^3)$. Since $\psi(X) = O(X^2)$, this implies that $\psi(X(r_0,\theta)) = O((\theta-\theta_0)^6)$ does not contribute either to the third derivative $\frac{\partial^3 h}{\partial \theta^3}$.

There remains to compute $\frac{\partial^3 Y}{\partial \theta^3}(r_0,\theta_0)$.

Since $e(X,Y) = 1 + O(X^2+Y^2) = 1 + O((\theta-\theta_0)^4)$, we can forget this term when differentiating the expression for Θ twice. Since $\frac{\partial X}{\partial \theta} = O((\theta-\theta_0)^2)$, $\frac{\partial Y}{\partial \theta} = \frac{1}{2}\frac{\partial^3 Y}{\partial \theta^3}(r_0,\theta_0)(\theta-\theta_0)^2 + O((\theta-\theta_0)^3)$ and $\frac{\partial Y}{\partial r}(r_0,\theta_0) = 0$, we conclude

$$(**)\, \frac{\partial^3 h}{\partial \theta^3}(r_0,\theta_0) = \frac{\partial^3 Y}{\partial \theta^3}(r_0,\theta_0) = 2\frac{\partial^2 \Theta}{\partial \theta^2}(r_0,\theta_0).$$ ∎

Proposition 13 *Assume that the Riemannian metric $g \in \mathcal{M}$. Then at each point of B_1, the function cut is smooth and has a nondegenerate critical point.*

Proof. We show that for $\theta_0 \in B_1$ and $r_0 = cut(\theta_0) = conj(\theta_0)$, $\dfrac{\partial^2 \Theta}{\partial \theta^2}(r_0,\theta_0) \neq 0$.

It is folklore that for a generic metric in dimension 2, the singularities of the conjugate locus are ordinary cusps tangent to the cutlocus at endpoints. This can be extracted from [B3]. On page 112, M. Buchner gives a model for the energy functional on the space of paths from x near the minimizing segment joining x to and endpoint y. In fact, M. Buchner uses a finite dimensional approximation $B = \mathbf{R}^n \times \mathbf{R}^2$ where the endpoint map (on paths) corresponds to the projection on the second factor (in some local coordinates (u,v) near y). The model is

$$H(x_1,\ldots,x_n,u,v) = x_1^4 + ux_1^2 + vx_1 + \sum_{i=2}^{n} x_i^2.$$

In a neighborhood of y, with y excluded, the cutlocus is the set of (u,v) such that H, as a function of $(x_1,\ldots,x_n)$ has two minima. This leads to $v = 0$ and $u < 0$.

Near y, the conjugate locus is the set of (u,v) such that H, as a function of $(x_1,\ldots,x_n)$ has a degenerate critical point. The equations for such a degenerate critical point are

$$x_i = 0,\ i \geq 2, \quad 4x_1^3 + 2ux_1 + v = 0, \quad 12x_1^2 + 2u = 0.$$

Eliminating x_1 gives the equation of the conjugate locus

$$8u^3 + 27v^2 = 0.$$

This is an ordinary cusp, with the cutlocus as a half tangent at the vertex y.

The conjugate locus is also the image of the map $\theta \mapsto \exp_x(conj(\theta),\theta))$. Let us show that

$$\frac{d^3}{d\theta^3}v(\exp_x(conj(\theta),\theta)))(\theta_0) \neq 0.$$

Let $T(s)$ denote the affine line (in the (u,v) coordinates) tangent to the conjugate locus at $(-\frac{3}{2}s^2, -s^3)$. Its slope is s. For $\epsilon > 0$ small, let $(\epsilon, U(s))$ be the intersection of $T(s)$ with the line $u = \epsilon$. Then $U(s)$ is a smooth function of s with $U'(0) \neq 0$. If we replace affine lines by geodesics in some smooth riemannian metric, then for ϵ small enough the function U still satisfies $U'(0) \neq 0$. Since the line $u = \epsilon$ is away from the cutlocus, and transverse to the minimizing geodesic from x to y,

$$\exp_x^{-1}(\epsilon, U(s)) = (r(s),\theta(s)) = (conj(\theta(s)),\theta(s))$$

where $s \mapsto \theta(s)$ is smooth with $\theta'(0) \neq 0$. Since $\dfrac{\partial^3 v}{\partial s^3}(0) \neq 0$ and lower order derivatives vanish, we get $\dfrac{d^3 v}{d\theta^3}(\theta_0) \neq 0$.

