Metric problems concerning nilpotent groups

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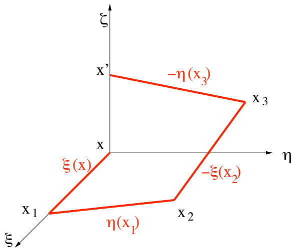
We present four open problems about maps between nilpotent groups.

1. (Continuation of Tessera's talk). Heisenberg groups do not biLipschitz embed in ℓ_2. But snowflaked versions do. Want bounds on distorsions, dimensions of such embeddings.
2. Gromov's Hölder homeomorphism problem and a variant.

Definition

Heisenberg group *Heis*^{3} *is the 3-dimensional Lie group with Lie algebra spanned by* ξ, η *and* ζ *with* $[\xi, \eta] = \zeta$. *The left-invariant vectorfields* ξ *and* η *span a plane field* H, Carnot-Carathéodory distance $d_{cc}(x, x')$ *is* inf *of length of curves tangent to* H *joining* x *to* x'. Dilation δ_t *is automorphism induced by* $\delta_t(\xi) = t\xi$, $\delta_t(\eta) = t\eta$, $\delta_t(\zeta) = t^2\zeta$. *It multiplies Carnot distances by* t.

Finiteness of Carnot distance follows from picture:



Translation and dilation invariance implies

1. $d_{cc}(x, x\exp(t^2\zeta)) = t d_{cc}(1, \exp(\zeta)) = \mathrm{const.}\, t$.
2. $\mathrm{volume} B(x,r) = r^4 \mathrm{volume} B(x,1) = \mathrm{const.}\, r^4$, thus Hausdorff dimension is 4.
3. The same number of balls of radius $r/2$ suffice to cover every ball $B(x,r)$.

Heisenberg group in its Carnot-Carathéodory metric gives a sharp approximation of the word metric on the *integral Heisenberg group*

$$Heis_{\mathbb{Z}} = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \; ; x,\, y,\, z \in \mathbb{Z} \}.$$

In general, if Γ is a finitely generated nilpotent group, given a finite generating system, the word metric space (Γ, d_w) admits an asymptotic cone, which is a *Carnot group*. A Carnot group is a nilpotent Lie group equipped with a Carnot-Carathéodory metric homogeneous under dilations.

Definition

The doubling dimension $\dim(X)$ *of a (doubling) metric space* X *is the least* d *such that for all* R*, every* R*-ball can be covered by* 2^d $R/2$*-balls.*

Example

1. $\mathbb{R}^n$, *Heis*^{n} *(n odd) have doubling dimension linear in n.*
2. The Internet network equipped with its latency metric is believed to have low dimension.

Theorem

(Assouad 1983). *For every* $\epsilon \in (0,1)$ *and* $d > 0$*, there exist* $D(d,\epsilon)$ *and* $N(d,\epsilon)$ *such that for every* d*-dimensional metric space* X*, the* snowflaked metric $(X, d_X^{1-\epsilon})$ *embeds in* ℓ_2^N *with distorsion* $\leq D$.

So snowflaked Heisenberg group does biLipschitz embed in ℓ_2.

Question

Give sharp bounds on D *and* N.

Theorem

(Gupta, Krauthgamer, Lee 2003; Lee, Mendel, Naor 2004). *In Assouad's theorem, one can take* $N = O(\frac{d \log d}{\epsilon})$ *and* $D = O(\frac{d}{\sqrt{\epsilon}})$.

Unclear wether dimension bound is sharp or not.

Question

What is the minimal dimension $N(\epsilon)$ *of a Euclidean space in which* $(Heis^3, d_{cc}^{1-\epsilon})$ *admits a biLipschitz embedding ?*

Remark

$N(\epsilon) > 4$.

Indeed, the Hausdorff dimension of $(Heis^3, d_{cc}^{1-\epsilon})$ is $\frac{4}{1-\epsilon} > 4$.

