Quasiconformal mappings and manifolds of negative curvature

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In 1928, H.Grötzsch [19] observed that the classical Liouville and Picard theorems - nonexistence of entire functions which are bounded or, more generally, omit more than two points - extended to a class of non holomorphic maps. A holomorphic bijection between plane domains is conformal with respect to Euclidean metric : its differential at each point is a similitude, i.e., an isometry times a homothety. H. Grötzsch considered maps whose differential sits at a bounded distance from similitudes. If, at some point x, the differential takes a circle to an ellipse with axes a and b, he defined the distorsion $Q(x)$ as the least number such that

$$1/Q \leq a/b \leq Q .$$

He furthermore allowed a discrete set of singular points where the map is a ramified covering. He showed that the Picard theorem extends to the class of maps defined on the whole plane which have bounded distorsion.

Nowadays, these maps are called *quasiregular* maps. A smooth diffeomorphism with bounded distorsion is called *quasiconformal*, and the word "quasiregular" includes maps which are not 1-1 and whose differential may vanish.

It is clear that one can define quasiconformality, at least for smooth diffeomorphisms between Riemannian manifolds in any dimension. This notion plays a crucial role in G.D. Mostow's rigidity theorem for compact Riemannian manifolds of constant sectional curvature -1 and dimension ≥ 3. This theorem states that two such manifolds, if diffeomorphic, have to be isometric. The argument involves quasiconformal mappings in the following way. A diffeomorphism δ between two quotients D/Γ and D/Γ' of the unit disk D in R^n lifts to a quasiconformal mapping Δ of D. Δ extends by continuity to a quasiconformal homeomorphism f of the unit sphere S^{n-1}. It satisfies

$$f \cdot g = \delta_*(g) \cdot f \quad , \quad g \in \Gamma \qquad (*)$$

where δ_* denotes the isomorphism induced by δ on the fundamental groups Γ and Γ'. Equation $(*)$, together with some regularity of f implies that f is in fact conformal. The corresponding hyperbolic isometry of D descends to an isometry between the quotients.

In this talk, we investigate whether the first steps of the argument carry over to general simply connected manifolds of negative curvature. In this class, there is a natural notion of "ideal boundary" [7], thus two questions arise :

do quasiconformal mappings extend to this boundary ?
is the extension quasiconformal in some sense ?

In section 1, we discuss a generalization of Schwarz' Lemma which shows that a quasiconformal mapping f between manifolds of bounded negative curvature is a quasiisometry, i.e., satisfies metric inequalities

$$-C + d(x,y)/L \leq d(fx,fy) \leq Ld(x,y) + C$$

where C and L are large constants. Such a map extends to ideal boundaries. Our proof relies on conformal distances constructed by means of capacities. Section 2 contains capacity estimates which lead to a nice application : locally symmetric spaces do not admit very pinched metrics. The precise definition of quasiregular maps is delayed until section 3, where the conformal structure on the boundary of homogeneous Riemannian manifolds of negative curvature is described. In sections 4 and 5, we organise material found in several papers by Pekka Tukia and Dennis Sullivan, in particular, we observe that one can speak of conformal and quasiconformal mappings on the boundary of a manifold of negative curvature as soon as there is a cocompact isometry group.

I learned most of the material described here in conversations with Mikhail Gromov. I hope that, in the present paper, it is apparent how much I owe him. Section 2 was completed in Japan during the Symposium. It is a pleasure to thank the Taniguchi Foundation for its generous support.

1. Schwarz Lemma and conformal distances

The Schwarz Lemma claims that, if a holomorphic function f on the unit disk fixes the origin, i.e., $f(0) = 0$, and if $|f(z)| \leq 1$ for all z with $|z| \leq 1$, then $|f'(0)| \leq 1$. The normalization $f(0) = 0$ can be avoided by expressing the result in terms of the *hyperbolic* metric d of the unit disk.

1 *Schwarz Lemma*.- Let the unit disk be equipped with its hyperbolic metric d of constant curvature -1. Then any holomorphic map of the disk to itself is distance decreasing, i.e.,

$$d(f(x),f(y)) \leq d(x,y)$$

In this form, the lemma belongs to Riemannian geometry. This is even more clear with L. Ahlfors' extension of Schwarz' Lemma to surfaces of variable curvature.

2 *Theorem* [1] .-Let S be a surface endowed with a Riemannian metric of curvature ≤ -1. Any holomorphic function on the disk with values in S is distance decreasing.

It is apparent in L. Ahlfors' paper [1] that Schwarz' Lemma only depends on the isoperimetric behaviour of the target surface S. Let us define the *isoperimetric profile* $I(v)$ of a Riemannian manifold M as follows. For a real number $v \leq \mathrm{Volume}(M)$, let $I(v)$ be the infimum of the volumes of the hypersurfaces in M which bound a compact domain of volume v.

$$I(v) = \inf \{\mathrm{vol}(\delta D) : D \text{ compact}, \mathrm{vol}(D) = v\} .$$

Remember that the Classical Isoperimetric Inequality states that the isoperimetric profile of Euclidean space R^n has the form

$$I_{eucl}(v) = (c_n\ v)^{n-1/n}$$

for a sharp constant c_n.

3 *Theorem* (M. Gromov [18], compare I.G. Reshetnjak [49]).- Let N be a complete n-dimensional Riemannian manifold whose isoperimetric profile I satisfies the following two inequalities :

(i) for small v, N is at least as good as Euclidean space, i.e., if one writes

$$I(v)^{n/n-1} = c_n v(1 + \tau(v)) ,$$

then the integral

$$\int_0 T(v)^{-} \, dv/v$$

should be finite.

(ii) for large v, N is strictly better than Euclidean space, i.e., the integral

$$\int^{+\infty} I(v)^{-n/n-1} \, dv$$

should be finite.

Let M be a Riemannian manifold with sectional curvature $\geq -a^2$. Then any conformal (resp. quasiregular) immersion of M to N is Lipschitz (resp. Hölder continuous) with constants which depend only on a, the function I and the deviation from conformality.

