Quasiisometric embeddings of the Heisenberg group into L^p, after Cheeger, Kleiner, Lee, Naor

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Pledge in favor of analysis on metric spaces

Bourdon and Pajot's quasiisometric rigidity of Fuchsian buildings

Let X be the ideal boundary of a Fuchsian building. They produce a differential for quasiMöbius selfhomeomorphisms f of X, show that it has to be a similarity, and conclude that f is conformal, i.e. preserves a crossratio.

Bonk and Kleiner's strategy for Cannon's conjecture.

Let G be a hyperbolic group whose ideal boundary is a topological 2-sphere.

- Find a Loewner metric in the conformal gauge by minimizing Assouad dimension over Ahlfors regular metrics.
- Show that such a metric is quasiMöbius to the round metric on the 2-sphere (done).

One has to investigate degenerating sequences of metrics in presence of a group action that dilates. Differentiability properties should show up.

Results

Theorem
(J. Cheeger, B. Kleiner, 2006). *The Heisenberg group* $(\mathbb{H}, d)$ *does not quasiisometricly embed in* L^1*.*

Theorem
(J. Lee, A. Naor, 2006). *There is an equivalent metric* d' *on the Heisenberg group such that* $(\mathbb{H}, d'^{1/2})$ *isometricly embeds into* L^2*.*

The proof of Cheeger and Kleiner's theorem seems to me relevant to the present conference, since it develops a new concept of differentiability on metric spaces.

Plan of lectures

1. *Motivation from geometric group theory*
2. *Motivation from computer science*
3. *Earlier non embedding results*
4. *Proof of non embedding to* L^1

Definition

A map $f : X \to Y$ between metric spaces is a uniform embedding *if there exist a constant C and a function ϕ that tends to $+\infty$ at ∞ such that for all x, $x' \in X$,*

$$\phi(d(x,x')) \leq d(f(x), f(x')) \leq C\, d(x,x').$$

Theorem

(G. Yu, 2000). *Groups with uniform embeddings into Hilbert spaces satisfy the coarse Baum-Connes conjecture.*

Theorem

(M. Gromov, 2003). *There exist finitely presented groups with no uniform embeddings into Hilbert spaces.*

This motivates the following quantitative measurement of uniform embeddings.

Definition

The largest possible ϕ in the above definition is called the compression *of f.*

Question

Given a metric space X and a class of metric spaces $\mathcal{Y}$, what is the largest possible compression over all maps f of X to elements of $\mathcal{Y}$?

Examples of Hilbert and ℓ^p embeddings

Theorem

(J. Bourgain, 1986, R. Tessera, 2006). *Let ϕ be the compression of some ℓ^p-embedding of a regular tree. Then* $\int_1^{+\infty} (\frac{\phi(t)}{t})^{\max\{2,p\}} \frac{dt}{t} < +\infty$.

Conversely, every increasing function satisfying $\int_1^{+\infty} (\frac{\phi(t)}{t})^p \frac{dt}{t} < +\infty$ *is bounded from above by the compression of some ℓ^p embedding of an arbitrary space in the following list : regular trees, cocompact lattices in connected Lie groups, hyperbolic groups, wreath products $F \wr \mathbb{Z}$ for finite F.*

Example

(Folklore). *Trees embed into ℓ^p with compression $\phi(t) \geq t^{1/p}$.*

Indeed, fix an origin $o \in T^0$ and isometricly embed the tree T to $\ell^1(T^0)$ by mapping a vertex x to the characteristic function of the geodesic from o to x. Then map $\ell^1(T^0)$ to $\ell^p(T^0)$ in the obvious manner. This has compression $t^{1/p}$.

Question

Are all useful uniform ℓ^p embeddings obtained in this manner ?

For a while, it might have seemed so. Yu's *Property A*, a sufficient condition for a metric space to embed uniformly into L^2, works this way.

For several basic problems, the best known algorithm relies on efficiently embedding some finite metric space simultaneously in Euclidean space and in L^1. Which shall give an example soon, `Sparsest Cut`.

