

Flexibility of surface groups in semi-simple Lie groups

I. Kim, P. Pansu

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Definition

Let G be the group of real points of an algebraic group. Let Γ be a finitely generated group. Say a non Zariski dense homomorphism $\rho : \Gamma \rightarrow G$ is flexible if it is a limit of Zariski dense homomorphisms, locally rigid otherwise.

Problem

Determine which homomorphisms of surface groups to almost simple Lie groups are locally rigid.

Plan of lecture

We survey (global) rigidity results concerning surface groups in semisimple Lie groups.

We complement them with new flexibility results.

Let X be a Hermitian symmetric space, with Kähler form Ω (the metric is normalized so that the minimal sectional curvature equals -1). Let Σ be a closed surface of negative Euler characteristic, let $\Gamma = \pi_1(\Sigma)$ act isometrically on X . Pick a smooth equivariant map $\tilde{f} : \tilde{\Sigma} \rightarrow X$.

Definition

Define the Toledo invariant of the action $\rho : \Gamma \rightarrow \text{Isom}(X)$ by

$$T_\rho = \frac{1}{2\pi} \int_{\Sigma} \tilde{f}^* \Omega.$$

Then

1. T_ρ depends continuously on ρ .
2. There exists $\ell_X \in \mathbb{Q}$ such that $T_\rho \in \ell_X \mathbb{Z}$.
3. $|T_\rho| \leq |\chi(\Sigma)| |\text{rank}(X)|$.

Example

When $X = H_{\mathbb{C}}^1$ is a line, inequality $|T_{\rho}| \leq |\chi(\Sigma)|$ is due to J. Milnor (1958).
Furthermore $\ell_X = 1$, T takes all integer values between $-|\chi(\Sigma)|$ and $|\chi(\Sigma)|$.

Theorem

(W. Goldman, 1980). Let $X = H_{\mathbb{C}}^1$. The level sets of T coincide with the connected components of the character variety $\chi(\Gamma, PU(1, 1))$. Furthermore $|T_{\rho}| = |\chi(\Sigma)|$ if and only if $\rho(\Gamma)$ is discrete and cocompact in $PU(1, 1) = \text{Isom}(H_{\mathbb{C}}^1)$.

Note that all components of $\chi(\Gamma, PU(1, 1))$ have the same dimension $3|\chi(\Sigma)|$.

Theorem

(D. Toledo, 1979, 1989). Let $X = H_{\mathbb{C}}^n$ have rank 1. Then $|T_{\rho}| \leq |\chi(\Sigma)|$. Furthermore, $|T_{\rho}| = |\chi(\Sigma)|$ if and only if $\rho(\Gamma)$ stabilizes a complex geodesic $H_{\mathbb{C}}^1$ in X and acts cocompactly on it.

It follows that, for $n \geq 2$, different components of $\chi(\Gamma, PU(n, 1))$ can have different dimensions.

Definition

Actions ρ such that $|T_\rho| = |\chi(\Sigma)|\text{rank}(X)$ are called maximal representations.

Example

Pick cocompact actions ρ_1, \dots, ρ_r of Γ on $H_{\mathbb{C}}^1$. Then the direct sum representation on the polydisk $(H_{\mathbb{C}}^1)^r$ is maximal. When the polydisk is embedded in a larger symmetric space of rank r , it remains maximal. It follows that all Hermitean symmetric spaces admit maximal representations.

Proposition

(Toledo, 1987). In case X is Siegel's upper half space (i.e. $\text{Isom}(X) = \text{Sp}(n, \mathbb{R})$), such actions can be deformed to become Zariski dense.

In fact, this can be achieved by bending (Burger, Iozzi, Wienhard, 2005).
But this may fail for other Hermitean symmetric spaces.

Theorem

(L. Hernández Lamóneda, 1991, S. Bradlow, O. García-Prada, P. Gothen, 2003).
Maximal reductive representations of Γ to $PU(p, q)$, $p \leq q$, can be conjugated into $P(U(p, p) \times U(q - p))$.