Remark.- This theorem is sharp under the condition that a sharp isoperimetric inequality is used. For example, in order to conclude from it that isometries are the only conformal self maps of a rank one symmetric space M (which in fact is known, see [28]), one needs that, among domains in M of a given volume, balls have minimum boundary volume. This is known yet only when the sectional curvature is constant.

4 Corollary.- Let M, M' be complete simply connected Riemannian manifolds with bounded negative sectional curvature, i.e.,

$$-a^2 \leq K \leq -b^2 < 0$$

Then any quasiconformal diffeomorphism of M onto M' is a quasiisometry, i.e., satisfies inequalities of the form

$$-C + d(m,m')/L \leq d(fm,fm') \leq L\, d(m,m') + C$$

for some constants C and L which depend only on a, b and the deviation from conformality.

Proof.- It is known that the assumption on M implies a linear isoperimetric inequality (see [16] chap. 6), thus condition (ii) is satisfied. Condition (i) follows from a very general principle : as far as small volumes are concerned, all Riemannian manifolds behave almost like Euclidean space, see [4]. Finally, the Hölder condition for f and f^{-1}, as well as any relation of the type $d(fx,fy) \geq \xi(d(x,y))$, for some homeomorphism ξ of $\mathbb{R}_+$, implies that f is a quasiisometry.∎

Let us now return to H. Grötsch's original motivation for the study of quasiregular mappings. Schwarz' Lemma implies Liouville's theorem as follows. A bounded entire function f is interpreted as a quasiregular map f from $\mathbb{C}$ to the disk. Let us compose f with homotheties of $\mathbb{C}$, and restrict them to the unit disk in $\mathbb{C}$. With respect to hyperbolic metrics, all these maps should be uniformly Hölder, thus equicontinuous. Considering larger and larger homotheties at a point z shows that $f'(z)$ vanishes, thus f is constant. The same argument gives Picard's theorem, once one observes that the plane minus two points admits a conformal complete metric with bounded negative curvature, the quasihyperbolic metric which will be defined in § 10.

However, the isoperimetric method does not provide any extension of Picard's theorem to dimensions ≥ 3. The following theorem by S. Rickman requires some more value distribution theory.

5 Theorem ([50]).- For $n \geq 3$, $q \geq 3$, there is a lower bound $K(n,q)$ for the deviation from conformality of quasiregular maps on $\mathbb{R}^n$ which omit at least q values.

It is striking that there exist such maps in dimension 3 [52]. The preceding theorem has a quantitative version ([51]) : A quasiregular map on the disk which omits enough points $a_1,\dots,a_q$ is Lipschitz (as far as large distances are concerned) with respect to a complete conformal metric on the complement of these points. This metric is not the quasihyperbolic metric.

Question.- What are the invariance properties of this metric ?

The Schwarz Lemma, as a tool to prove nonexistence of holomorphic maps from $\mathbb{C}^n$ to certain complex manifolds, has had many developments (see [25]). The idea is to construct functorial pseudo-distances on complex manifolds, for which holomorphic maps are distance decreasing. Such a distance has to be identically zero for $\mathbb{C}^n$ thus no holomorphic map can exist from $\mathbb{C}^n$ to a manifold for which the pseudo-distance is a distance ("hyperbolic" complex manifolds). There are analogues in projective [26] and affine geometry [66].

In conformal geometry, functorial pseudo-distances can be constructed using capacities. Conformal capacity has been used in the plane for a long time. Its introduction in higher dimensions is due to Ch. Loewner [34].

6 Definition.- A condenser in a manifold is a triple (C,B_0,B_1) where C is open, B_0 and B_1, the "plates", are closed and contained in the closure of C. We shall admit $B_0 = \infty$ when C is unbounded. Assume that the manifold is Riemannian. The conformal capacity of a condenser is the infimum of the volumes of the conformal metrics on C for which the plates stay at distance 1 apart, i.e.,

$$\operatorname{dist}(B_1,B_0) \geq 1 .$$

7 Definition.- Let M be Riemannian manifold. We define two conformal pseudo-distances α and β as follows : for two points x and y,

$$\alpha(x,y)^{-n} = \inf\{\operatorname{cap}(M,B_0,B_1) : B_0 \text{ (resp. } B_1) \text{ is connected, unbounded and contains } x \text{ (resp. } y)\}$$

$$\beta(x,y) = \inf\{\operatorname{cap}(M,B,\infty) : B \text{ is compact, connected and contains } x \text{ and } y)\}$$

Both these quantities enter as tools in a number of papers (for references, see the papers of M. Vuorinen [68] and H. Tanaka [54]). They have been studied for their own sake by J. Ferrand [31] and I.S. Gál [8] respectively.

8 Properties.-

i) α and β are conformal invariants

ii) a quasiconformal homeomorphism is bi-Lipschitz with respect to α and β.

iii) a quasiregular map is Lipschitz with respect to β.

9 Example.- For two point homogeneous spaces, the conformal pseudo--distances and the symmetric Riemannian metric d are functionally dependant, i.e., $\beta = \xi(d)$, but the function ξ is more or less unknown. In the constant curvature case, F.W. Gehring [9] has shown that, in the definition of α or β, the minimizing condenser is the one whose plates are geodesic segments (known as the Teichmüller - resp. Grötsch - condenser). This is unknown for other rank one (non compact) symmetric spaces. This yields inequalities of the form

$$A\, d \leq \xi(d) \leq A\, d + B$$

where A and B are constants depending only on the dimension, see [68]. A similar inequality is obtained for α thanks to the identity

$$\alpha(x,y)^{-n} = 2^{n-2}\, \xi(\log(1+2t+\sqrt{1+t})) \quad \text{where} \quad 2t = 1/\cosh(\tfrac{1}{2}d(x,y))-1 .$$

In dimension 2, more is known. Indeed, the Schwarz- Christoffel formula for the conformal mapping of the half-space onto the Teichmüller condenser leads to a representation of ξ by means of an elliptic integral, see [30].

