INTRODUCTION TO L^2 BETTI NUMBERS

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I hope the present notes will motivate readers to enter the theory of Von Neumann algebras further. An excellent source is Mikhael Shubin's lecture notes [1993]. I have found them very helpful and I would like to thank Mikhael Shubin for sending his manuscript to me. It is a pleasure to express again my thanks to the organizers of the Riemannian geometry workshop.

CHAPTER 1

Introduction

1.1

Let X be a finite simplicial complex. The Betti numbers

$$b_i(X) = \dim H^k(X.\mathbf{R})$$

are homotopy invariants of X. The Euler characteristic of X is given by

$$\chi(X) = \sum_i (-1)^i b_i(X).$$

Let $\hat{X}$ be a covering of X of degree d. The Euler characteristics satisfy

$$\chi(\hat{X}) = d\chi(X).$$

However, individual Betti numbers do not obey such a simple rule. In general,

$$b_i(\hat{X}) \neq d\, b_i(X).$$

In these notes, we explain how to construct a variant of Betti numbers called L^2-Betti numbers, which shares many properties of ordinary Betti numbers, i.e.,

B1 Homotopy invariance: *For each i, $L^2b_i(X)$ is a homotopy invariant of X;*

B2 Euler characteristic: $\chi(X) = \sum_i (-1)^i L^2 b_i(X)$;

but has a simple transformation rule under finite coverings:

B3 Finite coverings: *if $\hat{X} \to X$ is a d-fold covering, then $L^2b_i(\hat{X}) = d\, L^2 b_i(X)$.*

What will be lost is integrality: L^2Betti numbers are by definition real numbers. They are conjectured to be always rational. Property B3 suggests the following definition for L^2Betti numbers:

B4 Continuity: *if $\hat{X}_j \to X$ is a d_j-fold covering, and if the sequence $\hat{X}_j \to X$ converges to the universal covering in the following sense: every loop in X lifts to an open path in some $\hat{X}_j$, then*

$$L^2 b_i(X) = \lim_{j\to\infty} \frac{1}{d_j} b_i(\hat{X}_j).$$

However, not every complex admits such a tower of finite coverings (it is the case, if and only if, the fundamental group of X is *residually finite*).

The actual definition involves L^2 cohomology of the universal covering $\tilde{X}$ of X. Although the main idea is present in Murray-Von Neumann's theory of type II factors, (Murray and Von Neumann [1943]) the concept is due to M. Atiyah [1976], and I.M. Singer [1977], (property B4 has been proved only recently by W. Lück [1993]).

L^2 Betti numbers are useful tools for topology, as shown by J. Cheeger and M. Gromov [1986], and W. Lück (to appear). In these notes, we shall describe a

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striking application to a problem in Riemannian geometry that goes back to H. Hopf and maybe even to S.S. Chern.

1.1.1 H. Hopf's problem Does the sign of sectional curvature determine the sign of Euler characteristic?

In two dimensions, the Gauss-Bonnet formula immediately implies that a compact manifold which admits a positively (resp. negatively) curved metric has positive (resp. negative) Euler characteristic.

In four dimensions, one can still check that positive (resp. negative) sectional curvature implies that the Gauss-Bonnet integrand is pointwise positive (J. Milnor). This is a computation in linear algebra. However, in higher dimensions, it is known that the sign of sectional curvature only does not determine the sign of the Gauss-Bonnet integrand; see for example Geroch [1976], even in the Kähler case (Bourguignon and Polombo [1981]). Nevertheless, the following question is still open:

Let M be a compact negatively curved riemannian manifold of dimension $2m$. Is it true that $(-1)^m\chi(M) > 0$?

M. Gromov has given a positive answer in the case the metric is Kähler. His solution uses L^2 Betti numbers and their relation with L^2-cohomology of the universal covering, which we now explain.

1.1.2 L^2 cohomology Let $\tilde{M}$ be a Riemannian manifold. The Riemannian metric allows to impose decay conditions at infinity such as square integrability. Let $L^2\Omega^*$ denote the space of differential forms on $\tilde{M}$ which are in L^2as well as their exterior derivative. Then the exterior differential $d : L^2\Omega^* \to L^2\Omega^*$ is bounded. L^2 cohomology is the quotient $L^2H^*(\tilde{M}) = \mathrm{Ker}\, d/\mathrm{Im}\, d$. Note that $\mathrm{Im}\, d$ is not always a closed subspace in $\mathrm{Ker}\, d$. This leads to the definition of *reduced* L^2 cohomology where $\mathrm{Im}\, d$ is replaced by its closure $L^2\overline{H}^*(\tilde{M}) = \mathrm{Ker}\, d/\overline{\mathrm{Im}\, d}$. If $\tilde{M}$ is complete, integration by parts works, and shows that L^2 harmonic forms are closed and coclosed. A standard argument shows that every class in $L^2\overline{H}^*(\tilde{M})$ contains a unique L^2 harmonic form, thus reduced L^2 cohomology is isomorphic to the space $L^2\mathcal{H}^*(\tilde{M})$ of L^2 harmonic forms.

If $\tilde{M}$ covers a compact manifold M, then any Riemannian metric on M gives rise to a Riemannian metric on $\tilde{M}$, and L^2 cohomology of $\tilde{M}$ is independent on the particular choice. Then L^2 Betti numbers of M can be defined in terms of the action of $\pi_1(M)$ on $L^2\mathcal{H}^*(\tilde{M})$. In particular, one has:

B5 Harmonic forms: *$L^2b_i(M)$ vanishes if and only if $L^2\mathcal{H}^i(\tilde{M})$ vanishes.*

1.1.3 J. Dodziuk and I.M. Singer's approach Properties 2 and 5 imply that a positive answer to H. Hopf's question would follow from a vanishing theorem for $L^2\mathcal{H}^i(\tilde{M})$, $i \neq m$, and a non vanishing theorem for $L^2\mathcal{H}^m(\tilde{M})$. J. Dodziuk and I.M. Singer asked whether $L^2\mathcal{H}^i(\tilde{M}) = 0$, $i \neq m$ for the universal covering of every compact negatively curved manifold. This question is still open too. The answer is positive for rotationnally symmetric metrics (Dodziuk [1977]) and symmetric spaces (Borel [1983]). It is known that a stronger pinching condition on sectional curvature implies vanishing of $L^2\mathcal{H}^i(\tilde{M})$, $i \neq m$ (H. Donnelly and F. Xavier, [1984]). However such a pinching condition is needed, as examples by M. Anderson [1984] show, they are simply connected manifolds without compact quotients.

Gromov [1991] introduces the class of *Kähler hyperbolic* manifolds, i.e., complete Kähler manifolds for which the Kähler form is the differential of a bounded

form. He observes that, for the universal covering of a compact negatively curved manifold, every closed bounded form is the differential of a bounded form. He proves that, for a Kähler hyperbolic $2m$-manifold $\tilde{M}$, $L^2\mathcal{H}^i(\tilde{M}) = 0$, $i \neq m$. The argument is elementary, and yields a lower bound on the L^2-spectrum of the Laplacian on forms on $\tilde{M}$, orthogonally to L^2harmonic m-forms. The last step, non vanishing of $L^2H^*(\tilde{M})$, is less elementary. It relies on a new avatar of the L^2 index theorem.

CHAPTER 2

Von-Neumann Dimension

2.1

Let Γ be a discrete group. We are about to attach a real number $\dim_\Gamma$ to certain unitary representations V of Γ. The construction fits naturally into the theory of Von Neumann algebras, see Murray and Von Neumann [1943], Atiyah [1976], Shubin [1993]. Instead of sketching the general theory, we shall follow a pedestrian route.

The idea is to renormalize infinite dimensions by dividing out the order of the group. Thus, typically the dimension of the regular representation of a group Γ should be 1. Recall that the main role played by dimension in usual linear algebra is in the following lemma. *If $L : V \to W$ is linear and $\dim V > \dim W$ then L has a non trivial kernel.* The proof amounts to constructing a complement V' of $\operatorname{Ker} L$ in V, observing that $L_{|V'}$ is an isomorphism of V' onto $\operatorname{Im} L$, and the formula $\dim \operatorname{Ker} L + \dim V' = \dim V$.

