Then f_1, f_2 and g_2 vanish on Q by construction. Further, outside a compact set, $g_1 = \psi^1 \frac{\partial \psi^2}{\partial y_1}$ where $\psi_{y_1}(x_2, y_2) = (\psi^1, \psi^2)$. Hence g_1 is independent of x_1 and has compact support with respect to the other variables. Therefore $d(g_1 y_1')$ is bounded and we may take $\beta'' = \beta' - d(g_1 y_1')$. ∎

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Differential forms and connections adapted to a contact structure, after M. Rumin[1]

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Michel Rumin is a student of Mikhael Gromov, who asked him the following question : Let M be a manifold with contact structure ξ, E a vector bundle over M. A *partial connection* on E is a covariant derivative $\nabla_v e$ defined for smooth sections e of E but only for vectors v in ξ. In particular, parallel translation is defined only along Legendrian curves, that is curves which are tangent to ξ. Can one define the curvature of such a connection ?

Gromov provided the following hint : For an ordinary connection A, curvature arises in the asymptotics of holonomy around short loops. A loop encompasses a certain "span" (a 2-vector, see below), quadratic in length, and holonomy deviates from the identity by an amount proportional to curvature times span, that is, quadratic in length. In case M has dimension 3 and carries a contact structure, then every Legendrian loop has essentially zero area. Gromov conjectured that, in this case, curvature should arise as the cubic term in the asymptotic expansion of holonomy.

Michel Rumin has found a notion of curvature for partially defined connections along the above lines. The point is to understand the exterior differential for a partially defined 1-form. In fact, M. Rumin constructs a substitute for the de Rham complex : a locally exact complex of hypoelliptic operators naturally attached to a contact manifold (M, ξ) of dimension $2m+1$. The operator which sends m-forms to m+1-forms is new. It is of second order.

[1]Research partially supported by the EEC under contract SC1–0105–C, GADGET

In this lecture, after some comments on the asymptotics of holonomy, I explain M. Rumin's construction. Then I describe an application of M. Rumin's ideas to analysis on complex hyperbolic space : boundary values of L^2 harmonic forms. I learned most of the material presented here thanks to P.Y. Gaillard, V. Goldshtein, J. Heber, P. Julg and M. Rumin.

1 Asymptotic expansion of holonomy

In this section, I explain how the asymptotic expansion of the holonomy of a smooth connection specializes to Legendrian curves in a contact manifold.

1.1 The classical formula.– I define the "span", i.e., the algebraic area spanned by a loop c in a vector space $V = \mathbf{R}^n$. Let $\alpha \in \Lambda^2 V^*$ be a 2-form, viewed as a translation invariant differential form on V. Then $d\alpha = 0$, and there exists a 1-form β such that $d\beta = \alpha$. The linear functional

$$\alpha \mapsto \int_c \beta$$

on $\Lambda^2 V^*$ corresponds to a unique 2-vector span(c).

Now let M be a differentiable manifold, $x \in M$, $c \subset M$ a small loop through x. Using coordinate charts ϕ we define various covectors

$$\mathrm{span}_\phi(c) = \mathrm{span}(\phi(c)) \in \Lambda^2 T_x M$$

which coincide up to an error of size $o(\mathrm{area}(c))$, where area(c), the "geometric area", is the least area of a surface spanned by c. Thus it makes sense to state

1.2 Fact .–*Let D be a connection on some vector bundle E over a manifold M. For $x \in M$, c a short loop through x, let $Hol(D, c)$ denote the holonomy of the connection D along c (an endomorphism of the fibre E_x). Then, as the length of c tends to 0,*

$$Hol(D, c) = 1 + \langle F^D, \mathrm{span}(c)\rangle + o(\mathrm{area}(c)),$$

where F^D is the curvature of D at x, an $End(E_x)$-valued 2-form on $T_x M$.

1.3 The case of contact manifolds.– Geometric areas will be measured relative to Carnot-Caratheodory metrics. Given a norm on the plane field ξ, a Carnot-Caratheodory metric is defined by minimizing the length of Legendrian curves between two points.

In dimension $2m+1 \geq 5$, every Legendrian curve bounds a Legendrian surface, which has Hausdorff dimension 2. Thus we define the geometric area area(c) to be the infimum of the 2-dimensional Hausdorff measures of surfaces whose boundary is c. However, in dimension 3, all smooth surfaces have Hausdorff dimension 3, and we take 3-dimensional Hausdorff measure in the definition of geometric area. Note that contact transformations are Lipschitz with respect to Carnot-Caratheodory metrics, so they preserve the rough size of geometric areas.