The best known approximate algorithm for `Sparsest Cut`, `SDP` (to be explained next), yields in the worst case, for an n-vertex graph, an answer which is wrong by a factor equal to L_n defined as follows.

Notation

Consider all n-point metric spaces (X, d) *such that* $(X, d^{1/2})$ *embeds isometricly into Euclidean space. Let* L_n *be the smallest L such that every such metric space admits an L-Lipschitz and distance nondecreasing embedding into* L^1*.*

This has lead M.X. Goemans (in 1997) and N. Linial (in 2002) to ask wether L_n could be bounded independantly of n.

In 2005, S. Khot and N. Vishnoi provided a counterexample.

It turns out that balls in the Cayley graph of the integral Heisenberg group provide nicer counterexamples.

Sparsest Cut

Problem

Sparsest Cut, i.e. computing the Cheeger constant of a finite graph.

A *cut* in a weighted graph G is a partition of vertices as $G^0 = S \cup \bar{S}$.

$$\Phi(S) = \frac{\#\partial S}{\#S\,\#\bar{S}},$$

and define the *Cheeger constant* $\Phi^*(G) = \min_{\emptyset \subsetneq S \subsetneq G^0} \Phi(S)$.

Computing Φ^* is NP-hard. Computing an approximate solution S is a basic step in several useful algorithms.

Observe that

$$\Phi(S) = \frac{\sum_{edges\ uv} m(uv)|1_S(u) - 1_S(v)|}{\sum_u \sum_v |1_S(u) - 1_S(v)|}.$$

Let $d(u, v) = |1_S(u) - 1_S(v)|$. This is a semi-metric on G^0, induced by a map to the 2-point set $\{0, 1\}$. The convex hull of such semi-metrics consists exactly of all semi-metrics induced by maps to some L^1-space. Therefore

$$\begin{aligned} \Phi^* &= \min_{d \text{ embeddable into } L^1} \frac{\sum_{edges\ uv} m(uv)d(u,v)}{\sum_u \sum_v d(u,v)} \\ &= \min\{ \sum_{edges\ uv} m(uv)d(u,v) \mid d \text{ embeddable into } L^1, \sum_u \sum_v d(u,v) = 1\}. \end{aligned}$$

Unfortunately, deciding wether a metric on a finite set is embeddable into L^1 is NP-complete.

Relax the problem by removing the L^1-embeddability condition. This leads to a linear programming problem, denoted by LP, for which there exist efficient algorithms. Let Φ^{LP} denote the infimum.

Theorem

(J. Bourgain, 1985). *Every n-point metric space embeds into L^2 (and thus into L^1) with distorsion at most $O(\log(n))$.*

Corollary

(N. Linial, E. London, Y. Rabinovich, 1995). *Bourgain's theorem can be made algorithmic, thus*

$$\Phi^{LP} \leq \Phi^* \leq C \log(n)\Phi^{LP},$$

showing that LP computes efficiently Φ^* up to a $\log(n)$ factor.

Proof. The L^1-embeddable metric d' which is $O(\log(n))$ close to the solution d of LP satisfies

$$\Phi^* \leq \Phi(d') \leq C \log(n)\Phi^* = C \log(n)\Phi^{LP}.$$

Remark

Bourgain's theorem is sharp, as shown by bounded degree expanders.

Observe that

$$\Phi(S) = \frac{\sum_{edges\ uv} m(uv)|1_S(u) - 1_S(v)|^2}{\sum_u \sum_v |1_S(u) - 1_S(v)|^2}.$$

Relax the problem by allowing real valued functions $x : G^0 \to \mathbb{R}$ instead of $\{0,1\}$-valued ones, keeping the constraint, satisfied by characteristic functions, that for all u, v and $w \in G^0$,

$$|x(u) - x(v)|^2 \leq |x(u) - x(w)|^2 + |x(w) - x(v)|^2.$$

This leads to a semidefinite programming problem, denoted by SDP. Let $\Phi^{SDP} = \Phi(x)$ be the minimum. There exist again efficient algorithms for computing Φ^{SDP}. Set $d(u,v) = |x(u) - x(v)|^2$. This is a distance on G^0, and $d^{1/2}$ is induced by a map to Euclidean space. Therefore

$$\Phi^{SDP} = \min\{\sum_{edges\ uv} m(uv)d(u,v) \,|\, d \text{ distance}, \sqrt{d} \text{ embeddable into } L^2, \sum_u \sum_v d(u,v) = 1\}.$$

Clearly,

$$\Phi^{SDP} \leq \Phi^* \leq L_n \Phi^{SDP},$$

showing that Φ^* can be efficiently computed up to a L_n factor.