In an infinite dimensional situation, one will often encounter dense images, so we want that renormalized dimension be equal for 2 spaces when one is isomorphic to a dense subspace in the other. Furthermore, in order that property B3 of L^2 Betti numbers holds, a simple transformation rule for change of group to a finite index subgroup should be available. In summary, the renormalized dimension $\dim_\Gamma$ should have the following properties.

D1 Positivity: *$\dim_\Gamma(V) \geq 0$, and $\dim_\Gamma(V) = 0$ iff $V = 0$;*

D2 Invariance: *if V is isomorphic to a dense subspace of W, then $\dim_\Gamma(V) = \dim_\Gamma(W)$;*

D3 Additivity: *if Z is the orthogonal direct sum of V and W, then $\dim_\Gamma(Z) = \dim_\Gamma(V) + \dim_\Gamma(W)$;*

D4 Continuity: If V_j is a decreasing sequence of Γ-invariant subspaces, then:

$$\dim_\Gamma(\cap_j V_j) = \lim_{j\to\infty} \dim_\Gamma(V_j);$$

D5 Finite index subgroups: *if $\Gamma' \subset \Gamma$ is a subgroup with index d, then any representation V of Γ becomes a representation of Γ' and $\dim_{\Gamma'}(V) = d \dim_\Gamma(V)$;*

D6 Normalization: $\dim_\Gamma(\ell^2(\Gamma)) = 1$.

2.1.1 Example: Invariant subspaces in $\ell^2(\mathbf{Z})$. $\ell^2(\mathbf{Z})$ consists of doubly infinite series $(a_n)_{n\in\mathbf{Z}}$ of complex numbers such that $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$. The group $\mathbf{Z}$ acts on it by shifting indices. The Fourier transform is an isomorphism of $\ell^2(\mathbf{Z})$ with $L^2(S^1)$ where $n \in \mathbf{Z}$ acts on $L^2(S^1)$ by $f(z) \mapsto z^n f(z)$.

If $A \subset S^1$ is measurable, then the subspace E_A of functions that vanish outside A is closed, invariant. Assuming that $\dim_\Gamma(E_A)$ makes sense, let us deduce from the axioms only that it is equal to the measure of A.

Indeed, according to axioms D3 and D4, $A \mapsto \dim_\Gamma(E_A)$ is a countably additive measure. The transformation law D5 applied to $\Gamma' = p\mathbf{Z}$ shows that this probability measure is invariant under translations τ of order p (since $E_{\tau A}$ is isomorphic to E_A under $p\mathbf{Z}$), and thus it coincides with Lebesgue measure.

As a consequence, measure theory on the circle should be a particular case of our Γ-dimension theory. This is important for intuition.

2.1.2 Finite groups When Γ is finite, axioms D5 and D6 force

$$\dim_\Gamma(V) = \frac{\dim V}{\mathrm{Card}\,\Gamma}.$$

Here is a fancy way to rewrite this formula. Assume that V is finite dimensional. Then V is contained in the sum of finitely many copies of the regular representation $\ell^2(\Gamma)$, in other words, $V \subset \ell^2(\Gamma)\otimes F$ where Γ acts trivially on F. The space $\ell^2(\Gamma)\otimes F$ admits a natural Γ-invariant orthonormal basis $(e_{\gamma,i}) = \gamma e_i$ where $\gamma \in \Gamma$ and (e_i) is a basis of F. Let Π denote the orthogonal projection of $\ell^2(\Gamma)\otimes F$ onto V. Then

$$\begin{aligned}\dim V &= \mathrm{Trace}\,\Pi \\ &= \textstyle\sum_\gamma (\sum_i \langle e_{\gamma,i}|\Pi|e_{\gamma,i}\rangle) \\ &= \mathrm{Card}\,\Gamma\ \textstyle\sum_i \langle e_{1,i}|\Pi|e_{1,i}\rangle\end{aligned}$$

since the sum within parentheses does not depend on γ. This sum will be taken as a definition of $\dim_\Gamma(V)$ for infinite Γ. Unfortunately, the definition applies only to spaces embeddable in $\ell^2(\Gamma)\otimes F$ and this is a serious restriction when Γ is infinite. In Proposition 2.6, an intrinsic characterization of these special representations will be given, together with an intrinsic expression for $\dim_\Gamma$.

2.1.3 The Γ-trace

Definition 2.1 Let Γ be a discrete group acting by unitary transformations on a Hilbert space H. H is a free Γ-module if it admits a Γ-invariant Hilbert basis, i.e., a Hilbert basis $(e_i)_{i\in I}$ such that Γ acts freely on I and $e_{\gamma i} = \gamma e_i$. If P is a Γ-equivariant operator on H, we define its Γ-trace by

$$\mathrm{Trace}_\Gamma P = \sum_{i\in\Gamma\backslash I} \langle e_i|P|e_i\rangle,$$

where the notation means that one representative is chosen in each Γ-orbit in I.

If V is a Γ-invariant subspace in H, we define its Γ-dimension as the Γ-trace of the orthogonal projector of H onto the closure of V.

For non-negative hermitian Γ-equivariant operators P, $\mathrm{Trace}_\Gamma P$ does not depend on the choice of Γ-invariant basis. Indeed, if U is Γ-equivariant and unitary, then $\mathrm{Trace}_\Gamma UPU^* = \mathrm{Trace}_\Gamma P$. This follows from the following easy lemma by setting $A = U\sqrt{P}$.

Lemma 2.2 *If A is a bounded Γ-equivariant operator, and if $\mathit{Trace}_\Gamma A^*A$ is finite, then so is $\mathit{Trace}_\Gamma AA^*$ and*

$$\mathit{Trace}_\Gamma(A^*A) = \mathit{Trace}_\Gamma(AA^*).$$

More generally, let us prove a precised version of axiom D2.

Lemma 2.3 *Let V and W be invariant subspaces in free Γ-modules. Assume there exists a closed densely defined injective Γ-equivariant operator $A : V \to W$ with dense image. Assume that $\dim_\Gamma V$ is finite. The so is $\dim_\Gamma W$ and $\dim_\Gamma V = \dim_\Gamma W$.*

Proof By taking the direct sum of ambient spaces for V and W, we can assume that V and W sit in the same free Γ-module H.

Let $\Pi_{\overline{V}} : H \to \overline{V}$ denote the orthogonal projector onto the closure of V. We use the polar decomposition $B = US$ of the closed unbounded densely defined operator $B = A\Pi_{\overline{V}}$. Here S is non negative self adjoint and U is a partial isometry, i.e., $U^*U = \Pi_{\overline{V}}$. By uniqueness, U and S are Γ-equivariant. Now

$$UU^* = \Pi_{\overline{W}}$$

so

$$\dim_\Gamma W = \mathrm{Trace}_\Gamma UU^* = \mathrm{Trace}_\Gamma U^*U = \dim_\Gamma V.$$

□

Axioms D1 (positivity), D3 (additivity), D5 (transformation under finite index subgroups) and D6 (normalization) are clear. Combined with Lemma 2.4 they lead to the expected generalization of finite dimensional linear algebra.

D7 Invariance of index. *Let V and W have finite Γ-dimension. Let $L : V \to W$ be a bounded Γ-equivariant operator. Then*

$$\dim_\Gamma \mathrm{Ker}\, L - \dim_\Gamma \mathrm{Ker}\, L^* = \dim_\Gamma V - \dim_\Gamma W.$$

Axiom D4 follows from the following *dominated convergence* result (see Atiyah [1976] and Shubin [1993]).