Recall that the Heisenberg group N of dimension $2m+1$ is the simply connected group attached to the following Lie algebra $\mathcal{N}$:

$$\mathcal{N} = \mathbf{R}^{2m} \oplus \mathbf{R}$$

with center $\mathbf{R}$ and the Lie bracket is given by the symplectic form $\mathbf{R}^{2m} \times \mathbf{R}^{2m} \to \mathbf{R}$. The left-invariant plane field generated by the factor $\mathbf{R}^{2m}$ is a contact structure ξ_0. Every contact manifold is locally isomorphic to the Heisenberg group (N, ξ_0).

We define the "algebraic area" spanned by a Legendrian loop c in the Heisenberg group as before, but replacing the symplectic vector space, which is the local model for manifolds, by the Heisenberg group, which is the local model for contact manifolds.

In dimensions $2m+1 \geq 5$, the closed left invariant 2-forms on N all are pulled-back from $N/[N,N] = \mathbf{R}^{2m} = \xi_0$. For such an $\alpha = d\beta$, the formula

$$\langle \mathrm{span}(c), \alpha\rangle = \int_c \beta$$

defines a 2-covector span$(c) \in \Lambda^2\xi_0$, equivariantly with respect to Heisenberg automorphisms, and also, up to an error controlled by geometric area, with respect to contact diffeomorphisms.

Note that, if τ denotes some left-invariant 1-form whose kernel is the canonical contact structure ξ_0, then

$$\langle \mathrm{span}(c), d\tau\rangle = 0.$$

In dimension 3, all left-invariant 2-forms are closed, and one gets a 2-covector span$(c) \in \Lambda^2\mathcal{N}$.

In both cases, one has

1.4 FACT .– *Let D be a connection on some vector bundle E over a contact manifold (M,ξ). Let $x \in M$, and c be a short Legendrian loop through x. Then, as the length of c tends to 0,*

$$Hol(D,c) = 1 + \langle F^D, \mathrm{span}\,(c)\rangle + o(\mathrm{area}(c)).$$

In dimensions $2m+1 \geq 5$, geometric area is quadratic in length (Y. Eliashberg, unpublished) and we see that the projection of F^D in

$$\Lambda^2\xi^*/\mathbf{R}d\tau$$

only depends on the restriction of D to the ξ directions, i.e., it is an invariant attached to the partial connection $\nabla = D_{|\xi}$. We call it the *curvature* of ∇. It has $(m(m-1)/2 \;-1)(\mathrm{rk}\,E)^2$ independant components.

In dimension 3, the geometric area is cubic in length, and we cannot ignore the vertical components of span$\,(c)$. We define the *curvature* of $\nabla = D_{|\xi}$ to be the projection of F^D mod multiples of the symplectic structure $d\tau$. It has $2(\mathrm{rk}\,E)^2$ independant components.

In the next section, these definitions will be shown to fit into the formalism of Rumin differential forms.

2 RUMIN'S COMPLEX

It is a substitute for the de Rham complex, where 1-forms are replaced by *partial* 1-forms, i.e., sections of the dual of a contact structure.

Let (M,ξ) be a $2m+1$-dimensional contact manifold. Let Ω^* denote the graded algebra of smooth differential forms. Let $\mathcal{I}^*$ be the graded differential ideal generated by contact forms (i.e., 1-forms τ whose kernel is the contact hyperplane ξ) and $\mathcal{J}^*$ the annihilator of $\mathcal{I}^*$ (i.e., forms α such that $\alpha\wedge\beta = 0$ for all $\beta\in\mathcal{I}^*$). It is again a graded differential ideal.

Once a contact form τ is chosen, the elements of $(\Omega^*/\mathcal{I}^*)\oplus\mathcal{J}^*$ identify with sections of subbundles or quotients of $\Lambda^*\xi^*$. Indeed, let

$$L\;:\Lambda^k\xi^* \to \Lambda^{k+2}\xi^*$$

denote exterior multiplication by $d\tau_{|\xi}$,

$$E^k = \Lambda^k\xi^*/\mathrm{im}\,L,$$

for $k \leq m$, and

$$E^k = \ker L \subset \Lambda^k\xi^*,$$

for $k \geq m$. Then $(\Omega^k/\mathcal{I}^k)\oplus\mathcal{J}^k$ coincides with smooth sections of E^k if $k\leq m$, and with smooth sections of E^{k-1} multiplied by τ if $k\geq m+1$. In particular, the elements in $\Omega^1/\mathcal{I}^1$ are sections of E^1, i.e., partial 1-forms.