It is obviously important to know for which manifolds the functions α and β really are distances, i.e., are positive. This has been studied by M. Vuorinen in the case of Euclidean domains. Then things are easier since one can use direct comparison with the ball. This is the reason why one introduces the following distances.

10 Definition [12].- Let Ω be an open, connected subset of Euclidean space. The quasihyperbolic metric k_Ω on Ω is obtained from Euclidean metric by a conformal factor equal to the inverse of the Euclidean distance to the boundary $\delta\Omega$. The distance j_Ω is defined by
$$j_\Omega(x,y) = \log(1 + |x-y|/d(\{x,y\},\delta\Omega))$$

11 Theorem [68].- For a domain Ω in R^n , the pseudo-distances α , β , k_Ω and j_Ω are linked by the following inequalities.
$$j_\Omega \le k_\Omega$$
$$\beta \le \Phi_1(k_\Omega) \le A\, k_\Omega + B$$
if $\delta\Omega$ is connected, $j_\Omega \le C\,\beta$
$$\exp(D\, k_\Omega + E) \le \alpha \le \Phi_2(j_\Omega)$$
where A, B, C, D, E are constants, Φ_1 and Φ_2 are homeomorphisms of R_+ depending only on the dimension.

These inequalities show that, for most Euclidean domains the conformal invariants α and β are distances. This is not that clear for general Riemannian manifolds. Again, isoperimetric inequalities give a useful criterion concerning β .

12 Theorem ([43], compare with [16]). Assume that the n-dimensional complete Riemannian manifold M satisfies a strictly stronger isoperimetric inequality than Euclidean n-space, i.e., condition (ii) in Theorem 3. Then the conformal pseudo-distance β is a distance.

If furthermore M has bounded geometry, i.e., bounded sectional curvature and a positive injectivity radius, and if for example one has an isoperimetric inequality of type
$$\mathrm{vol}(\delta D) \ge \mathrm{const.}\ \mathrm{vol}(D)^{1-\tau} \quad \text{with} \quad \tau < 1/n \ ,$$
then, for all $x, y \in M$,
$$\beta(x,y) \ge A\, d(x,y)^{1-n\tau} - B$$
for some constants A and B .

13 As a consequence, all simply connected manifolds of bounded negative sectional curvature, all non almost abelian solvable Lie groups (N. Varopoulos [64] and [65]) have a conformal distance. They do not admit quasiregular maps from Euclidean space, or, more generally, from any manifold with vanishing β . Any quasiconformal mapping between two such manifolds is a quasiisometry, as defined in the introduction.

14 For manifolds of bounded negative curvature, Theorem 12 is reasonably sharp. Indeed, the reverse inequality
$$\beta(x,y) \le A\, d(x,y) + B$$
holds. To check this, one merely needs compute the L^n norm of the gradient for some function of the distance to the geodesic segment through x and y .

On the other hand, let M be the 3-dimensional Heisenberg group, i.e., the simply connected nilpotent nonabelian group in dimension 3. Then an isoperimetric inequality holds with exponent $\tau = 1/4$ [43], thus Theorem 12 yields $\beta \ge d^{1/4}$. One easily sees that, conversely,
$$\liminf_{d(x,y)\to+\infty} \beta(x,y)/d(x,y) = 0 \ .$$
Indeed, given a left-invariant metric g , split g orthogonally as
$$g = g_H + g_Z$$
where Z is the direction of the center and H its orthogonal complement. Define a group automorphism δ_t by
$$\delta_t = t\ \mathrm{id}_H + t^2\ \mathrm{id}_Z$$
so that δ_t is a homothety by a factor t of g onto the metric
$$g^t = g_H + t^2\, g_Z \ .$$

y . Then $d(\delta_t x,\delta_t y) = t\ d(x,y)$, whereas
$$\beta(\delta_t x,\delta_t y) = t\ \beta^t(x,y) \le t\ \mathrm{Cap}^t(M,\sigma,\infty) \ .$$
When t goes to infinity, the metrics g^t "converge" to a Carnot metric d_∞ (see section 3) with Hausdorff dimension equal to 4. The capacity $\mathrm{Cap}^t(M,\sigma,\infty)$ converges to a corresponding capacity $\mathrm{Cap}_{\infty}{}^{\infty}(M,\sigma,\infty)$ which vanishes, since σ has codimension 3 with respect to d_∞ .

Question.- Determine the asymptotics of β on nilpotent groups.

To show that the conformal invariant α is non trivial, one merely needs produce a condenser with finite capacity.

15 Proposition.- (see section 2) Let M be an n-dimensional simply connected Riemannian manifold with pinched negative curvature
$$-1 \le K < -1 + 1/n \ .$$
Its conformal pseudo-distance α is a distance.

On the other hand, it is very likely that the invariant α vanishes for a smaller pinching, in particular for quaternionic hyperbolic spaces (compare Corollary 21 of section 2). This would show that no general inequality holds between α and β .

16 In [63] chapter17, J. Väisälä introduces property P1 for a domain Ω in R^n : Ω has property P1 at a boundary point b if, for all connected subsets E and F of Ω containing b in their closure, the capacity $\mathrm{cap}(\Omega,E,F)$ is infinite. He then shows that, if Ω has property P1 and if Ω' is an other domain which is finitely connected at each boundary point, then every quasiconformal mapping of Ω onto Ω' extends continuously to the boundary. A Riemannian manifold has vanishing invariant α (the class H_o of [31]) if and only if it has property P1 at ∞ . It would be interesting to have natural examples of manifolds with this property.

One can find other kinds of conformally invariant distances in the litterature. The Yamabe conjecture – in a conformal class of Riemannian metrics, find one, often unique, with constant scalar curvature – produces a conformally invariant Riemannian metric. It has been studied exactly for this purpose by Ch. Loewner and L. Nirenberg [35] on Euclidean domains. Another method consists in starting with some Riemannian metric and normalizing it by a clever conformal factor, a suitable power of the length $|W|$ of the Weyl tensor, see [42]. Both these tricks have pseudoconformal analogues, see [23] and [6]. These metrics behave badly under quasiconformal mappings. Indeed, there are quasiconformal mappings which are not locally Lipschitz, and thus, not Lipschitz under any Riemannian metric.