Lemma 2.4 *Let S be of Γ-trace class and P_j be operators with uniformly bounded operator norms. Assume that P_j weakly converge to P. Then*

$$\mathit{Trace}_\Gamma SP = \lim_{j\to\infty} \mathit{Trace}_\Gamma SP_j.$$

2.1.4 Square integrable representations At last, we give an intrinsic definition for $\dim_\Gamma$. Let V be some unitary representation of a group Γ. Denote by

$$B(v,w) = \sum_{\gamma\in\Gamma} \langle v|\gamma w\rangle^2.$$

This is not always finite. Say that $v \in V$ is a L^∞*vector* if there exists a constant C such that $B(v,w) \le C \parallel w \parallel^2$ for all $w \in V$ (the terminology is inspired by the case of $\ell^2(\mathbf{Z}) = L^2(S^1)$).

Definition 2.5 (Compare Dixmier [1964], Chapter 14) Say V is a square integrable representation if its L^∞vectors are dense.

If V admits a Γ-invariant basis (e_i), then $B(e_i,w) \le \parallel w \parallel^2$ for all w in V. As a consequence, the set of L^∞vectors is dense in V. More generally, all invariant subspaces in free Γ-modules are square integrable representations. The converse is true.

Proposition 2.6 *If V is a separable square integrable representation of Γ, then V embeds in a free Γ-module and*

$$dim_{\Gamma} V = \sup \sum_i \| v_i \|^2$$

over collections (v_i) of vectors which, together with their Γ-translates, generate a dense subspace in V, and are normalized by

$$\sum_i B(v_i, w) \leq \| w \|^2 \qquad \textit{for all } w \in V.$$

Proof Choose any countable collection v_i of L^∞vectors which, together with their Γ-translates, generate a dense subspace in V. Eventually rescale each of them so that the series $\sum_i \| v_i \|^2_\infty \leq 1$. Let F be the abstract Hilbert space with a countable Hilbert basis e_i. The linear Γ equivariant map $L : \ell^2(\Gamma) \otimes F \to V$ taking e_i to v_i is bounded with norm 1. It has a dense image. Thus its adjoint L^* embeds V into $\ell^2(\Gamma) \otimes F$. Let $L^* = US$ be the polar decomposition of L^*. The Γ-dimension of $L^*(V)$ is

$$\begin{aligned} \dim_{\Gamma} L^* V &= \mathrm{Trace}_{\Gamma}\, UU^* \\ &= \textstyle\sum_i \langle e_i | UU^* | e_i \rangle \\ &= \textstyle\sum_i \| U^* e_i \|^2 \\ &= \textstyle\sum_i \| S^{-1} v_i \|^2 \\ &\geq \textstyle\sum_i \| v_i \|^2 \end{aligned}$$

with equality if one replaces the collection v_i by $S^{-1} v_i$. □

2.1.5 Naive dimension Clearly, if V is representation of Γ and if a dense subspace in V is generated by finitely many L^∞vectors (and their Γ-translates), then $\dim_{\Gamma} V$ is finite. The converse is not true in general (consider the **Z**-module $\oplus L^2([0, 2^{-i}])$).

CHAPTER 3

Simplicial L^2 Betti Numbers

3.1

3.1.1 Simplicial cohomology Let $\tilde{X}$ be a locally finite simplicial complex. A k-cochain is a function which attaches a real number to each k-simplex of $\tilde{X}$. They form a vector space denoted by $C^k(\tilde{X}, \mathbf{R})$. The coboundary operator d is dual to the boundary ∂ which attaches to a simplex σ a combination of its faces $\partial\sigma = \sum_i (-1)^i \partial_i \sigma$.

There is a natural Hilbert space structure on $C^k(\tilde{X}, \mathbf{R})$, where

$$\| c \|^2 = \sum_{\sigma \text{ a } k\text{-simplex in } \tilde{X}} |c(\sigma)|^2.$$

Then the coboundary is a bounded operator on $L^2C^*(\tilde{X}, \mathbf{R})$, and we denote its adjoint by d^*.

Assume that $\tilde{X}$ is the universal covering of a finite complex X with fundamental group Γ. Then as representations of Γ,

$$L^2C^*(\tilde{X}, \mathbf{R}) = \ell^2(\Gamma) \otimes C^*(X, \mathbf{R}).$$

Indeed, choose for each simplex σ a lifting $\tilde{\sigma} \in \tilde{X}$. Then each cochain c of $\tilde{X}$ defines a sequence of cochains c_γ on X by $c_\gamma = c(\gamma\tilde{\sigma})$. However, note that the coboundary operator does not split.

Thus one can view the space $L^2\mathcal{H}^* = \mathrm{Ker}\,(d + d^*)$ of L^2 harmonic cocycles on $\tilde{X}$ as a Γ-invariant subspace in the free Γ-module $\ell^2(\Gamma) \otimes C^*(X, \mathbf{R})$ and assign it a Von Neumann dimension.

Definition 3.1 The simplicial L^2 Betti numbers are $L^2b_i(X) = \dim_{\Gamma} L^2\mathcal{H}^i$.

Next we check that this definition fulfills the rules set in the introduction. Rule B3 follows from the corresponding property D5 of Von-Neumann dimension. B1 and B5 will be established in the next paragraph. For B4, we refer to W. Lück's [1993] paper. Next we prove B2.

Proposition 3.2 *Simplicial L^2 Betti numbers compute the Euler characteristic*

$$\chi(X) = \sum_i (-1)^i L^2 b_i(X).$$

Proof Consider the bounded Γ-equivariant operator

$$L = d + d^* : L^2C^{even}(\tilde{X}) \to L^2C^{odd}(\tilde{X}).$$

According to property D7 of Von Neumann dimension,

$$\begin{aligned}\textstyle\sum_i(-1)^i L^2 b_i(X) &= \dim_\Gamma \mathrm{Ker}\, L - \dim_\Gamma \mathrm{Ker}\, L^* \\ &= \dim_\Gamma L^2 C^{even}(\tilde{X}) - \dim_\Gamma L^2 C^{odd}(\tilde{X}) \\ &= \dim C^{even}(X) - \dim C^{odd}(X) \\ &= \chi(X).\end{aligned}$$

□

Remark 3.3 The simplical Laplacian belongs in a special class of Γ-invariant operators in $\ell^2(\Gamma)\otimes F$ called *finite equations on* Γ. These are finite sums of operators of the form $R_\gamma \otimes M$ where R_γ is right translation by γ and $M \in \mathrm{End}\, F$ is a matrix. One expects that the space of L^2 solutions of such special operators have integer Γ-dimension.

This is easy when $\Gamma = \mathbf{Z}$. Indeed, via Fourier transform $\ell^2(\mathbf{Z}) = L^2(S^1)$, such an operator P translates into a square matrix $M(z)$ that depends polynomially on $z \in S^1$ and acts on vector-valued functions on the circle by multiplication. The L^2 kernel of P consists of L^2 sections of the measurable subbundle $\mathrm{Ker}\, M(z)$, whose rank jumps only at finitely many points. Thus $\dim_{\mathbf{Z}} L^2\mathrm{Ker}\, P = \max \dim \mathrm{Ker}\, M(z)$ is an integer.

This particular example is not so useful as far as Betti numbers are concerned, since spaces with abelian fundamental groups have vanishing L^2 Betti numbers.

CHAPTER 4

Homotopy Invariance

4.1

The homotopy invariance of L^2 Betti numbers has first been established by J. Dodziuk [1977]. We recall the sheaf theoretic proof of homotopy invariance of simplicial cohomology, and check that the proof still applies in an L^2 context.

4.1.1 Sheaves of Hilbert spaces We first recall the classical sheaf theory terminology. See Godement [1964].

A sheaf over a manifold is *fine* if its sections can be multiplied by smooth partitions of unity. A sheaf over a simplicial complex is *fine* if its sections can be multiplied by continuous piecewise linear functions.

A complex of sheaves $0 \to \mathcal{A}^{-1} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \cdots$ is a *resolution* of $\mathcal{A}^{-1}$ if it is locally exact.

A covering $\mathcal{U} = (U_\alpha)$ is *acyclic* with respect to the complex of sheaves $0 \to \mathcal{A}^{-1} \to \mathcal{A}^0 \to \cdots$ if the sequence is exact on each open set (U_α) and on all intersections of (U_α)'s.