Definition

Say a Hermitean symmetric space is of tube type if it can be realized as a domain in \mathbb{C}^n of the form $\mathbb{R}^n + iC$ where $C \subset \mathbb{R}^n$ is a proper open cone.

Example

Siegel's upper half spaces and Grassmannians with isometry groups $PO(2, q)$ are of tube type.

The Grassmannian $\mathcal{D}_{p,q}$, $p \leq q$, with isometry group $PU(p, q)$ is of tube type iff $p = q$.

The Grassmannian with isometry group $SO^(2n)$ is of tube type iff n is even.*

The exceptional Hermitean symmetric space of dimension 27 is of tube type, the other one (of dimension 16) is not.

Products of tube type spaces are of tube type, so polydisks are of tube type.

Lemma

All maximal tube type subsymmetric spaces in a Hermitean symmetric space are conjugate.

Example

The maximal tube type subsymmetric space in $\mathcal{D}_{p,q}$ is $\mathcal{D}_{p,p}$.

Theorem

(Burger, Iozzi, Wienhard, 2003). *Let Γ be a closed surface group and X a Hermitian symmetric space. Every maximal representation $\Gamma \rightarrow \text{Isom}(X)$ stabilizes a tube type subsymmetric space Y and is Zariski dense in $\text{Isom}(Y)$.*

In particular, maximal representations of surface groups in non tube type Hermitian symmetric spaces are globally rigid.

Example

In case X is the n -ball $\mathcal{D}_{1,n}$ (resp. $\mathcal{D}_{p,q}$), one recovers Toledo's (resp. Barlow et al.) results.

Proof.

1. Surfaces admit ideal triangulations.
2. Ideal triangles in Hermitian symmetric spaces have their endpoints on the Shilov boundary, in general position.
3. The Kähler area of an ideal triangle either takes a full interval of values (non tube type case) or finitely many values (tube type case), in both cases bounded by $\frac{1}{2}\text{rank}(X)$ (Clerc-Ørsted).
4. Equality holds iff the triangle is contained in a subsymmetric space of tube type.

Problem

Characterize locally rigid actions of closed surface groups on (non necessarily Hermitean) symmetric spaces.

All previously known examples of flexibility can be obtained by bending representations.

Example

(Burger, Iozzi, Wienhard). *In a tube type Hermitean symmetric space, a surface group stabilizing a maximal polydisk and acting diagonally on it is flexible.*

Theorem

Amenable representations of surface groups into semisimple Lie groups are flexible, provided genus is high enough, and up to restricting to a finite index subgroup.

Theorem

Let G be a simple Lie group. Let $H \subset G$ be a semisimple subgroup. Let Γ be the fundamental group of a closed surface of genus $\geq \dim(G)^2$. Let $\rho : \Gamma \rightarrow H$ be a Zariski dense homomorphism.

- 1. Let \mathfrak{s} be a semisimple Levi summand of the centralizer of H . Then ρ is flexible in the semisimple group $G' = H \exp(\mathfrak{s})$, i.e., can be approximated by Zariski dense homomorphisms $\rho' : \Gamma \rightarrow G'$.*
- 2. If such a ρ' is locally rigid in G , then the symmetric space of noncompact type associated to G' is Hermitean of tube type, and the representation $\rho' : \Gamma \rightarrow G'$ is maximal.*
- 3. If the centralizer \mathfrak{z} of G' in G is 1-dimensional, then ρ' is locally rigid in G if and only if G' is Hermitean of tube type and the induced symplectic representation on $\mathfrak{g}/(\mathfrak{z} \oplus \mathfrak{g}')$ is maximal.*

1. In case G is of Hermitean type and the Zariski closure of the homomorphism is semisimple with a , this is an exact converse to the result by Burger et al (up to the restriction on genus).

Example

If $G = SU(p, q)$, $H = SU(p, p)$, then $G' = S(U(p, p) \times U(q - p))$ has a 1-dimensional centralizer. Only one pair of nonzero opposite roots. If genus is $\geq (p + q)^2$, Zariski dense representations to $SU(p, p)$ are locally rigid in $SU(p, q)$ if and only if they are maximal.