To study a conformally flat manifold M , one may be tempted to imitate the construction of the Kobayashi holomorphic or projective distances, i.e., define
$$\sigma(x,y) = \inf\{d(z,t) : z\ ,\ t \text{ in the disk } D \subset R^n\ ,$$
$$\text{there exists a conformal embedding } f : D \hookrightarrow M$$
$$\text{such that } f(z) = x\ ,\ f(t) = y\}\ ,$$
and make a distance out of the function σ . For Euclidean domains, inequalities as in Theorem 11 hold, but it is unclear for which conformally flat manifolds this distance vanishes.

2. Capacity estimates

In this section, we prove two inequalities concerning conformal not vanish. The proof applies to capacities with arbitrary exponents, which is also of interest. (Remember that capacity originates from electrostatic capacity in physics, which, in dimension 3, has exponent 2, see [47]).

17 Definition.–Let (C, B_o, B_1) be a condenser in a Riemannian manifold. Let p be a positive real number. The p-capacity is

$$\inf \{ \int_C |du|^p : u \text{ smooth on } C \text{, extends continuously to values } 0 \text{ on } B_o \text{ and } 1 \text{ on } B_1 \}$$

Conformal capacity is obtained for p equal to the dimension (see [10] for the equivalence with definition 6).

Imitating J. Ferrand's definition of conformal distance α, we introduce a notion of critical exponent for a simply connected Riemannian manifold of non positive sectional curvature.

18 Definition.– Let M be an open Riemannian manifold. Its critical exponent $p(M)$ is the least exponent $q > n-1$ such that there exist condensers with connected and unbounded plates and which have a finite q-capacity.

If furthermore M has non positive sectional curvature, it has an Eberlein-O'Neill boundary δM. One can consider condensers of type (M,x,y) where x and y are points at infinity. We define the modified exponent $p(M)$ as the infimum of exponents q such that there exist points $x, y \in \delta M$ with $\mathrm{Cap}_q(M,x,y) < +\infty$.

It is likely that $p(M) = p(M)$. On the other hand, $p(M) > \dim M$ implies that the conformal distance α does not vanish. Thus Proposition 15 follows from the following Lemma.

19 Lemma.– Let M be simply connected and have bounded negative curvature $-a^2 \leq K \leq -b^2 < 0$. Then both $p(M)$ and $p(M) \leq (n-1)a/b$.

Proof.– We exhibit a condenser which has finite q-capacity for all $q > (n-1)a/b$. We choose B_o and B_1 to be two opposite rays on a geodesic τ. Let m be a point on τ between B_o and B_1. Let u be any function on M which is constant on rays through m. We claim that du is L^q-integrable outside a neighbourhood of m. Indeed, Rauch's comparison theorem and $K \leq -b^2$ yield

$$du \leq e^{-br}$$

on the sphere S_r of center m and radius r, whereas $K \geq -a^2$ implies

$$\mathrm{vol}(S_r) \leq e^{(n-1)ar}.$$

Thus

$$\int_M |du|^q \leq \int^{+\infty} e^{-qbr} \mathrm{vol}(S_r)\, dr \leq \int^{+\infty} e^{((n-1)a-qb)r}\, dr$$

is finite if $q > (n-1)a/b$. ∎

In fact, if we denote the volume entropy by

$$h_{vol}(M) = \limsup_{r \to +\infty} \log \mathrm{vol}(S_r)/r ,$$

we have proven the following inequality

$$K \leq -b^2 \Rightarrow p(M) \leq h_{vol}(M)/b .$$

For example, volume entropy for a rank one symmetric space with sectional curvature normalized by $-4 \leq K \leq -1$ and dimension n is equal to $n + k - 2$, where $k = 2$ for complex hyperbolic spaces, $k = 4$ for quaternionic hyperbolic spaces, $k = 8$ for Cayley hyperbolic plane. Thus these space have $p \leq n + k - 2$. This inequality is sharp.

20 Lemma.– Let M be a rank one symmetric space with sectional curvature normalized by $-4 \leq K \leq -1$. For each $n-1 < q < n + k - 2$, there exists a positive constant $c_{n,q}$ such that, if h denotes a horofunction attached to a point at infinity x, and B is any closed subset of M, then

$$\mathrm{Cap}_q(M,B,x) \geq c_{n,q}\, \mathrm{length}(h(B)) .$$

21 Corollary.– For such a symmetric space, $p = n + k - 2$.

Indeed, for $q < n + k - 2$, $x, y \in \delta M$, u smooth on M with $u(x) = 1$, $u(y) = 0$, h a horofunction attached to x or y one has

$$2 \int_M |du|^q \geq c_{n,q}\, 2^{-q}[\mathrm{length}(h\{u>½\}) + \mathrm{length}(h\{u<½\})] = +\infty . ∎$$

22 Corollary.– Let M be a compact quotient of a rank one symmetric space of dimension $n \geq 2k$. Then M does not admit any metric with pinching better than $(n-1/n+k-2)^2$.

Notice that the sharp result – 1/4 is the best possible pinching – is already known in dimension 4 (M. Ville [67]).

The invariants p and p can be defined in a combinatorial way for nets. It is likely that they are quasiisometry invariants (see the work of M. Kanai [24] for 2-capacities). If this is true, then the conclusion of Corollary 22 extends to all compact manifolds whose fundamental group is isomorphic to a cocompact subgroup of $U(m,1)$, $Sp(m,1)$ or F_4^{-20}.

Question (Gromov).– Compare $p(M)$ with the exponents for which L^q or $L^{q,\infty}$-cohomology of M in degree 1 vanishes.