There is an induced complex

$$d_\xi\;:(\Omega^*/\mathcal{I}^*)\oplus\mathcal{J}^* \to (\Omega^*/\mathcal{I}^*)\oplus\mathcal{J}^*.$$

One easily checks that d_ξ is locally exact at $\Omega^k/\mathcal{I}^k$ (resp. at $\mathcal{J}^k$) if $k<m$ (resp. if $k>m+1$). This has been observed independantly by V. Ginzburg [5], and generalized by Zhong Ge, [15].

2.1 THEOREM (M. Rumin, [14]).– *There exists a linear second order operator*

$$d_R\;:\Omega^m/\mathcal{I}^m \to \mathcal{J}^{m+1}$$

such that the sequence

$$0\to\mathbf{R}\to\Omega^0\xrightarrow{d_\xi}\Omega^1/\mathcal{I}^1\xrightarrow{d_\xi}\cdots\xrightarrow{d_\xi}\Omega^m/\mathcal{I}^m\xrightarrow{d_R}\mathcal{J}^{m+1}\xrightarrow{d_\xi}\cdots\xrightarrow{d_\xi}\mathcal{J}^{2m+1}\to 0$$

is a locally exact complex, i.e., a resolution of the constant sheaf $\mathbf{R}$.

We explain this in 3 dimensions. If $\alpha\in\Omega^1$ is a 1-form, there is a unique choice of a function f so that $d(\alpha+f\tau)$ vanishes on ξ. Take

$$d_R(\alpha \;mod\; \mathcal{I}^1) = d(\alpha+f\tau)\in\mathcal{J}^2.$$

The function f is determined by the equation

$$d\alpha_{|\xi} + f\,d\tau_{|\xi} = 0$$

and depends on first derivatives of α (in the ξ directions only), thus d_R involves two derivatives in the ξ directions.

Given a metric on ξ, there is a normalization of the contact form τ so that $|d\tau_{|\xi}| = 1$. One gets a pointwise inner product on $\Omega^*/\mathcal{I}^*\oplus\mathcal{J}^*$, and a Hodge operator

$$*\;:\Omega^k/\mathcal{I}^k \to \mathcal{J}^{2m+1-k}$$

such that $*1 = \tau\wedge(d\tau)^m$ and

$$\langle\alpha,\beta\rangle * 1 = \alpha\wedge*\beta.$$

2.2 THEOREM (M. Rumin, [14]).– *Given a metric on ξ, put $d_\xi^* = -*d_\xi*$ (resp. $d_R^* = -*d_R*$). This is a formal adjoint to d_ξ (resp. d_R).*

The laplacians

$$(n-k)d_\xi d_\xi^* + (n-k-1)d_\xi^* d_\xi \quad\text{on}\quad (\Omega^k/\mathcal{I}^k)\oplus\mathcal{J}^k \quad\text{for}\quad k\neq m, m+1,$$

$$(d_\xi d_\xi^*)^2 + d_R^* d_R \quad\text{on}\quad \Omega^m/\mathcal{I}^m,$$

$$(d_\xi^* d_\xi)^2 + d_R d_R^* \quad\text{on}\quad \mathcal{J}^{m+1},$$

are *maximally hypoelliptic*.

Maximal hypoellipticity means that, given vector fields X_i tangent to ξ, one locally has estimates of the form

$$\| X_1X_2\alpha \|_2 \leq C(\| \Delta\alpha \|_2 + \| \alpha \|_2)$$

for the second order Laplacian, and similarly for the fourth order Laplacian.

There is an analogue in contact geometry of the principal symbol, which gives a criterion for hypoellipticity, see [11]. However this criterion becomes effective only when combined with clever Bochner type formulas, see [14]. Simultaneously, various vanishing theorems are obtained. They include the following important feature : on a CR manifold whose Webster torsion vanishes (these are integrability conditions on the metric, analogous to the Kähler condition for Hermitian metrics), the above Laplacians preserve the bidegree (the unusual choice of coefficients is essential).