Theorem

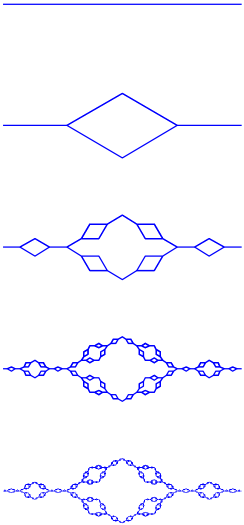
(Gupta, Krauthgamer, Lee 2003; Lee, Mendel, Naor 2004). *In Assouad's theorem, one can take* $N = O(\frac{d \log d}{\epsilon})$ *and* $D = O(\frac{d}{\sqrt{\epsilon}})$.
For fixed d, the distorsion bound is sharp (Lee, Mendel, Naor 2004)*:*
$(1-\epsilon)$*-snowflaked Laakso spaces require distorsion* $\Omega(\frac{1}{\sqrt{\epsilon}})$ *when embedded in* ℓ_2.

Dependance on d ? Heisenberg groups do not help. Indeed, *(Lee, Naor 2006)*: $(Heis^n, d_{cc}^{1-\epsilon})$ embed in ℓ_2 with distorsion $O(\frac{1}{\sqrt{\epsilon}})$ independant on n.

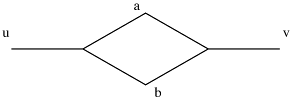
Question

What is the minimal distorsion of a biLipschitz embedding of $(Heis^3, d_{cc}^{1-\epsilon})$ *in* ℓ_2 *?*

Laakso graphs



Proof *(Lee, Mendel, Naor 2004)* that $1-\epsilon$-snowflaked Laakso graph $(G_j, d_j^{1-\epsilon})$ requires distorsion $D_j \geq \Omega(\frac{1}{\sqrt{\epsilon}})$ when embedded in ℓ_2.



By induction on j. Show that $D_j^2 \geq 4^{-\epsilon} D_{j-1}^2 + \frac{1}{4}$.
In rescaled Laakso graph, $uavb$ is a unit square with a diagonal of length 2. When mapped to a quadrilateral $u'a'v'b'$ in Euclidean space in a distance nondecreasing manner, parallelogram inequality

$$|u'-v'|^2 + |a'-b'|^2 \leq |u'-a'|^2 + |a'-v'|^2 + |v'-b'|^2 + |b'-u'|^2$$

implies

$$4\frac{|u'-v'|^2}{|u-v|^2} + \frac{|a'-b'|^2}{|a-b|^2} \leq \frac{|u'-a'|^2}{|u-a|^2} + \frac{|a'-v'|^2}{|a-v|^2} + \frac{|v'-b'|^2}{|v-b|^2} + \frac{|b'-u'|^2}{|b-u|^2},$$

or

$$4^{1-\epsilon}\frac{|u'-v'|^2}{|u-v|^{2(1-\epsilon)}} + \frac{|a'-b'|^2}{|a-b|^{2(1-\epsilon)}} \leq \frac{|u'-a'|^2}{|u-a|^{2(1-\epsilon)}} + \frac{|a'-v'|^2}{|a-v|^{2(1-\epsilon)}} + \frac{|v'-b'|^2}{|v-b|^{2(1-\epsilon)}} + \frac{|b'-u'|^2}{|b-u|^{2(1-\epsilon)}}$$

Question

(Gromov 1993). *Let G be a Carnot group of dimension n. For which $\alpha \in (0,1)$ does there exist locally a homeomorphism $\mathbb{R}^n \to G$ which is C^α-Hölder continuous ?*

Definition

Let X, Y be metric spaces. Let $Holder(X,Y) = \sup\{\alpha \in (0,1) \,|\, \exists$ locally a C^α-Hölder continuous homeomorphism $X \to Y$ whose inverse is Lipschitz $\}$.

Example

If G is a r-step Carnot group, the exponential map $\mathfrak{g} = Lie(G) \to G$ is locally $C^{1/r}$-Hölder continuous and its inverse is Lipschitz. Thus $Holder(\mathbb{R}^n, G) \geq 1/r$.