Estimating L_n

Theorem

(S. Arora, J. Lee, A. Naor, 2005). *Let (X, d) be an n-point metric space. Assume that $(X, d^{1/2})$ embeds isometricly into L^2. Then (X, d) also embeds into L^2 with distorsion $O(\sqrt{\log(n)}\log(\log(n)))$. In other words, $L_n = O(\sqrt{\log(n)}\log(\log(n)))$.*

Remark

This is nearly sharp, since the vertex set of the ℓ^1 n-dimensional cube cannot embed into L^2 with distorsion $< \sqrt{n}$ (Enflo, 1969).

Corollary

$L_n = O(\sqrt{\log(n)}\log(\log(n)))$.

Indeed, L^2 embeds isometricly into L^1.

Remark

Non-embeddability of Heisenberg group implies some lower bound on L_n. Cheeger, Kleiner and Naor claim that it can be made effective. Conjecturally, $L_n = \Omega(\log(\log(n))^\delta)$ for some $\delta > 0$.

Conclusion

The SDP approach gives the best known solution for the general `Sparsest Cut`*. In case all weights are equal,* S. Arora, E. Hazan, S. Kale (2004) *give a different polynomial algorithm which computes Φ^* up to an $O(\sqrt{\log(n)})$ factor.*

Previously known embedding facts about Heisenberg group and other PI spaces

Theorem

(Semmes, 1996). *$\mathbb{H}$ has no quasiisometric embeddings into finite dimensional Banach spaces.*
(Pauls, 2001). *$\mathbb{H}$ has no quasiisometric embeddings into Hilbert spaces, or more generally, $CAT(0)$ spaces.*
(Cheeger-Kleiner, 2006). *$\mathbb{H}$ has no quasiisometric embeddings into Banach spaces which have the Radon-Nikodym property (this includes separable dual spaces).*

Definition

(Heinonen-Koskela, 1996). *Say a metric measure space is* PI *if it is doubling and satisfies a $(1, p)$-Poincaré inequality*

$$\fint_B |f - \fint_B f| \leq \text{const. diameter}(B) \left(\fint_{2B} |\nabla f|^p \right)^{1/p},$$

for all upper gradients $|\nabla f|$ of f.

Theorem

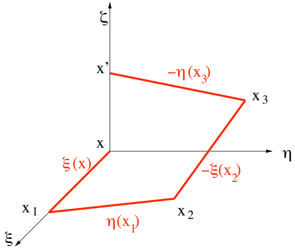
(Cheeger-Kleiner, 2006). *PI spaces whose tangent cones have Hausdorff dimension greater that their topological dimension have no biLipschitz embeddings into Banach spaces which have good finite dimensional approximations. This includes ℓ^1. However, among such spaces, there are examples which embed into L^1.*

Facts about Heisenberg group

Definition

Heisenberg group $\mathbb{H}$ *is the 3-dimensional Lie group with Lie algebra spanned by* ξ, η *and* ζ *with* $[\xi, \eta] = \zeta$. *The left-invariant vectorfields* ξ *and* η *span a plane field* H, Carnot distance $d(x, x')$ *is* inf *of length of curves tangent to* H *joining* x *to* x'. Dilation δ_t *is automorphism induced by* $\delta_t(\xi) = t\xi$, $\delta_t(\eta) = t\eta$, $\delta_t(\zeta) = t^2\zeta$. *It multiplies Carnot distances by* t.