Let us compare Fermi coordinates to u and v. Since the cutlocus is contained in $v = 0$, the differential dY is colinear to dv at y. Therefore there exists a nonzero constant λ such that $Y = \lambda v + O(u^2 + v^2)$.

Now we argue by contradiction. Let $r_0 = cut(\theta_0) = conj(\theta_0)$. Assume that $\frac{\partial^2\Theta}{\partial\theta^2}(r_0, \theta_0)$ vanishes. Since $\frac{\partial\Theta}{\partial r}(r_0, \theta_0) \neq 0$ and $conj'(\theta_0) = 0$, differentiating the identity $\Theta(conj(\theta), \theta) = 0$ twice gives $conj''(\theta_0) = 0$. Let us denote $Y(conj(\theta), \theta) = Y(\theta)$ for short. Then

$$\frac{d^2Y}{d\theta^2}(\theta_0) = \frac{\partial^2 Y}{\partial\theta^2}(r_0, \theta_0) = 0$$

and since $\frac{\partial Y}{\partial r}(r_0, \theta_0) = 0$,

$$\frac{d^3Y}{d\theta^3}(\theta_0) = \frac{\partial^3 Y}{\partial\theta^3}(r_0, \theta_0).$$

Since $u = O((\theta - \theta_0)^2)$ and $v = O((\theta - \theta_0)^3)$, $Y = \lambda v + O((\theta - \theta_0)^4)$. Therefore

$$\frac{d^3Y}{d\theta^3}(\theta_0) = \lambda \frac{d^3 v}{d\theta^3}(\theta_0) \neq 0,$$

a contradiction since, according to (**),

$$\frac{\partial^3 Y}{\partial\theta^3}(r_0, \theta_0) = 2\frac{\partial^2\Theta}{\partial\theta^2}(r_0, \theta_0)$$

and this was assumed to vanish. With Lemma 12, this shows that for a metric in $\mathcal{M}$, for each $\theta_0 \in B_1$, $cut''(\theta_0) \neq 0$. ∎

7 Proof of Theorem 1

7.1 Singular values are genericly distinct

Here we check that genericly the singular values of the distance to x on $Cutlocus(x)$ are distinct.

For a metric $g \in \mathcal{M}$, triple points are not conjugate to x along minimizing geodesics, and do not sit on geodesic loops. Therefore, they vary smoothly with the metric (as transverse intersection points) as does their distance to x, and the corresponding set of directions $B_3(g) \subset S^1$.

The other singular values of the distance to x on $Cutlocus(x)$ are the critical values of the cut function. For a metric $g \in \mathcal{M} \cap \mathcal{M}'$, the critical points are nondegenerate, they vary smoothly with the metric in a neighborhood of g and so do the values of cut on them.

Lemma 14 *Fix a metric $g_0 \in \mathcal{M} \cap \mathcal{M}'$ and a number R which is not a singular value of the distance to x on $Cutlocus(x)$ for the metric g_0. Then for g close to g_0 the singular values below R $r_1(g), \dots, r_k(g)$ of the distance to x on $Cutlocus(x)$ vary smoothly with g and the map*

$$g \mapsto (r_1(g), \dots, r_k(g))$$

is a submersion to $\mathbf{R}^k$.

Proof. Each $r_j(g)$ is equal to $cut_g(\theta)$ for $\theta = \theta_j(g) \in B_1(g)$, resp. for $\theta_j(g)$, $\theta_j'(g) \in B_2(g)$, resp. for $\theta_j(g)$, $\theta_j(g)$, $\theta_j''(g) \in B_3(g)$. Denote $r_j = cut_{g_0}(\theta_j(g_0))$.