Back to connections : A partial connection is a Lie algebra valued partial 1-form A and one can make sense of the curvature $dA+A\wedge A$ as was done for dA. In dimensions $2m+1\geq 5$, it is a Lie algebra valued 2-form on ξ mod $d\tau$, in dimension 3, it is a Lie algebra valued 2-form vanishing on ξ, as announced in the previous section.

3 L^2-Harmonic forms on complex hyperbolic space

Complex hyperbolic $m+1$-space is a complete symmetric Kähler manifold, isometric to the unit ball of $\mathbf{C}^{m+1}$ equipped with its Bergman metric. It is the symmetric space of the simple Lie group $SU(m+1,1)$. It is a generalization of the unit disk in $\mathbf{C}$, equipped with its Poincaré metric, which is the symmetric space of $SU(1,1)=PSL(2,\mathbf{R})$.

The new feature when $m\geq 1$ is that the boundary S^{2m+1} inherits a canonical contact structure. At a point $x\in S^{2m+1}$, the contact plane ξ_x is the maximal complex vector subspace in $T_xS^{2m+1}\subset T_x\mathbf{C}^{m+1}$.

Complex hyperbolic $m+1$-space has L^2-cohomology in the middle dimension $m+1$ and in each type (p,q), $p+q=m+1$. Following recent work by Pierre Julg and Michel Rumin ([9]), we explain that L^2-harmonic forms have boundary values, which realize an isomorphism with an explicit space of Rumin differential forms on the boundary, the sphere S^{2m+1} equipped with its canonical contact structure.

3.1 The case of real hyperbolic space.– Let us first explain the corresponding theory for real hyperbolic $2m+2$-space, i.e., the space form of constant sectional curvature -1. L^2-harmonic forms in the middle dimension are conformally invariant, so we can replace hyperbolic space minus one point with $S^{2m+1}\times\mathbf{R}_+$ in a product metric, a situation which has been studied by Atiyah-Patodi-Singer, [1].

Since the Hodge $*$ commutes with the Laplacian and $*^2=(-1)^{m+1}$, harmonic forms split into self-dual and anti-self-dual forms ($*\alpha=\pm i_m\alpha$ where $i_m=1$ if $m+1$ is even, $i_m=i$ if $m+1$ is odd). The equations for a closed self dual (resp. anti-self-dual) form α on $S^{2m+1}\times\mathbf{R}_+$ can be viewed as an ODE in the $y\in\mathbf{R}_+$ variable. Splitting $\alpha=a\pm i_m(*a)\wedge dy$, it reads

$$da=0 \quad\text{and}\quad \frac{\partial}{\partial y}a=-\pm i_m d*a. \tag{1}$$

This equation has constant operator coefficients and explicit solutions in terms of data at $y=0$, i.e., along the boundary, are easily found. This leads to

3.2 Proposition .– *A closed, self-dual (resp. anti-self-dual) $m+1$-form α on real hyperbolic $2m+2$-space has a boundary value $BV(\alpha)$, which is a closed $m+1$-form on S^{2m+1}, $\pm$-invariant under the operator*

$$F=\operatorname{sign}A,\quad A=i_m d*_{|\ker d}.$$

The L^2 norm translates into a Sobolev norm on the boundary :

$$\| \alpha \|_2=\| |A|^{-1/2}\alpha_{|\partial} \|_2 .$$

The L^2 norm of harmonic forms is recovered as follows (P. Julg) : Forms α smooth up to the boundary are dense in L^2 solutions of (1). Choose a smooth form β such that $d\beta=\alpha$ on $S^{2m+1}\times\mathbf{R}_+$ and $d*\beta=0$ on S^{2m+1}, then

$$\begin{aligned}\| \alpha \|_2^2 &= \int_{S^{2m+1}\times\mathbf{R}_+}\pm i_m\alpha\wedge\alpha\\ &=\int_{S^{2m+1}}\beta\wedge\pm i_m\alpha\\ &=\int_{S^{2m+1}}\langle\alpha,\pm i_m(-1)^{m+1}*\beta\rangle\\ &=\int_{S^{2m+1}}\langle\alpha,\pm A^{-1}\alpha\rangle\\ &=\| |A|^{-1/2}(\alpha_{|S^{2m+1}}) \|_2^2 .\end{aligned}$$

3.3 Problem.– Since conformal mappings of S^{2m+1} extend as isometries of real hyperbolic space, we observe that both the operator F and the norm

$$\| \, |A|^{-1/2}\alpha \, \|_2$$

on closed $m+1$-forms are Möbius invariants. More generally, since every quasiconformal mapping of S^{2m+1} extends to a quasiisometry of real hyperbolic space, the norm on closed $m+1$-forms is quasiinvariant under quasiconformal mappings. To what extent is the operator F invariant under quasiconformal mappings ?