Proof of Lemma 20 .– Let us foliate the symmetric space by parallel horospheres N centered at x – the levels of the horofonction h. Let u be a function on M which takes value 1 on B and extends by continuity to value 0 at x. By the coarea formula, it suffices to uniformly estimate

$$\int_N |du_{|N}|^q .$$

For the horospheres which hit B – a set of measure length $h(B)$ – the function $u_{|N}$ takes the value one on N, thus the integral is greater that some capacity $\mathrm{Cap}_q(N,\text{point},\infty)$. This capacity does not depend on the particular point, since N is homogeneous. It does not depend on the particular horosphere, since they are all pairwise isometric. It is non zero for two reasons.

i) Since the exponent $q > n-1$, the Sobolev embedding of $W^{1,q}$ into $C^{1-(n-1)/q}$ allows one to replace the point by a ball of finite size as a plate of the condenser.

ii) The horosphere N is isometric to a nilpotent Lie group with left-invariant metric, whose isoperimetric profile satisfies $I(v) \geq \text{const.}\, v^{p-1/p}$ where $p = n + k - 2$ (N. Varopoulos [65]), thus Theorem 12 applies. ∎

For M a symmetric space and x, y points at infinity, we show that $\mathrm{Cap}_q(M,x,y) = +\infty$ for all $q < n + k - 2$. Question.– What happens when q is equal to the critical exponent ?

3. Regularity properties of quasiconformal mappings.

Early, it has turned out to be necessary to consider quasiregular mappings which are not of class C^1. In Teichmüller's theory (see [55], [3], [5]) one obtains as solutions of a variational problem mappings which are smooth except at a finite number of points. Furthermore, in the deformation theory of Riemann surfaces, there definitely occur quasiconformal mappings which are nowhere smooth, as we shall see below. We give two equivalent definitions of quasiregular maps. A quasiconformal mapping is a quasiregular

23 Analytic definition.- (C.B. Morrey [37]) A continuous map f between Riemannian manifolds of dimension $n \geq 2$ is K-quasiregular if it admits a differential df in the sense of distributions which is a locally L^n-integrable function and satisfies

$$|df|^n \leq K \, \mathrm{Jac}(f) .$$

The number K is only one of the various ways to measure the deviation from conformality, i.e., the distance between the differential df and the similitudes. In terms of the eigenvalues $\mu_1^2,\dots,\mu_n^2$ of the endomorphism ${}^t df \cdot df$, one has

$$K = \mu_n^n/\mu_1 \dots \mu_n .$$

An equally satisfactory quantity is

$$Q = \mu_n/\mu_1$$

which satisfies

$$\log Q \leq \log K \leq (n-1) \log Q .$$

For a linear map A between Euclidean spaces, the number Q has a metric interpretation : Given a ball B , its image is pinched between two balls $B(s)$ and $B(S)$ - i.e.,

$$B(s) < AB < B(S)$$

such that $Q = S/s$.

More generally, if f is a continuous, discrete, open map between Riemannian manifolds, x is a point and r is small enough, one can define the ratio $Q_f(x,r) = S/s$ where S is the minimum radius of a ball centered at $f(x)$ which contains $fB(x,r)$, and s is the maximum radius of a ball centered at $f(x)$ which is contained in $fB(x,r)$.

24 Metric Definition (M.A.Lavrentiev [29]) A continuous map between Riemannian manifolds is quasiregular if it is orientation preserving, open, discrete, and if

$$Q_f(x) = \limsup_{r \to 0} \; Q_f(x,r)$$

is bounded.

There is a third characterization of quasiconformality by means of capacities, [46], [2], [36]. The fact that, in dimensions ≥ 2, these definitions coincide is a series of theorems by I.N. Pesin [45], J.A. Jenkins [22], F.W. Gehring - J. Väisälä [14], O. Martio - S. Rickman - J. Väisälä [36]. This is the conclusion of longstanding efforts to determine to which class of regularity quasiregular maps exactly belong. This regularity is expressed by the following properties.

25 Properties.- In dimensions ≥ 2, quasiregular mappings are absolutely continuous on lines, i.e., in a coordinate patch, a quasiregular map is absolutely continuous on almost every line. As a consequence, they send Lebesgue null sets to null sets.
Quasiregular mappings have a differential almost everywhere, which is L^n integrable.

These properties have turned out to be essential in G.D. Mostow's rigidity theorem for compact manifolds of constant sectional curvature.

26 Theorem [39].- If two compact Riemannian manifolds of dimension ≥ 3 , with constant sectional curvature -1 , are diffeomorphic, then they are isometric.

27 Here is a sketch of the proof. A diffeomorphism between two such manifolds lifts to a quasiconformal mapping f of the universal covers, i.e., the unit disk in $\mathbb{R}^n$. Let us denote by Γ and Γ' the fundamental groups of the compact manifolds. They act conformally on the disk. The diffeomorphism induces an isomorphism $i : \Gamma \approx \Gamma'$ and, for $g \in \Gamma$, one has

$$f \cdot g = i(g) \cdot f$$

The quasiconformal mapping f extends to the unit sphere (Property P1 of § 16) and the extension, still denoted by f ,is a quasiconformal homeomorphism of the (n-1)-sphere (Schwarz' reflection principle). We now show, following P. Tukia [60], that f is a conformal mapping of the sphere. This is due to the fact that the action of Γ on the sphere is highly transitive, and necessitates little regularity of f . Still, it fails when $n = 2$. Choose the upper half-space model, and normalize f so that $f(0) = 0$ and $f(\infty) = \infty$. Consider the 1-parameter group of homotheties h_t . Since $\Gamma < O(n,1)$ is cocompact, there exist elements $g_t \in \Gamma$ such that $h_t^{-1} \cdot g_t = k_t$ are bounded in $O(n,1)$ Then one can write $i(g_t) = h_s \cdot j_t$ with j_t and the ratio s/t bounded. The conjugacy condition now reads

$$h_t^{-1} \cdot f \cdot h_t = h_{s/t} \cdot j_t \cdot f \cdot k_t^{-1} .$$

Choose subsequences such that s/t , k_t and j_t converge. If f is differentiable at 0 ,then in the limit $k \cdot f \cdot j = df(0)$ is linear. From there on, it is easy to show that f is conformal.▪