Given $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \mathbf{R}^k$, we define a metric g_ϵ using the polar coordinates fo g_0. g_ϵ equals g_0 except in a small neighborhood of $(\frac{1}{2}r_j, \theta_j(g_0))$ where

$$g_\epsilon = (1 + \epsilon_j \chi_j(r)\eta_j(\theta))dr^2 + \Theta(r, \theta)^2 d\theta^2.$$

Here $\chi_j \geq 0$ is a function with support in a small interval centered at $\frac{1}{2}r_j$ such that $\int_0^{r_j} \chi_j(r)\, dr = 1$ and $\eta_j \geq 0$ is a function with support in a small interval centered at $\theta_j(g_0)$ such that $\eta(\theta_j(g_0)) = 1$. In case of double (resp. triple) points, the same function χ_j and the same parameter ϵ_j are used in a neighborhood of $\theta_j'(g_0)$, $\theta_j''(g_0)$.

The geodesics with initial speeds close to θ_j, θ_j', θ_j'' remain geodesics in the metric g_ϵ, only the parameter when it approaches the cutlocus has changed by adding a constant ϵ_j. Therefore the points of $B_1(g_0)$, $B_2(g_0)$ remain critical points of cut for the metric g_ϵ. Since $B_1(g_\epsilon)$, $B_2(g_\epsilon)$ vary smoothly with ϵ, we conclude that $B_1(g_\epsilon) = B_1(g_0)$, $B_2(g_\epsilon) = B_2(g_0)$. Similarly, triple points for g_0 are joined to x by 3 geodesics of equal length with respect to g_ϵ. Since there are finitely many such points, depending smoothly on ϵ, they do not move at all.

Therefore, $r_j(g_\epsilon) = \int_0^{r_j}(1 + \epsilon_j\chi_j(r))\, dr = r_j + \epsilon_j$ and this is clearly a submersion to $\mathbf{R}^k$. ∎

Proposition 15 *For a residual subset $\mathcal{M}'' \subset \mathcal{M} \cap \mathcal{M}'$, the volume growth is smooth away from a locally finite set $\Sigma \subset R_+$, and for each $r \in \Sigma$ only one event occurs. The sphere of radius r contains exactly one endpoint, or one double point, or one triple point of $Cutlocus(x)$.*

Indeed, for $R > 0$ let $\mathcal{M}''_R$ be the set of metrics in $\mathcal{M} \cap \mathcal{M}'$ such that all singular values $r_j < R$ are distinct. According to Lemma 14, $\mathcal{M}''_R$ is the complement of a finite union of submanifolds in $\mathcal{M} \cap \mathcal{M}'$, thus it is residual. Then $\mathcal{M}'' = \bigcap_{R \to +\infty} \mathcal{M}''_R$ is residual. ∎

7.2 End of the proof of Theorem 1

Singularity at $r_0 = 0$. Let i be the injectivity radius at x. For $r \in [0, i[$, $v'(r) = \int_{S^1} \Theta(r, \theta)\, d\theta$. The function $R(r, \theta) = R(\exp_x(r, \theta))$ is smooth on $\mathbf{R}^2$. Then Θ, the solution of

$$\frac{\partial^2\Theta}{\partial r^2}(r, \theta) + R(r, \theta)\Theta(r, \theta) = 0, \quad \Theta(0, \theta) = 0, \quad \frac{\partial\Theta}{\partial r}(0, \theta) = 1,$$

is a smooth function on $\mathbf{R}^2$. Therefore v coincides on $[0, i[$ with a smooth function defined on $\mathbf{R}$.