Proposition

Let G have dimension n and Hausdorff dimension Q. Then $Holder(\mathbb{R}^n, G) \leq \frac{n}{Q}$.

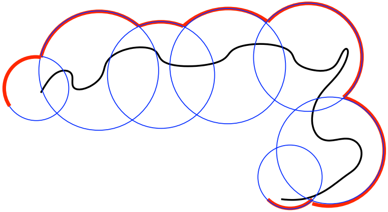
Proposition

Let G have dimension n and Hausdorff dimension Q. Then $Holder(\mathbb{R}^n, G) \leq \frac{n-1}{Q-1}$.

Proof. Use the Varopoulos (1985) isoperimetric inequality for piecewise smooth domains $D \subset M$,

$$vol(D)^{Q-1/Q} \leq const.\, \mathcal{H}^{Q-1}(\partial D).$$

It follows that the boundary of any non smooth domain Ω has Hausdorff dimension at least $Q-1$. Indeed, cover $\partial\Omega$ with balls B_j and apply (*) to $\Omega \cup \bigcup B_j$. This gives a lower bound on $\mathcal{H}^{Q-1}(\partial(\bigcup B_j)) \leq \sum \mathcal{H}^{Q-1}(\partial B_j) \leq const. \sum \mathrm{diameter}(B_j)^{Q-1}$.



Proposition

(Gromov 1993). *Let $n = 2m+1$, let $Heis^n$ denote n-dimensional Heisenberg group. Let $V \subset Heis^n$ be a subset of topological dimension $m+1$. Then the Hausdorff dimension of V is at least $m+2$. It follows that $Holder(\mathbb{R}^n, Heis^n) \leq \frac{m+1}{m+2}$.*

Proof. According to topological dimension theory (Alexandrov), there exists an m-dimensional polyhedron P and a continuous map $f : P \to Heis^n$ such that every map sufficiently C^0-close to f hits V.

Gromov approximates f with piecewise *horizontal* maps which sweep an open set U. This gives rise to a local projection $p : U \to \mathbb{R}^{m+1}$ such that for every ball B, the tube $p^{-1}(p(B))$ has volume $\leq$ const. $\mathrm{diameter}(B)^{m+2}$.

Cover V with balls B_j. The corresponding tubes $T_j = p^{-1}(p(B_j))$ cover U. Then the volume of U is less than $\sum \mathrm{diameter}(B_j)^{m+2}$, which shows that $dim_{Hau}(V) \geq m+2$.

Theorem

(Gromov 1993). *Let G be a generic Carnot group of dimension n, Hausdorff dimension Q, with an h-dimensional distribution. Let $k \leq h$ be such that $h-k \geq (n-h)k$. Then $Holder(\mathbb{R}^n, G) \leq \frac{n-k}{Q-k}$.*

Curvature pinching

Definition

Let M be a Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ-pinched if sectional curvature ranges between -1 and δ. Define the optimal pinching $\delta(M)$ of M as the least $\delta \geq -1$ such that M is quasiisometric to a δ-pinched simply connected Riemannian manifold.

Example

Rank one symmetric spaces of noncompact type are hyperbolic spaces over the reals $H^n_{\mathbb{R}}$, the complex numbers $H^m_{\mathbb{C}}$, the quaternions $H^m_{\mathbb{H}}$, and the octonions $H^2_{\mathbb{O}}$.
Real hyperbolic space has sectional curvature -1. Other rank one symmetric spaces are $-\frac{1}{4}$-pinched.

Question

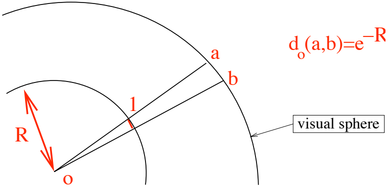
What is the optimal pinching of $H^m_{\mathbb{C}}$?

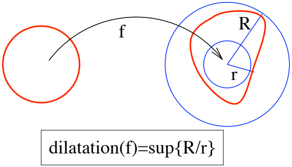
Definition

Say two geodesic rays in a Riemannian manifold are asymptotic *if their Hausdorff distance is finite. The* visual boundary *of a negatively curved manifold is the set of asymptoticity classes of geodesic rays.*

Facts.