28 In [40], [41], G.D. Mostow generalized the rigidity theorem to all locally symmetric spaces without 2-dimensional factors. The argument in the rank one case also relies on the theory of quasiconformal mapping, but in a slightly extended context. Indeed, the first steps are the same. A symmetric space of rank one is a simply connected Riemannian manifold M with negative sectional curvature. As such, it admits an "ideal boundary", defined by means of asymptotic geodesics [7]. The lift of a diffeomorphism - in fact, of any homotopy equivalence - is a quasiisometry(as defined in the introduction). It extends to a homeomorphism of the ideal boundary (a fact which can be traced back to M. Morse [38]). This extension is not quasiconformal with respect to any Riemannian metric on δM . Indeed, this fails to be true even for isometries : the analogue of the homotheties in the upper half space model for hyperbolic geometry is a 1-parameter group δ_t of isometries whose action on δM can be written, in suitable coordinates x_i , y_j ,

$$\delta_t(x_i) = t \, x_i \; , \quad \delta_t(y_j) = t^2 \, y_j \, .$$

The plane with equations $dy_j = 0$ at the origin is part of a distribution of planes V which is invariant under the isometry group. The boundary extensions of isometries are conformal on the subbundle V and only there.

29 Let us define a family of distances on δM adapted to the situation. Fix a point $x \in M$. There is a unique Euclidean metric g_x on the subbundle V which is invariant under the isometries fixing x . It allows one to define the length of curves tangent to V , and we set, for two points p , q in δM ,

$$d_x(p,q) = \inf \{\text{length } c : c \text{ joins } p \text{ to } q \text{ in the boundary, } c \text{ is tangent to } V \}$$

This number is finite since the distribution V is non integrable, and defines a distance on δM . When x varies, the distance d_x changes conformally, i.e., a small d_x-ball is very close to a d_y-ball. Thus we have defined a conformal structure (in a generalized sense) on the boundary δM .

Now the boundary extension of a quasiisometry of M is a quasiconformal mapping with respect to any of the metrics d_x . Here we take the metric definition for quasiregular maps, which is meaningful for arbitrary metric spaces. The class of maps obtained coincides with G.D. Mostow's "quasiconformal mappings over a division algebra" [40]. These maps are absolutely continuous on a suitable class of "lines" [41] and almost everywhere differentiable [44] in a sense which we explain below. Thus P. Tukia's argument, as well as G.D. Mostow's, extends to prove the rigidity theorem in rank one.▪

Let M be a rank one symmetric space with isometry group G . To a choice of a point x in M and a boundary point $p \in \delta M$, there

K is the stabilizer of x

N is simply transitive on $\delta M - p$

A is a one-parameter group of translations along the geodesic through x and p.

In the constant curvature case, N is abelian and A consists of homotheties. In the other cases, N is two-step nilpotent, its Lie algebra splits as

$$n = V + [n,n]$$

and the element δ_t of A acts on n by multiplication by t on V and t^2 on $[n,n]$. Thus the ideal boundary of a rank one symmetric space identifies with a nilpotent Lie group. The results of absolute continuity and differentiability of quasi- conformal mappings will in fact apply to the whole class of Carnot groups, which we define now.

30 Definition.– A Carnot group is a simply connected nilpotent Lie group whose Lie algebra n splits as

$$n = V_1 \oplus \ldots \oplus V_r \quad \text{where} \quad [V_i, V_j] = V_{i+j} .$$

A Carnot group N admits a one-parameter group of homotheties

$$\delta_t \in \mathrm{Aut}(N) \ , \ \delta_t \text{ is multiplication by } t^i \text{ on } V_i .$$

By a norm, we mean a left-invariant distance on N which is homogeneous of degree one under the group of homotheties. Particular norms are the Carnot metrics : given a Banach space structure on V_1, one can define the length of curves in N which are tangent to the left-invariant subbundle of TN generated by V_1. One defines quasiconformal mappings using the metric definition. The class obtained does not depend on the particular choice of norm.

A continuous map f between Carnot groups N and N' equipped with homotheties $\{\delta_t\}$ and $\{\delta'_t\}$ is said to be δ-differentiable at x if the limit

$$Df(x)\mu = \lim_{t \to 0} \delta'^{-1}_t (f(x)^{-1} f(x \delta_t \mu)$$

exists for all μ.

A line is an orbit of a left-invariant vector field which is tangent to V_1.

For a smooth function u on N, let

$$du(x) = \sup \{ wu(x) : w \in V_1 \ , \ |w| = 1 \} .$$

We define the p-capacity of a condenser (C, B_0, B_1) as the infimum of the integrals (with respect to Haar measure)

$$\int_C |du|^p$$

over all smooth functions u on C which tend to 0 on B_0 and to 1 on B_1. Conformal capacity is obtained for p equal to the group's Hausdorff dimension

$$p = \Sigma \dim V_i$$

31 Theorem [44].– A quasiconformal homeomorphism between open subsets of Carnot groups admits almost everywhere a δ-differential which is a group isomorphism intertwining the two one-parameter groups of homotheties.

It is absolutely continuous on almost every line [41] and, as a consequence, it send null-sets to null-sets.

1-quasiconformal mappings preserve conformal capacities, and K-quasiconformal mappings multiply them at most by K (for a suitable measurement K of the deviation from conformality).

In other words, a big part of the analytic theory of quasiconformal mappings in Euclidean space can be carried out on Carnot groups. However, it seems to be harder to obtain capacity estimates. For example, the condenser between two concentric balls has positive capacity $c(r)$, depending only on the ratio r of the radii. This is sufficient to prove that 1-quasiconformal mappings are Lipschitz and to

However, one needs further information – still unknown – on the function $c(r)$ to conclude that quasiconformal mappings are Hölder continuous. It is also unclear whether the condenser whose plates are two arbitrary curves has a non-zero capacity.