Next we add L^2 norms and uniformity in these notions.

When we deal with a sheaf $\mathcal{A}$ of Hilbert spaces we assume that

(a) if $U \subset V$ and f is a section of $\mathcal{A}$) on V, then

$$|f_{|U}|_U \le |f|_V \;;$$

(b) if f is a section of $\mathcal{A}$ on $U \cup V$, then

$$|f|_{U\cup V} \le \mathrm{const.}\,(|f_{|U}|_U + |f_{|V}|_V).$$

A covering $\mathcal{U} = (U_\alpha)$ of of a metric space X is *uniform* if

(1) there exists an $\epsilon > 0$ such that the sets

$$\tilde{U}_\alpha^\epsilon = \{x \in U_\alpha \mid d(x, M \setminus U_\alpha) > \epsilon\}$$

still cover X;

(2) each set U_α intersects only a bounded number of $U_{\alpha'}$;

(3) the diameter of U_α is bounded.

A covering $\mathcal{U}$ is *uniformly acyclic* with respect to the complex of sheaves

$$0 \to \mathcal{A}^{-1} \xrightarrow{d} \mathcal{A}^0 \xrightarrow{d} \cdots$$

if on all intersections, there is a bounded operator (with a uniform bound on the norm) which solves $d\beta = \alpha$ when $d\alpha = 0$.

Uniform coverings exist on Riemannian manifolds of bounded geometry (i.e., with curvature bounded from above and injectivity radius bounded from below) and simplicial complexes of bounded geometry (i.e., each simplex meets a bounded number of simplices). These assumptions are automaticly satisfied for spaces which isometricly cover a compact space.

Theorem 4.1 (de Rham, sheaf theoretic version) *Let*

$$0 \to \mathbf{R} \to \mathcal{A}^0 \to \mathcal{A}^1 \to \cdots$$

be a fine resolution of the constant sheaf $\mathbf{R}$ *over* X. *Let* $\mathcal{U}$ *be an acyclic covering of* X. *Then the cohomology of the complex of global sections of* $\mathcal{A}^*$ *on* X *coincides with the Čech cohomology of the covering, i.e., the simplicial cohomology of the nerve of the covering* $\mathcal{U}$. L^2 case: *If one adds* uniformity *in all assumptions, the conclusion is that the complexes* $L^2\mathcal{A}^*$ *and* $L^2\check{C}^*(\mathcal{U})$ *are bounded homotopy equivalent. In particular, they have isomorphic* L^2 *cohomology and reduced* L^2 *cohomology.*

Given a metric space X equipped with a measure, we apply Theorem 4.1 to the resolution of constants by *Alexander-Spanier cochains of size* η. Such a i-cochain is a function on $i+1$-tuples of points in X with pairwise distances less than η. The L^2 norm comes from the product measure on the space of admissible simplices.

Lemma 4.2 *The Alexander-Spanier* L^2 *cohomology of the unit simplex in* $\mathbf{R}^n$ *is independant of the size* η. *As a consequence (take* $\eta = 2$*), it vanishes except in degree zero.*

Proof Subdivide a large simplex into smaller simplices. □

Let X be a simplicial complex of bounded geometry. Then the covering by stars of vertices is uniform and uniformly acyclic with respect to the resolution of constants by Alexander-Spanier cochains of size 1/2. Thus L^2 cohomology can be computed via Alexander-Spanier cochains. Since this is natural under uniformly continuous maps, and in particular, maps lifted from a compact quotient, we conclude that L^2 cohomology of the universal covering is a homotopy invariant. This proves property B1 of L^2 Betti numbers.

4.1.2 Differential forms Given a Riemannian manifold of bounded geometry, we apply Theorem 4.1 to the resolution of constants by differential forms.

Lemma 4.3 *The De Rham* L^2 *cohomology of the unit ball in* $\mathbf{R}^n$ *vanishes except in degree zero.*

Proof Use Poincaré's homotopy formula. Fixing an origin p, use polar coordinates, write a form $\alpha = a + b \wedge dr$ and set $h_p(\alpha) = \int b\, ds$. Then $\int h_p\, dp$ is L^2 bounded and solves $d\beta = \alpha$. □

As a consequence, if M is a compact manifold, L^2 cohomology of the universal covering can be computed via differential forms. This proves property B5 of L^2 Betti numbers. For oriented n-manifolds, the Hodge $*$ operator sends L^2 harmonic i-forms to L^2 harmonic $n-i$-forms isomorphicly, and we obtain one more formal property of L^2 Betti numbers. **B6** Poincaré duality: *if* M *is a compact* n*-dimensional manifold, then*

$$L^2 b_i(M) = L^2 b_{n-i}(M).$$

CHAPTER 5

Invariants of Discrete Groups

5.1

Let Γ be a discrete group. Assume that there exist finite simplicial complexes X such that $\tilde{X}$ is contractible and $\pi_1(X) = \Gamma$. Then we can define L^2 Betti numbers for Γ as the L^2 Betti numbers of any such X.

A closer look into the proof of Theorem 4.1 shows that, for *any* finite complex X, $L^2H^1(\tilde{X})$ only depends on $\pi_1(X)$. The same holds for the *exact* cohomology in degree 2 (exact L^2cohomology is the kernel of $L^2H^*(\tilde{X}) \to H^*(\tilde{X}, \mathbf{R})$). In particular L^2b_1 and EL^2b_2 make sense for all finitely presented groups.

Any infinite group has $L^2b_0 = 0$. Further vanishing results, combined with the formal rules B1 to B6, sometimes enable one to compute L^2Betti numbers.

5.1.1 Example: Free groups The free group F_n on n letters has

$$L^2b_0(F_n) = 0,$$

$$L^2b_i(F_n) = 0$$

for $i \geq 2$ and

$$L^2b_1(F_n) = -\chi(F_n) = n - 1.$$

In particular, a regular tree which is not a line has non-vanishing $L^2\overline{H}^1$. This fact also follows directly from uniformization.

This is easy to see directly. Indeed, let T be a tree, e an edge which divides T into infinite subtrees T_0 and T_1. Let u be the function on vertices (i.e. the 0-cochain) which takes value 0 on vertices of T_0 and 1 on vertices of T_1. Its coboundary du is supported on e and thus L^2, but no solution of $dv = du$ can be in L^2. This shows that $L^2H^1 \neq 0$.

If a tree has at least 3 edges at each vertex, then Sobolev inequality holds: for compactly supported v, $\| v \| \leq$ const. $\| dv \|$ (this is an isoperimetric property, see for example Gromov, Lafontaine and Pansu [1981], Chapter 6). This in turn implies that $dL^2\Omega^0 \subset L^2\Omega^1$ is closed, and one concludes that $L^2\overline{H}^1 = L^2H^1 \neq 0$.

For $n = 1$ we observe that $L^2\overline{H}^1(\mathbf{Z}) = 0$ whereas $L^2H^1(\mathbf{Z}) \neq 0$. More generally, the discrepancy $L^2\overline{H}^1 \neq L^2H^1$ characterizes *amenable* groups; see R. Brooks [1981].

5.1.2 Example: Surface groups Let Γ be the fundamental group of a compact surface M of negative Euler characteristic. Then $L^2b_i(\Gamma) = 0$ for $i \geq 3$. Poincaré duality implies $L^2b_2(\Gamma) = L^2b_0(\Gamma) = 0$ and thus $L^2b_1(\Gamma) = -\chi(\Gamma)$. Again this implies that $\tilde{M}$ has non vanishing $L^2\mathcal{H}^1$.

This follows directly from uniformization. Indeed, the space $L^2\mathcal{H}^1$ of L^2 harmonic 1-forms is a conformal invariant. Now conformally, $\tilde{M}$ coincides with the unit disk D in $\mathbf{C}$, where the form dz is harmonic and L^2. In fact, L^2harmonic

1-forms on D are differentials of harmonic functions, which in turn are determined by their boundary values on the circle. Up to constants, $L^2\mathcal{H}^1(D)$ identifies with the Sobolev space of functions on the circle with a half derivative in L^2.