- The visual boundary, seen from a point o, is a sphere (use polar coordinates).
- It carries a *visual metric* d_o.
- Different visual metrics d_o and $d_{o'}$ are equivalent.
- Quasiisometries between negatively curved Riemannian manifolds induce *quasisymmetric* maps between ideal boundaries.





Example

If M is a rank one symmetric space, the visual metrics on its ideal boundary are locally equivalent to Carnot-Carathéodory metrics on Carnot groups.

Proposition

Let M be a simply connected δ-pinched Riemannian manifold. Equip the ideal boundary ∂M of M with a visual metric. The natural homeomorphism $S^{n-1} \to \partial M$ is C^α with $\alpha = \sqrt{-\delta}$, and its inverse is Lipschitz. Therefore $Holder(\mathbb{R}^{n-1}, \partial M) \geq \sqrt{-\delta}$.

Indeed, geodesics from a unit ball to a point come together exponentially fast, with exponents ranging from $\sqrt{-\delta}$ to 1 (Rauch comparison theorem, 1950's).

Question

Let G be a Carnot group. Let $\alpha > 1/2$. Does there exist quasisymmetricly equivalent metrics on G which locally admit C^α homeomorphisms from Euclidean space ? With Lipschitz inverses ?

If no, then optimal pinching of $H_{\mathbb{C}}^m$ is $-\frac{1}{4}$.

Definition

Let $f : X \to Y$ be a homeomorphism. The conformal Hölder exponent $CHolder(f)$ of f is the supremum of α's such that for all $\ell > 0$, there exists $L > 0$ such that for all x, x', x'' in X,

$$d(f(x), f(x'')) \leq \ell\, d(f(x), f(x')) \Rightarrow d(x, x'') \leq L\, d(x, x')^{\alpha}.$$

Let $CHolder(X, Y)$ denote the supremum of α's such that there locally exist homeomorphisms $X \to Y$ with conformal Hölder exponents $\geq \alpha$.

Lemma

1. If $f : X \to Y$ is C^{α} and f^{-1} is C^{β}, then $CHolder(f) \geq \alpha\beta$. In particular, $Holder(X, Y) \leq CHolder(X, Y)$.
2. Let $f : X \to Y$ and $g : Y \to Z$ be homeomorphisms. Assume that g is quasisymmetric. Then $CHolder(g \circ f) = CHolder(f)$.

Definition

Let X be a compact metric space. The conformal dimension *of X is the infimum of Hausdorff dimensions of metric spaces quasisymmetricly equivalent to X.*

Example

(Pansu 1990). *Carnot groups have conformal dimension equal to their Hausdorff dimension.*

Corollary

Let G be a Carnot group of dimension n and Hausdorff dimension Q. Then $CHolder(\mathbb{R}^n, G) \leq \frac{n}{Q}$.

Theorem

(Pansu 2009). *The optimal pinching of $H^2_{\mathbb{C}}$ is $-\frac{1}{4}$.*

Gives some hope for $CHolder(\mathbb{R}^3, Heis^3) = \frac{1}{2}$ and therefore $Holder(\mathbb{R}^3, Heis^3) = \frac{1}{2}$.

Metric geometry, algorithms and groups

Paris, january 10th - april 8th, 2011

Institut Henri Poincaré

Organizers: Guy Kindler (Jerusalem), James Lee (U. Washington), Claire Mathieu (Brown), Ryan O'Donnell (Carnegie Mellon), Pierre Pansu (Paris-Sud/ENS), Nicolas Schabanel (LIAFA-CNRS), Lior Silberman (Vancouver)

Workshops:
january 17-21 : embeddings, algorithms, complexity

march 21-25 : expanders, derandomization

In between : courses (Valette, Linial, Pisier, Maurey, Breuillard, Cordero-Erausquin, Lang, Wenger, Silberman, Peres, Gamburd, Wigderson + computer scientists)

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