32 A new feature of the nilpotent theory is that, in general, there are no quasiconformal mappings at all. The reason is that there is too little choice for differentials. Indeed, these should live in the group $\mathrm{Aut}_\sigma(N)$ of automorphism of N which commute with the homotheties. In the abelian case, this is the whole linear group, and every smooth diffeomorphism is locally quasiconformal. The Iwasawa component of $U(n,1)$ is the Heisenberg group. The group $\mathrm{Aut}_\sigma(N)$ consists of homotheties times symplectic $2n-2$ by $2n-2$ matrices ; a smooth diffeomorphism is locally quasiconformal if and only if it is a contact transformation, i.e., it preserves the plane distribution V [25]. This still produces an infinite dimensional group of quasiconformal mappings. In contrast, when N is the Iwasawa component of $Sp(n,1)$, $n \geq 2$, the group $\mathrm{Aut}_\sigma(N)$ consists of homotheties and a compact group $Sp(n-1)Sp(1)$. In this case, any quasiconformal mapping is 1-quasiconformal.

33 Corollary [44].– The elements of $Sp(n,1)$ are the only global quasiconformal self-maps of the boundary of quaternionic hyperbolic n-space.

In the same vein, if $\mathrm{Aut}_\sigma(N)$ consists of homotheties only – a case which definitely occurs, see [44] – then any (even local) quasiconformal mapping of N is the restriction of a translation or homothety.

It would be interesting to have a local version of the preceding corollary. This amounts to prove that local 1-quasiconformal mappings are smooth. Proofs of this fact in the Euclidean case are due to I.M. Reshetnjak [48] and F.W. Gehring [11]. They rely on non-linear elliptic regularity theory. In the nilpotent case, the corresponding equations are hypoelliptic and the necessary regularity is not yet available.

34 There is much room left for further generalizations, since the metric definition for quasiregular maps can be taken using arbitrary metric spaces.

i) The solvable Lie groups which admit left-invariant metrics with strictly negative curvature have been classified by E. Heintze [21]. They are of the form AN where N is a nilpotent Lie group and A a one-parameter group of contracting automorphisms. The data of N, A together with a norm is a very natural generalization of a Banach space with its homotheties. One can speak of differentiability and of quasiconformal mappings, a class which will not depend on the particular norm. A new feature is that these groups are not length spaces (cf [16]), i.e., the distance between two points is not the length of any curve joining them. In fact, a group N admits a length norm if and only if it is a Carnot group.

ii) A question of Gromov : as far as I know, nothing is known about quasiconformal mappings between separable Hilbert spaces. What about Liouville theorem ? Notice that traditional methods use integration, and thus do not extend to infinite dimensions.

iii) Lift the assumption of discreteness and you can speak of quasiregular maps between manifolds of different dimensions. Is Liouville theorem still true ?

4. Quasiconformal groups

35 Definition [13].- A quasiconformal group on a metric space is a group of uniformly quasiconformal homeomorphisms.

36 We are concerned with the following question, originally due to F.W. Gehring and B.P. Palka : when is a quasiconformal group of the standard sphere (resp., ideal boundary of a manifold of negative curvature) quasiconformally conjugate to a group of conformal transformations ?

A general method to adress this question is due to D. Sullivan [53]. He observes that every quasiconformal group leaves invariant at least one measurable conformal structure. Indeed, at each point, the space of conformal structure on the tangent space identifies with $Gl(n)/CO(n)$, which admits an invariant metric with non-positive sectional curvature. The set of pull-backs of the standard structure by the elements of the group is bounded, thus one can attach to it a unique point, the center of the smallest ball which contains it, for example (see [59]). These data form a measurable conformal structure, almost everywhere invariant under the group. Notice that this argument carries over to Carnot groups, since all what is needed is the a. e. differentiability of quasiconformal mappings. In this case, by a measurable conformal structure, we mean the data at each point of a Euclidean metric on the left-invariant subbundle generated by V_1. The metric should depend measurably on the point.

37 In dimension 2, the sphere has only one conformal structure, by Riemann's mapping theorem. (The extension to measurable conformal structures is due to C.B. Morrey [37]). Thus the invariant conformal structure is quasiconformally conjugate to the standard one, and we are done [57]. As a consequence, all quasiconformal groups in dimension 2 are known.

38 The argument fails in higher dimensions, and in fact, P. Tukia has constructed domains in R^n which admit a transitive, connected quasiconformal group which is not isomorphic to any subgroup of $O(n,1)$ [58]. Thus one needs extra assumptions. A typical argument is as follows : assume that the group contains an element g which is expanding at the point x. Assume that the invariant conformal structure is smooth at x. Normalize is so that it coincides with the standard structure at x. Using iterates of g, one sees that some neighborhood U of x can be mapped conformally to smaller and smaller neighborhoods of x, which are very close to a standard disk. One concludes that U itself is conformally equivalent to a standard disk. In fact, as proved by P. Tukia, this argument works under no regularity assumption on the invariant conformal structure.

39 Theorem [59].- Let μ be a measurable conformal structure on the sphere. If its conformal group is cocompact in the space of triples of distinct points (for example, if it comes from a cocompact group of quasiisometries of the disk), then μ is the standard conformal structure.

Clearly, this applies to Carnot groups too.

D. Sullivan has a different result : let μ be invariant under a discrete subgroup Γ of $O(n,1)$. Then μ is standard under a weaker assumption on Γ : that it approaches almost every point horosphericly [53]. The assumption in P.Tukia's theorem is conical approach a.e.

Since the connected subgroups of $O(n,1)$, $U(n,1)$,.. are known, this leads to a a method to decide where two homogeneous Riemannian manifolds are quasiconformally equivalent. For the case of Euclidean domains, see the forthcoming work by F.W. Gehring and G. Martin.

40 Corollary.- Let N be a Carnot group with homothety group A, let M denote the group AN endowed with a left-invariant Riemannian metric of negative sectional curvature. Let M' be a simply connected Riemannian manifold with negative sectional curvature and cocompact isometry group. Assume that M and M' are quasiconformally equivalent. Then $Isom(M)$ and $Isom(M')$ are cocompact subgroups in a common topological group.

Notice that, if $Isom(M')$ is discrete, we may conclude that both $Isom(M)$ and $Isom(M')$ are subgroups of a simple Lie group $O(n,1)$, $U(n,1)$...