5.1.3 Example: Free abelian groups Since tori admit finite self coverings of any degree, property B3 implies that all their Betti numbers vanish. As a consequence, $L^2b_i(\mathbf{Z}^n) = 0$ for all n.

More generally, *if $\pi_1(X)$ has an infinite center, then all its L^2Betti numbers vanish.*

We give a short sketch of the argument, taken from M. Gromov [1993].

First step. The center acts trivially on L^2cohomology. Indeed, a central element γ acts by an isometry of $\tilde{X}$ which translates points a bounded amount ϵ. This number may be large. Nevertheless, since $\tilde{X}$ is uniformly contractible, its L^2cohomology can be computed using Alexander-Spanier cochains of arbitrarily large size. Then for any Alexander-Spanier simplex σ, $\gamma_*\sigma - \sigma$ bounds a prism of size ϵ and dually $\gamma^* - 1$ is null homotopic.

Second step. Any infinite proper isometric action on $\tilde{X}$ acts non trivially on L^2harmonic cocycles. Indeed, for each L^2cocycle c, there is a compact set K that contains more than half of the square L^2norm. If $\gamma K \cap K = \emptyset$, then $\gamma^* c \neq c$.

This gives a rather short proof of the fact that $\chi(\Gamma) = 0$ when Γ has a finite $K(\Gamma, 1)$ and a non trivial center, compare Rosset [1984]. A drawback is the assumption made that Γ has a finite $K(\Gamma, 1)$. Cheeger and Gromov [1986] have been able to remove it by enlarging the definition of L^2Betti numbers.

5.1.4 Amenable groups The general idea is that L^2Betti numbers vanish unless the group has a large "ideal boundary". For example, infinite amenable groups have vanishing L^2Betti numbers (Cheeger and Gromov [1986]). Amenable means that there are finite subsets $A \in \Gamma$ with $\#\partial A/\#A$ arbitrarily small. On the opposite, hyperbolic groups give many examples of non vanishing L^2b_i for some i. However, note that fundamental groups of compact odd dimensional real hyperbolic manifolds have all L^2Betti numbers zero. (See Lott and Lück [1991] for a computation of L^2Betti numbers of 3-manifolds.)

CHAPTER 6

Atiyah's L^2 Index Theorem

6.1

For compact manifolds, one can view the formula $\chi(M) = \sum_i (-1)^i \dim \mathcal{H}^i(M)$ as a special case of the index theorem. Indeed, it is a topological expression for the index of the elliptic operator $d + d^* : \Omega^{even}(M) \to \Omega^{odd}(M)$. Our formula $\chi(M) = \sum_i (-1)^i dim_\gamma L^2\mathcal{H}^i(\tilde{M})$ is also a special case of a more general L^2index theorem. A variant of it will be used in section 9 to obtain a non vanishing result for L^2solutions of a perturbation of $d + d^*$.

Theorem 6.1 *(M. Atiyah [1976]) Let M be a compact manifold, P a determined elliptic operator on sections of certain bundles over M. Denote by $\tilde{P}$ its lift to the universal covering $\tilde{M}$. Let $\Gamma = \pi_1(M)$. Then the L^2kernel of $\tilde{P}$ has a finite Γ-dimension, and*

$$L^2 Index_\Gamma \tilde{P} = dim_\gamma L^2 Ker \tilde{P} - dim_\gamma L^2 Ker \tilde{P}^* = Index P.$$

Proof *Finiteness of dimension.* The orthogonal projector Π onto the L^2kernel of $\tilde{P}$ has a smooth (vector-valued) kernel $p(x, y)$, since it satisfies a determined elliptic system of equations. Fix a fundamental domain U for Γ in $\tilde{M}$ and write

$$L^2(\tilde{M}, \text{bundle}) = \ell^2(\Gamma) \otimes L^2(U, \text{bundle}).$$

Then $\mathrm{Trace}_\Gamma \Pi = \mathrm{Trace}\, \pi \Pi i$ where $i : L^2(U) \to L^2(\tilde{M})$ is injection and $\pi : L^2(\tilde{M}) \to L^2(U)$ is orthogonal projection. Now $\mathrm{Trace}_\Gamma \Pi$ is the trace of an operator $L^2(U) \to L^2(U)$ whose kernel is p restricted to $U \times U$. Thus

$$\mathrm{Trace}_\Gamma \Pi = \int_U \mathrm{Trace}\, p(x, x)\, dx \qquad (*)$$

is finite.

Index formula. The index can be expressed in terms of a parametrix, i.e., an operator Q such that

$$QP = 1 - S_0, \quad PQ = 1 - S_1$$

where S_0 and S_1 have smooth kernels. The formula

$$\mathrm{Index}\, P = \mathrm{Trace}\, S_0 - \mathrm{Trace}\, S_1,$$

which is obvious for Green's function $Q = P^{-1} : \mathrm{Im}\, P \to \mathrm{Ker}\, P^\perp, = 0$ on $\mathrm{Im}\, P^\perp$, is in fact true for all parametrices. The rather formal proof extends to the case where traces are replaced by Γ-traces. Now on the compact manifold M, one can choose a parametrix whose kernel is supported in a neighborhood of the diagonal. This kernel lifts to $\tilde{M}$ into a parametrix for $\tilde{P}$. The numbers $L^2\mathrm{Index}_\Gamma \tilde{P}$ and $\mathrm{Index}\, P$ are then obtained by integration of equal functions on U and M.

See Atiyah [1985] for a complete proof. A slightly different proof will be given in Chapter 10. □

Example 6.2 The theorem applies to the operator $P=\bar{\partial}\colon \Omega^0(\Sigma) \to \Omega^{0,1}(\Sigma)$ on a compact Riemann surface Σ of genus g. Its adjoint is $\bar{\partial} : \Omega^{1,0} \to \Omega^{1,1}$. Then $\dim \mathrm{Ker}\, P = 1$ and $\dim \mathrm{Ker}\, P^* = g$. If $g > 1$, one concludes that $\tilde{\Sigma}$ admits non trivial L^2holomorphic 1-forms. Again, this is clear from uniformization. In the boundary value description of 5.2, these 1-forms correspond to $H^{1/2}$ functions with all negative Fourier coefficients vanishing.

6.1.1 Proportionnality principle A consequence of the formula (*) for Γ dimensions as integrals of Schwartz kernels is the following **B7** Proportionnality principle: *if M and M' are compact manifolds which have isometric universal coverings, then for all i*

$$\frac{L^2 b_i(M)}{\mathrm{Volume}\, M} = \frac{L^2 b_i(M')}{\mathrm{Volume}\, M'}.$$

CHAPTER 7

L^∞ Cohomology and Negative Curvature

7.1

When is a closed bounded form on a Riemannian manifold the differential of a bounded form? This is true exactly if L^∞cohomology vanishes. Observe that, if $\tilde{M}$ covers a compact manifold M, then $L^\infty H^*(\tilde{M})$ does not depend on the choice of a metric on M. More, as already mentionned for L^2cohomology, $L^\infty H^1(\tilde{M})$ and exact $EL^\infty H^2(\tilde{M})$ only depend on the fundamental group of M.

Proposition 7.1 *Let $\tilde{M}$ be a complete, simply connected manifold with sectional curvature $K \leq -\epsilon^2 < 0$. Then $L^\infty H^*(\tilde{M}) = 0$, i.e., every closed bounded form on $\tilde{M}$ is the differential of a bounded form.*

Proof Fix an origin p. According to the Cartan-Hadamard theorem (Ballman, Gromov, and Schröder [1985]) for every point x, there is a unique geodesic segment joining x to p. Denote by $Z(x)$ its unit tangent vector at x. Denote by $\tau_s(x)$ the point at distance s from x on this segment. The Rauch comparison theorem implies that τ_s contracts at least by a factor $e^{-\epsilon s}$ for large s.