Singularity at a point $r_0 \in \Sigma_1$. Say that $r_0 \in \Sigma_1$ if the sphere $\partial B(x, r_0)$ contains an endpoint of $Cutlocus(x)$. If $g \in \mathcal{M}''$, then $cut^{-1}(r_0)$ is a finite set in a neighborhood of which cut is smooth and has exactly one critical point, a nondegenerate local minimum θ_0. At that point, $\Theta(r_0, \theta_0) = 0$ but $\frac{\partial\Theta}{\partial r}(r_0, \theta_0) \neq 0$. Therefore below r_0, v' is smooth and above r_0

$$v'(r) = \int_{\{cut > r\}} \Theta(r, \theta)\, d\theta = (r - r_0)h(\sqrt{r - r_0})$$

where h is smooth with $h(0) = 0$ and $h'(0) < 0$, see Lemme 1 in [GP2].

Singularity at a point $r_0 \in \Sigma_2$. Say that $r_0 \in \Sigma_2$ if the sphere $\partial B(x, r_0)$ contains a double point y of $Cutlocus(x)$ which is a critical point of the distance to x. If $g \in \mathcal{M}''$, then $cut^{-1}(r_0)$ is a finite set in a neighborhood of which cut is smooth and has exactly two critical points θ_0 and θ_0' which are nondegenerate. At these points, $\Theta(r_0, \cdot) \neq 0$. θ_0 (or θ_0') is a local maximum of cut if and only if y is a local maximum of the distance to x on $Cutlocus(x)$. If this is the case, then v' equals a smooth function plus a smooth function of $\sqrt{r_0 - r}$ (extended by zero above r_0). Otherwise, θ_0 and θ_0' are local minima, v' equals a smooth function plus a smooth function of $\sqrt{r - r_0}$.

Singularity at a point $r_0 \in \Sigma_3$. Say that $r_0 \in \Sigma_3$ if the sphere $\partial B(x, r_0)$ contains a triple point y of $Cutlocus(x)$. If $g \in \mathcal{M}''$, then $cut^{-1}(r_0)$ is a finite set of the form $A \cup \{\theta_0, \theta_0', \theta_0''\}$. The cut function is smooth with nonvanishing derivative at each point of A. At θ_0, θ_0', θ_0'', cut is the minimum of two smooth functions with distinct, nonvanishing derivatives. Let us show that θ_0, θ_0', θ_0'' are local maxima of cut. The values of cut near θ_0 correspond to values of the distance to x along $Cutlocus(x)$ near the triple point y. Each arc of $Cutlocus(x)$ makes an acute angle with at least one of the minimizing geodesics from y to x. Therefore, the distance to x strictly decreases along the cutlocus. One concludes that v' is smooth both below and above r_0, but its left and right derivatives satisfy $v_\ell''(r_0) < v_r''(r_0)$. ∎

8 Appendix: Genericly all geodesic loops are nondegenerate

We prove the following result which was needed in Proposition 9.

Theorem 5 *Let M be a manifold, x a point in M. There is a residual set $\mathcal{M}'$ of complete metrics on M such that x is a regular value of* $\exp_x$.

Remark 16 *For a metric in $\mathcal{M}'$, the energy functional on loops based at x has only nondegenerate critical points.*

Proof. The set $\mathcal{M}'$ of complete metrics on M such that x is a regular value of $\exp_x$ is a G_δ in fine C^∞ topology. It suffices to show that $\mathcal{M}'$ is dense.

8.1 Reduction to a transversality assertion

Recall the Hamiltonian description of the geodesic flow, cf. [KT]. A metric g determines a dual norm on covectors, and the Hamiltonian $H_g(\xi) = \frac{1}{2}|\xi|^2$ on T^*M. Let ω denote the canonical symplectic structure on T^*M. The geodesic flow is the Hamiltonian vector field X_g on T^*M defined by

$$\iota_{X_g}\omega = dH_g.$$

Let $\phi_g : T^*M \to T^*M$ denote the time 1 map of the geodesic flow. Then x is a regular value of $\exp_{x,g}$ if and only if the restriction of ϕ_g to T_x^*M is transversal to the submanifold $T_x^*M \subset TM$. We fix a nonzero covector $v \in T_x^*M$. We show that the map

$$\phi : \{g\} \to T^*M, \quad \phi(g) = \phi_g(v)$$

is transverse to T_x^*M. It follows that

$$\tilde{\phi} : \{g\} \times T_x^*M \to T^*M, \quad \phi(g, v) = \phi_g(v)$$

is transverse to T_x^*M (we already know that $\tilde{\phi}$ is transverse to T_x^*M at $(g, 0)$ for every g). Finally, it will follow that for a dense set of metrics g, ϕ_g is transverse to T_xM.