3.4 The case of complex hyperbolic space.– A similar computation can be done in the complex case. Harmonic $m+1$-forms split into types and primitive components. Only primitive forms can be in L^2. A conformal change leads to a metric on $S^{2m+1} \times \mathbf{R}_+$ of the form

$$g_{|\xi} + y^{-2} g_{|\xi^\perp} + dy^2$$

where ξ, the complex tangent to S^{2m+1} in the embedding of S^{2m+1} in $\mathbf{C}^{m+1}$, is the canonical contact structure. The ODE for ∂ and $\overline{\partial}$-closed primitive forms does not have constant, nor even commuting coefficients. Splitting forms on S^{2m+1} according to $\xi \oplus \xi^\perp$ looks hopeless since the splitting is not invariant under the exterior differential. Nevertheless, Rumin's complex precisely extracts the part of d that preserves the splitting.

It turns out that the ODE, when rephrased in terms of Rumin's d_R and $*$ operators, can be reduced to scalar equations. These equations are singular at $y=0$. Still, their L^2 solutions are determined by their values at $y=0$. One concludes

3.5 Theorem (P. Julg,[9]).– *There exists a boundary value operator BV on L^2 harmonic $m+1$-forms on complex hyperbolic $m+1$-space, with values in (non smooth) closed partial $m+1$-forms on S^{2m+1} (i.e., elements of $\mathcal{J}m+1 \cap \ker d_\xi$). It is an isometry for the norm*

$$\| \, |A|^{-1/2}\alpha \, \|_2$$

where $A = i_m d_R *_{|\ker d_\xi}$.

The boundary value operator BV sends the Hodge $i_m *$ to the operator $F = \operatorname{sign} A$.

The finer splitting of L^2 harmonic forms into complex types $\mathcal{H}^{p,q}$ seems to translate as follows. Since the contact hyperplane ξ carries a complex structure, Rumin forms of degree $k \geq m+1$ split into types,

$$\mathcal{J}^k = \bigoplus_{p+q=k-1} \mathcal{J}^{p,q}.$$

Then

- for $p,\ q \geq 2$, $BV(\mathcal{H}^{p,q})$ consists of closed forms in $\mathcal{J}^{p-1,q} \oplus \mathcal{J}^{p,q-1}$;
- $BV(\mathcal{H}^{m+1,0})$ (resp. $BV(\mathcal{H}^{0,m+1})$) consists of closed forms in $\mathcal{J}^{m,0}$ (resp. in $\mathcal{J}^{0,m}$) ;
- $BV(\mathcal{H}^{m,1})$ (resp. $BV(\mathcal{H}^{1,m})$) is the L^2-orthogonal complement of $BV(\mathcal{H}^{m+1,0})$ (resp. of $BV(\mathcal{H}^{0,m+1})$) inside closed forms in $\mathcal{J}^{m,0} \oplus \mathcal{J}^{m-1,1}$ (resp. $\mathcal{J}^{0,m} + \mathcal{J}^{1,m-1}$), unless $m=1$;
- when $m=1$, $BV(\mathcal{H}^{1,1})$ is the orthogonal complement of $BV(\mathcal{H}^{2,0}) \oplus BV(\mathcal{H}^{0,2})$ in $\mathcal{J}^2 = \mathcal{J}^{1,0} \oplus \mathcal{J}^{0,1}$.

3.6 The ring of representations of $SU(m+1,1)$.– The ring $R(G)$ consists of equivalence classes of formal differences of G-modules with a finite difference of dimension, i.e., of Fredholm G-modules.

P. Julg and G. Kasparov ([10]) prove that $R(SU(m+1,1)) = R(U(m+1))$.

Theorem 3.5 is a crucial tool in the proof. Indeed, it allows them to construct a representative of an important element γ of $R(SU(m+1,1))$ as a representation of $SU(m+1,1)$ on a module over the algebra of continuous functions on the compactification $\overline{X}$ of complex hyperbolic space – forms on X plus Rumin forms on ∂X – which implies that $\gamma = 1$.