Let α be a closed bounded k-form on $\tilde{M}$. To solve $d\beta = \alpha$, use Poincaré's formula

$$\beta_T = \int_0^T \tau_{-s}^*(i_Z\alpha)\, ds.$$

Then $d\beta_T = \alpha - \tau_{-T}^*\alpha$. Since

$$\| \tau_{-s}^*(i_Z\alpha) \|_\infty \leq e^{-(k-1)\epsilon s} \| \alpha \|_\infty,$$

the form β_T converges as $T \to +\infty$ to a form β such that $\| \beta \|_\infty \leq \frac{1}{(k-1)\epsilon} \| \alpha \|_\infty$. Also $\tau_{-T}^*\alpha$ tends to zero so $d\beta = \alpha$. □

CHAPTER 8

A Vanishing Theorem for Kähler Hyperbolic Manifolds

8.1

A Riemannian manifold is *Kähler* if it admits a parallel 2-form ω of unit norm whose stabilizer in the orthogonal group of some tangent space is a conjugate of the unitary group. The Kähler form ω is then closed.

Definition 8.1 (M. Gromov) A Kähler manifold is Kähler hyperbolic if the Kähler form is the differential of a bounded form.

Hermitian symmetric spaces are Kähler hyperbolic. According to Proposition 7.1, if M is a compact Kähler manifold, and if M admits some metric of negative sectional curvature, then its universal cover $\tilde{M}$ is Kähler hyperbolic. Kähler hyperbolicity implies hyperbolicity in the sense of Kobayashi (a purely holomorphic notion).

A key fact is that multiplication with the Kähler form ω commutes with the Laplacian; see Weil [1958]. If $\omega = db$, b bounded, then for every L^2 harmonic form α, $\omega \wedge \alpha = d(b \wedge \alpha)$ is zero in L^2 cohomology, and thus vanishes. However, linear algebra (Weil [1958]) says that multiplication by ω is injective on i-forms if $i < \frac{1}{2}\dim \tilde{M}$. One concludes that $L^2\mathcal{H}^i(\tilde{M}) = 0$ for $i < \frac{1}{2}\dim \tilde{M}$.

A little bit more is true.

Theorem 8.2 *Let $\tilde{M}$ be a complete Kähler hyperbolic manifold of real dimension $2m$. Then $L^2\Omega^*(\tilde{M})$ splits orthogonally as $L^2\Omega^*(\tilde{M}) = L^2\mathcal{H}^m \oplus E$ and the Laplacian is invertible on E. In particular, $L^2\mathcal{H}^i(\tilde{M}) = 0$ for $i \neq m$.*

Proof Let α be a i-form, $i \neq m$. We prove that $\| \alpha \|^2 \leq \text{const.} \langle \Delta\alpha | \alpha \rangle$. Eventually replacing α by $*\alpha$, we can assume that $i < m$. We shall use the integration by parts $\| d\beta \|^2 + \| d^*\beta \|^2 = \langle \Delta\beta | \beta \rangle$ and the fact that wedging with ω in degrees $< m$ is an isometric injection.

Write

$$\begin{aligned}
\| \alpha \|^2 &= \| \alpha \wedge \omega \|^2 \\
&= \langle \alpha \wedge \omega | \alpha \wedge db \rangle \\
&= \langle \alpha \wedge \omega | d(\alpha \wedge b) \rangle \pm \langle \alpha \wedge \omega | d\alpha \wedge b \rangle \\
&\leq \text{const.} \| \alpha \| (\| d^*(\alpha \wedge \omega) \| + \| d\alpha \|).
\end{aligned}$$

Since

$$\| d\alpha \|^2 \leq \langle \Delta\alpha | \alpha \rangle$$

and

$$\begin{aligned}
\| d^*(\alpha \wedge \omega) \|^2 &\leq \langle \Delta(\alpha \wedge \omega) | \alpha \wedge \omega \rangle \\
&= \langle \Delta\alpha \wedge \omega | \alpha \wedge \omega \rangle \\
&= \langle \Delta\alpha | \alpha \rangle,
\end{aligned}$$

we get $\| \alpha \|^2 \leq$ const. $\langle \Delta\alpha | \alpha \rangle$ where const. $= \| b \|_\infty$.

These estimates imply that for all $i < m$ the image of $d : L^2\Omega^i \to L^2\Omega^{i+1}$ is closed. Indeed, on $L^2\Omega^0$ one has $\| \alpha \| \leq$ const. $\| d\alpha \|$ which implies that $dL^2\Omega^0$ is closed in $L^2\Omega^1$. Then given $\alpha \in dL^2\Omega^1$, $\alpha = d\beta$, there is a 1-form $\tilde\beta$ of minimal norm in $\beta + dL^2\Omega^0$. This $\tilde\beta$ is coclosed and so satisfies $\| \tilde\beta \| \leq$ const. $\| \alpha \|$. This proves that $dL^2\Omega^1$ is closed in $L^2\Omega^2$, and so on by induction.

The Hodge decomposition

$$L^2\Omega^m = L^2\mathcal{H}^m \oplus \overline{dL^2\Omega^{m-1}} \oplus \overline{d^*L^2\Omega^{m+1}}$$

holds in general on complete manifolds. Here it becomes

$$E = dL^2\Omega^{m-1} \oplus *dL^2\Omega^{m-1}.$$

Since the Hodge $*$ and the differential d commute with the Laplacian, the inequality $\| \alpha \|^2 \leq$ const. $\langle \Delta\alpha | \alpha \rangle$ holds on E. This implies that Δ is invertible on E. □

Remark 8.3 There is an obvious generalization of the previous argument to *quaternionic Kähler hyperbolic* manifolds. A $4m$-dimensional Riemannian manifold is *quaternionic Kähler* if it admits a parallel 4-form ω whose stabilizer in the orthogonal group of some tangent space is conjugate to the maximal subgroup $Sp(m)Sp(1)$. The needed linear algebra is due to E. Bonan [1982].

CHAPTER 9

Non Vanishing Theorems for L^2 Cohomology

9.1

Following M. Gromov, we shall prove that, if M is compact with a Kähler hyperbolic universal cover $\tilde M$, then $\chi(M) \neq 0$. In view of 8.2, it is enough to show that the Laplacian Δ is not invertible on $L^2\Omega^*(\tilde M)$. There is a general conjecture (due to J. Lott and christened "zero in the spectrum conjecture" by M. Gromov [1993]) saying that for every compact manifold M, Δ is not invertible on $L^2\Omega^*(\tilde M)$. We shall establish this conjecture in a few cases below.

M. Gromov's method amounts to construct arbitrarily small perturbations of $d + d^*$ on $L^2\Omega^*(\tilde M)$ with a non trivial L^2 kernel. For this, he applies the L^2 index theorem to a *twisted* $d + d^*$, i.e., to vector valued differential forms. The idea — expressing an operator P as a deformation of an other operator Q which has non zero index in order to obtain upper bounds on eigenvalues of P — goes back to Randol [1974].

Let $(L, \nabla) \to M$ be a vector bundle equipped with a hermitian metric and hermitian connection ∇. Then there is an induced exterior differential d^∇ on $\Omega^*(M) \otimes L$. If $P = d^\nabla + (d^*)^\nabla$, then Atiyah-Singer's index theorem states

$$\mathrm{Index}\,(P) = \int_M \mathcal{L}_M \smile \mathrm{Ch}\,(L).$$

Here $\mathcal{L}_M$ is Hirzebruch's $\mathcal{L}$ class,

$$\mathcal{L}_M = 1 + \cdots + e(M)$$

where $1 \in H^0(M)$ and $e(M) \in H^{\dim M}(M)$ is the Euler class. When L is a complex line bundle, $\mathrm{Ch}\,L = \exp(c_1(L))$.

We want (L, ∇) to be a small perturbation of a trivial bundle, and also that $c_1(L) \neq 0$. This cannot be realized on M itself, since $c_1(L)$ is an integral class, but on some finite cover it is sometimes possible.