Finiteness of Carnot distance follows from picture:



Remark

1. $d(x, x\exp(t^2\zeta)) = td(1, \exp(\zeta)) = \mathrm{const.}\, t$.
2. $\mathrm{volume}B(x,t) = t^4 \mathrm{volume}B(x,1) = \mathrm{const.}\, t^4$, *thus Hausdorff dimension is 4.*

1. *Horizontal derivatives exist almost everywhere.*

   This is what the Radon-Nikodym property is good for.

2. *They are approximately continuous at a.e. point.*

   This is general for doubling spaces.

3. *At such a point x, $d(f(x), f(x')) = o(d(x, x'))$ if x' belongs to the vertical line through x.*

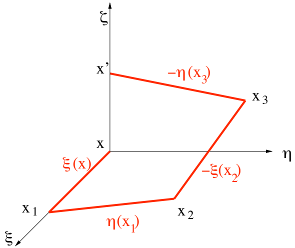
Let $x' = x \exp(t^2\zeta)$, so that $d(x, x') \sim t$. Join x to x' along integral curves of ξ, η, $-\xi$, $-\eta$, with endpoints $x = x_0$, x_1, x_2, x_3, $x_4 = x'$. Then

$$f(x_1) - f(x) \sim t\xi f(x), \quad f(x_3) - f(x_2) \sim -t\xi f(x),$$

thus

$$f(x_1) - f(x) + f(x_3) - f(x_2) = o(t).$$

Idem $f(x_2) - f(x_1) + f(x') - f(x_3) = o(t)$. Summing up yields $f(x') - f(x) = o(t)$.



Remark

L^1 is not Radon-Nikodym.

Indeed, $t \mapsto 1_{[0,t]}$, $\mathbb{R}_+ \to L^1(\mathbb{R}_+)$, is isometric, but nowhere differentiable.

Scheme of proof

of L^1-non embeddability.

1. To a map $f : X \to L^1(Y, \nu)$, there corresponds a canonical measured family of subsets $S \subset X$, generalizing level sets.
2. f has bounded variation if and only if almost every S has finite perimeter.
3. *(Franchi, Serapioni, Serra-Cassano, 2001).* If $S \subset \mathbb{H}$ has finite perimeter, then at almost every boundary point, δ_t-blown up copies of S converge to vertical half-spaces.
4. Near good points, at small scales, f nearly factorizes via $\mathbb{H}/[\mathbb{H}, \mathbb{H}]$, preventing f from being biLipschitz.

Cut semi-metrics

Definition

An elementary cut semi-metric *on X is $d_S(x,x') = |1_S(x) - 1_S(x')|$ for some cut $S \subset X$. A* cut semi-metric *is a sum of elementary cut semi-metrics, i.e.*

$$d(x,x') = \int_{\{S\}} d_S(x,x')\, d\mu_d(S)$$

for some positive measure μ_d on the set of cuts.

Lemma

(P. Assouad, 1977). *A semi-metric d on X is induced by a map $f : X \to L^1(Y,\nu)$ if and only if it is a cut semi-metric.*

Corollary

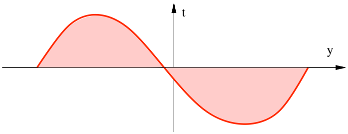
L^1-embeddable semi-metrics on X are closed under pointwise convergence.

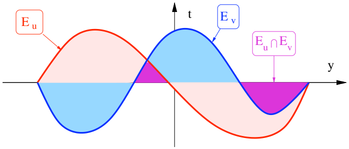
In particular, a quasiisometric embedding of $\mathbb{H}$ to L^1 gives rise to biLipschitz embeddings.

Proof of Assouad's lemma, $\Leftarrow$. Assume d is a cut semi-metric. Fix $o \in X$. Let $S(x)$ be the set of cuts separating x from o. Then $x \mapsto 1_{S(x)}$ embeds (X,d) isometricly in $L^1(\{S\},\mu_d)$.

Definition

The epigraph *of a function* $u : Y \to \mathbb{R}$ *is* $E_u = \{(y,t) \in Y \times \mathbb{R} \,|\, t^{-1}u(y) > 1\}$.





