L^p-cohomology and curvature pinching

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Goal: explain how a tool from quantitative topology (L^p-cohomology) helps in solving a question in Riemannian geometry (optimal curvature pinching).

Definition

Let M be a Riemannian manifold. Let $p > 1$. L^p-cohomology *of M is the cohomology of the complex of L^p-differential forms on M whose exterior differentials are L^p as well,*

$$\begin{aligned} H^{k,p} &= \text{closed } k\text{-forms in } L^p / d((k-1)\text{-forms in } L^p), \\ R^{k,p} &= \text{closed } k\text{-forms in } L^p / \text{closure of } d((k-1)\text{-forms in } L^p), \\ T^{k,p} &= \text{closure of } d((k-1)\text{-forms in } L^p)/d((k-1)\text{-forms in } L^p). \end{aligned}$$

$R^{k,p}$ *is called the* reduced cohomology. $T^{k,p}$ *is called the* torsion.

Remark

If M is compact, L^p-cohomology equals cohomology.

Remark

For (uniformly) contractible spaces, L^p-cohomology is quasiisometry invariant. Wedge product $\alpha, \beta \mapsto \alpha \wedge \beta$ induces cup-product $[\alpha] \smile [\beta] : H^{k,p} \times H^{k',p} \to H^{k+k',p/2}$ in a quasiisometry invariant manner.

Definition

$H^{k,p}$ = closed k-forms in L^p/$d((k-1)-$forms in L^p),
$R^{k,p}$ = closed k-forms in L^p/ closure of $d((k-1)$-forms in L^p),
$T^{k,p}$ = closure of $d((k-1)$-forms in L^p)/$d((k-1)$-forms in L^p).

Example

The real line $\mathbb{R}$.

$H^{0,p} = 0$.

$R^{1,p} = 0$, since every function in $L^p(\mathbb{R})$ can be approximated in L^p with derivatives of compactly supported functions. Therefore $H^{1,p}$ is only torsion.

$T^{1,p}$ is non zero and thus infinite dimensional. Indeed, the 1-form $\frac{dt}{t}$ (cut off near the origin) is in L^p for all $p > 1$ but it is not the differential of a function in L^p.

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Example

The real hyperbolic plane $H^2_{\mathbb{R}}$.

Here $H^{0,p} = 0 = H^{2,p}$ for all p.
If $p = 2$, since the Laplacian on L^2 functions is bounded below, $T^{1,2} = 0$. Therefore

$$\begin{aligned} H^{1,2} &= R^{1,2} \\ &= \{L^2 \text{ harmonic 1-forms}\} \\ &= \{\text{harmonic functions } h \text{ on } H^2_{\mathbb{R}} \text{ with } \nabla h \in L^2\}/\mathbb{R}\,. \end{aligned}$$

Using conformal invariance, switch from hyperbolic metric to euclidean metric on the disk D.

$$\begin{aligned} H^{1,2} &= \{\text{harmonic functions } h \text{ on } D \text{ with } \nabla h \in L^2\}/\mathbb{R} \\ &= \{\text{Fourier series } \Sigma a_n e^{in\theta} \text{ with } a_0 = 0, \Sigma |n|\, |a_n|^2 < +\infty\}, \end{aligned}$$

which is Sobolev space $H^{1/2}(\mathbb{R}/2\pi\mathbb{Z})$ mod constants.

Proposition

Let M be a simply connected negatively curved Riemannian manifold. Functions u on M whose differential belongs to L^p have boundary values u_∞ on the visual boundary. The cohomology class $[du] \in H^{1,p}(M)$ vanishes if and only if u_∞ is constant.

Indeed, since volume in polar coordinates grows exponentially, and $L^p(e^t\,dt) \subset L^1(dt)$, the radial derivative belongs to L^1, so $u_\infty(\theta) = \lim_{t\to\infty} u(\theta, t)$ exists a.e. If $u_\infty = 0$, Sobolev inequality $\|u\|_{L^p} \leq \|du\|_{L^p}$ applies, and $[du] = 0$.

This suggests

Definition

(Bourdon-Pajot 2004). *For a negatively curved manifold M, define the* Royden algebra *$\mathcal{R}_p(M)$ as the space of L^∞ functions u on M such that $du \in L^p$, modulo $L^p \cap L^\infty$ functions.*

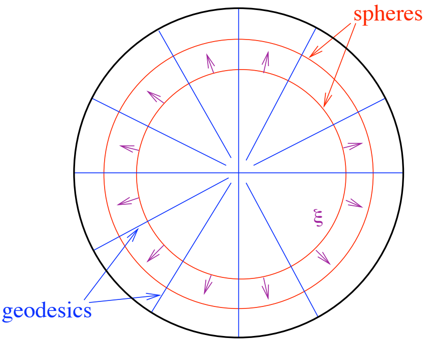
Then $\mathcal{R}_p(M)$ identifies with an algebra of functions on the visual boundary of M. If M is homogeneous, $\mathcal{R}_p(M)$ is a (possibly anisotropic) Besov space.