9.1.1 Riemann surface case Let Σ be a compact Riemann surface. Let us do as if we did not know that $\chi(\Sigma) \neq 0$ and give a complicated proof of it. This proof will be a model for higher dimensional situations.

9.1.2 Claim For every $\epsilon > 0$, there exists a finite cover $\Sigma_\epsilon \to \Sigma$ and an ϵ-contracting map $f_\epsilon : \Sigma_\epsilon \to \mathbf{C}P^1$ of degree 1.

Proof Equip Σ with a metric of curvature 0 or -1. Since $\pi_1(\Sigma)$ is residually finite, there exists a finite covering Σ_ϵ which contains an isometricly embedded hyperbolic disk D_ϵ of radius π/ϵ. The inverse of the exponential map is a length non increasing map to a Euclidean disk of the same radius. Apply a homothety to a Euclidean disk of radius π, then use the exponential map to $\mathbf{C}P^1$ equipped with its contant curvature 1 metric. Extend this map to a constant outside of D_ϵ. $\square$

Consider the tautological line bundle $L_0 \to \mathbf{C}P^1$ as a line subbundle in a trivial $\mathbf{C}^2$-bundle over $\mathbf{C}P^1$ with trivial connection. Let ∇_0 be the induced connection. The map f_ϵ pulls back (L_0, ∇_0) to a hermitian connected line bundle $(L_\epsilon, \nabla_\epsilon)$ on Σ_ϵ. Let

$$P_\epsilon = d^{\nabla_\epsilon} + (d^*)^{\nabla_\epsilon}.$$

If we assume that $\chi(\Sigma) = 0$ then the Atiyah-Singer formula reads $\mathrm{Index}\,(P_\epsilon) = c_1(L_\epsilon) \neq 0$. The L^2 index theorem applies to the lift $\tilde{P}_\epsilon$ of P_ϵ to the universal cover $\tilde{\Sigma}_\epsilon$, which is invariant under $\Gamma_\epsilon = \pi_1(\Sigma_\epsilon)$.

$$L^2\mathrm{Index}_{\Gamma_\epsilon} \tilde{P}_\epsilon = L^2\mathrm{Index}_{(P_\epsilon)} \neq 0.$$

This implies that either $\tilde{P}_\epsilon$ or its adjoint (which is the same operator with L_0 replaced by its dual) has a non trivial L^2 kernel.

Let us show that $\tilde{P}_\epsilon$ is indeed a small perturbation of $\widetilde{d+d^*}$, the ordinary Dirac operator on scalar differential forms on $\tilde{\Sigma}$.

L_ϵ is a line subbundle in a trivial $\mathbf{C}^2$ bundle over Σ_ϵ that varies slowly, i.e., has small second fundamental form B_ϵ. The induced connection is

$$\nabla_\epsilon = \overline{\nabla_\epsilon} + B_\epsilon$$

where $\overline{\nabla_\epsilon}$ is the trivial connection and $|B_\epsilon| \leq \mathrm{const.}\,\epsilon$. Extending B_ϵ to all of $\mathbf{C}^2$ by zero, we view ∇_ϵ as a connection on $\mathbf{C}^2$, which leaves the subbundle L_ϵ parallel. Let $Q_\epsilon = \widetilde{d^{\nabla_\epsilon} + (d^*)^{\nabla_\epsilon}}$ be the corresponding $\mathbf{C}^2$ valued Dirac operator on $\tilde{\Sigma}$. Then Q_ϵ contains a unitary equivalent copy of $\tilde{P}_\epsilon$, so it has a non trivial L^2 kernel. Also $Q_\epsilon = (\widetilde{d+d^*}) \otimes \mathbf{C}^2 + \tilde{b}_\epsilon$ with $\| \tilde{b} \|_\infty \leq \mathrm{const.}\,\epsilon$. Since ϵ can be arbitrarily small, this implies that $\widetilde{d+d^*}$ is not invertible.

9.1.3 Case of enlargeable manifolds Say a compact $2m$-dimensional manifold is *enlargeable* if for all $\epsilon > 0$, there exists a finite cover $\hat{M}_\epsilon \to M$ which admits an ϵ-contracting map of non zero degree to the sphere S^{2m} (M. Gromov and B. Lawson [1980] assume that $\hat{M}_\epsilon$ is spin but this is unnecessary here).

Pulling back a bundle $E \to S^{2m}$ with $c_{2m}(E) \neq 0$, we obtain again a bundle over $\hat{M}_\epsilon$, embedded in a trivial bundle, with small second fundamental form, and the argument goes through. This proves the following lemma.

Theorem 9.1 *If M is compact and enlargeable, then the Laplacian is not invertible on $L^2\Omega^*(\tilde{M})$. In other words, $L^2H^*(\tilde{M}) \neq 0$.*

9.1.4 Hypereuclidean manifolds General non positively curved manifolds are enlargeable only when their fundamental group is residually finite. Nevertheless, Theorem 9.1 can be extended to non residually finite cases (Gromov [1993]). Say $\tilde{M}$ is *hypereuclidean* if it admits ϵ contracting maps to the sphere which are constant at infinity. The above argument applies to hypereuclidean manifolds, but a new index theorem is needed: relative index as in Gromov and Lawson [1983] or Connes' index theorem for foliations. The zero in the spectrum conjecture, as well as the positive scalar curvature problem, is a motivation for developping more and more sophisticated versions of the index theorem.

9.1.5 Kähler hyperbolic case Let M be a compact Kähler manifold, with exact Kähler form $\omega = db$ on $\tilde{M}$. Let $\Gamma = \pi_1(M)$. For each ϵ, $\nabla_\epsilon = d + i\epsilon b$ (here $i = \sqrt{-1}$) is a unitary connection on the trivial line bundle $L = \tilde{M} \times \mathbf{C}$. One can try to make it Γ-invariant by changing to a non trivial action of Γ on $\tilde{M} \times \mathbf{C}$, i.e., setting, for $\gamma \in \Gamma$,

$$\gamma(\tilde{x}, z) = (\gamma\tilde{x}, e^{iu(\gamma,\tilde{x})}z).$$

We want $\gamma^*\nabla_\epsilon = \nabla_\epsilon$,i.e., $du = -(\gamma^*b - b)$. Since $d(\gamma^*b - b) = \gamma^*\omega - \omega = 0$, there always exists a solution $u(\gamma, \cdot)$, well defined up to a constant.

However, one cannot adjust the constants to obtain an action (if so, one would get a line bundle on M with curvature $\epsilon\omega$ and first Chern class $\frac{\epsilon}{2\pi}[\omega]$). The obstruction lives in $H^2(\Gamma, \mathbf{R})$. This means that the action is only defined on a central extension, we call this a *projective representation*.

Definition 9.2 Let G_ϵ be the subgroup of $\mathrm{Diff}(\tilde{M} \times \mathbf{C})$ formed by maps g which are linear unitary on fibers, preserve the connection ∇_ϵ and cover an element of Γ.

By construction we have an exact sequence

$$1 \to U(1) \to G_\epsilon \to \Gamma \to 1.$$

Since sections of the line bundle $\tilde{M} \times \mathbf{C} \to \tilde{M}$ can be viewed as $U(1)$ equivariant functions on $\tilde{M} \times U(1)$, the operator $\tilde{P}_\epsilon = d^{\nabla_\epsilon} + (d^*)^{\nabla_\epsilon}$ can be viewed as a G_ϵ invariant operator on the Hilbert space H of $U(1)$ equivariant basic L^2 differential forms on $\tilde{M} \times U(1)$.

We shall soon define a projective Von-Neumann dimension for invariant subspaces in a projective representation of Γ, and state an index theorem that will apply as follows to the present situation.