8.2 Computation of the differential $d\phi$

Let $s \mapsto g_s$ be a one parameter family of metrics and $\xi \in T_x^*M$, let $\xi_s(t) = \phi_{g_s}(tv)$ denote the trajectory of v under the geodesic flow of g_s, i.e.,

$$\frac{d\xi_s}{dt}(t) = X_{g_s}(\xi_s(t)).$$

The derivative $\sigma(t) = \frac{d\xi_s(t)}{ds}_{|s=0}$ satisfies the variational equation. In coordinates, it reads

$$\frac{d\sigma}{dt}(t) = d_{\xi_0(t)}X_{g_0}(\sigma(t)) + \frac{dX_{g_s}}{ds}(\xi_0(t)).$$

More invariantly, the vectorfield σ along ξ satisfies

$$\mathcal{L}_X\sigma = \frac{dX_{g_s}}{ds}_{|s=0}.$$

Let $h = \frac{dg_s}{ds}_{|s=0}$. Let σ be the solution of the variational equation with initial condition $\sigma(0) = 0$. Then $d_g\phi(h) = \sigma(1)$.

8.3 The equation of Riemannian variations

The next task is to understand which vectorfields Y along a trajectory ξ are *Riemannian variations*, i.e. are equal to $\frac{dX_{g_s}}{ds}_{|s=0}$ for some one parameter family of Riemannian metrics.

Let δ_λ denote multiplication by λ in the fibers. Let Z be the radial vertical vectorfield $Z(\xi) = \xi$ whose flow is δ_{e^t}. The canonical symplectic form ω is homogeneous of degree one, $\mathcal{L}_Z\omega = \omega$.

Any geodesic flow is homogeneous of degree one, $\mathcal{L}_ZX = X$. Therefore, the candidates Y for variations must be homogeneous of degree one, i.e. $\mathcal{L}_ZY = Y$.

Any geodesic flow is symplectic, $\mathcal{L}_X\omega = 0$. Therefore, the candidates Y for variations must be symplectic. In particular, $d(\iota_Y\omega)(X, Z) = 0$. This leads to the differential equation

$$(***) \qquad \omega(\mathcal{L}_XY, Z) = \omega(X, Y).$$

Conversely, given a vectorfield Y along a trajectory ξ which satisfies equation (***), let us extend Y by homogeneity as a vectorfield along the surface $S : (t, \lambda) \mapsto \delta_\lambda(\xi(t))$. Then $\mathcal{L}_ZY = Y$ and $d(\iota_Y\omega)_{|S} = 0$. Let f be a function on S such that $df_{|S} = \iota_Y\omega_{|S}$. Then f can be extended in a neighborhood of S, as a homogeneous function $f \circ \delta_\lambda = \lambda^2 f$ which satisfies $df = \iota_Y\omega$ at each point of ξ. If Y has compact support on ξ (i.e. vanishes in a neighborhood of the endpoints), then f can be chosen to have compact support, provided

$$\int_\xi \iota_Y\omega = 0.$$

The next lemma allows to replace f by a smooth field of quadratic forms. Therefore equation (***) characterizes Riemannian variations.