Step 1. *For q small, closed L^q 2-forms admit boundary values.*
Use the radial vectorfield $\xi = \frac{\partial}{\partial r}$ in polar coordinates and its flow ϕ_t, whose derivative is controlled by sectional curvature.

Use Poincaré's homotopy formula :
For α a closed 2-form in L^q,

$$\phi_t^* \alpha = \alpha + d\left(\int_0^t \phi_s^* \iota_\xi \alpha \, ds\right)$$

has a limit as $t \to +\infty$ under some curvature pinching assumption.



Step 2. *Boundary value determines cohomology class.*

Theorem

If $\dim(M) = 4$, M is δ-pinched and $q < 1 + 2\sqrt{-\delta}$, then a boundary value operator is defined, it injects $H^{2,q}$ into closed forms on the boundary. In particular, $T^{2,q} = 0$.

δ-pinched means sectional curvature $\in [-1, \delta]$.

Definition

Let M be a Riemannian manifold. Let $-1 \leq \delta < 0$. Say M is δ-pinched *if sectional curvature ranges between -1 and δ. Define the* optimal pinching $\delta(M)$ *of M as the least $\delta \geq -1$ such that M is quasiisometric to a δ-pinched simply connected Riemannian manifold.*

Example

Let $M = \mathbb{R}^4$ with metric $dt^2 + e^t dx^2 + e^t dy^2 + e^{2t} dz^2$. Then $\delta(M) = -\frac{1}{4}$.

Proof. M is $-\frac{1}{4}$-pinched. $T^{2,q}(M) \neq 0$ for $2 < q < 4$.

Theorem

If $\dim(M) = 4$, M is δ-pinched and $q < 1 + 2\sqrt{-\delta}$, then $T^{2,q} = 0$.

Remark

For (uniformly) contractible spaces, L^p-cohomology is quasiisometry invariant.

Definition

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Example

Let $M = \mathbb{R}^4$ with metric $dt^2 + e^t dx^2 + e^t dy^2 + e^{2t} dz^2$. Then $\delta(M) = -\frac{1}{4}$.

M is isometric to a left-invariant metric on the Lie group $\mathbb{R} \ltimes \mathbb{R}^3$.

Example

Let $M = \mathbb{R}^4$ with metric $dt^2 + e^t dx^2 + e^t dy^2 + e^{2t}(dz - xdy)^2$. Then M is $-\frac{1}{4}$-pinched, but $T^{2,q}(M) = 0$ for $2 \leq q < 4$.

M is isometric to a left-invariant metric on the Lie group $\mathbb{R} \ltimes$ *Heis*.
M is isometric to *complex hyperbolic plane* $H^2_{\mathbb{C}}$.

Complex hyperbolic space $H^m_{\mathbb{C}}$ is a metric on the ball in $\mathbb{C}^m$ which is invariant under all holomorphic automorphisms. In $H^m_{\mathbb{C}}$, every geodesic is contained in a complex line (a totally geodesic plane of curvature -1). Every plane orthogonal to a complex line integrates into a totally geodesic plane of curvature $-\frac{1}{4}$.
Spheres in $H^m_{\mathbb{C}}$ are homogeneous under conjugates of $U(m)$. Horospheres are homogeneous under Heisenberg group $Heis^{m-1}$.
There are quaternionic $H^m_{\mathbb{H}}$ and octonionic $H^2_{\mathbb{O}}$ siblings.
All are $-\frac{1}{4}$-pinched.
Together with ordinary hyperbolic space $H^n_{\mathbb{R}}$, these constitute the list of all negatively curved symmetric spaces (i.e. geodesic inversion is an isometry), E. Cartan, 1925.
And also the list of all noncompact Riemannian manifolds with 2-point transitive isometry groups, J. Tits, 1955.
All have compact quotients.

Question

Is it true that the optimal pinching of $H^m_{\mathbb{C}}$, $H^m_{\mathbb{H}}$ $(m \geq 2)$ and $H^2_{\mathbb{O}}$ is $-\frac{1}{4}$?

If one sticks to Riemannian manifolds admitting a compact quotient, the answer has been known since the 1980's, including the equality case.

Fact

Let N be a compact quotient of $H^m_{\mathbb{C}}$, $H^m_{\mathbb{H}}$ ($m \geq 2$) or $H^2_{\mathbb{O}}$. If a metric on N is $-\frac{1}{4}$-pinched, then it lifts to a symmetric metric.

This is due to

- M. Ville, 1984 for $H^2_{\mathbb{C}}$ (estimate on a characteristic class),
- L. Hernández-Lamoneda, 1991, and independently S.T. Yau and F. Zheng, 1991 for $H^m_{\mathbb{C}}$,
- N. Mok, Y.T. Siu and S.K. Yeung, 1993, and independently J. Jost and S.T. Yau, 1993 for other spaces (harmonic maps).

Theorem

The optimal pinching of $H^2_{\mathbb{C}}$ is equal to $-\frac{1}{4}$.