Theorem 9.3 *The operator $\tilde{P}_\epsilon$ has a finite projective L^2 index given by*

$$L^2\mathit{Index}_{G_\epsilon}(\tilde{P}_\epsilon) = \int_M \mathcal{L}_M \smile \exp(\frac{\epsilon}{2\pi}[\omega]).$$

This number is a polynomial in ϵ whose highest degree term is

$$\int_M (\frac{\omega}{2\pi})^m \neq 0$$

($\dim M = 2m$) thus for ϵ small enough, $\tilde{P}_\epsilon$ has a non zero L^2 kernel. By construction, $\tilde{P}_\epsilon$ is an ϵ-small perturbation of $\widetilde{d+d^*}$, so $\widetilde{d+d^*}$ is not invertible.

Corollary 9.4 *Let M be a $2m$-dimensional compact Kähler manifold. Assume that its universal cover $\tilde{M}$ is Kähler hyperbolic. Then $\tilde{M}$ admits non zero L^2 harmonic m-forms.*

CHAPTER 10

L^2 Index for Projectively Invariant Operators

Let Γ be a discrete group.

Definition 10.1 A projective representation of Γ is the data of a central extension $1 \to U(1) \to G \to \Gamma \to 1$ and a unitary representation of G on a Hilbert space H, such that the center $U(1)$ acts on H by multiplication.

Warning. This terminology (projective taken in the sense of homomorphism into $PU(H)$) should not be confused with the notion of a projective module from commutative algebra.

A collection $(e_i)_{i\in I}$ is a *G-invariant Hilbert basis* of H if G acts on I, $e_{gi} = ge_i$ and if the choice of one representative for each $U(1)$ orbit yields a Hilbert basis of H.

Definition 10.2 The G-trace of a G equivariant operator P on H is the number

$$\mathrm{Trace}_G P = \sum_{i\in G\backslash I} \langle e_i|P|e_i\rangle.$$

Example 10.3 Let $\tilde{M}$ cover a compact manifold M. Let E' and F' be vector-bundles on M equipped with hermitian metrics, let $P' : C^\infty(M, E') \to C^\infty(M, F')$ be a determined elliptic operator.

Let L be a line bundle on $\tilde{M}$ equipped with a metric and a unitary connection ∇ with Γ-invariant curvature form ω. We assume that, as in Chapter 9, a central extension G of Γ lifts to ∇-preserving transformations of the bundle L. Then G acts on the twisted bundles $E = \tilde{E}' \otimes L$, $F = \tilde{F}' \otimes L$ and the preserves the twisted operator $P = P' \otimes \nabla$ over $\tilde{M}$. We call such an operator a *projectively invariant operator on* $\tilde{M}$.

A G-invariant Hilbert basis of the space H of L^2 sections of E on $\tilde{M}$ is obtained as follows: pick a fundamental domain U for Γ in $\tilde{M}$, a Hilbert basis (e_j) of L^2 sections of E' over U, and its translates by G.

Let Π be the orthogonal projection on the L^2 kernel of P. Then Π is G-equivariant, it has a smooth kernel

$$p(x,y) \in \mathrm{Hom}\,(E'_x \otimes L_x, E'_y \otimes L_y).$$

Along the diagonal, $p(x,x) \in \mathrm{End}\, E'_x$. Then

$$\mathrm{Trace}_G \Pi = \int_M \mathrm{Trace}\, p(x,x)\, dx.$$

Theorem 10.4 *Let M be a compact manifold. Let P be a projectively invariant operator on $\tilde{M}$ arising from a determined elliptic operator P' on M. Then the G-index of P is given by*

$$L^2 Index_G\, P = \int_M \mathcal{I}_{P'} \wedge \exp(\frac{\omega}{2\pi})$$

where $\mathcal{I}_{P'}$ denotes the Atiyah-Bott-Patodi index form of P'.

Proof Theorem 10.4 is a particular case of the L^2 index theorem for G-invariant operators, where G is a Lie group (with countably many components) acting properly and freely on $\tilde{M}$ with a compact quotient. Although they do not state the theorem in this generality (and we won't either), A. Connes and H. Moscovici [1982] provide all the necessary ingredients.

The proof follows the heat equation method of M. Atiyah, R. Bott and V. Patodi [1973]. For simplicity let us denote by Δ either P^*P or PP^*. One first shows that $e^{-t\Delta}$ has a finite G-trace. Then, that

$$L^2\mathrm{Index}_G\, P = \mathrm{Trace}_G\, e^{-tP^*P} - \mathrm{Trace}_G\, e^{-tPP^*}.$$

Finally, that $\mathrm{Trace}_G\, e^{-t\Delta}$ has an asymptotic expansion as t tends to zero of the following form

$$\mathrm{Trace}_G\, e^{-t\Delta} = \sum_{j=-m}^{k} t^{\frac{j}{2}} \int_{\tilde{M}} f\mu_j + o(t^k)$$

where $\dim M = 2m$, μ_k is a quantity locally computable from the symbol of P in any coordinate chart and f any cut-off function on $\tilde{M}$ such that for all $x \in \tilde{M}$, $\int f(g^{-1}x)\, dg = 1$. The formula for μ_j is the same as in the compact case, and thus for $P = \tilde{P'} \otimes \nabla$ one finds

$$\mu_0(P^*P) - \mu_0(PP^*) = \text{top degree component of } \mathcal{I}_{P'} \wedge \exp(\frac{\omega}{2\pi}),$$

which proves the theorem. □

Here is how the necessary information on the heat kernel is obtained. The main technical trick is an averaging procedure

$$\mathrm{Av}\,T = \int_G g^{-1}Tg\, dg,$$

first defined for nonnegative self adjoint operators T with compactly supported Schwartz kernel, then extended by continuity to a domain which is invariant under left or right multiplication with bounded G-invariant operators. Its key properties are is

$$\mathrm{Trace}_G\, \mathrm{Av}\,T = \mathrm{Trace}\,T \quad \text{and} \quad \mathrm{Av}\,(RTQ) = R\mathrm{Av}\,(T)Q$$

for Q and R bounded, G-invariant.

If f is a non negative compactly supported function such that for all $x \in \tilde{M}$,

$$\int f(g^{-1}x)\, dg = 1,$$

then $f\,e^{-t\Delta} \in \mathrm{Dom\,Av}$ and $e^{-t\Delta} = \mathrm{Av}\, f\, e^{-t\Delta}$ so

$$\mathrm{Trace}_G\, e^{-t\Delta} = \mathrm{Trace}\, f\, e^{-t\Delta}$$

is finite.

The McKean-Singer cancellation formula

$$L^2\mathrm{Index}_G\, P = \mathrm{Trace}_G\, e^{-tP^*P} - \mathrm{Trace}_G\, e^{-tPP^*}$$

follows, as in the compact case, from the fact that, orthogonally to $\mathrm{Ker}\,\Delta$, P (or rather the unitary part of its polar decomposition) conjugates e^{-tP^*P} to e^{-tPP^*}.

The asymptotic expansion for $\mathrm{Trace}_G\, e^{-t\Delta}$ follows from the corresponding expansion for $f\,e^{-t\Delta}$ (see for example Gilkey [1984] Chapter 3). Note that P. Gilkey's method for constructing a parametrix Q_λ (holomorphic in λ) for $\Delta - \lambda$ involves cut-offs anyway. For each k, one obtains operators Q_λ which have compactly supported kernels and satisfy

$$Q_\lambda(\Delta - \lambda) = f + R_\lambda$$

where R_λ is smoothing with L^1 norm

$$\| R_\lambda \|_{\mathrm{tr}} \leq \mathrm{const.}\,(1 + |\lambda|^{-k-1}).$$

A contour integration (Mellin transform) yields an operator

$$E(t) = \frac{1}{2i\pi} \int e^{-t\lambda} Q_\lambda\, d\lambda$$

which has a smooth compactly supported kernel, satisfies

$$\| E(t) - f\,e^{-t\Delta} \|_{\mathrm{tr}} \leq \mathrm{const.}\, t^k$$

and whose trace has an asymptotic expansion with locally computable coefficients. This gives the asymptotic expansion for $\mathrm{Trace}\, f\, e^{-t\Delta}$ and thus for $\mathrm{Trace}_G\, e^{-t\Delta}$.

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