# $L^p$-cohomology and curvature pinching

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|---|---|---|
| *topological space* | $\longrightarrow$ | *cohomology* |
| *manifold* | $\longrightarrow$ | *de Rham cohomology* |
| *metric space* | $\longrightarrow$ | *cohomology with decay condition* |
| *Riemannian manifold* | $\longrightarrow$ | *de Rham cohomology with decay condition* |

### Definition

*Let $M$ be a Riemannian manifold. Let $p > 1$. $L^p$-*cohomology *of $M$ is the cohomology of the complex of $L^p$-differential forms on $M$ whose exterior differentials are $L^p$ as well,*

$$\begin{aligned} H^{k,p} &= \textit{closed } k\textit{-forms in } L^p / d((k-1)\textit{-forms in } L^p), \\ R^{k,p} &= \textit{closed } k\textit{-forms in } L^p / \textit{closure of } d((k-1)\textit{-forms in } L^p), \\ T^{k,p} &= \textit{closure of } d((k-1)\textit{-forms in } L^p) / d((k-1)\textit{-forms in } L^p). \end{aligned}$$

$R^{k,p}$ *is called the* reduced cohomology*. $T^{k,p}$ is called the* torsion*.*

# Example : the real hyperbolic plane $H^2_{\mathbb{R}}$

Here $H^{0,p} = 0 = H^{2,p}$ for all $p$.
If $p = 2$, since the Laplacian on $L^2$ functions is bounded below, $T^{1,2} = 0$. Therefore

$$\begin{aligned} H^{1,2} &= R^{1,2} \\ &= \{L^2 \text{ harmonic 1-forms}\} \\ &= \{\text{harmonic functions } h \text{ on } H^2_{\mathbb{R}} \text{ with } \nabla h \in L^2\}/\mathbb{R}\,. \end{aligned}$$

Using conformal invariance, switch from hyperbolic metric to euclidean metric on the disk $D$.

$$\begin{aligned} H^{1,2} &= \{\text{harmonic functions } h \text{ on } D \text{ with } \nabla h \in L^2\}/\mathbb{R} \\ &= \{\text{Fourier series } \Sigma a_n e^{in\theta} \text{ with } a_0 = 0, \Sigma|n|\,|a_n|^2 < +\infty\}, \end{aligned}$$

which is Sobolev space $H^{1/2}(\mathbb{R}/2\pi\mathbb{Z})$ mod constants.

More generally, for $p > 1$, $T^{1,p} = 0$ and $H^{1,p}$ is equal to the Besov space $B^{1/p}_{p,p}(\mathbb{R}/2\pi\mathbb{Z})$ mod constants.

# Example : the real line $\mathbb{R}$

$H^{0,p} = 0$.

$R^{1,p} = 0$,
since every function in $L^p(\mathbb{R})$ can be approximated in $L^p$ with derivatives of compactly supported functions. Therefore $H^{1,p}$ is only torsion.

$T^{1,p}$ *is non zero and thus infinite dimensional.*
Indeed, the 1-form $\frac{dt}{t}$ (cut off near the origin) is in $L^p$ for all $p > 1$ but it is not the differential of a function in $L^p$.

# What are our favourite spaces ?

- $L^p$-cohomology has been used (L. Saper, S. Zucker) to study manifolds with thin ends, e.g. locally symmetric spaces of finite volume. The answer is related to the topology of a compactification.

- In this talk : manifolds with large ends, e.g. symmetric spaces themselves. $L^p$-cohomology is related to analytic features of a compactification.

In conclusion,

- $L^p$-cohomology tastes like harmonic analysis.
- We shall apply it to a problem in Riemannian geometry.
- Ideas from algebraic topology play a role.

### Remark

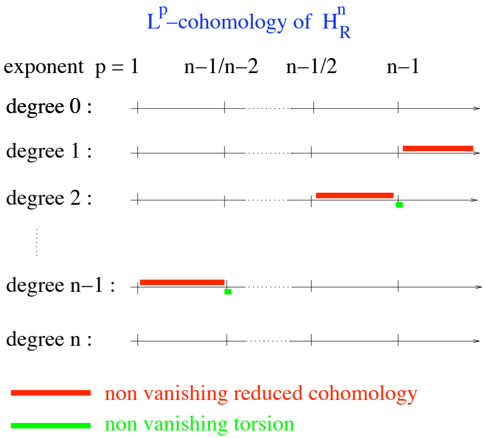
*Rank one symmetric spaces of noncompact type are hyperbolic spaces over the reals $H^n_{\mathbb{R}}$, the complex numbers $H^m_{\mathbb{C}}$, the quaternions $H^m_{\mathbb{H}}$, and the octonions $H^2_{\mathbb{O}}$.*
*Real hyperbolic space has sectional curvature $-1$. Other rank one symmetric spaces are $-\frac{1}{4}$-pinched, i.e. their sectional curvature ranges between $-1$ and $-\frac{1}{4}$.*