Lemma 17 *Let N be the image in T^*M of a smooth nowhere vanishing section on some open set U. Let f be a smooth function on T^*U which is homogeneous of degree 2 along the fibers. Then there exists a smooth function h on T^*U, homogeneous quadratic along the fibers, such that $dh = df$ at each point of N. If f has compact support, so does h.*

Proof of Lemma. Setting $h = f$ on N gives h the correct derivatives in the direction of N. There remains to adjust the fiber derivatives, i.e., to construct on each fiber T_y^*M a quadratic form h_y with prescribed 1-jet at a nonzero point $\xi = N \cap T_y^*M$. If h is viewed as a symmetric map $h : T_y^*M \to T_yM$, the equation is $h(\xi) = d_\xi f$. This is a linear system of maximal rank. Its solutions form a smooth bundle of affine spaces, which admits a smooth section. Since f is homogeneous of degree 2, any such section h will automaticly satisfy

$$f(\xi) = \frac{1}{2}\langle \xi, d_\xi f \langle = \frac{1}{2}\langle \xi, h\xi \langle = h(\xi).$$

Where f vanishes, h can be chosen to vanish as well. ∎

8.4 Solving the equations

Lemma 18 *Let $\xi : [0,1] \to T^*M$ be a trajectory of the geodesic flow $X = X_g$ whose projection γ to M has no selfintersection. Let $\eta_0 \in T_{\xi(0)}T^*M$, $\eta_1 \in T_{\xi(1)}T^*M$ satisfy*

$$\omega(X, \eta_0) = \omega(X, \eta_1) = 0.$$

There exists a one parameter family of metrics g_s supported in an arbitrary small neighborhood of a proper subarc of γ such that the solution σ of the variational equation $\mathcal{L}_X\sigma = (dX_{g_s}/ds)_{|s=0}$, satisfies $\sigma(0) = \eta_0$ and $\sigma(1) = \eta_1$.

Proof. By linearity, we can assume that η_0 and η_1 are very small. Therefore the whole discussion takes place in a tiny tubular neighborhood of the trajectory ξ. Let us choose coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ in which $\xi(0) = 0$, $X = \frac{\partial}{\partial q_1}$ and $\omega = \sum dp_i \wedge dq_i$. This is possible since a nonvanishing hamiltonian vectorfield has no local invariants. We view a vectorfield Y along ξ as a map $[0,1] \to \mathbf{R}^{2n}$ and interpret $\mathcal{L}_X Y = Y'(t)$ as a derivative. A solution Y of the equation of Riemannian variations which satisfies the extra equation $\omega(X, Y) = 0$ becomes a map $[0,1] \to \mathbf{R}^{2n-1}$ which satisfies $\omega(Z, Y') = 0$, i.e., Y is tangent to the contact structure $\ker \iota_Z\omega$.

In the chosen coordinates, the variational equation reads $\sigma' = Y$ with boundary conditions in $\mathbf{R}^{2n-1}$. The problem reduces to finding a closed contact curve in $\mathbf{R}^{2n-1}$ with prescribed barycenter. It obviously admits solutions Y with compact support in $]0,1[$. Since $\omega(X, Y) \equiv 0$, the condition $\int_\xi \iota_Y\omega = 0$ is satisfied, and one can integrate $\iota_Y\omega$ into a homogeneous function f with the required support. Lemma 17 then produces a field h of quadratic forms, and the family of metrics on TM dual to $H_s = H + sh$ has the required properties. ∎

8.5 Proof of Theorem 5

Let g be a Riemannian metric on M such that $\phi(g) \in T_xM$. Let $\xi(t)$ be the trajectory of v under the geodesic flow X_g, so that $\phi(g) = \xi(1)$. Let $\gamma(t) = \pi(\xi(t))$ be the corresponding geodesic loop. Choose a, $b \in]0,1[$, $a < b$ such that

- the arc $\gamma([a,b])$ has no selfintersection;

- $\gamma([a,b])$ is disjoint from $\gamma([0,a[)$ and $\gamma(]b,1])$

Let $\eta \in T_{\xi(1)}T^*M$ be a tangent vector such that $\omega(X, \eta) = 0$. Let $\sigma(t)$ be a smooth vectorfield along ξ such that $\sigma(0) = 0$, $\sigma(1) = \eta$ and σ satisfies the variational equation with vanishing right hand side $\mathcal{L}_X\sigma = 0$ on $]0,a[$ and on $]b,1[$. Then

$$\frac{d}{dt}\omega(X, \sigma) = \mathcal{L}_X\omega(X, \sigma) = \omega(X, \mathcal{L}_X\sigma) = 0$$