Lemma

If $u, v \in L^1(Y,\nu)$, $\|u-v\|_{L^1} = (\nu \otimes dt)(E_u \Delta E_v)$.

Proof of Assouad's lemma, $\Rightarrow$. Let $f : X \to L^1(Y,\nu)$. To each $(y,t) \in Y \times \mathbb{R}$, there corresponds a cut $S(y,t) = \{x \in X \,|\, (y,t) \in E_{f(x)}\} = \{x \in X \,|\, t^{-1}f(x)(y) > 1\}$. Let $\mu_f = S_*(\nu \otimes dt)$. Then for all x, $x' \in X$,

$$\begin{aligned} d(f(x),f(x')) &= \|f(x)-f(x')\|_{L^1} = (\nu \otimes dt)(E_{f(x)} \Delta E_{f(x')}) \\ &= \int_{Y \times \mathbb{R}} |1_{E_{f(x)}}(y,t) - 1_{E_{f(x')}}(y,t)|\, d\nu(y)\, dt \\ &= \int_{Y \times \mathbb{R}} |1_{S(y,t)}(x) - 1_{S(y,t)}(x')|\, d\nu(y)\, dt \\ &= \int_{\{S\}} |1_S(x) - 1_S(x')|\, d\mu_f(S). \end{aligned}$$

Definition

Let X be a PI space, Y a metric space. Say an L^1 map $f : X \to Y$ has bounded variation *if it is the L^1-limit of a sequence of locally Lipschitz functions h_i with $\mathrm{Lip} h_i$ bounded in L^1. The* inf *of limits of $\int \mathrm{Lip} h_i$ over all L^1-approximating sequences h_i is called the* variation *of f.*

Say a subset $S \subset X$ has finite perimeter *if its characteristic function 1_S has bounded variation. Its* perimeter *equals the variation of 1_S.*

For real valued BV functions h, the *coarea formula* reads

$$\mathrm{variation}(h) = \int_{\mathbb{R}} \mathrm{perimeter}(\{h > t\})\, dt.$$

This extends to maps $X \to L^1(Y, \nu)$ as follows.

Theorem

(Cheeger-Kleiner). *Let X be PI. Let $f \in L^1(X, L^1(Y, \nu))$. Then f has bounded variation if and only if μ_f-every cut has finite perimeter. Furthermore*

$$\int_{\{S\}} \mathrm{perimeter}(S)\, d\mu_f(S) = \int_Y \mathrm{variation}(f(\cdot, y))\, d\nu(y) \leq \mathrm{const.}\ \mathrm{variation}(f).$$

Bad points

From now on, $X = \mathbb{R}^n$ or $X = \mathbb{H}$. A *half-space* in $\mathbb{H}$ is the inverse image of $\mathbb{R}_+$ by a group homomorphism $\mathbb{H} \to \mathbb{R}$ (it has to be vertical).

Notation

For $S \subset X$, $x \in \partial S$, let $\alpha(S, x, r) = \min_{H \text{ half-space thru } x} \fint_{B(x,r)} |1_S - 1_H|$. Say a point x is (ϵ, R)-bad for S if there exists $r < R$ such that $\alpha(S, x, r) > \epsilon$. Denote the set of (ϵ, R)-bad points for S by $Bad_{\epsilon,R}(S)$.

For each S, the perimeter of S is a measure $\|\partial S\|$ on X, supported on the boundary of S. One can restrict it to bad points. Given a measure μ on finite perimeter cuts, the *total bad perimeter measure* $\lambda_{\mu,\epsilon,R}$ is defined on continuous functions u by

$$\int_X u(x)\, d\lambda_{\mu,\epsilon,R}(x) = \int_{\{S\}} \int_{Bad_{\epsilon,R}(S)} u(x)\, d\|\partial S\|(x)\, d\mu(S).$$

Theorem

(Franchi, Serapioni, Serra-Cassano, 2001). *Let S be a set of finite perimeter in $\mathbb{H}$. The mass of the perimeter measure restricted to $Bad_{\epsilon,R}(S)$ tends to 0 as R tends to 0.*

Corollary

Let μ be a measure on finite perimeter sets. The mass of $\lambda_{\mu,\epsilon,R}$ tends to 0 as R tends to 0.