### Definition

*Define the* optimal pinching $\delta(M)$ *of a Riemannian manifold $M$ as the least $\delta \geq -1$ such that $G$ is bi-Lipschitz equivalent to a $\delta$-pinched Riemannian manifold.*

### Question

*Is it true that the optimal pinching of $H^m_{\mathbb{C}}$, $H^m_{\mathbb{H}}$ $(m \geq 2)$ and $H^2_{\mathbb{O}}$ is $-\frac{1}{4}$ ?*


L^p–cohomology of H^n_R
exponent p = 1
n–1/n–2
n–1/2
n–1
degree 0 :
degree 1 :
degree 2 :
degree n–1 :
degree n :
non vanishing reduced cohomology
non vanishing torsion

### Theorem

*If $M^n$ is simply connected and $\delta$-pinched for some $\delta \in [-1, 0)$, then*

$$p < 1 + \frac{n-k}{k-1}\sqrt{-\delta} \quad \Rightarrow \quad T^{k,p}(M) = 0.$$

This is sharp. For instance, let $n = 4$, $k = 2$, $\delta = 1/4$ and consider the semidirect product $G = \mathbb{R}^3 \rtimes_\alpha \mathbb{R}$ where $\alpha = diag(1, 1, 2)$.

- It admits a $-\frac{1}{4}$-pinched left-invariant Riemannian metric, therefore $\delta(G) \leq -\frac{1}{4}$.
- It has $T^{2,p}(G) \neq 0$ for $2 < p \leq 4$. This implies that $\delta(G) = -\frac{1}{4}$.

### Remark

*Complex hyperbolic plane $H^2_{\mathbb{C}}$ is isometric to $G' = Heis^3 \rtimes_\alpha \mathbb{R}$ where $\alpha = diag(1, 1, 2)$ and Heis denotes the Heisenberg group. Therefore it is very close to $G$.*

### Theorem

$T^{2,p}(H^2_{\mathbb{C}}) = 0$ *for* $2 < p < 4$.

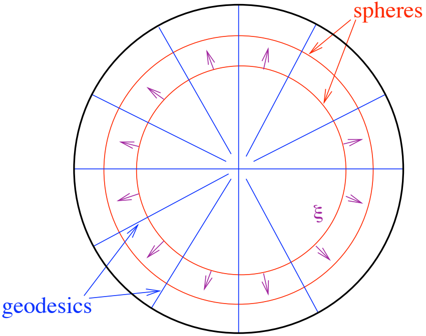
# Proof of torsion comparison theorem

Step 1. *For $p$ large, closed $L^p$ forms admit boundary values.*
Use the radial vectorfield $\xi = \frac{\partial}{\partial r}$ in polar coordinates and its flow $\phi_t$, whose derivative is controlled by sectional curvature.

Use Poincaré's homotopy formula :
For $\alpha$ a closed $k$-form in $L^p$,

$$\phi_t^* \alpha = \alpha + d \left( \int_0^t \phi_s^* \iota_\xi \alpha \, ds \right)$$

has a limit as $t \to +\infty$ under the assumptions of the theorem.




Step 2. *Boundary value determines cohomology class.*
This boundary value map injects $H^{k,p}$ into a function space of closed forms on the ideal boundary, showing that $H^{k,p}$ is Hausdorff.

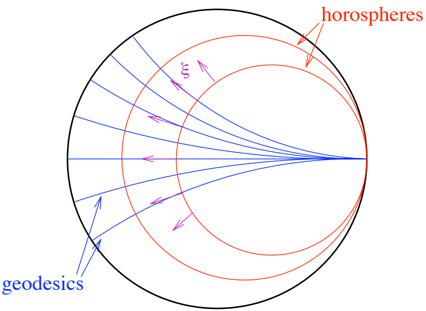
# Proof of torsion vanishing for $H^2_{\mathbb{C}}$

Switch point of view. Use horospherical coordinates. View $H^2_{\mathbb{C}}$ as a product $Heis \times \mathbb{R}$. Prove a Künneth type theorem.