2. Viewed as a rigidity result, the last statement is not fully satisfactory. Indeed, ρ' could be flexible while ρ is locally rigid.
3. Work in progress : it should be possible to merge both theorems into a more general one where no assumption is made on representations.
4. The restriction on genus is probably irrelevant.
5. Non constructive : deformations are not given by explicit formulae nor geometric constructions.

Theorem

(W. Goldman, 1985). *If Γ is a surface group and ρ is reductive, then, in a neighborhood of the conjugacy class of ρ , $\text{Hom}(\Gamma, G)/G$ is analytically equivalent to*

$$\{u \in H^1(\Gamma, \mathfrak{g}_{\text{ad} \circ \rho}) \mid [u, u] = 0\} / Z_G(\rho(\Gamma)).$$

Here, bracket denotes cup-product $H^1(\Gamma, \mathfrak{g}_{\text{ad} \circ \rho}) \rightarrow H^2(\Gamma, \mathfrak{g}_{\text{ad} \circ \rho})$.

Remark

This can prove flexibility without providing explicit deformations.

The dimension of $H^1(\Gamma, \mathfrak{g}_{\text{ad} \circ \rho})$ can be computed via Euler characteristic and Poincaré duality: $H^2(\Gamma, \mathfrak{g}) = (H^0(\Gamma, \mathfrak{g}^*))^*$.

Cup-products can be computed thanks to

Theorem

(W. Meyer, 1972). *Let (E, Ω) be a flat symplectic vector bundle over Σ . The quadratic form $Q(a) = \int_{\Sigma} \Omega(a \smile a)$ on $H^1(\Sigma, E)$ is nondegenerate of signature $4c_1(E, \Omega)$.*

Notation

$$\chi(\Gamma, G) = \text{Hom}(\Gamma, G)/G.$$

Remark

- ▶ *The dimension of $\chi(\Gamma, G)$ at points with trivial centralizers is $|\chi(\Sigma)|\dim(G)$.*
- ▶ *If the genus of Σ is large enough, non Zariski dense homomorphisms form a subset of $\chi(\Gamma, G)$ of dimension less than $|\chi(\Sigma)|\dim(G)$.*
- ▶ *Therefore it is sufficient to prove density of smooth points in neighborhoods of homomorphisms with nontrivial centralizers.*

Remark

Semisimple centralizers \mathfrak{z} are treated by writing explicit cohomology classes u such that $[u, u] = 0$, $[\cdot, \cdot]$ is a submersion at u and u has trivial stabilizer in \mathfrak{z} .

Indeed, Γ acts trivially on \mathfrak{z} . Pick suitable $u \in H^1(\Gamma) \otimes \mathfrak{z} = H^1(\Gamma, \mathfrak{z}) \subset H^1(\Gamma, \mathfrak{g})$.

This allows to jump from H to $G' = H \exp(\mathfrak{s})$, whose centralizer is split abelian.

If centralizer \mathfrak{z} is split abelian, $\mathfrak{g} \otimes \mathbb{C}$ splits under \mathfrak{z} into G' -invariant root spaces \mathfrak{g}_λ . $H^1(\Gamma, \mathfrak{g}) \otimes \mathbb{C}$ splits accordingly.

Lemma

$[\cdot, \cdot]$ vanishes on each $H^1(\Gamma, \mathfrak{g}_\lambda)$.

$H^1(\Gamma, \mathfrak{g}_\lambda)$ and $H^1(\Gamma, \mathfrak{g}_\mu)$ are orthogonal with respect to $[\cdot, \cdot]$ unless $\lambda + \mu = 0$.