so that $\omega(X, \sigma(a)) = \omega(X, \sigma(b)) = 0$. According to Lemma 18, there exists a smooth field of quadratic forms h, which vanishes in a neighborhood of $\gamma([0,a]) \cup \gamma([b,1])$, with hamiltonian flow Y, such that the solution $\tilde{\sigma}$ of the variational equation $\mathcal{L}_X\tilde{\sigma} = Y$ along $\xi([a,b])$ satisfies $\tilde{\sigma}(a) = \sigma(a)$ and $\tilde{\sigma}(b) = \sigma(b)$. Note that Y vanishes along $\xi([0,a])$ and $\xi([b,1])$. Putting σ and $\tilde{\sigma}$ together produces a solution of $\mathcal{L}_X\sigma = Y$ along all of $\xi([0,1])$. Therefore $\eta = d_g\phi(h)$.

We conclude that the image of $d_g\phi$ contains $\ker \iota_X\omega$, a subspace which is transverse to the fiber T_x^*M. ∎

Remark 19 *If v generates a periodic geodesic, then the image of $d_g\phi$ equals* $\ker \iota_X\omega$.

Indeed, for any one parameter family of metrics g_s with derivative h at 0, $H_s(\xi_s(1)) - H_s(\xi_s(0)) = 0$. Differentiating with respect to s at 0 gives

$$\begin{aligned} h(\xi(1)) - h(\xi(0)) &= d_{\xi(1)}H(\sigma(1)) - d_{\xi(0)}H(\sigma(0)) \\ &= \iota_X\omega(\sigma(1)) - \iota_X\omega(\sigma(0)) \\ &= \iota_X\omega(d_g\phi(h)). \end{aligned}$$

Thus $d_g\phi(h) \in \ker \iota_X\omega$ if $\xi(1) = \xi(0)$, i.e. if v is periodic. Otherwise, ϕ is a submersion at g.

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A Combinatorial Approach to 3-manifolds

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This paper describes joint work with Riccardo Benedetti. Our main result is a *combinatorial realization* of some categories of 3-manifolds with extra structure. Let us first explain what this means exactly. Let $\mathcal{T}$ be a collection of topological objects (for us, 3-dimensional manifolds with a certain extra structure), regarded up to a suitable equivalence relation. The first ingredient of a realization will be an explicitly described set $\mathcal{S}$ of finite combinatorial objects (typically, finite graphs with certain properties and decorations), together with an effective mapping $\Psi : \mathcal{S} \to \mathcal{T}$, called the *reconstruction map*, whose image is the whole of $\mathcal{T}$. The second ingredient of the realization is the *calculus*, namely an explicitly described finite set of local moves on $\mathcal{S}$ with the property that two elements of $\mathcal{S}$ have the same image in $\mathcal{T}$ under Ψ if and only if they are related to each other by a finite combination of the moves of the calculus.

A combinatorial realization which has been known for a long time is the presentation of links up to isotopy in S^3 by means of planar diagrams, where the calculus is generated by Reidemeister moves. In 3-manifold topology some examples of combinatorial representation are known which satisfy at least some of the requirements which we have stated. Let us remark however that the requirements of *finiteness* and *locality* of the calculus are somewhat restrictive. For instance the presentation of closed connected oriented 3-manifolds via longitudinal Dehn surgery on framed links in S^3, with either of the two versions of the Kirby calculus, does not satisfy the requirements. If we use the version of the calculus which includes the band move, then we have a non-local move, whereas, if we use the generalized Kirby move, then we have indeed local moves, but we actually have to take into account infinitely many different ones, parametrized by the number of strands which link the curl removed by the move. In [1], using a slight refinement of the theory of standard spines and an appropriate graphic encoding, we have produced a combinatorial realization of the category of compact connected 3-manifolds with non-empty boundary, and a refined version of the same realization for the oriented case.

We will describe now the topological objects of which we provide a combinatorial realization. By a 3-manifold we will always mean a *connected, compact* and *oriented*