5. Global characterizations of quasiconformal mappings

41 I want to emphasize the fact that the conformality or quasi-conformality of a homeomorphism of a manifold M can be checked from its behaviour under conjugacy with conformal mappings of M. This applies only when the conformal group of M is large enough. Therefore, in the sequel, S denotes either the boundary of a rank one symmetric space (i.e., a sphere with an exotic conformal structure) or a Carnot group N. We denote by G its "conformal group", i.e., a simple group $O(n,1)$, $U(n,1)$, $Sp(n,1)$, F_4^{-20} in the symmetric case, the group MAN where M is maximal compact in $Aut_\delta(N)$ in the general case. Let us begin with a consequence of the preceding discussion.

42 Corollary (see [62] for an elementary proof in the case of Euclidean space) A quasiconformal group on S which contains a cocompact subgroup of the conformal group G consists only of 1-quasiconformal mappings (conformal in the symmetric case).

One of the applications of the methods of section 1, especially Theorem 3, is to equicontinuity properties of "normalized" quasi-conformal mappings (see also [63], chap. 20). Given balls $D_1, D_2 << D_3$ and a point $x \in D_1$, we say that a homeomorphism f of the sphere S is normalized if

$$f(x) = x \quad \text{and} \quad D_2 < f(D_1) < D_3$$

then the normalized quasiconformal mappings of S with a given distorsion are equicontinuous. Any quasiconformal mapping can be normalized - depending only on its distorsion - by multiplying it with suitable elements of the conformal group. Thus one can state : if Q denotes the set of quasiconformal mappings on S with deviation from conformality less than K, then $Q < GB$ where B is a compact subset of the homeomorphism group of the sphere. A kind of converse is true.

43 Proposition.- Let Γ be a cocompact subgroup of the conformal group G. A homeomorphism f of S is quasiconformal if and only if $f\Gamma < \Gamma B$ where B is a compact set of homeomorphisms of S.

44 Corollary.- Let Γ be cocompact in G. A homeomorphism f is 1-quasiconformal (conformal in the symmetric case) if and only if Γ is cocompact in the closed subgroup of $Homeo(S)$ generated by Γ and f.

45 Remark.- Since the boundary of a simply connected manifold of negative curvature can be reconstructed functorially from any discrete cocompact group of isometries, this corollary shows that the conformal group G can be recovered from any discrete cocompact subgroup. By the way, this is Mostow's rigidity theorem : any isomorphism between

lattices extends to an isomorphism between the Lie groups. This leads us to a notion of conformal mappings between the boundaries of the universal covers of arbitrary compact manifolds with negative curvature.

46 *Definition*.- Let M , M' be simply connected manifolds with negative sectional curvature. Let Γ , Γ' be cocompact groups of isometries. A homeomorphism $f : \delta M \quad \delta M'$ is said to be *conformal* if the multiplicative set generated by Γ , f , f^{-1} and Γ' is contained in $\Gamma B \cup \Gamma' B'$ where B, $B' < \mathrm{Homeo}(\delta M)$, $\mathrm{Homeo}(\delta M')$ are compact. The map f is called *quasiconformal* if $f\Gamma < \Gamma' B$ where $B < \mathrm{Homeo}(\delta M')$ is compact.

47 *Theorem* (P. Tukia [61]).- With the above definition, a quasiconformal mapping extends to a quasiisometry of M onto M', well-defined modulo maps with bounded displacement. Conversely, a quasiisometry extends to a quasiconformal mapping.

48 *Question*.- Does the conformal group of δM preserve some conformal class of distances ? In case the group Γ is smooth with respect to some differentiable structure on δM , is there any relation between the quasiconformal group as defined above and the quasiconformal group attached to the smooth structure ? The case of symmetric spaces already shows that there is no inclusion.

49 *Invariants of patterns of points*.- It is easy to construct conformal invariants of a finite number of points on the standard sphere : given k distinct points, take the various simplices they generate in hyperbolic space, choose a combination of volumes, mutual angles and distances. I claim that any such invariant is quasi-invariant under quasiconformal mappings of the sphere. This just follows from compactness modulo conformal normalization.

Conversely, quasiconformal mappings of the standard sphere can be defined using any conformal invariant of 4 points $(.,.,.,.)$ which tends to infinity when exactly two points become close to each other. Indeed, given a homeomorphism f , use a conformal mapping so that f fixes three given points a, b and c . Then the range of variation of $f(d)$ is controlled by the 4-points invariant $(a,b,c,f(d))$, and a modulus of continuity at d is given by the invariant $(a,b,f(d),f(e))$. Thus the set of normalized homeomorphisms which almost preserve $(.,.,.,.)$ is compact, and so uniformly quasiconformal. In particular, a group G' of homeomorphisms of S^n strictly containing $O(n,1)$ cannot preserve such an invariant. Does this imply some dynamical property of the action of G' on quadruples of distinct points ?

There are two famous examples of conformal invariants of several points. First, the cross-ratio on the two-sphere. There, the invariant has values in the two-sphere itself. Second, the volume of the hyperbolic simplex generated by $n+2$ points on the n-sphere, which enters M. Gromov's proof of Mostow's rigidity, see [56], chap. 6. It characterizes conformal mappings. Indeed, a hyperbolic simplex is regular (i.e., has maximal symmetry) if and only if it has maximal volume [20]. A homeomorphism which preserves regular simplices preserves the pattern generated by one of them by reflection across the faces, which is dense in the sphere, so the homeomorphism extends to an isometry of hyperbolic space. I do not know whether quasiconformal mappings can be characterized by this invariant of $n+2$ points.

50 One may wonder whether the conformal mappings on the boundary of a manifold of negative curvature as defined in § 46 preserve some kind of capacity. One can characterize the class of semi-open function on δM whose derivative is L^n-integrable, since they have adequate compactness properties, see [31], but it is unclear whether one can

reconstruct the whole *Royden algebra* of continuous functions with L^n-integrable derivative, together with its norm

$$\|u\|_{L^\infty} + \|du\|_{L^n} .$$

It is known (see [32], [33]) that this algebra completely determines the conformal structure.

Question.- Can one concoct a Royden algebra for δM out of the Royden algebra of M ? Some answer must exist already for the disk.

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