3.7 Poisson transform.– The results 3.2 and 3.5 provide us with a Poisson transform for closed middle degree forms, whose inverse is given by taking boundary values, in an L^2 setting.

More generally, one may naively wonder wether there is a Poisson transform for differential forms on symmetric spaces G/K with the following properties :

- it is G-equivariant,
- it commutes with the exterior differential,
- it coincides with the ordinary Poisson transform for functions,
- its inverse amounts to take some kind of boundary value,
- its image consists of all harmonic forms on G/K.

P.Y. Gaillard has studied in [4] the case of real hyperbolic space, i.e., $G = SO(n,1)$ (see also [8]). The Poisson transform takes forms on the boundary isomorphicly onto coclosed harmonic forms, and commutes with exterior derivative. (There is however an exception : in dimension $2m+1$, the Poisson transform kills coclosed m-forms on the boundary, and thus reaches only closed and coclosed m-forms on hyperbolic space). In general, Poisson transforms have boundary values only in degrees strictly less than half the dimension.

It is likely that there is an analogous Poisson transform for differential forms on complex hyperbolic space. Obviously, Rumin differential forms and modified exterior differential should be used. Also, Poisson transforms are probably automaticly primitive, ∂ and $\overline{\partial}$-coclosed.

4 L^p-COHOMOLOGY

The proof of P. Julg's theorem involves several magic identities satisfied by special functions. We present now a direct argument that shows that L^2-harmonic forms on complex hyperbolic space are representable by partial boundary values. It turns out that the method applies to L^p-cohomology as well.

Recall that the L^p-cohomology $H_p^*(X)$ of a Riemannian manifold X is the cohomology of the complex $(\Omega_{(p)}^*(X), d)$ where $\Omega_{(p)}^*(X)$ is the space of differential forms α with $|\alpha| \in L^p$ and $|d\alpha| \in L^p$. In general, the image

$$d(\Omega_{(p)}^{k-1}(X)) \subset \Omega_{(p)}^k(X)$$

is not closed, and one defines *reduced* L^p-cohomology as the quotient

$$\overline{H}_p^k(X) = \Omega_{(p)}^k(X)/\overline{d\Omega_{(p)}^{k-1}(X)}$$

by the closure of the image of d.

If $H_p^*(X) = \overline{H}_p^*(X)$, i.e., if the image $d(\Omega_{(p)}^{k-1}(X))$ is closed in $\Omega_{(p)}^k(X)$, we say that X *has only reduced L^p-cohomology in degree k*. This property is invariant under coarse quasiisometries, like those arising from isomorphisms of cocompact isometry groups.

For $p = 2$, there is exactly one L^2-harmonic form in each reduced L^2-cohomology class, i.e., the space of L^2-harmonic forms is isomorphic to $\overline{H}_2^*(X)$.

We explain next that on a negatively curved manifold, a closed form in L^p often admits a boundary value. We start again with the easier case of real hyperbolic space, which has been computed independently by V. Goldshtein, V. Kuz'minov and I. Shvedov, [6].

4.1 L^p-cohomology of real hyperbolic space.–

One uses the decomposition of real hyperbolic n-space X as a warped product

$$X = \mathbf{R}_+ \times_{\sinh r} S^{n-1}.$$

Split a k-form α as

$$\alpha = a + b \wedge dr$$

where a and b are viewed as functions on $\mathbf{R}_+$ with values in L^p-differential forms on the sphere S^{n-1}. The L^p norm of α is roughly the norm of a in $L^p(e^{(n-1-kp)r}\,dr)$ plus the norm of b in $L^p(e^{(n-1-(k-1)p)r}\,dr)$. The form α is closed iff a is closed and $\frac{\partial}{\partial r}a = \pm db$, which can be written

$$\frac{\partial}{\partial r}d^{-1}a = \pm d^{-1}db$$

where d^{-1} takes exact k-forms to coexact $k-1$-forms. Thus d^{-1} denotes the pseudo-inverse of d.

If $p < n - 1/k - 1$, $L^p(e^{(n-1-(k-1)p)r}\,dr) \subset L^1(dr)$ so $d^{-1}\alpha$ converges in L^p as $r \to +\infty$, and a converges to a distribution $a(\infty) = BV(\alpha)$.