Approximating a cut measure by a half-space cut measure

Let μ be a measure on finite perimeter cuts, let d_μ be the corresponding cut semi-metric. Given $x \in X$ and $r > 0$, let $\delta^*_{x,r} d_\mu$ denote the distance composed with the dilation by r around point x.

Theorem

(Cheeger-Kleiner). *For a.e. $x \in X$, there exist measures μ_r supported on half-space cuts such that $\|\frac{1}{r}\delta^*_{x,r} d_\mu - d_{\mu_r}\|$ tends to 0 in $L^1(X \times X)$.*

Proof. Differentiating the total bad perimeter measure yields near a.e. point x a set of almost full measure in $B(x, r)$ of points where μ-most cuts are close to half-spaces measurewise. For each such cut S, select the closest half-space $HS(S)$, let $\mu_r = (\delta_{x,r})_* \mu$.

Proof of nonembedding theorem. If $f : \mathbb{H} \to L^1$ is biLipschitz, with cut-measure μ, $d_\mu(x', x'') = d(f(x'), f(x'')) \geq \mathrm{const}.d(x', x'')$, thus

$$\frac{1}{r}\delta^*_{x,r} d_\mu(x', x'') \geq \mathrm{const}.d(x', x'').$$

On the other hand, a cut semi-metric concentrated on half-spaces satisfies

$$d_{\mu_r}(x', x'') = d_{\mu_r}(x' \mod Z(\mathbb{H}), x'' \mod Z(\mathbb{H})).$$

Such semi-metrics cannot be L^1-close.

We sketch the proof of Franchi, Serapioni and Serra-Cassano's rectifiability theorem. Again, $X = \mathbb{R}^n$ or $\mathbb{H}$. Fix a finite perimeter set $S \subset X$. To define a unit normal along the boundary, the divergence formula is used. In $\mathbb{H}$, divergence is defined as follows : for $\phi = \phi_\xi \xi + \phi_\eta \eta$, $div(\phi) = \xi\phi_\xi + \eta\phi_\eta$.

Lemma

(De Giorgi, 1954). *There exists a measurable horizontal unit vectorfield ν such that for all smooth compactly supported horizontal vectorfields ϕ,*

$$-\int_S div(\phi) = \int_X \langle \nu, \phi \rangle d\|\partial S\|.$$

Proof. For Lipschitz functions h, divergence formula holds and implies

$$|\int_S h\, div(\phi)| \leq \|\phi\|_{L^\infty} \int_X \mathrm{Lip} h,$$

showing that $\phi \mapsto -\int_S div(\phi)$ is a (vector valued) Radon measure with mass no greater than variation(h). This fact extends to arbitrary functions of bounded variation like 1_S. In this case, the Radon measure is absolutely continuous with respect to perimeter measure, with ν as a density (Riesz).

Lemma

(Ambrosio, 2001). *At $\|\partial S\|$-a.e. point x, the $\|\partial S\|$-mass of a ball $B(x,r)$ is $\sim r^3$.*

According to Federer, this implies that averages of the unit normals converge.

By compactness, dilates $\delta_{x,1/r}(S)$ subconverge to some locally finite perimeter set E as $r \to 0$. Unit normals also subconverge $\|\partial S\|$-a.e., showing that the horizontal unit normal of E is a.e. constant.

Lemma

A set E with locally finite perimeter whose horizontal unit normal of E is a.e. equal to ξ is a half-space

Indeed, travelling from the origin both sides along η-orbits and positively along ξ-orbits can reach exactly all points of a (vertical) half-space. This completes the proof.

More is known.

Theorem

(Franchi, Serapioni and Serra-Cassano, 2001). *Finite perimeter sets in $\mathbb{H}$ are, up to sets of vanishing 3-dimensional Hausdorff measure, countable unions of compact pieces of surfaces defined by C^1 equations $g = 0$ where the horizontal gradient of g does not vanish.*