Step 1. For $p \notin \{4/3, 2, 4\}$, differential forms $\alpha$ on $H^2_{\mathbb{C}}$ split into components $\alpha_+$ and $\alpha_+$ which are contracted (resp. expanded) by $\phi_t$. Then

$$h_t : \alpha \mapsto \int_0^t \phi_s^* \iota_\xi \alpha_+ \, ds - \int_{-t}^0 \phi_s^* \iota_\xi \alpha_- \, ds$$

converges as $t \to +\infty$ to a bounded operator $h$ on $L^p$. $P = 1 - dh - hd$ retracts the $L^p$ de Rham complex onto a complex $\mathcal{B}$ of differential forms on $Heis^3$ with missing components and weakly regular coefficients.

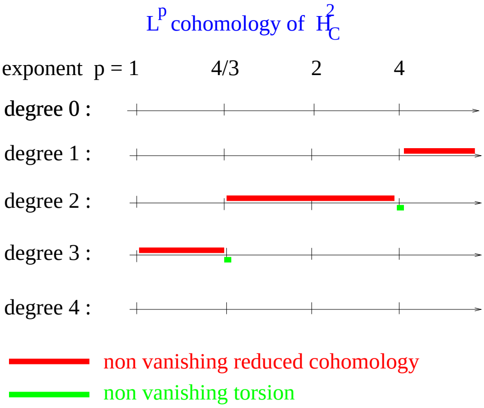



Step 2. If $2 < p < 4$, this complex is nonzero in degrees 1 and 2. Let $\tau$ denote the invariant contact form on $Heis^3$. Then $\mathcal{B}^1$ consists of 1-forms which are multiples of $\tau$. Since

$$d(f\tau) = df \wedge \tau + f \, d\tau,$$

$d(f\tau)$ determines $f$, therefore $d : \mathcal{B}^1 \to \mathcal{B}^2$ has closed range.

# $L^p$-cohomology of $H^2_{\mathbb{C}}$



# Subalgebra theorem

### Definition

(Bourdon-Pajot 2004). *For a negatively curved manifold $M$, define the* Royden algebra $\mathcal{R}_p(M)$ *as the space of bounded functions $u$ on $M$ such that $du \in L^p$, modulo $L^p \cap L^\infty$ functions.*

Obviously, $u \in \mathcal{R}_p(M)$ implies $[du] \in H^{1,p}(M)$.
Recall that cup-product $\smile$ is well defined : $H^{k,q}(M) \times H^{\ell,r}(M) \to H^{k+\ell,p}(M)$ provided $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

### Theorem

(Work in progress). *Assume $M^n$ is simply connected and $\delta$-pinched for some $\delta \in (-1,0)$. Assume*

$$p < 1 + \frac{n-k}{k-1}\sqrt{-\delta}, \quad r < 1 + \frac{n-k+1}{k-2}\sqrt{-\delta} \quad \textit{and} \quad \frac{1}{p} = \frac{1}{q} + \frac{1}{r}.$$

*Pick a class $\kappa \in H^{k-1,r}(M)$. Then the set of $u \in \mathcal{R}_q(M)$ such that $[du] \smile \kappa = 0$ in $H^{k,p}(M)$ is a subalgebra.*

### Example

*$n = 4$, $k = 2$, $q = r = 2p$. For $\delta < -\frac{1}{4}$, the theorem applies for some $p > 2$.*

### Proposition

(Work in progress). *Let $2 < p < 4$. There exists a conical open set $U \subset H^2_{\mathbb{C}}$, a class $\kappa \in H^{1,2p}(U)$ and a function $u \in \mathcal{R}_{2p}(U)$ such that $[du] \smile \kappa = 0$ in $H^{2,p}(U)$ but $[d(u^2)] \smile \kappa \neq 0$ in $H^{2,p}(U)$.*

### Remark

*This should imply that the optimal pinching for $H^2_{\mathbb{C}}$ is $-\frac{1}{4}$, provided subalgebra theorem localizes on conical open sets.*

**Proof**

If $\alpha \in \mathcal{B}^2$ is a 2-form, then $\alpha \in d\mathcal{B}^1$ implies that there exists a function $f$ such that $\alpha = df \wedge \tau + f\, d\tau$. Then $f = \dfrac{\tau \wedge \alpha}{\tau \wedge d\tau}$. Thus $\alpha$ satisfies a linear differential equation

$$\alpha = d\left(\frac{\tau \wedge \alpha}{\tau \wedge d\tau}\tau\right).$$

If $du \wedge \beta$ is a solution, $d(u^2) \wedge \beta$ is not a solution, unless $\beta$ is proportional to $du$.

### Example

*In coordinates, if $\tau = dz - x\, dy$, one can take $u = x$, $\beta = dy$.*