If $\alpha \in d(\Omega_{(p)}^{k-1})$, or if $p \leq n - 1/k$, then $BV(\alpha) = 0$.

Conversely, if $BV(\alpha) = 0$, then α admits a primitive β in L^p. Indeed, the Poincaré homotopy formula

$$\beta(r) = -\int_0^{+\infty} b(r+s)\,ds$$

(no dr component) solves $d\beta = \alpha$ and is in L^p (Hardy inequality) for $p < n - 1/k - 1$.

In conclusion, for real hyperbolic n-space, L^p-cohomology in degree k vanishes for $p \leq n-1/k$, and, for $n-1/k < p < n-1/k-1$, it is isomorphic to a certain function space of closed k-forms on S^{n-1}. In particular, it is a Hausdorff Banach space, thus, for such values of p and k, real hyperbolic space has only reduced cohomology. The L^p norm can be recovered in terms of boundary values - up to a constant, see [12] for the case when $k = 1$.

For $p = n - 1/k - 1$, reduced cohomology vanishes but L^p-cohomology is huge.

The same argument applies to manifolds with variable curvature. Indeed, what matters is the Lie derivative of the metric on forms under the radial vector field $\frac{\partial}{\partial r}$. This is controlled by sectional curvature. This leads to the following comparison result (Jens Heber's help was instrumental in obtaining the sharp curvature assumption).

4.2 THEOREM [13].– *Let X be a complete simply connected Riemannian manifold of dimension n with negatively δ-pinched sectional curvature, i.e., $-1 \leq K \leq \delta < 0$. For all*

$$p < 1 + \frac{n-k}{k-1}\sqrt{-\delta},$$

an L^p closed k-form admits a boundary value, which determines its cohomology class. In particular, X has only reduced L^p-cohomology in degree k.

4.3 L^2-cohomology of complex hyperbolic plane.–

We now check that the L^2-cohomology of complex hyperbolic plane in degree 2 is a limiting case of the above comparison theorem. Indeed, the theorem applies to L^p closed 2-forms on complex hyperbolic plane, for all $p < 2$: there exists a boundary value, which determines the L^p-cohomology class.

For $p \geq 2$, the boundary value does not exist any more, but a partial boundary operator will replace it, at least when $p < 4$

The complex hyperbolic plane in polar coordinates is not a warped product : the metric on spheres increases at different speeds along the factors of the splitting

$$TS^3 = \xi \oplus \xi^{\perp}.$$

Accordingly, a 2-form has to be split into four components

$$\alpha = a + a' d\tau + b \wedge dr + b' \tau \wedge dr.$$

The L^p norm of α in roughly the sum of the norms of $a \in L^p(e^{(4-3p)r}\, dr)$, $a' + b' \in L^p(e^{(4-2p)r}\, dr)$, $b \in L^p(e^{(4-p)r}\, dr)$. For $p = 2$, the limiting exponent 0 for b' prevents one from having an ordinary boundary value as in the preceding paragraph.

If we view the forms $a + da' \wedge \tau \in \mathcal{J}^2$ and $b \in \Omega^1/\mathcal{I}^1$ as elements of Rumin's complex, the equation $d\alpha = 0$ implies

$$d_{\xi}(a + da' \wedge \tau) = 0$$

and

$$\frac{\partial}{\partial r}(a + da' \wedge \tau) = d_R b$$

which implies that $a + da' \wedge \tau$ converges (when $p < 4$), this is our partial boundary value $BV(\alpha)$. It factors through reduced L^p-cohomology , and is injective on the reduced cohomology.

It turns out that the complex hyperbolic plane has only reduced cohomology in degree 2. This is a special case of a theorem of A. Borel, [2]. It also follows from estimates on the spectrum of the Laplacian. Indeed, for L^2-functions and 1-forms, the spectrum of the Laplacian is bounded below, [3]. This implies an estimate of the form

$$\| \beta \|_2^2 \leq C(\| d\beta \|_2^2 + \| \delta\beta \|_2^2)$$

for compactly supported 1-forms β, which therefore implies that the image $d(\Omega^1_{(2)}(\mathbf{C}H^2))$ is closed in $\Omega^2_{(2)}(\mathbf{C}H^2)$. We conclude that our partial boundary value BV is injective on L^2-cohomology .

This elementary approach cannot give the finer information on complex types contained in theorem 3.5.

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