On each $\mathfrak{g}_{\lambda, \mathbb{R}} = \mathfrak{g} \cap (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda})$, all ad_Z , $Z \in \mathfrak{z}$ are proportional. Therefore the corresponding alternating forms $(X, Y) \rightarrow Z \cdot [X, Y]$ are proportional to a single G' -invariant symplectic form Ω_λ . On $H^1(\Gamma, \mathfrak{g}_{\lambda, \mathbb{R}})$, all $Z \cdot [\cdot, \cdot]$ are proportional to the quadratic form $Q_\lambda(u, u) = \int_\Sigma \Omega_\lambda(u \smile u)$. Let $\rho_\lambda : \Gamma \rightarrow Sp(\mathfrak{g}_{\lambda, \mathbb{R}}, \Omega_\lambda)$ denote the composed symplectic linear representation, and E_λ the corresponding symplectic vectorbundle over Σ . Meyer's formula yields

Lemma

If $\lambda \neq 0$, Q_λ is nondegenerate and its index is equal to $4c_1(E_\lambda)$. Therefore

$$4|c_1(E_\lambda)| \leq \dim(H^1(\Gamma, \mathfrak{g}_\lambda)) = -\chi(\Sigma)\text{rank}(E_\lambda).$$

In particular, Q_λ is definite if and only if ρ_λ is a maximal representation.

If all inequalities are strict, pick in each $H^1(\Gamma, \mathfrak{g}_\lambda)$, $\lambda \neq 0$, a nonzero u_λ such that $Q_\lambda(u_\lambda) = 0$. Then $u = \sum u_\lambda$ satisfies $[u, u] = 0$, $[\cdot, \cdot]$ is a submersion at u , u has trivial stabilizer in \mathfrak{z} . Therefore u represents a smooth point of $\chi(\Gamma, G)$ nearby ρ' .

If $\dim(\mathfrak{z}) = 1$, then the scalar quadratic form $[\cdot, \cdot] = \sum_{\lambda > 0} \lambda Q_\lambda$ is definite if and only if

$$4 \left| \sum_{\lambda > 0} c_1(E_\lambda) \right| = -\chi(\Sigma) \sum_{\lambda > 0} \text{rank}(E_\lambda) = -\chi(\Sigma) \dim(\mathfrak{g}/(\mathfrak{z} \oplus \mathfrak{g}')),$$

i.e. if and only if the symplectic representation on $\mathfrak{g}/(\mathfrak{z} \oplus \mathfrak{g}')$ is maximal.

Remark

$\mathfrak{g}/(\mathfrak{z} \oplus \mathfrak{g}')$ is the tangent space to the adjoint orbit of \mathfrak{z} , equipped with the Kirillov-Kostant-Souriau symplectic structure.

The fact that G' has to be Hermitean of tube type follows from the following theorem.

Theorem

(Burger, Iozzi, Wienhard, 2007). *Let S be a semisimple Lie group whose symmetric space is Hermitean. Let $\rho : \Gamma \rightarrow S$ be a maximal representation of a surface group. Then ρ is tight. Its Zariski closure G' is reductive of Hermitian type. The embedding $G' \hookrightarrow S$ is tight. If S is of tube type, so is G' .*

In semisimple Lie groups $G = KAN$, maximal amenable subgroups are, up to conjugacy, of the form $G' = K'A'N'$ where $A' \subset A$, $N' \subset N$, the centralizer of A can be written MA , and $K' = K \cap M$. Furthermore, \mathfrak{a}' is generated by its intersection with the Weyl chamber \mathfrak{a}^+ of \mathfrak{a} .

When conjugated by a one parameter subgroup of $\exp(\mathfrak{a}^+)$, a homomorphism $\rho : \Gamma \rightarrow G'$ converges to a homomorphism $\rho' : \Gamma \rightarrow K'A'$. Thus density of Zariski dense homomorphisms in a neighborhood of ρ' implies the same property for ρ .

Up to finite index, one can deform ρ' so that its Zariski closure contains K' . Indeed, there is no rigidity in compact connected semisimple Lie groups. Write K'_0 as a central extension of a semisimple group. Since such central extensions are classified by discrete H^2 groups, small deformations lift from base to extension, so Zariski dense deformations exist in K'_0 as well. To pass from K' to K'_0 may require restricting to a finite index subgroup.

Once this is done, the centralizer of ρ' is A' , which is split abelian. Since one can use equivariant maps which factor through flats to define the bundles E_λ , their first Chern classes vanish, no maximal representation is encountered, and flexibility holds.