Scheme of proof

- *Recall Royden algebras $\mathcal{R}_p(M)$, $p > 1$, are quasi-isometry invariants.*
- *Given $u \in \mathcal{R}_p$, define a vectorsubspace $\mathcal{S}_p(u) \subset \mathcal{R}_p$, in a quasi-isometry invariant manner.*
- *If M is δ-pinched and $p < 2 + 4\sqrt{-\delta}$, then for every u, $\mathcal{S}_p(u)$ is a subalgebra of $\mathcal{R}_p$.*
- *If $M = H^2_{\mathbb{C}}$, for all $p \in (4, 8)$, there exists (locally) $u \in \mathcal{R}_p$ such that $\mathcal{S}_p(u)$ is not a subalgebra of $\mathcal{R}_p$.*

Definition

Let M be a simply connected negatively curved manifold, let $p > 4$, let $u \in \mathcal{R}_p(M)$. Define

$$\mathcal{S}_p(u) = \{v \in \mathcal{R}_p(M) \,|\, [dv] \smile [du] = 0 \in H^{2,p/2}(M)\}.$$

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Theorem

If M is 4-dimensional, δ-pinched and $p < 2 + 4\sqrt{-\delta}$, then for every u, $\mathcal{S}_p(u)$ is a subalgebra of $\mathcal{R}_p(M)$.

Proof. Let $v, v' \in \mathcal{S}_p(u)$. Then $[dv] \smile [du]$ vanishes if and only if its boundary value $dv_\infty \wedge du_\infty = 0$ a.e. Then $v'_\infty dv_\infty \wedge du_\infty + v_\infty dv'_\infty \wedge du_\infty = 0$ a.e., showing that $[d(vv')] \smile [du] = 0$, i.e. $vv' \in \mathcal{S}_p(u)$.

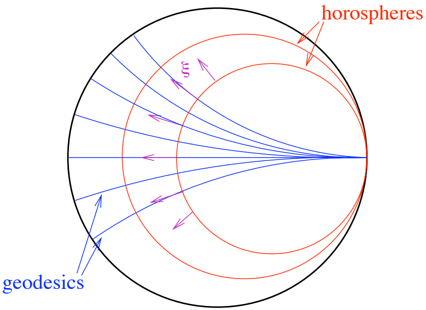
Now we compute $H^{2,q}(H^2_{\mathbb{C}})$ for $2 < q = p/2 < 4$.

Step 1. *Switch point of view. Use horospherical coordinates. View $H^2_{\mathbb{C}}$ as a product $\mathbb{H}^1 \times \mathbb{R}$. Prove a Künneth type theorem.*

For $q \notin \{4/3, 2, 4\}$, differential forms α on $H^2_{\mathbb{C}}$ split into components α_+ and α_+ which are contracted (resp. expanded) by ϕ_t. Then

$$h_t : \alpha \mapsto \int_0^t \phi_s^* \iota_\xi \alpha_+ \, ds - \int_{-t}^0 \phi_s^* \iota_\xi \alpha_- \, ds$$

converges as $t \to +\infty$ to a bounded operator h on L^q. $P = 1 - dh - hd$ retracts the L^q de Rham complex onto a complex $\mathcal{B}$ of differential forms on $\mathbb{H}^1$ with missing components and weakly regular coefficients.



Step 2. *If $2 < q < 4$, this complex is nonzero in degrees 1 and 2.*
$\mathcal{B}^1$ consists of 1-forms which are multiples of the left-invariant contact form τ on $\mathbb{H}^1$.

Step 3. *If $2 < q < 4$, vanishing of degree 2 cohomology classes is characterized by a differential equation.*
If $\alpha \in \mathcal{B}^2$ is a 2-form, then $\alpha \in d\mathcal{B}^1$ if and only if α satisfies the linear differential equation

$$\alpha = d(\frac{\tau \wedge \alpha}{\tau \wedge d\tau}\tau).$$

If $dv \wedge du$ is a solution, $d(v^2) \wedge du$ is not a solution, unless dv is proportional to du.

Failure of the subalgebra theorem for $H^2_{\mathbb{C}}$.
In coordinates (x, y, z) on $\mathbb{H}^1$, one can take (locally) $u = y$ and $v = x$. Then $dv \wedge du = -d\tau$ belongs to $d\mathcal{B}^1$, whereas $d(v^2) \wedge du$ does not. So for $4 < p = 2q < 8$, $\mathcal{S}_p(u)$ is not (locally) a subalgebra of $\mathcal{R}_p(H^2_{\mathbb{C}})$.

Other rank one symmetric spaces.
The comparison theorem works for all of them: in the definition of $\mathcal{S}_\kappa$, replace du by a cohomology class κ of degree 1, resp. 3 resp. 7. Steps 1 and 2 of the L^q computation in degree 2 resp. 4 resp. 8 are unchanged. It turns out that for all spaces but $H^2_{\mathbb{C}}$, the differential equation of Step 3 is a consequence of $d\alpha = 0$, so $\mathcal{S}_\kappa$ is an algebra in these cases.